Belief revision in a temporal framework

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Abstract

We study a branching-time temporal logic of belief revision where the interaction of belief and information is modeled explicitly. The logic is based on three modal operators: a belief operator, an information operator and a next-time operator. We consider three logics of increasing strength. The first captures the most basic notion of minimal belief revision. The second characterizes the qualitative content of Bayes' rule. The third is the logic proposed in [8], where some aspects of its relationship with the AGM theory of belief revision were investigated. We further explore the relationship to AGM with the help of semantic structures that have been used in the rational choice literature. Further strengthening of the logic are also investigated.

1 Introduction

Since the foundational work of Alchourrón, Gärdenfors and Makinson [1], the theory of belief revision has been a very active area of research. Recently several authors have been attempting to re-cast belief revision within a modal framework. Pioneering work in this new area was done by Segerberg ([32], [33]) in the context of dynamic doxastic logic, Board [5] in the context of multi-agent doxastic logic and van Benthem [3] in the context of dynamic epistemic logic. Much progress has been made both in dynamic epistemic logic (see, for example, [2], [15], [16] and the recent survey in [17]) as well as in dynamic doxastic logic (see [27]). Another very active area of research has been iterated belief revision (see, for example, [11], [14], [29], [30]).

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This paper joins the recent attempts to establish a qualitative view of belief revision in a modal framework, by continuing the study of belief revision within a temporal framework that was first proposed in [8]. Since belief revision deals with the interaction of belief and information over time, branching-time temporal logic seems a natural setting for a theory of belief change. On the semantic side we consider branching-time frames with the addition of a belief relation and an information relation for every instant t. We thus extend to a temporal setting the standard Kripke [26] semantics used in the theory of static belief pioneered by Hintikka [23]. On the syntactic side we consider a propositional language with a next-time operator, a belief operator and an information operator. Three logics of increasing strength are studied. The first is a logic that expresses the most basic notion of minimal belief revision. The second captures the qualitative content of Bayes' rule, thus generalizing the two-date result of [6] to a branching-time framework. The third logic is the logic proposed in [8], where some aspects of the relationship between that logic and the AGM theory of belief revision were investigated. In this paper we provide frame characterization results for all three logics and we further investigate the relationship between the strongest of the three logics and the notion of AGM belief revision functions. We do so with the help of semantic structures that have been used in the rational choice literature. We call these structures one-stage revision frames and show that there is a correspondence between the set of one-stage revision frames and the set of AGM belief revision functions. Further strengthening of the logic are also investigated.

While the structures that we consider accommodate iterated belief revision in a natural way, we do not attempt to axiomatize iterated revision in this paper. First steps in this direction have been taken in [35].

We provide frame characterization results and do not address the issue of completeness of our logics. Completeness of the basic logic with respect to a more general class of temporal belief revision frames (where the set of state is allowed to change over time) is proved in [9]; that result has been extended in [35] to the set of frames considered in this paper.

2 Temporal belief revision frames

We consider the semantic frames introduced in [8], which are branching-time structures with the addition of a belief relation and an information relation for every instant t.

A *next-time branching frame* is a pair $\langle T, \rightarrowtail \rangle$ where T is a non-empty, countable set of instants and $\rightarrowtail$ is a binary relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,

(1)	backward uniqueness	if $t_1 \rightarrowtail t_3$ and $t_2 \rightarrowtail t_3$ then $t_1 = t_2$
(2)	acyclicity	if $\langle t_1, ..., t_n \rangle$ is a sequence with $t_i \rightarrowtail t_{i+1}$ for every $i = 1, ..., n-1$, then $t_n \neq t_1$.

The interpretation of $t_1 \rightarrowtail t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2: every instant has at most a unique immediate predecessor but can have several immediate successors.

Definition 1 *A* temporal belief revision frame *is a tuple* $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$ *where* $\langle T, \rightarrowtail\rangle$ *is a next-time branching frame,* Ω *is a non-empty set of states (or possible worlds) and, for every* $t \in T$, $\mathcal{B}_t$ *and* $\mathcal{I}_t$ *are binary relations on* Ω*.*

The interpretation of $\omega\mathcal{B}_t\omega'$ is that at state ω and time t the individual considers state ω' possible (an alternative expression is "ω' is a doxastic alternative to ω at time t"), while the interpretation of $\omega\mathcal{I}_t\omega'$ is that at state ω and time t, according to the information received, it is possible that the true state is ω'. We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{B}_t\omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega : \omega\mathcal{I}_t\omega'\}.$$

Figure 1 illustrates a temporal belief revision frame. For simplicity, in all the figures we assume that the information relations $\mathcal{I}_t$ are equivalence relations (whose equivalence classes are denoted by rectangles) and the belief relations $\mathcal{B}_t$ are serial, transitive and euclidean[1] (we represent this fact by enclosing states in ovals and, within an equivalence class for $\mathcal{I}_t$, we have that - for every two states ω and ω' - $\omega' \in \mathcal{B}_t(\omega)$ if and only if ω' belongs to an oval).[2] For example, in Figure 1 we have that $\mathcal{I}_{t_1}(\gamma) = \{\alpha, \beta, \gamma\}$ and $\mathcal{B}_{t_1}(\gamma) = \{\alpha, \beta\}$.

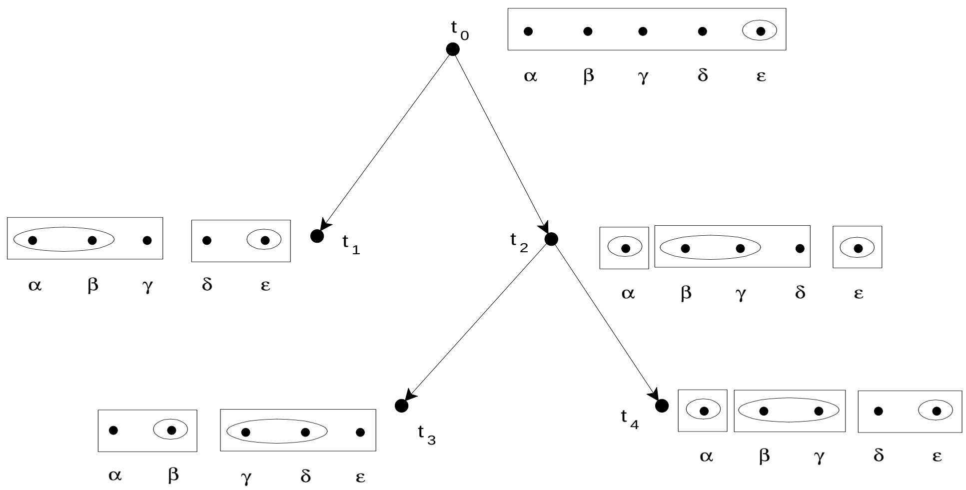

Figure 1

[1] $\mathcal{B}_t$ is serial if, $\forall\omega \in \Omega$, $\mathcal{B}_t(\omega) \neq \varnothing$; it is transitive if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega)$; it is euclidean if $\omega' \in \mathcal{B}_t(\omega)$ implies that $\mathcal{B}_t(\omega) \subseteq \mathcal{B}_t(\omega')$.

[2] Note, however, that our results do *not* require $\mathcal{I}_t$ to be an equivalence relation, nor do they require $\mathcal{B}_t$ to be serial, transitive and euclidean.

Temporal belief revision frames can be used to describe either a situation where the objective facts describing the world do not change − so that only the beliefs of the agent change over time − or a situation where both the facts and the doxastic state of the agent change. In the literature the first situation is called belief revision, while the latter is called belief update (see [25]). We shall focus on belief revision.

On the syntactic side we consider a propositional language with five modal operators: the next-time operator $\bigcirc$ and its inverse $\bigcirc^{-1}$, the belief operator B, the information operator I and the "all state" operator A. The intended interpretation is as follows:

$\bigcirc\phi$:	"at every next instant it will be the case that ϕ"
$\bigcirc^{-1}\phi$:	"at every previous instant it was the case that ϕ"
$B\phi$:	"the agent believes that ϕ"
$I\phi$:	"the agent is informed that ϕ"
$A\phi$:	"it is true at every state that ϕ".

The "all state" operator A is needed in order to capture the non-normality of the information operator I (see below). For a thorough discussion of the "all state" operator see Goranko and Passy [20].

Note that, while the other operators apply to arbitrary formulas, *we restrict the information operator to apply to Boolean formulas only*, that is, to formulas that do not contain modal operators. Boolean formulas are defined recursively as follows: (1) every atomic proposition is a Boolean formula, and (2) if ϕ and ψ are Boolean formulas then so are $\neg\phi$ and $(\phi \vee \psi)$. The set of Boolean formulas is denoted by Φ^B. Boolean formulas represent facts and, therefore, we restrict information to be about facts.[3]

Given a temporal belief revision frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$ one obtains a *model based on it* by adding a function $V : S \rightarrow 2^{\Omega}$ (where S is the set of atomic propositions and 2^{Ω} denotes the set of subsets of Ω) that associates with every atomic proposition p the set of states at which p is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition p depends only on the state and not on the time.[4] Given a model, a state ω, an instant t and a formula ϕ, we write $(\omega, t) \models \phi$ to denote that ϕ is true at state ω and time t. Let $\|\phi\|$ denote the truth set of ϕ, that is, $\|\phi\| = \{(\omega, t) \in \Omega \times T : (\omega, t) \models \phi\}$ and let $\lceil\phi\rceil_t \subseteq \Omega$ denote the set of states at which ϕ is true at time t, that is, $\lceil\phi\rceil_t = \{\omega \in \Omega : (\omega, t) \models \phi\}$. Truth of an arbitrary formula at a pair (ω, t) is defined recursively as follows:

[3] Zvesper [35] has recently proposed a version of our logic where the restriction to Boolean formulas is dropped.

[4] Belief update would require a valuation to be defined as a function $V : S \rightarrow 2^{\Omega\times T}$.

if $p \in S$,	$(\omega, t) \models p$ if and only if $\omega \in V(p)$;
$(\omega, t) \models \neg\phi$	if and only if $(\omega, t) \not\models \phi$;
$(\omega, t) \models \phi \vee \psi$	if and only if either $(\omega, t) \models \phi$ or $(\omega, t) \models \psi$ (or both);
$(\omega, t) \models \bigcirc\phi$	if and only if $(\omega, t') \models \phi$ for every t' such that $t \rightarrowtail t'$;
$(\omega, t) \models \bigcirc^{-1}\phi$	if and only if $(\omega, t'') \models \phi$ for every t'' such that $t'' \rightarrowtail t$;
$(\omega, t) \models B\phi$	if and only if $\mathcal{B}_t(\omega) \subseteq \lceil\phi\rceil_t$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{B}_t(\omega)$;
$(\omega, t) \models I\phi$	if and only if $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$, that is, if (1) $(\omega', t) \models \phi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \phi$ then $\omega' \in \mathcal{I}_t(\omega)$;
$(\omega, t) \models A\phi$	if and only if $\lceil\phi\rceil_t = \Omega$, that is, if $(\omega', t) \models \phi$ for all $\omega' \in \Omega$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq \lceil\phi\rceil_t$ we require equality: $\mathcal{I}_t(\omega) = \lceil\phi\rceil_t$. Thus our information operator is formally similar to the "all and only" operator introduced in Humberstone [24] and the "only knowing" operator studied in Levesque (see [28]), although the interpretation is different. It is also similar to the "assumption" operator used in Brandenburger and Keisler [12].

Remark 2 *The truth value of a Boolean formula does not change over time: it is only a function of the state. That is, fix an arbitrary model and suppose that* $(\omega, t) \models \phi$ *where* $\phi \in \Phi^B$*; then, for every* $t' \in T$, $(\omega, t') \models \phi$ *(for a proof see [8], p. 148).*

A formula ϕ is *valid in a model* if $\|\phi\| = \Omega \times T$, that is, if ϕ is true at every state-instant pair (ω, t). A formula ϕ is *valid in a frame* if it is valid in every model based on it.

3 The basic logic

The formal language is built in the usual way (see [4]) from a countable set of atomic propositions, the connectives $\neg$ and $\vee$ (from which the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators $\bigcirc$, $\bigcirc^{-1}$, B, I and A, with the restriction that $I\phi$ is a well-formed formula if and only if ϕ is a Boolean formula. Let $\Diamond\phi \overset{def}{=} \neg\bigcirc\neg\phi$, and $\Diamond^{-1}\phi \overset{def}{=} \neg\bigcirc^{-1}\neg\phi$. Thus the interpretation of $\Diamond\phi$ is "at *some* next instant it will be the case that ϕ " while the interpretation of $\Diamond^{-1}\phi$ is "at some immediately preceding instant it was the case that ϕ".

We denote by $\mathbb{L}_0$ the basic logic defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.

2. Axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A[5]: for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$

$$(\Box\phi \wedge \Box(\phi \to \psi)) \to \Box\psi \quad \text{(K)}$$

3. Temporal axioms relating $\bigcirc$ and $\bigcirc^{-1}$:

$$\begin{array}{ll} \phi \to \bigcirc\Diamond^{-1}\phi & (\mathrm{O}_1) \\ \phi \to \bigcirc^{-1}\Diamond\phi & (\mathrm{O}_2) \end{array}$$

4. Backward Uniqueness axiom:

$$\Diamond^{-1}\phi \to \bigcirc^{-1}\phi \quad \text{(BU)}$$

5. S5 axioms for A:

$$\begin{array}{ll} A\phi \to \phi & (\mathrm{T}_A) \\ \neg A\phi \to A\neg A\phi & (5_A) \end{array}$$

6. Inclusion axiom for B (note the absence of an analogous axiom for I):

$$A\phi \to B\phi \quad (\mathrm{Incl}_B)$$

7. Axioms to capture the non-standard semantics for I: for $\phi, \psi \in \Phi^B$ (recall that Φ^B denotes the set of Boolean formulas),

$$\begin{array}{ll} (I\phi \wedge I\psi) \to A(\phi \leftrightarrow \psi) & (\mathrm{I}_1) \\ A(\phi \leftrightarrow \psi) \to (I\phi \leftrightarrow I\psi) & (\mathrm{I}_2) \end{array}$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\phi, \quad \phi \to \psi}{\psi}$ (MP)

2. Necessitation for A, $\bigcirc$ and $\bigcirc^{-1}$: for every $\Box \in \{A, \bigcirc, \bigcirc^{-1}\}$, $\frac{\phi}{\Box\phi}$ (Nec).

Note that from MP, Incl_B and Necessitation for A one can derive necessitation for B. On the other hand, necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

Remark 3 *By MP, axiom K and Necessitation, the following is a derived rule of inference for the operators $\bigcirc$, $\bigcirc^{-1}$, B and A: $\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$ for $\Box \in \{\bigcirc, \bigcirc^{-1}, B, A\}$. We call this rule RK. On the other hand, rule RK is not a valid rule of inference for the operator I.*

[5] Axiom K for I is superfluous, since it can be derived from axioms I_1 and I_2 below (see [6] , p. 204).

4 The weakest logic of belief revision

Our purpose is to model how the beliefs of an individual change over time in response to *factual* information. Thus *the axioms we introduce are restricted to Boolean formulas*, which are formulas that do not contain any modal operators.

We shall consider axioms of increasing strength that capture the notion of minimal change of beliefs.

The first axiom says that if ϕ and ψ are facts (Boolean formulas) and - currently - the agent believes that ϕ and also believes that ψ and his belief that ϕ is non-trivial (in the sense that he considers ϕ possible) then - at every next instant - if he is informed that ϕ it will still be the case that he believes that ψ. That is, if at a next instant he is informed of some fact that he currently believes non trivially, then he *cannot drop* any of his current factual beliefs ('W' stands for 'Weak' and 'ND' for 'No Drop'):[6] if ϕ and ψ are Boolean,

$$(B\phi \wedge \neg B\neg\phi \wedge B\psi) \rightarrow \bigcirc(I\phi \rightarrow B\psi). \qquad (WND)$$

The second axiom says that if ϕ and ψ are facts (Boolean formulas) and - currently - the agent believes that ϕ and does not believe that ψ, then - at every next instant - if he is informed that ϕ it will still be the case that he does not believe that ψ. That is, at any next instant at which he is informed of some fact that he currently believes he *cannot add* a factual belief that he does not currently have ('W' stands for 'Weak' and 'NA' stands for 'No Add'):[7] if ϕ and ψ are Boolean,

$$(B\phi \wedge \neg B\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi). \qquad (WNA)$$

Thus, by WND, no belief can be dropped and, by WNA, no belief can be added, at any next instant at which the individual is informed of a fact that he currently believes.

An axiom is *characterized by* (or *characterizes*) a property of frames if it is valid in a frame if and only if the frame satisfies that property.

[6] It is shown in the Appendix that the following axiom (which says that if the individual is informed of some fact that he believed non-trivially at a previous instant then he must continue to believe every fact that he believed at that time) is equivalent to WND: if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \wedge I\phi \rightarrow B\psi.$$

This, in turn, is propositionally equivalent to $\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \rightarrow (I\phi \rightarrow B\psi)$.

[7] It is shown in the Appendix that the following is an equivalent formulation of WNA: if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \rightarrow \neg B\psi.$$

All the propositions are proved in the Appendix.

Proposition 4 *(1) Axiom WND is characterized by the following property:* $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,

$$\text{if } t_1 \rightarrowtail t_2, \mathcal{B}_{t_1}(\omega) \neq \varnothing \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_1}(\omega). \quad (P_{WND})$$

(2) Axiom WNA is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{B}_{t_2}(\omega). \quad (P_{WNA})$$

Let $\mathbb{L}_W$ (where 'W' stands for 'Weak') be the logic obtained by adding WND and WNA to $\mathbb{L}_0$. We denote this by writing $\mathbb{L}_W = \mathbb{L}_0 + WNA + WND$. The following is a corollary of Proposition 4.

Corollary 5 *Logic $\mathbb{L}_W$ is characterized by the class of temporal belief revision frames that satisfy the following property:* $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,

$$\text{if } t_1 \rightarrowtail t_2,\ \mathcal{B}_{t_1}(\omega) \neq \varnothing \text{ and } \mathcal{B}_{t_1}(\omega) \subseteq \mathcal{I}_{t_2}(\omega) \text{ then } \mathcal{B}_{t_1}(\omega) = \mathcal{B}_{t_2}(\omega).$$

The frame of Figure 1 violates the property of Corollary 5, since $t_2 \rightarrow t_3$, $\mathcal{B}_{t_2}(\alpha) = \{\alpha\} \subseteq \mathcal{I}_{t_3}(\alpha) = \{\alpha, \beta\}$ and $\mathcal{B}_{t_3}(\alpha) = \{\beta\} \neq \mathcal{B}_{t_2}(\alpha)$.

Logic $\mathbb{L}_W$ captures a weak notion of minimal change of beliefs in that it requires the agent not to change his beliefs if he is informed of some fact that he already believes. This requirement is stated explicitly in the following axiom ('WNC' stand for 'Weak No Change'): if ϕ and ψ are Boolean formulas,

$$(I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi)) \rightarrow (B\psi \leftrightarrow \Diamond^{-1}B\psi). \qquad (WNC)$$

WNC says that if the agent is informed of something that he believed non-trivially in the immediately preceding past, then he now believes a fact if and only if he believed it then.

Proposition 6 *WNC is a theorem of $\mathbb{L}_W$.*

We now turn to a strengthening of $\mathbb{L}_W$.

5 The logic of the Qualitative Bayes Rule

Logic $\mathbb{L}_W$ imposes no restrictions on belief revision whenever the individual is informed of some fact that he did not previously believe. We now consider a stronger logic than $\mathbb{L}_W$. The following axiom strengthens WND by requiring the individual not to drop any of his current factual beliefs at any next instant at which he is informed of some fact that he currently *considers possible* (without necessarily believing it: the condition $B\phi$ in the antecedent of WND is dropped): if ϕ and ψ are Boolean,

$$(\neg B\neg\phi \wedge B\psi) \rightarrow \bigcirc(I\phi \rightarrow B\psi). \qquad (ND)$$

The corresponding strengthening of WNA requires that if the individual considers it possible that $(\phi \wedge \neg\psi)$ then at any next instant at which he is informed that ϕ he does not believe that ψ:[8] if ϕ and ψ are Boolean,

$$\neg B \neg(\phi \wedge \neg\psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi). \qquad (NA)$$

One of the axioms of the AGM theory of belief revision (see [19]) is that information is believed. Such axiom is often referred to as "Success" or "Acceptance". The following axiom is a weaker form of it: information is believed when it is not surprising. If the agent considers a fact ϕ possible, then he will believe ϕ at any next instant at which he is informed that ϕ. We call this axiom *Qualified Acceptance* (QA): if ϕ is Boolean,

$$\neg B \neg\phi \rightarrow \bigcirc(I\phi \rightarrow B\phi). \qquad (QA)$$

Proposition 7 *(1) Axiom ND is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,*

if $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing$ then $\mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_1}(\omega)$. (P_{ND})

(2) Axiom NA is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,

if $t_1 \rightarrowtail t_2$ then $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \subseteq \mathcal{B}_{t_2}(\omega)$. (P_{NA})

(3) Axiom (QA) is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2 \in T$,

if $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing$ then $\mathcal{B}_{t_2}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$. (P_{QA})

We call the following property of temporal belief revision frames "Qualitative Bayes Rule" (QBR): $\forall t_1, t_2 \in T, \forall\omega \in \Omega$,

$$\text{if } t_1 \rightarrowtail t_2 \text{ and } \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega) \neq \varnothing \text{ then } \mathcal{B}_{t_2}(\omega) = \mathcal{B}_{t_1}(\omega) \cap \mathcal{I}_{t_2}(\omega). \qquad (QBR)$$

The expression "Qualitative Bayes Rule" is motivated by the following observation (see [6]). In a probabilistic setting, let P_{ω,t_1} be the probability measure over a set of states Ω representing the individual's beliefs at state ω and time t_1; let $F \subseteq \Omega$ be an event representing the information received by the individual at a

[8] Axiom NA can alternatively be written as $\Diamond(I\phi \wedge B\psi) \rightarrow B(\phi \rightarrow \psi)$, which says that if there is a next instant at which the individual is informed that ϕ and believes that ψ, then he must now believe that whenever ϕ is the case then ψ is the case. Another, propositionally equivalent, formulation of NA is the following: $\neg B(\phi \rightarrow \psi) \rightarrow \bigcirc(I\phi \rightarrow \neg B\psi)$, which says that if the individual does not believe that whenever ϕ is the case then ψ is the case, then - at any next instant - if he is informed that ϕ then he cannot believe that ψ.

later date t_2 and let P_{ω,t_2} be the posterior probability measure representing the revised beliefs at state ω and date t_2. Bayes' rule requires that, if $P_{\omega,t_1}(F) > 0$, then, for every event $E \subseteq \Omega$, $P_{\omega,t_2}(E) = \frac{P_{\omega,t_1}(E\cap F)}{P_{\omega,t_1}(F)}$. Bayes' rule thus implies the following (where $supp(P)$ denotes the support of the probability measure P):

$$\text{if } \; supp(P_{\omega,t_1}) \cap F \neq \varnothing, \text{ then } \; supp(P_{\omega,t_2}) = supp(P_{\omega,t_1}) \cap F.$$

If we set $\mathcal{B}_{t_1}(\omega) = supp(P_{\omega,t_1})$, $F = \mathcal{I}_{t_2}(\omega)$ (with $t_1 \rightarrowtail t_2$) and $\mathcal{B}_{t_2}(\omega) = supp(P_{\omega,t_2})$ then we get the Qualitative Bayes Rule as stated above. Thus in a probabilistic setting the proposition "at date t the individual believes ϕ" would be interpreted as "the individual assigns probability 1 to the event $\lceil\phi\rceil_t \subseteq \Omega$".

The following is a corollary of Proposition 7.

Corollary 8 *The conjunction of axioms ND, NA and QA characterizes the Qualitative Bayes Rule.*

The frame of Figure 1 violates QBR, since $t_2 \rightarrow t_3$, $\mathcal{B}_{t_2}(\delta) = \{\beta,\gamma\}$ and $\mathcal{I}_{t_3}(\delta) = \{\gamma,\delta,\varepsilon\}$, so that $\mathcal{B}_{t_2}(\delta) \cap \mathcal{I}_{t_3}(\delta) = \{\gamma\} \neq \varnothing$; however, $\mathcal{B}_{t_3}(\delta) = \{\gamma,\delta\} \neq \mathcal{B}_{t_2}(\delta) \cap \mathcal{I}_{t_3}(\delta)$. On the other hand, the frame of Figure 2 does satisfy QBR.

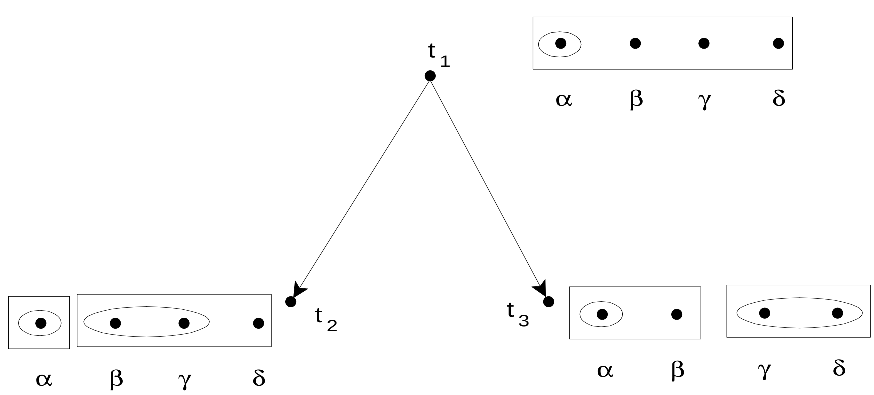

Figure 2

Let $\mathbb{L}_{QBR} = \mathbb{L}_0 + ND + NA + QA$.

Remark 9 *Logic $\mathbb{L}_{QBR}$ contains (is a strengthening of) $\mathbb{L}_W$. In fact, WND is a theorem of logic $\mathbb{L}_0 + ND$, since $(B\phi \wedge \neg B\neg\phi \wedge B\psi) \rightarrow (\neg B\neg\phi \wedge B\psi)$ is a tautology, and WNA is a theorem of logic $\mathbb{L}_0 + NA$ (see the Appendix).*

6 The logic of AGM

We now strengthen logic $\mathbb{L}_{QBR}$ by adding four more axioms.

The first axiom is the Acceptance axiom, which is a strengthening of Qualified Acceptance: if ϕ is Boolean,

$$I\phi \to B\phi. \tag{A}$$

The second axiom says that if there is a next instant where the individual is informed that $\phi \wedge \psi$ and believes that χ, then at every next instant it must be the case that if the individual is informed that ϕ then he must believe that $(\phi \wedge \psi) \to \chi$ (we call this axiom $K7$ because it corresponds to AGM postulate ($\circledast$7): see the next section): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I(\phi \wedge \psi) \wedge B\chi) \to \bigcirc(I\phi \to B\left((\phi \wedge \psi) \to \chi\right)). \tag{$K7$}$$

The third axiom says that if there is a next instant where the individual is informed that ϕ, considers $\phi \wedge \psi$ possible and believes that $\psi \to \chi$, then at every next instant it must be the case that if the individual is informed that $\phi \wedge \psi$ then he believes that χ (we call this axiom $K8$ because it corresponds to AGM postulate ($\circledast$8): see the next section): if ϕ, ψ and χ are Boolean formulas,

$$\Diamond(I\phi \wedge \neg B\neg(\phi \wedge \psi) \wedge B(\psi \to \chi)) \to \bigcirc(I(\phi \wedge \psi) \to B\chi). \tag{$K8$}$$

The fourth axiom says that if the individual receives consistent information then his beliefs are consistent, in the sense that he does not simultaneously believe a formula and its negation ('WC' stands for 'Weak Consistency'): if ϕ is a Boolean formula,

$$(I\phi \wedge \neg A\neg\phi) \to (B\psi \to \neg B\neg\psi). \tag{WC}$$

Proposition 10 *(1) axiom A is characterized by the following property:* $\forall\omega \in \Omega, \forall t \in T$,

$\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$. *($P_A$)*

(2) Axiom ($K7$) is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$,

if $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$ *and* $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ *then* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \subseteq \mathcal{B}_{t_3}(\omega)$. *($P_{K7}$)*

(3) Axiom ($K8$) is characterized by the following property: $\forall\omega \in \Omega$, $\forall t_1, t_2, t_3 \in T$,

if $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ *and* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$ *then* $\mathcal{B}_{t_3}(\omega) \subseteq \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$. *($P_{K8}$)*

(4) Axiom WC is characterized by the following property: $\forall\omega \in \Omega, \forall t \in T$, *if* $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$. *($P_{WC}$)*

Let $\mathbb{L}_{AGM} = \mathbb{L}_0 + A + ND + NA + K7 + K8 + WC$. Since QA can be derived from A, logic $\mathbb{L}_{AGM}$ contains (is a strengthening of) logic $\mathbb{L}_{QBR}$.

Definition 11 *An* $\mathbb{L}_{AGM}$-frame *is a temporal belief revision frame that satisfies the following properties:*
(1) the Qualitative Bayes Rule,
(2) $\forall \omega \in \Omega$, $\forall t \in T$, $\mathcal{B}_t(\omega) \subseteq \mathcal{I}_t(\omega)$,
(3) $\forall \omega \in \Omega$, $\forall t \in T$, *if* $\mathcal{I}_t(\omega) \neq \varnothing$ *then* $\mathcal{B}_t(\omega) \neq \varnothing$.
(4) $\forall \omega \in \Omega,\ \forall t_1, t_2, t_3 \in T$,
if $t_1 \rightarrowtail t_2,\ t_1 \rightarrowtail t_3,\ \mathcal{I}_{t_3}(\omega) \subseteq \mathcal{I}_{t_2}(\omega)$ *and* $\mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) \neq \varnothing$
then $\mathcal{B}_{t_3}(\omega) = \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega)$.
An $\mathbb{L}_{AGM}$-model *is a model based on an* $\mathbb{L}_{AGM}$*-frame.*

The frame of Figure 2 is not an AGM frame, although it satisfies QBR. In fact, we have that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\gamma) = \{\gamma, \delta\}$, $\mathcal{I}_{t_2}(\gamma) = \{\beta, \gamma, \delta\}$ and $\mathcal{B}_{t_2}(\gamma) = \{\beta, \gamma\}$, so that $\mathcal{I}_{t_3}(\gamma) \subseteq \mathcal{I}_{t_2}(\gamma)$ and $\mathcal{I}_{t_3}(\gamma) \cap \mathcal{B}_{t_2}(\gamma) = \{\gamma\} \neq \varnothing$ but $\mathcal{B}_{t_3}(\gamma) = \{\gamma, \delta\} \neq \mathcal{I}_{t_3}(\omega) \cap \mathcal{B}_{t_2}(\omega) = \{\gamma\}$.

Corollary 12 *It follows from Proposition 10 that logic* $\mathbb{L}_{AGM}$ *is characterized by the class of* $\mathbb{L}_{AGM}$*-frames.*

Some aspects of the relationship between logic $\mathbb{L}_{AGM}$ and the AGM theory of belief revision were investigated in [8]. In the next section we explore this relationship in more detail, with the help of structures borrowed from the rational choice literature.

7 Relationship to the AGM theory

We begin by recalling the theory of belief revision due to Alchourrón, Gärdenfors and Makinson [1], known as the AGM theory (see also [19]). In their approach beliefs are modeled as sets of formulas in a given syntactic language and belief revision is construed as an operation that associates with every deductively closed set of formulas K (thought of as the initial beliefs) and formula ϕ (thought of as new information) a new set of formulas $K^{\circledast}_{\phi}$ representing the revised beliefs.

7.1 AGM belief revision functions

Let S be a countable set of atomic propositions and $\mathcal{L}_0$ the propositional language built on S. Thus the set Φ_0 of formulas of $\mathcal{L}_0$ is defined recursively as follows: if $p \in S$ then $p \in \Phi_0$ and if $\phi, \psi \in \Phi_0$ then $\neg\phi \in \Phi_0$ and $\phi \vee \psi \in \Phi_0$.

Given a subset $K \subseteq \Phi_0$, its PL-deductive closure $[K]^{PL}$ (where 'PL' stands for Propositional Logic) is defined as follows: $\psi \in [K]^{PL}$ if and only if there exist $\phi_1, ..., \phi_n \in K$ such that $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \psi$ is a tautology (that is, a theorem of Propositional Logic). A set $K \subseteq \Phi_0$ is *consistent* if $[K]^{PL} \neq \Phi_0$ (equivalently, if there is no formula ϕ such that both ϕ and $\neg\phi$ belong to $[K]^{PL}$). A set

$K \subseteq \Phi_0$ is *deductively closed* if $K = [K]^{PL}$. A *belief set* is a set $K \subseteq \Phi_0$ which is deductively closed. The set of belief sets will be denoted by $\mathbb{K}$ and the set of consistent belief sets by $\mathbb{K}^{con}$.

Let $K \in \mathbb{K}^{con}$ be a consistent belief set representing the agent's initial beliefs. A *belief revision function for* K is a function

$$K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$$

that associates with every formula $\phi \in \Phi_0$ (thought of as new information) a set $K^{\circledast}(\phi) \subseteq \Phi_0$ (thought of as the new belief). It is common in the literature to use the notation $K^{\circledast}_{\phi}$ instead of $K^{\circledast}(\phi)$, but we prefer the latter. A belief revision function is called an *AGM revision function* if it satisfies the following properties, known as the AGM postulates: $\forall \phi, \psi \in \Phi_0$,

$(\circledast 1)$	$K^{\circledast}(\phi) \in \mathbb{K}$
$(\circledast 2)$	$\phi \in K^{\circledast}(\phi)$
$(\circledast 3)$	$K^{\circledast}(\phi) \subseteq [K \cup \{\phi\}]^{PL}$
$(\circledast 4)$	if $\neg\phi \notin K$, then $[K \cup \{\phi\}]^{PL} \subseteq K^{\circledast}(\phi)$
$(\circledast 5a)$	if ϕ is a contradiction then $K^{\circledast}(\phi) = \Phi_0$
$(\circledast 5b)$	if ϕ is not a contradiction then $K^{\circledast}(\phi) \neq \Phi_0$
$(\circledast 6)$	if $\phi \leftrightarrow \psi$ is a tautology then $K^{\circledast}(\phi) = K^{\circledast}(\psi)$
$(\circledast 7)$	$K^{\circledast}(\phi \wedge \psi) \subseteq [K^{\circledast}(\phi) \cup \{\psi\}]^{PL}$
$(\circledast 8)$	if $\neg\psi \notin K^{\circledast}(\phi)$, then $[K^{\circledast}(\phi) \cup \{\psi\}]^{PL} \subseteq K^{\circledast}(\phi \wedge \psi)$.

$(\circledast 1)$ requires the revised belief set to be deductively closed.

$(\circledast 2)$ requires that the information be believed.

$(\circledast 3)$ says that beliefs should be revised minimally, in the sense that no new formula should be added unless it can be deduced from the information received and the initial beliefs.

$(\circledast 4)$ says that if the information received is compatible with the initial beliefs, then any formula that can be deduced from the information and the initial beliefs should be part of the revised beliefs.

$(\circledast 5ab)$ require the revised beliefs to be consistent, unless the information ϕ is contradictory (that is, $\neg\phi$ is a tautology).

$(\circledast 6)$ requires that if ϕ is propositionally equivalent to ψ then the result of revising by ϕ be identical to the result of revising by ψ.

$(\circledast 7)$ and $(\circledast 8)$ are a generalization of $(\circledast 3)$ and $(\circledast 4)$ that

> "applies to *iterated* changes of belief. The idea is that if $K^{\circledast}(\phi)$ is a revision of K [prompted by ϕ] and $K^{\circledast}(\phi)$ is to be changed by adding further sentences, such a change should be made by using expansions of $K^{\circledast}(\phi)$ whenever possible. More generally, the minimal change of K to include both ϕ and ψ (that is, $K^{\circledast}(\phi \wedge \psi)$) ought to

be the same as the expansion of $K^{\circledast}(\phi)$ by ψ, so long as ψ does not contradict the beliefs in $K^{\circledast}(\phi)$" (Gärdenfors [19], p. 55).[9]

We now turn to a semantic counterpart to the AGM belief revision functions, which is in the spirit of Grove's [21] system of spheres. The structures we will consider are known in rational choice theory as *choice functions* (see, for example, [31] and [34]).

7.2 Choice structures and one-stage revision frames

Definition 13 *A* choice structure *is a quadruple* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ *where*

- Ω *is a non-empty set of* states*; subsets of* Ω *are called* events*.*
- $\mathcal{E} \subseteq 2^{\Omega}$ *is a collection of events (*2^{Ω} *denotes the set of subsets of* Ω*).*
- $\mathbf{R} : \mathcal{E} \to 2^{\Omega}$ *is a function that associates with every event* $E \in \mathcal{E}$ *an event* $\mathbf{R}_E \subseteq \Omega$ *(we use the notation* $\mathbf{R}_E$ *rather than* $\mathbf{R}(E)$*).*
- $\mathbb{O} \in \mathcal{E}$ *is a distinguished element of* $\mathcal{E}$ *with* $\mathbb{O} \neq \varnothing$*.*

In rational choice theory a set $E \in \mathcal{E}$ is interpreted as a set of available alternatives and $\mathbf{R}_E$ is interpreted as the subset of E which consists of those alternatives that could be rationally chosen. In our case, we interpret the elements of $\mathcal{E}$ as possible items of information that the agent might receive and the interpretation of $\mathbf{R}_E$ is that, if informed that event E has occurred, the agent considers as possible all and only the states in $\mathbf{R}_E$. For the distinguished element $\mathbb{O}$, we interpret $\mathbf{R}_{\mathbb{O}}$ as the *original* or *initial* beliefs of the agent.[10]

Note that we do not impose the requirement that $\Omega \in \mathcal{E}$.

Definition 14 *A* one-stage revision frame *is a choice structure* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ *that satisfies the following properties:* $\forall E, F \in \mathcal{E}$*,*

(BR1) $\mathbf{R}_E \subseteq E$,

(BR2) if $E \neq \varnothing$ then $\mathbf{R}_E \neq \varnothing$,

(BR3) if $E \subseteq F$ and $\mathbf{R}_F \cap E \neq \varnothing$ then $\mathbf{R}_E = \mathbf{R}_F \cap E$.[11]

(BR4) if $\mathbf{R}_{\mathbb{O}} \cap E \neq \varnothing$ then $\mathbf{R}_E = \mathbf{R}_{\mathbb{O}} \cap E$.

In the rational choice literature, (BR1) and (BR2) are taken to be part of the definition of a choice function, while (BR3) is known as Arrow's axiom (see

[9] The expansion of $K^{\circledast}(\phi)$ by ψ is $\left[K^{\circledast}(\phi) \cup \{\psi\}\right]^{PL}$.

[10] In the rational choice literature there is no counterpart to the distinguished set $\mathbb{O}$.

[11] It is proved in the Appendix that, in the presence of (BR1), (BR3) is equivalent to: $\forall E, F \in \mathcal{E}$,

$$\text{(BR3}'\text{)} \quad \text{if } R_F \cap E \neq \varnothing \text{ and } E \cap F \in \mathcal{E} \text{ then } R_{E \cap F} = R_F \cap E.$$

[34] p. 25). Property (BR4), which corresponds to our Qualitative Bayes Rule, has not been investigated in that literature.

The following is an example of a belief revision frame: $\Omega = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{E} = \{\{\alpha, \beta\}, \{\gamma, \delta\}, \{\alpha, \beta, \gamma\}\}$, $\mathbb{O} = \{\alpha, \beta, \gamma\}$, $\mathbf{R}_{\{\alpha,\beta\}} = \{\beta\}$, $\mathbf{R}_{\{\gamma,\delta\}} = \{\gamma\}$, $\mathbf{R}_{\{\alpha,\beta,\gamma\}} = \{\beta, \gamma\}$.

A *one-stage revision model* is a quintuple $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ where $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R} \rangle$ is a one-stage revision frame and $V : S \to 2^{\Omega}$ is a function (called a *valuation*) that associates with every atomic proposition p the set of states at which p is true. Truth of an arbitrary formula in a model is defined recursively as follows ($\omega \models \phi$ means that formula ϕ is true at state ω): (1) for $p \in S$, $\omega \models p$ if and only if $\omega \in V(p)$, (2) $\omega \models \neg\phi$ if and only if $\omega \not\models \phi$ and (3) $\omega \models \phi \vee \psi$ if and only if either $\omega \models \phi$ or $\omega \models \psi$ (or both). The truth set of a formula ϕ is denoted by $\|\phi\|$. Thus $\|\phi\| = \{\omega \in \Omega : \omega \models \phi\}$.

Given a one-stage revision model, we say that

(1) the agent *initially believes that* ϕ if and only if $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi\|$,

(2) the agent *believes that* ϕ *upon learning that* ψ if and only if $\|\psi\| \in \mathcal{E}$ and $\mathbf{R}_{\|\psi\|} \subseteq \|\phi\|$.

Definition 15 *A one-stage revision model is* comprehensive *if for every formula ϕ, $\|\phi\| \in \mathcal{E}$. It is* rich *if, for every finite set $P = \{p_1, ..., p_n, q_1, ..., q_m\}$ of atomic propositions, there is a state $\omega_P \in \Omega$ such that $\omega_P \models p_i$ for every $i = 1, ..., n$ and $\omega_P \models \neg q_j$ for every $j = 1, ..., m$.*

Thus in a comprehensive one-stage revision model every formula is a possible item of information. For example, a model based on a one-stage revision frame where $\mathcal{E} = 2^{\Omega}$ is comprehensive. In a rich model every formula consisting of a conjunction of atomic proposition or the negation of atomic propositions is true at some state.

7.3 Correspondence

We now show that the set of AGM belief revision functions corresponds to the set of comprehensive and rich one-stage revision models, in the sense that
(1) given a comprehensive and rich one-stage revision model, we can associate with it a consistent belief set K and a corresponding AGM belief revision function $K^{\circledast}$, and
(2) given a consistent belief set K and an AGM belief revision function $K^{\circledast}$ there exists a comprehensive and rich one-stage revision model whose associated belief set and AGM belief revision function coincide with K and $K^{\circledast}$, respectively.

Proposition 16 *Let $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ be a comprehensive one-stage revision model. Define $K = \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{O}} \subseteq \|\psi\|\}$. Then K is a consistent belief set. For every $\phi \in \Phi_0$ define $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$. Then the function $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ so defined satisfies AGM postulates ($\circledast$1)-($\circledast$5a) and ($\circledast$6)-($\circledast$8). If the model is rich then also ($\circledast$5b) is satisfied.*

Proposition 17 *Let $K \in \mathbb{K}$ be a consistent belief set and $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ be an AGM belief revision function (that is, $K^{\circledast}$ satisfies the AGM postulates ($\circledast$1)-($\circledast$8)). Then there exists a comprehensive and rich one-stage revision model $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ such that $K = \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{O}} \subseteq \|\psi\|\}$ and, for every $\phi \in \Phi_0$, $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$.*

7.4 Back to $\mathbb{L}_{AGM}$ frames

Given an $\mathbb{L}_{AGM}$ frame $\langle T, \rightarrowtail, \Omega, \{\mathcal{B}_t, \mathcal{I}_t\}_{t \in T} \rangle$ (see Definition 11) we can associate with every state-instant pair (ω_0, t_0) a one-stage revision frame (see Definition 14) $\langle \Omega^0, \mathcal{E}^0, \mathbb{O}^0, \mathbf{R}^0 \rangle$ as follows. Let $\overset{\rightarrowtail}{t_0} = \{t \in T : t_0 \rightarrowtail t\}$, then

- $\Omega^0 = \Omega$,
- $\mathcal{E}^0 = \left\{ E \subseteq \Omega : E = \mathcal{I}_t(\omega_0) \text{ for some } t \in \overset{\rightarrowtail}{t_0} \right\}$,
- $\mathbb{O}^0 = \mathcal{I}_{t_0}(\omega_0)$,
- $\mathbf{R}_{\mathbb{O}^0} = \mathcal{B}_{t_0}(\omega_0)$
- for every $E \in \mathcal{E}$, if $E = \mathcal{I}_t(\omega_0)$ (for some $t \in \overset{\rightarrowtail}{t_0}$) then $\mathbf{R}^0_E = \mathcal{B}_t(\omega_0)$,

By Property (2) of $\mathbb{L}_{AGM}$-frames the frame $\langle \Omega^0, \mathcal{E}^0, \mathbb{O}^0, \mathbf{R}^0 \rangle$ so defined satisfies property BR1 of the definition of one-stage revision frame, while Property (3) ensures that BR2 is satisfied, Property (4) ensures that BR3 is satisfied and Property (1) ensures that BR4 is satisfied.

Consider now the subset of the set of $\mathbb{L}_{AGM}$ frames consisting of those frames satisfying the following properties:

$$\forall t \in T, \forall \omega \in \Omega, \forall E \in 2^{\Omega} \backslash \{\varnothing\}, \quad \exists t' \in T : t \rightarrowtail t' \text{ and } \mathcal{I}_{t'}(\omega) = E. \qquad (P_{CMP})$$

and

$$\forall t \in T, \forall \omega \in \Omega, \ \ \mathcal{I}_t(\omega) \neq \varnothing. \quad \text{(seriality of } \mathcal{I}_t\text{)}$$

Let $\mathbb{L}_{comp}$ (“comp” stands for “comprehensive”) be $\mathbb{L}_{AGM} + CMP + I_{con}$ where CMP and I_{con} are the following axioms: for every Boolean ϕ

$$\neg A \neg \phi \to \Diamond I \phi. \qquad \text{(CMP)}$$

$$\neg I(\phi \wedge \neg \phi). \qquad (I_{con})$$

Axiom CMP says that, for every Boolean formula ϕ, if there is a state where ϕ is true, then there is a next instant where the agent is informed that ϕ, while axiom I_{con} rules out contradictory or inconsistent information.

Proposition 18 *Logic* $\mathbb{L}_{comp}$ *is characterized by the class of* $\mathbb{L}_{AGM}$*-frames that satisfy* P_{CMP} *and seriality of* $\mathcal{I}_t$.[12]

We can view logic $\mathbb{L}_{comp}$ as an axiomatization of the AGM belief revision functions. In fact, if we take any model based on a $\mathbb{L}_{comp}$ frame and any state-instant pair, the one-stage revision frame associated with it is such that $\mathcal{E} = 2^{\Omega} \backslash \{\varnothing\}$. Thus the corresponding one-stage revision model is comprehensive (see Definition 15) and therefore, by Proposition 16, the associated AGM belief revision function $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ satisfies AGM postulates ($\circledast$1)-($\circledast$5a) and ($\circledast$6)-($\circledast$8). Conversely, by Proposition 17, for every consistent belief set K and AGM belief revision function $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ there is a model based on an $\mathbb{L}_{comp}$ frame whose associated AGM belief revision function coincides with $K^{\circledast}$.[13]

$\mathbb{L}_{comp}$ models, however, are "very large" in that, for every state-instant pair and for every Boolean formula ϕ whose truth set is non-empty, there is a next instant where the agent is informed that ϕ. This requirement corresponds to assuming a complete belief revision policy for the agent, whereby the agent contemplates his potential reaction to every conceivable (and consistent) item of information. In a typical $\mathbb{L}_{AGM}$ frame, on the other hand, the items of information that the individual might receive at the next instant might be few, so that the agent's belief revision policy is limited to a few (perhaps the most likely) pieces of information. How does this limited belief revision policy associated with $\mathbb{L}_{AGM}$ frames relate to the AGM postulates for belief revision? The answer is given in the following proposition, which was proved in [8] (we have reworded it to fit the set-up of this section). We can no longer recover an entire AGM belief revision function from a model based on an arbitrary $\mathbb{L}_{AGM}$ frame. However we can recover, for every pair of Boolean formulas ϕ and ψ, the values $K^{\circledast}(\phi)$ and $K^{\circledast}(\phi \wedge \psi)$ of an AGM belief revision function whenever there is a next instant at which the agent is informed that ϕ and there is another next instant where he is informed that $(\phi \wedge \psi)$.

Proposition 19 *(A) Let* $K \subseteq \Phi^B$ *be a consistent and deductively closed set and let* $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ *be an AGM belief revision function. Fix arbitrary* $\phi, \psi \in \Phi^B$*. Then there is an* $\mathbb{L}_{AGM}$*-model,* $t_1, t_2, t_3 \in T$ *and* $\alpha \in \Omega$ *such that*

(A.1) $t_1 \rightarrowtail t_2$
(A.2) $K = \{\chi \in \Phi^B : (\alpha, t_1) \models B\chi\}$
(A.3) $(\alpha, t_2) \models I\phi$
(A.4) $K^{\circledast}(\phi) = \{\chi \in \Phi^B : (\alpha, t_2) \models B\chi\}$
(A.5) if ϕ *is consistent then* $(\beta, t) \models \phi$ *for some* $\beta \in \Omega$ *and* $t \in T$
(A.6) $t_1 \rightarrowtail t_3$
(A.7) $(\alpha, t_3) \models I(\phi \wedge \psi)$
(A.8) $K^{\circledast}(\phi \wedge \psi) = \{\chi \in \Phi^B : (\alpha, t_3) \models B\chi\}$

[12] Note that, given the non-standard validation rule for $I\phi$, the equivalence of axiom D ($I\phi \to \neg I \neg \phi$) and seriality of $\mathcal{I}_t$ breaks down. It is still true that if $\mathcal{I}_t$ is serial then the axiom $I\phi \to \neg I \neg \phi$ is valid, but the converse is not true (see [6], Footnote 25, p. 226).

[13] All we need to do in this respect is to eliminate the empty set from $\mathcal{E}$ in the proof of Proposition 17, that is, discard the possibility that ϕ is a contradiction.

(A.9) if $(\phi \wedge \psi)$ is consistent then $(\gamma, t') \models (\phi \wedge \psi)$ for some $\gamma \in \Omega$ and $t' \in T$.

(B) Fix an $\mathbb{L}_{AGM}$-model such that (1) for some $t_1, t_2, t_3 \in T$, $\alpha \in \Omega$ and $\phi, \psi \in \Phi^B$, $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$, (2) if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$ and (3) if $(\phi \wedge \psi)$ is not a contradiction then $(\gamma, t') \models (\phi \wedge \psi)$, for some $\gamma \in \Omega$ and $t' \in T$. Define $K = \left\{\chi \in \Phi^B : (\alpha, t_1) \models B\chi\right\}$. Then there exists an AGM belief revision function $K^{\circledast} : \Phi_0 \to 2^{\Phi_0}$ such that $K^{\circledast}(\phi) = \left\{\chi \in \Phi^B : (\alpha, t_2) \models B\chi\right\}$ and $K^{\circledast}(\phi \wedge \psi) = \left\{\chi \in \Phi^B : (\alpha, t_3) \models B\chi\right\}$. Furthermore, for every $\phi, \psi \in \Phi^B$, there exists an $\mathbb{L}_{AGM}$-model such that, for some $\alpha \in \Omega$ and $t_2, t_3 \in T$, (1) $(\alpha, t_2) \models I\phi$ and $(\alpha, t_3) \models I(\phi \wedge \psi)$, (2) if ϕ is not a contradiction then $(\beta, t) \models \phi$, for some $\beta \in \Omega$ and $t \in T$ and (3) if $(\phi \wedge \psi)$ is not a contradiction then $(\gamma, t') \models (\phi \wedge \psi)$, for some $\gamma \in \Omega$ and $t' \in T$.

8 Conclusion

We proposed a temporal logic where information and beliefs are modeled by means of two modal operators I and B, respectively. A third modal operator, the next-time operator $\bigcirc$, enables one to express the dynamic interaction of information and beliefs over time. The proposed logic can be viewed as a temporal generalization of the theory of static belief pioneered by Hintikka [23].

The combined syntactic-semantic approach of modal logic allows one to state properties of beliefs in a clear and transparent way by means of axioms and to show the correspondence between axioms and semantic properties. Natural extensions of our $\mathbb{L}_{AGM}$ logic would impose, besides consistency of information (axiom I_{con})[14], the standard KD45 axioms for belief (axiom 4: $B\phi \to BB\phi$ and axiom 5: $\neg B\phi \to B\neg B\phi$, while the D axiom: $B\phi \to \neg B\neg\phi$ would follow from axioms I_{con} and WC). Furthermore, one might want to investigate axioms that capture the notion of memory or recall, for instance $B\phi \to \bigcirc B \bigcirc^{-1} B\phi$ and $\neg B\phi \to \bigcirc B \bigcirc^{-1} \neg B\phi$ (the agent always remembers what he believed and what he did not believe in the immediately preceding past). Further strengthenings might add the requirement that information be correct ($I\phi \to \phi$) or the weaker requirement that the agent trusts the information source ($B \bigcirc (I\phi \to \phi)$). Another natural direction to explore is the axiomatization of *iterated revision*, a topic that has received considerable attention in recent years (see, for example, [11], [14], [29], [30]). Extensions of logic $\mathbb{L}_{AGM}$ that incorporate axioms for iterated revision have been recently investigated in [35]. Finally, another line of research, which is pursued in [10], deals with the conditions under which belief revision can be rationalized by a plausibility ordering on the set of states, in the sense that the set of states that are considered possible after being informed that ϕ coincides with the most plausible states that are compatible with ϕ.

[14] As pointed out by Friedman and Halpern [18], it is not clear how one could be informed of a contradiction.

A Appendix

Proof of the claim in Footnote 6, namely that axiom WND is equivalent to the following axiom: if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \wedge I\phi \to B\psi.$$

Derivation of WND from the above axiom ('PL' stands for 'Propositional Logic'):

1.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to (I\phi \to B\psi)$	above axiom, PL
2.	$\bigcirc\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to \bigcirc(I\phi \to B\psi)$	1, rule RK for $\bigcirc$
3.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to \bigcirc\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	Temporal axiom O_1
4.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to \bigcirc(I\phi \to B\psi)$	2,3, PL.

Derivation of the above axiom from WND:

1.	$(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to \bigcirc(I\phi \to B\psi)$	Axiom WND
2.	$\neg\bigcirc(I\phi \to B\psi) \to \neg(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	1, PL
3.	$\bigcirc^{-1}\neg\bigcirc(I\phi \to B\psi) \to \bigcirc^{-1}\neg(B\phi \wedge B\psi \wedge \neg B\neg\phi)$	2, rule RK for $\bigcirc^{-1}$
4.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to \Diamond^{-1}\bigcirc(I\phi \to B\psi)$	3, PL, definition of $\Diamond^{-1}$
5.	$\neg(I\phi \to B\psi) \to \bigcirc^{-1}\Diamond\neg(I\phi \to B\psi)$	Temporal axiom O_2
6.	$\Diamond^{-1}\bigcirc(I\phi \to B\psi) \to (I\phi \to B\psi)$	5, PL, definition of $\Diamond^{-1}$ and $\Diamond$
7.	$\Diamond^{-1}(B\phi \wedge B\psi \wedge \neg B\neg\phi) \to (I\phi \to B\psi)$	4, 6, PL. ■

Proof of the claim in Footnote 7, namely that axiom WNA is equivalent to the following axiom: if ϕ and ψ are Boolean,

$$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \to \neg B\psi.$$

Derivation of WNA from the above axiom:

1.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \to \neg B\psi$	above axiom
2.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \to (I\phi \to \neg B\psi)$	1, PL
3.	$\bigcirc\Diamond^{-1}(B\phi \wedge \neg B\psi) \to \bigcirc(I\phi \to \neg B\psi)$	2, rule RK for $\bigcirc$
4.	$(B\phi \wedge \neg B\psi) \to \bigcirc\Diamond^{-1}(B\phi \wedge \neg B\psi)$	Temporal axiom O_1
5.	$(B\phi \wedge \neg B\psi) \to \bigcirc(I\phi \to \neg B\psi)$	3, 4, PL.

Derivation of the above axiom from WNA:

1.	$(B\phi \wedge \neg B\psi) \to \bigcirc(I\phi \to \neg B\psi)$	Axiom WNA
2.	$\neg\bigcirc(I\phi \to \neg B\psi) \to \neg(B\phi \wedge \neg B\psi)$	1, PL
3.	$\bigcirc^{-1}\neg\bigcirc(I\phi \to \neg B\psi) \to \bigcirc^{-1}\neg(B\phi \wedge \neg B\psi)$	2, rule RK for $\bigcirc^{-1}$
4.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \to \Diamond^{-1}\bigcirc(I\phi \to \neg B\psi)$	3, PL and definition of $\Diamond^{-1}$
5.	$\neg(I\phi \to \neg B\psi) \to \bigcirc^{-1}\Diamond\neg(I\phi \to \neg B\psi)$	Temporal axiom O_2
6.	$\Diamond^{-1}\bigcirc(I\phi \to \neg B\psi) \to (I\phi \to \neg B\psi)$	5, PL, definition of $\Diamond^{-1}$ and $\Diamond$
7.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \to (I\phi \to \neg B\psi)$	4, 6, PL
8.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \to \neg B\psi$	7, PL. ■

Proof of Proposition 4. (1) Fix a frame that satisfies P_{WND}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models (B\phi \wedge B\psi \wedge \neg B\neg\phi)$. Since $(\alpha, t_1) \models \neg B\neg\phi$, there exists an $\omega \in \mathcal{B}_{t_1}(\alpha)$ such that $(\omega, t_1) \models \phi$. Thus $\mathcal{B}_{t_1}(\alpha) \neq \varnothing$. Fix an arbitrary $t_2 \in T$ such that $t_1 \to t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$.

Fix an arbitrary $\beta \in \mathcal{B}_{t_1}(\alpha)$. Since $(\alpha, t_1) \models B\phi$, $(\beta, t_1) \models \phi$. Since ϕ is Boolean, by Remark 2 $(\beta, t_2) \models \phi$. Hence $\beta \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by P_{WND}, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_1}(\alpha)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$. Then $\omega \in \mathcal{B}_{t_1}(\alpha)$ and, since $(\alpha, t_1) \models B\psi$, $(\omega, t_1) \models \psi$. Since ψ is Boolean, by Remark 2 $(\omega, t_2) \models \psi$. Thus $(\alpha, t_2) \models B\psi$.

Conversely, suppose that P_{WND} is violated. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha) \neq \varnothing$, $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_1}(\alpha)$. Let p and q be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$ and $||q|| = \mathcal{B}_{t_1}(\alpha) \times T$. Then $(\alpha, t_1) \models (Bp \wedge Bq \wedge \neg B\neg q)$. By hypothesis, there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_1}(\alpha)$, so that $(\beta, t_2) \nvDash q$. Hence $(\alpha, t_2) \nvDash Bq$ while $(\alpha, t_2) \models Ip$, so that $(\alpha, t_2) \nvDash Ip \rightarrow Bq$. Thus, since $t_1 \rightarrowtail t_2$, WND is falsified at (α, t_1).

(2) Fix a frame that satisfies P_{WNA}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models B\phi \wedge \neg B\psi$. Then there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \neg\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$. Fix an arbitrary $\omega \in \mathcal{B}_{t_1}(\alpha)$. Since $(\alpha, t_1) \models B\phi$, $(\omega, t_1) \models \phi$. Since ϕ is Boolean, by Remark 2 $(\omega, t_2) \models \phi$ and therefore $\omega \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and, by P_{WNA}, $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{B}_{t_2}(\alpha)$. Since $(\beta, t_1) \models \neg\psi$ and $\neg\psi$ is Boolean (because ψ is), by Remark 2 $(\beta, t_2) \models \neg\psi$. Since $\beta \in \mathcal{B}_{t_1}(\alpha)$ and $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{B}_{t_2}(\alpha)$, $\beta \in \mathcal{B}_{t_2}(\alpha)$ and therefore $(\alpha, t_2) \models \neg B\psi$.

Conversely, suppose that P_{WNA} is violated. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_1}(\alpha) \nsubseteq \mathcal{B}_{t_2}(\alpha)$. Let p and q be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$ and $||q|| = \mathcal{B}_{t_2}(\alpha) \times T$. Then $(\alpha, t_1) \models Bp \wedge \neg Bq$ and $(\alpha, t_2) \models Ip \wedge Bq$, so that, since $t_1 \rightarrowtail t_2$, $(\alpha, t_1) \models \neg \bigcirc (Ip \rightarrow \neg Bq)$. ■

Proof of Proposition 6. First of all, note that, since $\bigcirc^{-1}$ is a normal operator, the following is a theorem of $\mathbb{L}_0$ (hence of $\mathbb{L}_W$):

$$\Diamond^{-1}\chi \wedge \bigcirc^{-1}\xi \rightarrow \Diamond^{-1}(\chi \wedge \xi). \tag{1}$$

It follows from (1) and axiom BU that the following is a theorem of $\mathbb{L}_0$:

$$\Diamond^{-1}\chi \wedge \Diamond^{-1}\xi \rightarrow \Diamond^{-1}(\chi \wedge \xi). \tag{2}$$

The following is a syntactic derivation of WNC:

1.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \Diamond^{-1}B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge B\psi)$	Theorem of L_0 (see (2) above)
2.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge B\psi) \wedge I\phi \to B\psi$	Equivalent to WND (see Footnote 6)
3.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \Diamond^{-1}B\psi \wedge I\phi \to B\psi$	1, 2, PL
4.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (\Diamond^{-1}B\psi \to B\psi)$	3, PL
5.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	Theorem of L_0 (see (1) above)
6.	$\neg(B\phi \wedge \neg B\psi) \to \neg(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	Tautology
7.	$\bigcirc^{-1}\neg(B\phi \wedge \neg B\psi) \to \bigcirc^{-1}\neg(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi)$	6, rule RK for $\bigcirc^{-1}$
8.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi \wedge \neg B\psi) \to \Diamond^{-1}(B\phi \wedge \neg B\psi)$	7, PL, def. of $\Diamond^{-1}$
9.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \to \Diamond^{-1}(B\phi \wedge \neg B\psi)$	5, 8, PL
10.	$\Diamond^{-1}(B\phi \wedge \neg B\psi) \wedge I\phi \to \neg B\psi$	equivalent to WNA (see Footnote 7)
11.	$\Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \wedge \bigcirc^{-1}\neg B\psi \wedge I\phi \to \neg B\psi$	9, 10, PL
12.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (\bigcirc^{-1}\neg B\psi \to \neg B\psi)$	11, PL
13.	$(\bigcirc^{-1}\neg B\psi \to \neg B\psi) \to (B\psi \to \Diamond^{-1}B\psi)$	tautology and definition of $\Diamond^{-1}$
14.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (B\psi \to \Diamond^{-1}B\psi)$	12, 13, PL
15.	$I\phi \wedge \Diamond^{-1}(B\phi \wedge \neg B\neg\phi) \to (B\psi \leftrightarrow \Diamond^{-1}B\psi)$	4, 14, PL. ■

Proof of Proposition 7. (1) Fix a frame that satisfies P_{ND}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models \neg B\neg\phi \wedge B\psi$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg\phi$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi$. Since ϕ is Boolean, by Remark 2 $(\beta, t_2) \models \phi$ and, therefore, $\beta \in \mathcal{I}_{t_2}(\alpha)$. Thus $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$ and, by P_{ND}, $\mathcal{B}_{t_2}(\alpha) \subseteq \mathcal{B}_{t_1}(\alpha)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$. Then $\omega \in \mathcal{B}_{t_1}(\alpha)$ and, since $(\alpha, t_1) \models B\psi$, $(\omega, t_1) \models \psi$. Since ψ is Boolean, by Remark 2, $(\omega, t_2) \models \psi$. Hence $(\alpha, t_2) \models B\psi$.

Conversely, fix a frame that does not satisfy P_{ND}. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \neq \varnothing$ and $\mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_1}(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{B}_{t_1}(\alpha) \times T$ and $\|q\| = \mathcal{I}_{t_2}(\alpha) \times T$. Then $(\alpha, t_1) \models \neg B\neg q \wedge Bp$ and $(\alpha, t_2) \models Iq$. By hypothesis there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_1}(\alpha)$. Thus $(\beta, t_2) \not\models p$ and therefore $(\alpha, t_2) \models \neg Bp$. Hence $(\alpha, t_1) \models \neg \bigcirc (Iq \to Bp)$.

(2) Fix a frame that satisfies P_{NA}, an arbitrary model based on it and arbitrary $\alpha \in \Omega$, $t_1 \in T$ and Boolean formulas ϕ and ψ and suppose that $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$. Fix an arbitrary $t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and suppose that $(\alpha, t_2) \models I\phi$. Then $\mathcal{I}_{t_2}(\alpha) = \lceil\phi\rceil_{t_2}$. Since $(\alpha, t_1) \models \neg B\neg(\phi \wedge \neg\psi)$, there exists a $\beta \in \mathcal{B}_{t_1}(\alpha)$ such that $(\beta, t_1) \models \phi \wedge \neg\psi$. Since ϕ and ψ are Boolean, by Remark 2 $(\beta, t_2) \models \phi \wedge \neg\psi$. Thus $\beta \in \mathcal{I}_{t_2}(\alpha)$ and, by P_{NA}, $\beta \in \mathcal{B}_{t_2}(\alpha)$. Thus, since $(\beta, t_2) \models \neg\psi$, $(\alpha, t_2) \models \neg B\psi$.

Conversely, fix a frame that does not satisfy P_{NA}. Then there exist $\alpha \in \Omega$ and $t_1, t_2 \in T$ such that $t_1 \rightarrowtail t_2$ and $\mathcal{B}_{t_1}(\alpha) \cap \mathcal{I}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_2}(\alpha)$. Let p and q be atomic propositions and construct a model where $\|p\| = \mathcal{I}_{t_2}(\alpha) \times T$ and $\|q\| =$

$\mathcal{B}_{t_2}(\alpha)\times T$. Then $(\alpha,t_2)\models Ip\wedge Bq$ and, therefore, $(\alpha,t_1)\models\neg\bigcirc(Ip\to\neg Bq)$. Since $\mathcal{B}_{t_1}(\alpha)\cap\mathcal{I}_{t_2}(\alpha)\nsubseteq\mathcal{B}_{t_2}(\alpha)$ there exists a $\beta\in\mathcal{B}_{t_1}(\alpha)\cap\mathcal{I}_{t_2}(\alpha)$ such that $\beta\notin\mathcal{B}_{t_2}(\alpha)$. Thus $(\beta,t_1)\models p\wedge\neg q$. Hence $(\alpha,t_1)\models\neg B\neg(p\wedge\neg q)$, so that axiom NA is falsified at (α,t_1).

(3) Fix a frame that satisfies P_{QA}, an arbitrary model based on it and arbitrary $\alpha\in\Omega$, $t_1\in T$ and Boolean formula ϕ and suppose that $(\alpha,t_1)\models\neg B\neg\phi$. Then there exists a $\beta\in\mathcal{B}_{t_1}(\alpha)$ such that $(\beta,t_1)\models\phi$. Fix an arbitrary t_2 such that $t_1\rightarrowtail t_2$ and suppose that $(\alpha,t_2)\models I\phi$. Then $\mathcal{I}_{t_2}(\alpha)=\lceil\phi\rceil_{t_2}$. Since ϕ is Boolean and $(\beta,t_1)\models\phi$, by Remark 2 $(\beta,t_2)\models\phi$. Thus $\beta\in\mathcal{I}_{t_2}(\alpha)$ and, therefore, $\mathcal{B}_{t_1}(\alpha)\cap\mathcal{I}_{t_2}(\alpha)\neq\varnothing$. By P_{QA}, $\mathcal{B}_{t_2}(\alpha)\subseteq\mathcal{I}_{t_2}(\alpha)$. Thus $(\alpha,t_2)\models B\phi$. Hence $(\alpha,t_1)\models\bigcirc(I\phi\to B\phi)$.

Conversely, suppose that P_{QA} is violated. Then there exist $\alpha\in\Omega$ and $t_1,t_2\in T$ such that $t_1\rightarrowtail t_2$, $\mathcal{B}_{t_1}(\alpha)\cap\mathcal{I}_{t_2}(\alpha)\neq\varnothing$ and $\mathcal{B}_{t_2}(\alpha)\nsubseteq\mathcal{I}_{t_2}(\alpha)$. Let p be an atomic proposition and construct a model where $\|p\|=\mathcal{I}_{t_2}(\alpha)\times T$. Then $(\alpha,t_1)\models\neg B\neg p$ and $(\alpha,t_2)\models Ip$. By hypothesis, there exists a $\beta\in\mathcal{B}_{t_2}(\alpha)$ such that $\beta\notin\mathcal{I}_{t_2}(\alpha)$. Thus $(\beta,t_2)\nvDash p$ and therefore $(\alpha,t_2)\models\neg Bp$. Hence $(\alpha,t_1)\nvDash\bigcirc(Ip\to Bp)$. ■

Proof of the claim in Remark 9, namely that WNA is a theorem of logic $\mathbb{L}_0+NA$:

1.	$\neg B(\phi\to\psi)\to\bigcirc(I\phi\to\neg B\psi)$	Axiom NA (see Footnote 8)
2.	$B(\phi\to\psi)\to(B\phi\to B\psi)$	Axiom K for B
3.	$(B\phi\wedge\neg B\psi)\to\neg B(\phi\to\psi)$	2, PL
4.	$(B\phi\wedge\neg B\psi)\to\bigcirc(I\phi\to\neg B\psi)$	1, 3, PL. ■

Proof of Proposition 10. (1) The proof of this part is straightforward and is omitted.

(2)Fix a frame that satisfies property P_{K7}. Let α and t_1 be such that $(a,t_1)\models\Diamond(I(\phi\wedge\psi)\wedge B\chi)$, where ϕ, ψ and χ are Boolean formulas. Then there exists a t_3 such that $t_1\rightarrowtail t_3$ and $(\alpha,t_3)\models I(\phi\wedge\psi)\wedge B\chi$. Thus $\mathcal{I}_{t_3}(\alpha)=\lceil\phi\wedge\psi\rceil_{t_3}$. Fix an arbitrary t_2 such that $t_1\rightarrowtail t_2$ and suppose that $(\alpha,t_2)\models I\phi$. Then $\mathcal{I}_{t_2}(\alpha)=\lceil\phi\rceil_{t_2}$. Since ϕ and ψ are Boolean, by Remark 2, $\lceil\phi\wedge\psi\rceil_{t_3}=\lceil\phi\wedge\psi\rceil_{t_2}$. Thus, since $\lceil\phi\wedge\psi\rceil_{t_2}\subseteq\lceil\phi\rceil_{t_2}$, $\mathcal{I}_{t_3}(\alpha)\subseteq\mathcal{I}_{t_2}(\alpha)$. Hence by P_{K7}, $\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)\subseteq\mathcal{B}_{t_3}(\alpha)$. Fix an arbitrary $\beta\in\mathcal{B}_{t_2}(\alpha)$. If $(\beta,t_2)\models\neg(\phi\wedge\psi)$ then $(\beta,t_2)\models(\phi\wedge\psi)\to\chi$. If $(\beta,t_2)\models\phi\wedge\psi$, then, by Remark 2, $(\beta,t_3)\models\phi\wedge\psi$ and, therefore, $\beta\in\mathcal{I}_{t_3}(\alpha)$. Hence $\beta\in\mathcal{B}_{t_3}(\alpha)$. Since $(\alpha,t_3)\models B\chi$, $(\beta,t_3)\models\chi$ and, therefore, $(\beta,t_3)\models(\phi\wedge\psi)\to\chi$. Since $(\phi\wedge\psi)\to\chi$ is Boolean (because ϕ, ψ and χ are), by Remark 2, $(\beta,t_2)\models(\phi\wedge\psi)\to\chi$. Thus, since $\beta\in\mathcal{B}_{t_2}(\alpha)$ was chosen arbitrarily, $(\alpha,t_2)\models B((\phi\wedge\psi)\to\chi)$.

Conversely, suppose that P_{K7} is violated. Then there exist t_1, t_2, t_3 and α such that $t_1\rightarrowtail t_2$, $t_1\rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha)\subseteq\mathcal{I}_{t_2}(\alpha)$ and $\mathcal{I}_{t_3}(\alpha)\cap\mathcal{B}_{t_2}(\alpha)\nsubseteq\mathcal{B}_{t_3}(\alpha)$. Let p, q and r be atomic propositions and construct a model where $||p||=\mathcal{I}_{t_2}(\alpha)\times T$, $||q||=\mathcal{I}_{t_3}(\alpha)\times T$ and $||r||=\mathcal{B}_{t_3}(\alpha)\times T$. Then, $(\alpha,t_3)\models Br$ and, since $\mathcal{I}_{t_3}(\alpha)\subseteq\mathcal{I}_{t_2}(\alpha)$, $\mathcal{I}_{t_3}(\alpha)=\lceil p\wedge q\rceil_{t_3}$ so that $(\alpha,t_3)\models I(p\wedge q)$. Thus, since $t_1\to t_3$, $(\alpha,t_1)\models\Diamond(I(p\wedge q)\wedge Br)$. By construction, $(\alpha,t_2)\models Ip$. Since

$\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \nsubseteq \mathcal{B}_{t_3}(\alpha)$, there exists a $\beta \in \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$ such that $\beta \notin \mathcal{B}_{t_3}(\alpha)$. Thus $(\beta, t_2) \models \neg r$; furthermore, since $\beta \in \mathcal{I}_{t_3}(\alpha)$, $(\beta, t_3) \models p \wedge q$ and, by Remark 2, $(\beta, t_2) \models p \wedge q$. Thus, $(\beta, t_2) \nvDash (p \wedge q) \rightarrow r$. Since $\beta \in \mathcal{B}_{t_2}(\alpha)$ it follows that $(\alpha, t_2) \nvDash B((p \wedge q) \rightarrow r)$. Hence, since $t_1 \rightarrowtail t_2$, $(\alpha, t_1) \nvDash \bigcirc(Ip \rightarrow B((p \wedge q) \rightarrow r))$ so that axiom $K7$ is falsified at (α, t_1).

(3) Fix a frame that satisfies property P_{K8}. Let ϕ, ψ and χ be Boolean formulas and let α and t_1 be such that $(\alpha, t_1) \models \Diamond(I\phi \wedge \neg B \neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi))$. Then there exists a t_2 such that $t_1 \rightarrowtail t_2$ and $(\alpha, t_2) \models I\phi \wedge \neg B \neg(\phi \wedge \psi) \wedge B(\psi \rightarrow \chi)$. Thus $\mathcal{I}_{t_2}(\alpha) = \lceil \phi \rceil_{t_2}$ and there exists a $\beta \in \mathcal{B}_{t_2}(\alpha)$ such that $(\beta, t_2) \models \phi \wedge \psi$. Fix an arbitrary t_3 such that $t_1 \rightarrowtail t_3$ and suppose that $(\alpha, t_3) \models I(\phi \wedge \psi)$. Then $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$. Since $\phi \wedge \psi$ is a Boolean formula and $(\beta, t_2) \models \phi \wedge \psi$, by Remark 2, $(\beta, t_3) \models \phi \wedge \psi$ and therefore $\beta \in \mathcal{I}_{t_3}(\alpha)$. Hence $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$. Furthermore, since ϕ is Boolean, by Remark 2, $\lceil \phi \rceil_{t_3} = \lceil \phi \rceil_{t_2}$. Thus, since $\lceil \phi \wedge \psi \rceil_{t_3} \subseteq \lceil \phi \rceil_{t_3}$ it follows that $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$. Hence, by property P_{K8}, $\mathcal{B}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Fix an arbitrary $\gamma \in \mathcal{B}_{t_3}(\alpha)$. Then $\gamma \in \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$ and, since $(\alpha, t_2) \models B(\psi \rightarrow \chi)$, $(\gamma, t_2) \models \psi \rightarrow \chi$. Since $\psi \rightarrow \chi$ is a Boolean formula, by Remark 2 $(\gamma, t_3) \models \psi \rightarrow \chi$. Since $\gamma \in \mathcal{I}_{t_3}(\alpha)$ and $\mathcal{I}_{t_3}(\alpha) = \lceil \phi \wedge \psi \rceil_{t_3}$, $(\gamma, t_3) \models \psi$. Thus $(\gamma, t_3) \models \chi$. Hence $(\alpha, t_3) \models B\chi$.

Conversely, fix a frame that does not satisfy property P_{K8}. Then there exist t_1, t_2, t_3 and α such that $t_1 \rightarrowtail t_2$, $t_1 \rightarrowtail t_3$, $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$, $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$ and $\mathcal{B}_{t_3}(\alpha) \nsubseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Let p, q and r be atomic propositions and construct a model where $||p|| = \mathcal{I}_{t_2}(\alpha) \times T$, $||q|| = \mathcal{I}_{t_3}(\alpha) \times T$ and $||r|| = (\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)) \times T$. Then $(\alpha, t_2) \models Ip$ and, since $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$, if $\omega \in \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t) \models p \wedge q$ for every $t \in T$. Thus, since $\mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha) \neq \varnothing$, $(\alpha, t_2) \models \neg B \neg(p \wedge q)$. Fix an arbitrary $\omega \in \mathcal{B}_{t_2}(\alpha)$; if $\omega \in \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t_2) \models r$; if $\omega \notin \mathcal{I}_{t_3}(\alpha)$ then $(\omega, t_2) \models \neg q$; in either case $(\omega, t_2) \models q \rightarrow r$. Thus $(\alpha, t_2) \models B(q \rightarrow r)$. Hence $(\alpha, t_2) \models Ip \wedge \neg B \neg(p \wedge q) \wedge B(q \rightarrow r)$ and thus $(\alpha, t_1) \models \Diamond(Ip \wedge \neg B \neg(p \wedge q) \wedge B(q \rightarrow r))$. Since $\mathcal{I}_{t_3}(\alpha) = \lceil q \rceil_{t_3}$ and $\mathcal{I}_{t_2}(\alpha) = \lceil p \rceil_{t_2}$ and, by Remark 2 , $\lceil p \rceil_{t_2} = \lceil p \rceil_{t_3}$ and $\mathcal{I}_{t_3}(\alpha) \subseteq \mathcal{I}_{t_2}(\alpha)$, if follows that $\mathcal{I}_{t_3}(\alpha) = \lceil p \wedge q \rceil_{t_3}$, so that $(\alpha, t_3) \models I(p \wedge q)$. Since $\mathcal{B}_{t_3}(\alpha) \nsubseteq \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$, there exists a $\beta \in \mathcal{B}_{t_3}(\alpha)$ such that $\beta \notin \mathcal{I}_{t_3}(\alpha) \cap \mathcal{B}_{t_2}(\alpha)$. Then $(\beta, t_3) \models \neg r$ and therefore $(\alpha, t_3) \models \neg Br$. Thus $(\alpha, t_3) \nvDash I(p \wedge q) \rightarrow Br$ and hence, $(\alpha, t_1) \nvDash \bigcirc(I(p \wedge q) \rightarrow Br)$, so that axiom $K8$ is falsified at (a, t_1).

(4) Let ϕ be a Boolean formula, $\alpha \in \Omega$, $t \in T$ and suppose that $(\alpha, t) \models I\phi \wedge \neg A \neg \phi$. Then $\mathcal{I}_t(\alpha) = \lceil \phi \rceil_t$ and there exist $\beta \in \Omega$ that $(\beta, t) \models \phi$. Thus $\mathcal{I}_t(\alpha) \neq \varnothing$ and, by the above property, $\mathcal{B}_t(\alpha) \neq \varnothing$. Fix an arbitrary formula ψ and suppose that $(\alpha, t) \models B\psi$. Then, $\forall \omega \in \mathcal{B}_t(\alpha)$, $(\omega, t) \models \psi$. Since $\mathcal{B}_t(\alpha) \neq \varnothing$, there exists a $\gamma \in \mathcal{B}_t(\alpha)$. Thus $(\gamma, t) \models \psi$ and hence $(\alpha, t) \models \neg B \neg \psi$.

Conversely, fix a frame that does not satisfy property P_{WC}. Then there exist $\alpha \in \Omega$ and $t \in T$ such that $\mathcal{I}_t(\alpha) \neq \varnothing$ while $\mathcal{B}_t(\alpha) = \varnothing$. Let p be an atomic proposition and construct a model where $\|p\| = \mathcal{I}_t(\alpha) \times T$. Then $(\alpha, t) \models Ip$. Furthermore, since $\mathcal{I}_t(\alpha) \neq \varnothing$, there exists a $\beta \in \mathcal{I}_t(\alpha)$. Thus $(\beta, t) \models p$ and hence $(\alpha, t) \models \neg A \neg p$. Since $\mathcal{B}_t(\alpha) = \varnothing$, $(\alpha, t) \models B\psi$ for every formula ψ, so that $(\alpha, t) \models Bp \wedge B \neg p$. Thus WC is falsified at (α, t). ■

Proof of the claim in Footnote 11. (BR3$'$ $\Longrightarrow$ BR3). Fix arbitrary $E, F \in \mathcal{E}$ such that $E \subseteq F$ and $R_F \cap E \neq \varnothing$. Then $E \cap F = E$, so that $(E \cap F) \in \mathcal{E}$ and $R_{E\cap F} = R_E$. Thus, by (BR3$'$), $R_E = R_F \cap E$.

(BR3 + BR1 $\Longrightarrow$ BR3$'$). Let $E, F \in \mathcal{E}$ be such that $(E \cap F) \in \mathcal{E}$ and $R_F \cap E \neq \varnothing$. By (BR1), $R_F \subseteq F$ so that $R_F \cap F = R_F$. Hence

$$R_F \cap (E \cap F) = R_F \cap E. \qquad (\dagger)$$

Thus $R_F \cap (E \cap F) \neq \varnothing$. Hence, since $E \cap F \subseteq F$, it follows from (BR3) that $R_{E\cap F} = R_F \cap (E \cap F)$. Thus, by (†), $R_{E\cap F} = R_F \cap E$. ■

In order to prove Proposition 16 we need the following lemma. We shall throughout denote the complement of a set E by $\urcorner E$.

Lemma 20 *Let* $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ *be a* rich *belief revision model. The, for every formula* $\phi \in \Phi_0$, $\|\phi\| = \varnothing$ *if and only if* ϕ *is a contradiction (that is,* $\neg\phi$ *is a tautology).*

Proof. If ϕ is a tautology then $\|\phi\| = \Omega$. If ϕ is a contradiction then $\neg\phi$ is a tautology and thus $\|\neg\phi\| = \urcorner \|\phi\| = \Omega$, so that $\|\phi\| = \varnothing$. If ϕ is neither a tautology nor a contradiction then it is equivalent to a formula of the form $\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{m} Q_{ij}\right)\right)$ where each Q_{ij} is either an atomic proposition or the negation of a atomic proposition (see Hamilton [22], Corollary 1.20, p. 17). By definition of rich model, for every formula $\bigwedge_{j=1}^{m} Q_{ij}$, there is a state ω_i such that $\omega_i \models \bigwedge_{j=1}^{m} Q_{ij}$. Thus $\|\phi\| = \left\|\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{m} Q_{ij}\right)\right\| = \bigcup_{i=1}^{n}\left\|\bigwedge_{j=1}^{m} Q_{ij}\right\| \supseteq \{\omega_1, ..., \omega_n\} \neq \varnothing$. ■

Proof of Proposition 16. Let $\langle \Omega, \mathcal{E}, \mathbb{O}, \mathbf{R}, V \rangle$ be a comprehensive belief revision model and define $K = \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{O}} \subseteq \|\psi\|\}$. First we show that K is deductively closed, that is, $K = [K]^{PL}$. If $\psi \in K$ then $\psi \in [K]^{PL}$, because $\psi \to \psi$ is a tautology; thus $K \subseteq [K]^{PL}$. To show that $[K]^{PL} \subseteq K$, let $\psi \in [K]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology. Since $\|\phi_1 \wedge ... \wedge \phi_n\| = \|\phi_1\| \cap ... \cap \|\phi_n\|$, and $\phi_i \in K$ (that is, $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi_i\|$) for all $i = 1, ..., n$, it follows that $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi_1 \wedge ... \wedge \phi_n\|$. Since $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology, $\|(\phi_1 \wedge ... \wedge \phi_n) \to \psi\| = \Omega$, that is, $\|\phi_1 \wedge ... \wedge \phi_n\| \subseteq \|\psi\|$. Thus $\mathbf{R}_{\mathbb{O}}(\alpha) \subseteq \|\psi\|$, that is, $\psi \in K$. Next we show that $[K]^{PL} \neq \Phi_0$ (consistency). By definition of one-stage revision frame (see Definition 14), $\mathbb{O} \neq \varnothing$; thus, by property BR2, $\mathbf{R}_{\mathbb{O}} \neq \varnothing$. Choose an arbitrary atomic proposition $p \in S$. Then $\|(p \wedge \neg p)\| = \varnothing$ and therefore $\mathbf{R}_{\mathbb{O}} \nsubseteq \|(p \wedge \neg p)\|$, so that $(p \wedge \neg p) \notin K$. Since $K = [K]^{PL}$, $(p \wedge \neg p) \notin [K]^{PL}$.

Next we show that AGM postulates (⊛1)-(⊛5a) and (⊛6)-(⊛8) are satisfied. For every formula $\phi \in \Phi_0$, define $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$ (note that, since the model is comprehensive, for every $\phi \in \Phi_0$, $\|\phi\| \in \mathcal{E}$).

($\circledast$1) Fix an arbitrary $\phi \in \Phi_0$. We need to show that $\{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$ is deductively closed. We omit this proof since it is a repetition of the argument given above for K.

($\circledast$2) Fix an arbitrary $\phi \in \Phi_0$. We need to show that $\phi \in K^{\circledast}(\phi)$, that is, that $\mathbf{R}_{\|\phi\|} \subseteq \|\phi\|$. This is an immediate consequence of property BR1 of Definition 14.

($\circledast$3) Fix an arbitrary $\phi \in \Phi_0$. We need to show that $K^{\circledast}(\phi) \subseteq [K \cup \{\phi\}]^{PL}$. Let $\psi \in K^{\circledast}(\phi)$, that is, $\mathbf{R}_{\|\phi\|} \subseteq \|\psi\|$. First we show that $(\phi \to \psi) \in K$, that is, $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi \to \psi\| = \neg\|\phi\| \cup \|\psi\|$. If $\mathbf{R}_{\mathbb{O}} \subseteq \neg\|\phi\|$ there is nothing to prove. Suppose therefore that $\mathbf{R}_{\mathbb{O}} \cap \|\phi\| \neq \varnothing$. Then, by property BR4 of Definition 14,

$$\mathbf{R}_{\|\phi\|} = \mathbf{R}_{\mathbb{O}} \cap \|\phi\|. \tag{3}$$

Fix an arbitrary $\omega \in \mathbf{R}_{\mathbb{O}}$. If $\omega \notin \|\phi\|$ then $\omega \in \|\neg\phi\|$ and thus $\omega \in \|\phi \to \psi\|$; if $\omega \in \|\phi\|$, then, by (3), $\omega \in \mathbf{R}_{\|\phi\|}$ and thus, since $\mathbf{R}_{\|\phi\|} \subseteq \|\psi\|$, $\omega \in \|\psi\|$, so that $\omega \in \|\phi \to \psi\|$. Hence $(\phi \to \psi) \in K$. It follows that $\psi \in [K \cup \{\phi\}]^{PL}$.

($\circledast$4) Fix an arbitrary $\phi \in \Phi_0$. We need to show that if $\neg\phi \notin K$ then $[K \cup \{\phi\}]^{PL} \subseteq K^{\circledast}(\phi)$. Suppose that $\neg\phi \notin K$, that is, $\mathbf{R}_{\mathbb{O}} \not\subseteq \|\neg\phi\| = \neg\|\phi\|$, that is, $\mathbf{R}_{\mathbb{O}} \cap \|\phi\| \neq \varnothing$. Then by property BR4 of Definition 14,

$$\mathbf{R}_{\|\phi\|} = \mathbf{R}_{\mathbb{O}} \cap \|\phi\|. \tag{4}$$

Let $\chi \in [K \cup \{\phi\}]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K \cup \{\phi\}$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology. We want to show that $\chi \in K^{\circledast}(\phi)$, that is, $\mathbf{R}_{\|\phi\|} \subseteq \|\chi\|$. Since $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology, $\|(\phi_1 \wedge ... \wedge \phi_n) \to \chi\| = \Omega$, that is, $\|(\phi_1 \wedge ... \wedge \phi_n)\| \subseteq \|\chi\|$. If $\phi_i \in K$ for every $i = 1, ..., n$, then $\mathbf{R}_{\mathbb{O}} \subseteq \|(\phi_1 \wedge ... \wedge \phi_n)\|$ and thus $\mathbf{R}_{\mathbb{O}} \subseteq \|\chi\|$. Hence, by (4), $\mathbf{R}_{\|\phi\|} \subseteq \|\chi\|$. If, for some $j = 1, ..., n$, $\phi_j \notin K$, then we can assume (renumbering the formulas, if necessary) that $\phi_i \in K$, for every $i = 1, ..., n-1$, and $\phi_n \notin K$, which implies (since $\phi_i \in K \cup \{\phi\}$ for all $i = 1, ..., n$) that $\phi_n = \phi$. Since, by hypothesis, $(\phi_1 \wedge ... \wedge \phi_{n-1} \wedge \phi) \to \chi$ is a tautology and, furthermore, it is propositionally equivalent to $(\phi_1 \wedge ... \wedge \phi_{n-1}) \to (\phi \to \chi)$, $\|(\phi_1 \wedge ... \wedge \phi_{n-1}) \to (\phi \to \chi)\| = \Omega$, that is, $\|\phi_1 \wedge ... \wedge \phi_{n-1}\| \subseteq \|\phi \to \chi\|$, so that, since $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi_1 \wedge ... \wedge \phi_{n-1}\|$ (because $\phi_1, ..., \phi_{n-1} \in K$), $\mathbf{R}_{\mathbb{O}} \subseteq \|\phi \to \chi\|$. Thus $\mathbf{R}_{\mathbb{O}} \cap \|\phi\| \subseteq \|\phi\| \cap \|\phi \to \chi\| \subseteq \|\chi\|$. Hence, by (4), $\mathbf{R}_{\|\phi\|} \subseteq \|\chi\|$.

($\circledast$5a) If ϕ is a contradiction, $\|\phi\| = \varnothing$. By property BR1, $\mathbf{R}_{\|\phi\|} \subseteq \|\phi\|$. Hence $\mathbf{R}_{\|\phi\|} = \varnothing$ and, therefore, $K^{\circledast}(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\} = \Phi_0$.

($\circledast$6) If $\phi \leftrightarrow \psi$ is a tautology then $\|\phi \leftrightarrow \psi\| = \Omega$, that is, $\|\phi\| = \|\psi\|$. Hence $\mathbf{R}_{\|\phi\|} = \mathbf{R}_{\|\psi\|}$ and thus $K^{\circledast}(\phi) = \{\chi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\chi\|\} = \{\chi \in \Phi_0 : \mathbf{R}_{\|\psi\|} \subseteq \|\chi\|\} = K^{\circledast}(\psi)$.

($\circledast$7) Fix arbitrary $\phi, \psi \in \Phi_0$. We need to show that $K^{\circledast}(\phi \wedge \psi) \subseteq [K^{\circledast}(\phi) \cup \{\psi\}]^{PL}$. Let $\chi \in K^{\circledast}(\phi \wedge \psi)$, that is,

$$\mathbf{R}_{\|\phi \wedge \psi\|} \subseteq \|\chi\|. \tag{5}$$

First we show that $\mathbf{R}_{\|\phi\|} \subseteq \|(\phi \wedge \psi) \to \chi\| = \urcorner \|\phi \wedge \psi\| \cup \|\chi\|$. If $\mathbf{R}_{\|\phi\|} \subseteq \urcorner \|\phi \wedge \psi\|$ there is nothing to prove. Suppose therefore that $\mathbf{R}_{\|\phi\|} \cap \|\phi \wedge \psi\| \neq \varnothing$. Then, by property (BR3) (with $E = \|\phi \wedge \psi\|$ and $F = \|\phi\|$),

$$\mathbf{R}_{\|\phi\|} \cap \|\phi \wedge \psi\| = \mathbf{R}_{\|\phi \wedge \psi\|}. \tag{6}$$

Fix an arbitrary $\omega \in \mathbf{R}_{\|\phi\|}$. If $\omega \notin \|\phi \wedge \psi\|$ then $\omega \in \|\neg(\phi \wedge \psi)\|$ and thus $\omega \in \|(\phi \wedge \psi) \to \chi\|$; if $\omega \in \|\phi \wedge \psi\|$, then by (5) and (6), $\omega \in \|\chi\|$ so that $\omega \in \|(\phi \wedge \psi) \to \chi\|$. Hence $\mathbf{R}_{\|\phi\|} \subseteq \|(\phi \wedge \psi) \to \chi\|$, that is, $(\phi \wedge \psi \to \chi) \in K^{\circledast}(\phi)$. Since $(\phi \wedge \psi \to \chi)$ is tautologically equivalent to $(\psi \to (\phi \to \chi))$, and, by ($\circledast$1) (proved above), $K^{\circledast}(\phi)$ is deductively closed, $(\psi \to (\phi \to \chi)) \in K^{\circledast}(\phi)$. Furthermore, by ($\circledast$2) $\phi \in K^{\circledast}(\phi)$. Thus $\{\psi, (\psi \to (\phi \to \chi)), \phi\} \subseteq K^{\circledast}(\phi) \cup \{\psi\}$ and therefore $\chi \in [K^{\circledast}(\phi) \cup \{\psi\}]^{PL}$.

($\circledast$8) Fix arbitrary $\phi, \psi \in \Phi_0$. We need to show that if $\neg\psi \notin K^{\circledast}(\phi)$ then $[K^{\circledast}(\phi) \cup \{\psi\}]^{PL} \subseteq K^{\circledast}(\phi \wedge \psi)$. Suppose that $\neg\psi \notin K^{\circledast}(\phi)$, that is, $\mathbf{R}_{\|\phi\|} \nsubseteq \urcorner \|\psi\| = \|\neg\psi\|$, i.e. $\mathbf{R}_{\|\phi\|} \cap \|\psi\| \neq \varnothing$. Then by property (BR3$'$) (see footnote 11)

$$\mathbf{R}_{\|\phi \wedge \psi\|} = \mathbf{R}_{\|\phi\|} \cap \|\psi\|. \tag{7}$$

Let $\chi \in [K^{\circledast}(\phi) \cup \{\psi\}]^{PL}$, that is, there exist $\phi_1, ..., \phi_n \in K^{\circledast}(\phi) \cup \{\psi\}$ such that $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology. We want to show that $\chi \in K^{\circledast}(\phi \wedge \psi)$, that is, $\mathbf{R}_{\|\phi \wedge \psi\|} \subseteq \|\chi\|$. Since $(\phi_1 \wedge ... \wedge \phi_n) \to \chi$ is a tautology, $\|(\phi_1 \wedge ... \wedge \phi_n) \to \chi\| = \Omega$, that is, $\|(\phi_1 \wedge ... \wedge \phi_n)\| \subseteq \|\chi\|$. If $\phi_i \in K^{\circledast}(\phi)$ for every $i = 1, ..., n$, then $\mathbf{R}_{\|\phi\|} \subseteq \|(\phi_1 \wedge ... \wedge \phi_n)\|$ and thus $\mathbf{R}_{\|\phi\|} \subseteq \|\chi\|$. Hence, by (7), $\mathbf{R}_{\|\phi \wedge \psi\|} \subseteq \|\chi\|$. If, for some $j = 1, ..., n$, $\phi_j \notin K^{\circledast}(\phi)$, then we can assume (renumbering the formulas, if necessary) that $\phi_i \in K^{\circledast}(\phi)$, for every $i = 1, ..., n-1$, and $\phi_n \notin K^{\circledast}(\phi)$, which implies (since $\phi_i \in K^{\circledast}(\phi) \cup \{\psi\}$ for all $i = 1, ..., n$) that $\phi_n = \psi$. Since, by hypothesis, $(\phi_1 \wedge ... \wedge \phi_{n-1} \wedge \psi) \to \chi$ is a tautology and it is propositionally equivalent to $(\phi_1 \wedge ... \wedge \phi_{n-1}) \to (\psi \to \chi)$, $\|(\phi_1 \wedge ... \wedge \phi_{n-1}) \to (\psi \to \chi)\| = \Omega$, that is, $\|(\phi_1 \wedge ... \wedge \phi_{n-1})\| \subseteq \|\psi \to \chi\|$, so that, since $\mathbf{R}_{\|\phi\|} \subseteq \|(\phi_1 \wedge ... \wedge \phi_{n-1})\|$ (because $\phi_1, ..., \phi_{n-1} \in K^{\circledast}(\phi)$) $\mathbf{R}_{\|\phi\|} \subseteq \|\psi \to \chi\|$. Thus $\mathbf{R}_{\|\phi\|} \cap \|\psi\| \subseteq \|\psi\| \cap \|\psi \to \chi\| \subseteq \|\chi\|$. Hence, by (7), $\mathbf{R}_{\|\phi \wedge \psi\|} \subseteq \|\chi\|$.

Next we show that, if the model is rich, then ($\circledast 5b$) is satisfied.

($\circledast 5b$) If the model is rich and ϕ is not a contradiction, then by Lemma 20 $\|\phi\| \neq \varnothing$. Thus, by property BR2, $\mathbf{R}_{\|\phi\|} \neq \varnothing$. Fix an arbitrary $p \in S$. Since $\|p \wedge \neg p\| = \varnothing$, it follows that $\mathbf{R}_{\|\phi\|} \nsubseteq \|p \wedge \neg p\|$ and therefore $(p \wedge \neg p) \notin K^{\circledast}(\phi)$. Since, by ($\circledast$1) (proved above), $K^{\circledast}(\phi) = [K^{\circledast}(\phi)]^{PL}$, it follows that $[K^{\circledast}(\phi)]^{PL} \neq \Phi_0$. ■

Before proving Proposition 17 we note the following.

Definition 21 *A set* $\mathcal{E} \subseteq 2^{\Omega}$ *of events is called an* algebra *if it satisfies the following properties: (1)* $\Omega \in \mathcal{E}$*, (2) if* $E \in \mathcal{E}$ *then* $\neg E \in \mathcal{E}$ *and (3) if* $E, F \in \mathcal{E}$ *then* $(E \cup F) \in \mathcal{E}$*.*[15]

Remark 22 *In a belief revision frame where* $\mathcal{E}$ *is an algebra, property* $(BR3')$ *(see Footnote 11) is equivalent to:* $\forall \omega \in \Omega$, $\forall E, F \in \mathcal{E}$,

$(BR3'')$ if $R_F(\omega) \cap E \neq \varnothing$ then $R_{E \cap F}(\omega) = R_F(\omega) \cap E$.

Proof of Proposition 17. Let $\mathbb{M}$ be the set of maximally consistent sets (MCS) of formulas for a propositional logic whose set of formulas is Φ_0. For any $F \subseteq \Phi_0$ let $\mathbb{M}_F = \{\omega \in \mathbb{M} : F \subseteq \omega\}$. By Lindenbaum's lemma, $\mathbb{M}_F \neq \varnothing$ if and only if F is a consistent set, that is, $[F]^{PL} \neq \Phi_0$. To simplify the notation, for $\phi \in \Phi_0$ we write $\mathbb{M}_\phi$ rather than $\mathbb{M}_{\{\phi\}}$.
Define the following belief revision frame: $\Omega = \mathbb{M}$, $\mathcal{E} = \{\mathbb{M}_\phi : \phi \in \Phi_0\}$, $\mathbb{O} = \Omega$, $\mathbf{R}_\Omega = \mathbb{M}_K$ and, for every $\phi \in \Phi_0$,

$$\mathbf{R}_{\mathbb{M}_\phi} = \begin{cases} \varnothing & \text{if } \phi \text{ is a contradiction} \\ \mathbb{M}_\phi \cap \mathbb{M}_K & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing \\ \mathbb{M}_{K^{\circledast}(\phi)} & \text{if } \phi \text{ is consistent and } \mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing. \end{cases}$$

First of all, note that $\mathcal{E}$ is an algebra. (1) $\mathbb{M} \in \mathcal{E}$ since $\mathbb{M} = \mathbb{M}_{(p \vee \neg p)}$ where p is any atomic proposition. (2) Let $\phi \in \Phi_0$. Then $\mathbb{M}_\phi \in \mathcal{E}$ and $\neg \|\mathbb{M}_\phi\| = \{\omega \in \mathbb{M} : \phi \notin \omega\}$. By definition of MCS, for every $\omega \in \mathbb{M}$, $\phi \notin \omega$ if and only if $\neg\phi \in \omega$. Thus $\neg \|\mathbb{M}_\phi\| = \mathbb{M}_{\neg\phi} \in \mathcal{E}$. (3) Let $\phi, \psi \in \Phi_0$. Then $\mathbb{M}_\phi, \mathbb{M}_\psi \in \mathcal{E}$ and, by definition of MCS, $\mathbb{M}_\phi \cup \mathbb{M}_\psi = \mathbb{M}_{\phi \vee \psi} \in \mathcal{E}$.
Next we show that the frame so defined is indeed a one-stage revision frame, that is, it satisfies properties (BR1)-(BR4) of Definition 14.

(BR1) We need to shows that, for every $\phi \in \Phi_0$, $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \mathbb{M}_\phi$. If ϕ is a contradiction, then $\mathbb{M}_\phi = \varnothing$ and, by construction, $\mathbf{R}_{\mathbb{M}_\phi} = \varnothing$. If ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$ then $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K \subseteq \mathbb{M}_\phi$. If ϕ is consistent and $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$ then $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^{\circledast}(\phi)}$. Now, if $\omega' \in \mathbb{M}_{K^{\circledast}(\phi)}$ then $K^{\circledast}(\phi) \subseteq \omega'$ and, since by AGM postulate $(\circledast 2)$, $\phi \in K^{\circledast}(\phi)$, it follows that $\phi \in \omega'$, that is, $\omega' \in \mathbb{M}_\phi$. Hence $\mathbb{M}_{K^{\circledast}(\phi)} \subseteq \mathbb{M}_\phi$.

(BR2) We need to show that, for every $\omega \in \Omega$ and $\phi \in \Phi_0$, if $\mathbb{M}_\phi \neq \varnothing$ then $\mathbf{R}_{\mathbb{M}_\phi} \neq \varnothing$. Now, $\mathbb{M}_\phi \neq \varnothing$ if and only if ϕ is a consistent formula, in which case either $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K$ if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$ or $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^{\circledast}(\phi)}$ if $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$. In the latter case, by AGM postulate $(\circledast 5b)$, $K^{\circledast}(\phi)$ is a consistent set and therefore, by Lindenbaum's lemma, $\mathbb{M}_{K^{\circledast}(\phi)} \neq \varnothing$.

(BR3) Instead of proving (BR3) we prove the equivalent $(BR3'')$ (see Remark 22 and footnote 11), that is, we show that, for every $\phi, \psi \in \Phi_0$, if $\mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi \neq \varnothing$ then $\mathbf{R}_{\mathbb{M}_\phi \cap \mathbb{M}_\psi} = \mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi$. First note that, by definition of MCS, $\mathbb{M}_\phi \cap \mathbb{M}_\psi = \mathbb{M}_{\phi \wedge \psi}$. Since $\mathbf{R}_{\mathbb{M}_\phi} \neq \varnothing$, ϕ is a consistent formula and either $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K$, if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$, or $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^{\circledast}(\phi)}$, if $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$. Suppose first that

[15] Note that from (1) and (2) it follows that $\varnothing \in \mathcal{E}$ and from (2) and (3) it follows that if $E, F \in \mathcal{E}$ then $(E \cap F) \in \mathcal{E}$. In fact, from $E, F \in \mathcal{E}$ we get, by (2), $\neg E, \neg F \in \mathcal{E}$ and thus, by (3), $(\neg E \cup \neg F) \in \mathcal{E}$; using (2) again we get that $\neg(\neg E \cup \neg F) = (E \cap F) \in \mathcal{E}$.

$\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$. Then $\mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi = \mathbb{M}_\phi \cap \mathbb{M}_K \cap \mathbb{M}_\psi = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K \neq \varnothing$, and, thus, by construction, $\mathbf{R}_{\mathbb{M}_{\phi\wedge\psi}} = \mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K$. Thus $\mathbf{R}_{\mathbb{M}_\phi \cap \mathbb{M}_\psi} = \mathbf{R}_{\mathbb{M}_{\phi\wedge\psi}} = \mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi$. Suppose now that $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$. Then, by construction, $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^\circledast(\phi)}$ and, since $\mathbb{M}_{\phi\wedge\psi} \cap \mathbb{M}_K = \mathbb{M}_\phi \cap \mathbb{M}_\psi \cap \mathbb{M}_K \subseteq \mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$ we also have that $\mathbf{R}_{\mathbb{M}_{\phi\wedge\psi}} = \mathbb{M}_{K^\circledast(\phi\wedge\psi)}$. Thus we need to show that $\mathbb{M}_{K^\circledast(\phi\wedge\psi)} = \mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi$. By hypothesis, $\mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi \neq \varnothing$, that is, $\mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi \neq \varnothing$. This implies that $\neg\psi \notin K^\circledast(\phi)$.[16] Hence, by AGM postulates ($\circledast$7) and ($\circledast$8),

$$\left[K^\circledast(\phi) \cup \{\psi\}\right]^{PL} = K^\circledast(\phi \wedge \psi). \tag{8}$$

Let $\omega \in \mathbb{M}_{K^\circledast(\phi\wedge\psi)}$. Then $K^\circledast(\phi\wedge\psi) \subseteq \omega$ and, since $K^\circledast(\phi) \cup \{\psi\} \subseteq \left[K^\circledast(\phi) \cup \{\psi\}\right]^{PL}$, it follows from (8) that $K^\circledast(\phi) \cup \{\psi\} \subseteq \omega$. Thus $\omega \in \mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi$. Conversely, let $\omega \in \mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi$. Then $K^\circledast(\phi) \cup \{\psi\} \subseteq \omega$. Hence, by definition of MCS, $[K^\circledast(\phi) \cup \{\psi\}]^{PL} \subseteq \omega$. It follows from (8) that $K^\circledast(\phi \wedge \psi) \subseteq \omega$, that is, $\omega \in \mathbb{M}_{K^\circledast(\phi\wedge\psi)}$. Thus $\mathbb{M}_{K^\circledast(\phi\wedge\psi)} = \mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi$, that is, $\mathbf{R}_{\mathbb{M}_\phi \cap \mathbb{M}_\psi} = \mathbf{R}_{\mathbb{M}_\phi} \cap \mathbb{M}_\psi$.
(BR4) Since $\mathbb{O} = \Omega$ and, by construction, $\mathbf{R}_\Omega = \mathbb{M}_K$, we need to show that, for every formula ϕ, if $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$ then $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K$. But this is true by construction.

Consider now the model based on this frame given by the following valuation: for every atomic proposition p and for every $\omega \in \Omega$, $\omega \models p$ if and only if $p \in \omega$. It is well-known that in this model, for every formula ϕ,[17]

$$\|\phi\| = \mathbb{M}_\phi. \tag{9}$$

Note also the following (see Theorem 2.20 in Chellas, [13], p. 57): $\forall F \subseteq \Phi_0, \forall \phi \in \Phi_0$,

$$\phi \in [F]^{PL} \text{ if and only if } \phi \in \omega,\ \forall \omega \in \mathbb{M}_F. \tag{10}$$

We want to show that (1) $K = \{\psi \in \Phi_0 : \mathbf{R}_\mathbb{O} \subseteq \|\psi\|\}$ and, (2) for every $\phi \in \Phi_0$, $K^\circledast(\phi) = \{\psi \in \Phi_0 : \mathbf{R}_{\|\phi\|} \subseteq \|\psi\|\}$.

(1) By construction, $\mathbb{O} = \Omega$ and $\mathbf{R}_\Omega = \mathbb{M}_K$ and, by (9), for every formula ψ, $\|\psi\| = \mathbb{M}_\psi$. Thus we need to show that, for every formula ψ, $\psi \in K$ if and only if $\mathbb{M}_K \subseteq \mathbb{M}_\psi$. Let $\psi \in K$ and fix an arbitrary $\omega \in \mathbb{M}_K$. Then $K \subseteq \omega$ and thus $\psi \in \omega$, so that $\omega \in \mathbb{M}_\psi$. Conversely, suppose that $\mathbb{M}_K \subseteq \mathbb{M}_\psi$. Then $\psi \in \omega$, for every $\omega \in \mathbb{M}_K$. Thus, by (10), $\psi \in [K]^{PL}$. By AGM postulate ($\circledast$1), $K = [K]^{PL}$. Hence $\psi \in K$.

(2) Fix an arbitrary formula ϕ. First we show that $K^\circledast(\phi) \subseteq \{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{M}_\phi} \subseteq$

[16] Suppose that $\neg\psi \in K^\circledast(\phi)$. Then, for every $\omega \in \mathbb{M}_{K^\circledast(\phi)}$, $\omega \supseteq K^\circledast(\phi)$ and, therefore, $\neg\psi \in \omega$. But this implies that $\mathbb{M}_{K^\circledast(\phi)} \cap \mathbb{M}_\psi = \varnothing$.

[17] The proof is by induction on the complexity of ϕ. If $\phi = p$, for some sentence letter p, then the statement is true by construction. Now suppose that the statement is true of $\phi_1, \phi_2 \in \Phi_0$; we want to show that it is true for $\neg\phi_1$ and for $(\phi_1 \vee \phi_2)$. By definition, $\omega \models \neg\phi_1$ if and only if $\omega \not\models \phi_1$ if and only if (by the induction hypothesis) $\phi_1 \notin \omega$ if and only if, by definition of MCS, $\neg\phi_1 \in \omega$. By definition, $\omega \models (\phi_1 \vee \phi_2)$ if and only if either $\omega \models \phi_1$, in which case, by the induction hypothesis, $\phi_1 \in \omega$, or $\omega \models \phi_2$, in which case, by the induction hypothesis, $\phi_2 \in \omega$. By definition of MCS, $(\phi_1 \vee \phi_2) \in \omega$ if and only if either $\phi_1 \in \omega$ or $\phi_2 \in \omega$.

$\|\psi\|\}$. Let $\psi \in K^{\circledast}(\phi)$. If ϕ is a contradiction, $\mathbf{R}_{\mathbb{M}_\phi} = \varnothing$ and there is nothing to prove. If ϕ is consistent then two cases are possible: (i) $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$ and (ii) $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$. In case (i) $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^{\circledast}(\phi)}$. Since, by hypothesis, $\psi \in K^{\circledast}(\phi)$, $\mathbb{M}_{K^{\circledast}(\phi)} \subseteq \mathbb{M}_\psi$ and, by (9), $\mathbb{M}_\psi = \|\psi\|$. Thus $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \|\psi\|$. In case (ii), $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K$. First of all, note that $\mathbb{M}_\phi \cap \mathbb{M}_K = \mathbb{M}_{K\cup\{\phi\}}$. Secondly, it must be that $\neg\phi \notin K$.[18] Hence, by AGM postulates ($\circledast$3) and ($\circledast$4), $K^{\circledast}(\phi) = [K \cup \{\phi\}]^{PL}$. Since, by hypothesis, $\psi \in K^{\circledast}(\phi)$, $\psi \in [K \cup \{\phi\}]^{PL}$. Hence, by (10), $\psi \in \omega$, for every $\omega \in \mathbb{M}_{K\cup\{\phi\}}$. Thus $\mathbb{M}_{K\cup\{\phi\}} \subseteq \mathbb{M}_\psi$. Hence, since $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K = \mathbb{M}_{K\cup\{\phi\}}$, $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \mathbb{M}_\psi$. Next we show that $\{\psi \in \Phi_0 : \mathbf{R}_{\mathbb{M}_\phi} \subseteq \|\psi\|\} \subseteq K^{\circledast}(\phi)$. Suppose that $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \|\psi\| = \mathbb{M}_\psi$. If ϕ is a contradiction, then, by AGM postulate ($\circledast 5a$), $K^{\circledast}(\phi) = \Phi_0$ and, therefore, $\psi \in K^{\circledast}(\phi)$. If ϕ is not a contradiction, then either (i) $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$ or (ii) $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$. In case (i) $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_{K^{\circledast}(\phi)}$. Thus, since, by hypothesis, $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \mathbb{M}_\psi$, we have that $\mathbb{M}_{K^{\circledast}(\phi)} \subseteq \mathbb{M}_\psi$, that is, for every $\omega \in \mathbb{M}_{K^{\circledast}(\phi)}$, $\psi \in \omega$. By (10) $\psi \in [K^{\circledast}(\phi)]^{PL}$ and, by AGM postulate ($\circledast$1), $[K^{\circledast}(\phi)]^{PL} = K^{\circledast}(\phi)$. Thus $\psi \in K^{\circledast}(\phi)$. In case (ii), $\mathbf{R}_{\mathbb{M}_\phi} = \mathbb{M}_\phi \cap \mathbb{M}_K$. Thus, since, by hypothesis, $\mathbf{R}_{\mathbb{M}_\phi} \subseteq \mathbb{M}_\psi$, we have that $\mathbb{M}_\phi \cap \mathbb{M}_K \subseteq \mathbb{M}_\psi$, from which it follows (since $\mathbb{M}_\phi \cap \mathbb{M}_K = \mathbb{M}_{K\cup\{\phi\}}$) that $\mathbb{M}_{K\cup\{\phi\}} \subseteq \mathbb{M}_\psi$. This means that, for every $\omega \in \mathbb{M}_{K\cup\{\phi\}}$, $\psi \in \omega$. Hence, by (10), $\psi \in [K \cup \{\phi\}]^{PL}$. Since $\mathbb{M}_\phi \cap \mathbb{M}_K \neq \varnothing$, $\neg\phi \notin K$ and, therefore, by AGM postulates ($\circledast$3) and ($\circledast$4), $K^{\circledast}(\phi) = [K \cup \{\phi\}]^{PL}$. Thus $\psi \in K^{\circledast}(\phi)$. ■

Proof of Proposition 18. In view of Corollary 12 it is sufficient to show that (1) axiom CMP is characterized by property P_{CMP} and (2) I_{con} is characterized by seriality of $\mathcal{I}_t$.

(1) Fix an arbitrary model based on a frame that satisfies property P_{CMP}. Fix arbitrary $\alpha \in \Omega$, $t_0 \in T$ and Boolean formula ϕ and suppose that $(\alpha, t_0) \models \neg A \neg \phi$. Let $E = \lceil\phi\rceil_{t_0}$. Then $E \neq \varnothing$. We want to show that $(\alpha, t_0) \models \Diamond I\phi$. By property P_{CMP}, there exists a $t \in T$ such that $t_0 \rightarrowtail t$ and $\mathcal{I}_t(\omega) = E$. Since ϕ is Boolean, $\lceil\phi\rceil_{t_0} = \lceil\phi\rceil_t$. Thus $(\alpha, t) \models I\phi$ and hence $(\alpha, t_0) \models \Diamond I\phi$.

Conversely, fix a frame that violates property P_{CMP}. Then there exist $\alpha \in \Omega$, $t_0 \in T$ and $E \in 2^{\Omega}\backslash\{\varnothing\}$ such that, $\forall t \in T$, if $t_0 \rightarrowtail t$ then $\mathcal{I}_t(\omega) \neq E$. Construct a model where, for some atomic proposition p, $\|p\| = E \times T$. Then, $\forall t \in T$ with $t_0 \rightarrowtail t$, $(\alpha, t) \not\models Ip$. Thus $(\alpha, t_0) \not\models \Diamond Ip$.

(2) Fix an arbitrary model based on a frame where $\mathcal{I}_t$ is serial and suppose that $\neg I(\phi \wedge \neg\phi)$ is not valid, that is, for some $\alpha \in \Omega$, $t \in T$ and formula ϕ, $(\alpha, t) \models I(\phi \wedge \neg\phi)$. Then $\mathcal{I}_t(\alpha) = \lceil\phi \wedge \neg\phi\rceil_t$. But $\lceil\phi \wedge \neg\phi\rceil_t = \varnothing$, while by seriality $\mathcal{I}_t(\alpha) \neq \varnothing$, yielding a contradiction.

Conversely, fix a frame where $\mathcal{I}_t$ is not serial, that is, there exist $t \in T$ and $\alpha \in \Omega$ such that $\mathcal{I}_t(\alpha) = \varnothing$. Since, for every formula ϕ, $\lceil\phi \wedge \neg\phi\rceil_t = \varnothing$, it follows that $(\alpha, t) \models I(\phi \wedge \neg\phi)$ so that $\neg I(\phi \wedge \neg\phi)$ is not valid. ■

[18] If $\neg\phi \in K$ then $\neg\phi \in \omega$ for every $\omega \in \mathbb{M}_K$ and therefore $\mathbb{M}_\phi \cap \mathbb{M}_K = \varnothing$.

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