FORKING IN SHORT AND TAME ABSTRACT ELEMENTARY CLASSES

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Abstract. We develop a notion of forking for Galois-types in the context of Abstract Elementary Classes (AECs). Under the hypotheses that an AEC K is tame, type-short, and failure of an order-property, we consider

Definition 1. *Let $M_0 \prec N$ be models from K and A be a set. We say that the Galois-type of A over M* does not fork over M_0, *written $A \underset{M_0}{\perp} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that Galois-type of a over N^- is realized in M_0.*

Assuming property (E) (see Definition 3.3) we show that this non-forking is a well behaved notion of independence, in particular satisfies symmetry and uniqueness and has a corresponding U-rank. We find conditions for a universal local character, in particular derive superstability-like property from little more than categoricity in a "big cardinal". Finally, we show that under large cardinal axioms the proofs are simpler and the non-forking is more powerful.

In [BGKV] it is established that this notion of non-forking is the only independence relation possible.

Contents

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1. Introduction

Much of first order model theory has focused on Shelah's forking. In the last fifteen years, significant progress has been made towards understanding of unstable theories, especially simple theories (Kim [Ki98] and Kim and Pillay [KP97]), NIP theories (see surveys by Adler [Ad09] and Simon [Si]), and, most recently, NTP_2 (Ben-Yaacov and Chernikov [BYCh] and Chenikov, Kaplan and Shelah [CKS1007]).

In the work on classification theory for Abstract Elementary Classes (AECs), such a nicely behaved notion is not known to exist. However, much work has been done towards this goal. Already in 1970, Shelah [Sh3] introduced splitting as a weak independence notion for a nonelementary context that is now known as homogeneous model theory (see Grossberg and Lessmann [GrLe02] and Buechler and Lessmann [BuLe03]). For the more general cases of classes axiomatizeable by an $L_{\lambda^+,\omega}$ sentence or AECs, very little is known in this direction, although there have been several attempts.

In [Sh394], Shelah introduced analogues of splitting and strong splitting for AECs; splitting was combined with tameness by Grossberg and VanDieren in [GV06c] and [GV06a] to obtain upward categoricity transfer theorems. Makkai and Shelah [MaSh285] studied the case when a class is axiomatized by an $L_{\kappa,\omega}$ theory and κ is strongly compact. They managed to obtain an eventual categoricity theorem by introducing a forking-like relation on types. In this particular case, Galois types (defined in §2) can be identified with complete set of formulas taken from a fragment of $L_{\kappa,\kappa}$. In their paper, Makkai and Shelah assumed not only that κ is strongly compact but also that the class of structures is categorical in some λ^+ where $\lambda \geq \beth_{(2^\kappa)^+}$.

Motivated by a test question of Grossberg[1], Shelah in [Sh576], [Sh600], and [Sh705] (the last two appeared as chapters II and III of [Sh:h]) and Jarden and Shelah [JrSh875] have dealt with the problem whether $I(\lambda, K) = 1 \leq I(\lambda^+, K) < 2^{\lambda^+}$ implies existence of model of cardinality λ^{++}. While this question is still open (even under strong set-thoeretic assumptions), Shelah managed to get several approximations. For this, he needed to discover and develop a very rich conceptual infrastructure that occupies more than 500 pages. One of the more important notions is that of good λ-frame. This is a forking-like relation defined using Galois-types over models of cardinality λ. In establishing existence and basic properties, Shelah makes strong use of a "few models" hypotheses in several cardinals and also set-theoretic principles, such as cardinal arithmetic and the non-saturation of certain weak diamond ideals.

Our paper is an extension and generalization of the above results of Makkai and Shelah.

[1]This first appeared in Grossberg's 1980 MSc thesis and was inserted by him into [Sh88]. In fall 1994 Grossberg managed to convince Shelah that this is a central problem, see [Sh576].

We deal with a more general situation than [MaSh285], as our class is assumed to be an AEC that doesn't have a specific logic that axiomatizes it. Instead of categoricity, we assume Galois-stablity and, instead of their large cardinal assumption, we assume the (weaker) model theoretic properties of tameness and type-shortness. Our approach is orthogonal to Shelah's recent work on good λ-frames and we manage to obtain a forking notion on the class of all models above a natural threshold size (instead of models of a single cardinality). Instead of using $I(\lambda^{+n}, K) < 2^{\lambda^{+n}}$ for all $1 \leq n < \omega$, we assume the lack of an order property, which follows from few models in a single big cardinal. Unlike Shelah, our treatment does not make use of diamond-like principles as we work in ZFC. Also, our dependence relation is closer to the first-order notion than good frames, which allows us to mimic some first-order arguments.

Unfortunately, there is no free lunch and we pay for this luxury. Our payment is essentially in assuming tameness and type-shortness. As was shown by Boney in [Bonc], these assumptions are corollaries of certain large cardinal axioms, including the one assumed by Makkai and Shelah. It seems to be plausible that tameness and type shortness will be derived in the future from categoricity above a certain Hanf number that depends only on $LS(K)$.

In this paper, we introduce a notion that, like the one from [MaSh285], is an analogue of the first order notion of coheir. One of our main results is that, given certain model theoretic assumptions, this notion is in fact an independence notion.

Theorem (5.1)**.** *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ so that*

(1) *K is fully $< \kappa$-tame;*
(2) *K is fully $< \kappa$-type short;*
(3) *K doesn't have an order property; and*
(4) *$\perp$ satisfies existence and extension,*

then $\perp$ is an independence relation.

Sections 2, 3, and 4 give precise definitions and discussions of the terms in the theorem. This theorem also generalizes the work of [MaSh285] and [Sh472]. There, large cardinal axioms are used to prove the above conclusion for $L_{\kappa,\omega}$. As detailed in Section 8, Boney [Bonc] shows that the hypotheses of the above theorem also hold for any AEC with $LS(K) < \kappa$.

We also improve these papers by using purely model theoretic properties: tameness and type shortness. Section 5 gives some ZFC examples of AECs with these properties. Tameness was introduced in Grossberg and VanDieren [GV06b] as a locality property for the domains of types, and type shortness was introduced by the first author in [Bonc] as a dual property of tameness for the lengths of types. Together, tameness and type shortness give a locality condition for when an injection with domain not necessarily a model can be extended into a K embedding; see [Bonc].§3 for a longer discussion.

In a meeting at AIM that was dedicated to Classification Theory for AECs, there was a problem session moderated by Andres Villaveces [Vi06]. John Baldwin asked "Does Shelah's rank satisfy the Lascar inequalities, or is there another rank which does?" (in the context of Shelah's excellent classes). Theorem 7.7 provides an affirmative answer (for a much wider context). Another question asked by Baldwin and Grossberg at that meeting was "What is superstability for AECs?". While several approximations were offered by various authors, Theorem 7.9 provide the best known approximation to this question.

Section 2 gives the necessary background information for AECs. Section 3 gives a list of common axioms for independence relations and defines the froking relation that we will consider in this paper. Section 4 gives a fine analysis of when parameterized versions of the axioms from Section 3 hold about our forking relation. Section 5 gives the global assumptions that make our forking relation an independence relation. Section 6 introduces a notion that generalizes coheir and deduces local character of our forking from this and categoricity. Section 7 introduces a U rank and shows that it is well behaved. Section 8 continues the study of large cardinals from [Bonc] and shows that large cardinal assumptions simplify many of the previous sections.

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2. Preliminaries

The definition of an Abstract Elementary Class was first given by Shelah in [Sh88]. The definitions and concepts in this section are all part of the literature; in particular, see the books by Baldwin [Bal09] and Shelah [Sh:h], the article by Grossberg [Gro02], or the forthcoming book by Grossberg [Gro1X] for general information.

Definition 2.1. *We say that $(K, \prec_K)$ is an Abstract Elementary Class iff*

(1) *There is some language $L = L(K)$ so that every element of K is an L-structure;*
(2) *$\prec_K$ is a partial order on K;*
(3) *for every $M, N \in K$, if $M \prec_K N$, then M is an L-substructure of N;*

(4) $(K, \prec_K)$ *respects* L *isomorphisms: if* $f : N \to N'$ *is an* L *isomorphism and* $N \in K$*, then* $N' \in K$ *and if we also have* $M \in K$ *with* $M \prec_K N$*, then* $f(M) \prec_K N'$*;*
(5) (Coherence) *if* $M_0, M_1, M_2 \in K$ *with* $M_0 \prec_K M_2$*,* $M_1 \prec_K M_2$*, and* $M_0 \subseteq M_1$*, then* $M_0 \prec M_1$*;*
(6) (Tarski-Vaught axioms) *suppose* $\langle M_i \in K : i < \alpha\rangle$ *is a* $\prec_K$*-increasing continuous chain, then*
  (a) $\cup_{i<\alpha} M_i \in K$ *and, for all* $i < \alpha$*, we have* $M_i \prec_K \cup_{i<\alpha} M_i$*; and*
  (b) *if there is some* $N \in K$ *so that, for all* $i < \alpha$*, we have* $M_i \prec_K N$*, then we also have* $\cup_{i<\alpha} M_i \prec_K N$*.*
(7) (Lowenheim-Skolem number) $LS(K)$ *is the first infinite cardinal* $\lambda \geq |L(K)|$ *such that for any* $M \in K$ *and* $A \subset |M|$*, there is some* $N \prec_K M$ *such that* $A \subset |N|$ *and* $\|N\| \leq |A| + \lambda$*.*

Remark 2.2. *As is typical, we drop the subscript on* $\prec_K$ *when it is clear from context and abuse notation by calling* K *an AEC when we mean* $(K, \prec_K)$ *is an AEC. Also, we follow the convention of Shelah that, for* $M \in K$*, we differentiate between the model* M*, its universe* $|M|$*, and the cardinality of its universe* $\|M\|$*. Also, in this paper,* K *is always an AEC that has no models of size smaller than the Lowenheim-Skolem number.*

We will briefly summarize some of the basic notations, definitions, and results for AECs. As above, see [Gro1X] for a more detailed description and development.

Definition 2.3.
(1) *A* K embedding *from* M *to* N *is an injective* $L(K)$*-morphism* $f : M \to N$ *so* $f(M) \prec_K N$*.*
(2)
$$\begin{aligned} K_\lambda &= \{M \in K : \|M\| = \lambda\} \\ K_{\leq\lambda} &= \{M \in K : \|M\| \leq \lambda\} \end{aligned}$$
(3) K *has the* λ-amalgamation property *(*λ*-AP) iff for any* $M \prec N_0, N_1 \in K_\lambda$*, there is some* $N^* \in K$ *and* $f_i : M \to N_i$ *so that*
$$\begin{tikzcd} N_1 \arrow[r, "f_1"] & N^* \\ M \arrow[u] \arrow[r] & N_0 \arrow[u, "f_2"'] \end{tikzcd}$$
commutes. K *has the amalgamation property if this is true with no restriction on the cardinalities of the models.*
(4) K *has the* λ-joint embedding property *(*λ*-JEP) iff for any* $M_0, M_1 \in K_\lambda$*, there is some* $M^* \in K$ *and* $f_i : M_i \to M^*$*.* K *has the joint embedding property if this is true with no restriction on the cardinalities.*
(5) K *has* arbitrarily large models *iff* $K_\lambda \neq \emptyset$ *for all* $\lambda \geq LS(K)$*.*

(6) *K has* no maximal models *iff for all $M \in K$, there is $N \in K$ so $M \precneqq N$.*

Note that, if K has the joint embedding property, then having arbitrarily large models is equivalent to having no maximal models; see Baldwin, Larson, and Shelah [BLS1003].

We will use the above three assumptions in tandem throughout this paper. This allows us to make use of a monster model, as in the complete, first order setting; [Gro1X].4.4 gives details. The monster model $\mathfrak{C}$ is of large size and is universal and model homogeneous for all models that we consider. As is typical, we assume all elements come from the monster model.

We use a monster model to streamline our treatment. However, amalgamation is the only one of the properties that is crucial because it simplifies Galois types. Joint embedding and no maximal models are rarely used; one major exception is Proposition 5.3 in the discussion of the order property. After giving the definition of nonforking in the next section, we briefly detail the differences when we are not working in the context of a monster model.

In AECs, a consistent set of formulas is not a strong enough definition of type; any of the examples of non-tameness will be an example of this and it is made explicit in [BK09]. However, Shelah isolated a semantic notion of type in [Sh300] that Grossberg named *Galois type* in [Gro02] this can replace the first order notion.

We differ from the standard treatment of types in that we allow the length of our types to be possibly infinite. This has proven fruitful in Shelah and Makkai [MaSh285], Grossberg and VanDieren [GV06b], Shelah [Sh:h].V, and Boney [Bona], and is necessary for our work in the later sections.

Definition 2.4. *Let K be an AEC, $\lambda \geq LS(K)$, and $(I, <_I)$ be an ordered set.*

(1) *Let M in K and $\langle a_i \in \mathfrak{C} : i \in I\rangle$ be a sequence of elements. The Galois type of $\langle a_i \in \mathfrak{C} : i \in I\rangle$ over M is denoted $tp(\langle a_i \in \mathfrak{C} : i \in I\rangle/M)$ and is the orbit of $\langle a_i : i \in I\rangle$ under the action of automorphisms of $\mathfrak{C}$ fixing M. That is, $\langle a_i \in \mathfrak{C} : i \in I\rangle$ and $\langle b_i : i \in I\rangle$ have the same Galois type over M iff there is $f \in Aut_M \mathfrak{C}$ so that $f(a_i) = b_i$ for all $i \in I$.*
(2) *For $M \in K$, $S^I(M) = \{tp(\langle a_i : i \in I\rangle/M) : a_i \in \mathfrak{C}$ for all $i \in I\}$.*
(3) *Suppose $p = tp(\langle a_i : i \in I\rangle/M) \in S^I(M)$ and $N \prec M$ and $J \subset I$. Then, $p \upharpoonright N \in S^I(N)$ is $tp(\langle a_i : i \in I\rangle/N)$ and $p^{I'} \in S^{I'}(M)$ is $tp(\langle a_i : i \in I'\rangle/M)$.*
(4) *Given a Galois type $p \in S^I(M)$, then the domain of p is M and the length of p is I.*
(5) *If $p = tp(A/M)$ is a Galois type and $f \in Aut\, \mathfrak{C}$, then $f(p) = tp(f(A)/f(M))$.*

Remark 2.5.

(1) *We sometimes write that the type of two sets (say X and Y) are equal; given the above definitions, this really means there is some enumeration $X = \langle x_i : i \in I\rangle$ and $Y = \langle y_i : i \in I\rangle$ so that the types of the sequences are equal. If we reference some $x \subset X$ and the 'corresponding part of Y,'*

then this refers to $y \subset Y$ indexed by the same set that indexes x; that is, $y = \langle y_i : i \in I \text{ and } x_i \in x\rangle$.

(2) *Some authors place a g denoting 'Galois' in front of the above notions to differentiate them from the first order versions (ie, $gtp(a/M, N)$ and $gS(M)$); however, since we exclusively use Galois types in this paper, we omit this. Additionally, we will refer to 'Galois types' simply as 'types.'*

Along with types comes a notion of saturation. This is sometimes called 'Galois saturation', but, as above, we drop the adjective 'Galois'. A degree of saturation will be necessary when we deal with our independence relation, so we offer a definition here. Additionally, we include a lemma of Shelah that characterizes saturation by model homogeneity.

Definition 2.6. (1) *A model $M \in K$ is μ-saturated iff for all $N \prec M$ such that $\|N\| < \mu$ and $p \in S(N)$, we have that p is realized in M.*

(2) *A model $M \in K$ is μ-model homogeneous iff for all $N \prec M$ and $N' \succ N$ such that $\|N'\| < \mu$, there is $f : N' \to_N M$.*

Lemma 2.7 ([Sh576].0.26.1)**.** *Let $\lambda > LS(K)$ and $M \in K$ and suppose that K has the amalgamation property. Then M is λ-Galois saturated iff M is λ-model homogeneous.*

We conclude the preliminaries by recalling two locality properties that are key for this paper: tameness and type shortness. Tameness was first isolated by Grossberg and VanDieren [GV06b], although a weaker version had been used by Shelah [Sh394] in the midst of a proof. Later, Grossberg and VanDieren [GV06c] [GV06a] showed that a strong form of Shelah's Categoricity Conjecture holds for tame AECs.

Theorem 2.8. *Suppose K is an AEC that has amalgamation, joint embeddings, and no maximal models. If K is χ-tame and λ^+ categorical for $\lambda \geq LS(K)^+ + \chi$, then K is μ categorical for all $\mu \geq \lambda$.*

Type shortness was first defined by the first author in [Bonc] as a dual property for tameness. There, we derived type shortness and tameness from large cardinal hypotheses. Although there are many parameterizations of these definitions, we only highlight the most important ones.

Definition 2.9. (1) *K is (κ, λ)* tame for I length types *iff for any $M \in K_\lambda$ and $p \neq q \in S^I(M)$, there is some $N \in K_\kappa$ so that $N \prec M$ and $p \upharpoonright N \neq q \upharpoonright N$.*

(2) *K is* fully $< \kappa$ tame *iff K is $(< \kappa, \lambda)$ tame for I length types for all λ and I.*

(3) *K is (κ, λ)* type short over μ sized models *iff for any $M \in K_\mu$ and $p \neq q \in S^\lambda(M)$, there is some $I' \subset I$ of size κ so that $p^{I'} \neq q^{I'}$.*

(4) *K is* fully $< \kappa$ type short *iff K is $(< \kappa, \lambda)$ type short over μ sized models for all λ and μ.*

Our main hypothesis will be an AEC that is fully $< \kappa$ tame and $< \kappa$ type short; see Theorem 5.1. However, this is redundant as there is a relationship between tameness and type shortness. Recall the following theorem.

Theorem 2.10 ([Bonc].3.6)**.** *If K is $(< \kappa, \mu)$-type short over λ-sized models for $LS(K) \leq \lambda \leq \mu$ or over the empty set, then it is $(< \kappa + \lambda, \mu)$-tame for $\leq \mu$ length types.*

This means that, in particular, full $< \kappa$-type shortness implies full $< \kappa$-tameness. However, we continue to state both hypotheses for clarity.

3. Axioms of an independence relation and the definition of forking

The following hypothesis and definition of non-forking is central to this paper:

Hypothesis 3.1. *Assume that K has no maximal models and satisfies the λ-joint embedding and λ-amalgamation properties for all $\lambda \geq LS(K)$.*

Fix a cardinal $\kappa > LS(K)$. The nonforking is defined in terms of this κ and all subsequent uses of κ will refer to this fixed cardinal, until Section 8. If we refer to a model, tuple, or type as 'small,' then we mean its size is $< \kappa$, its length is of size $< \kappa$, or both its domain and its length are small.

Definition 3.2. *Let $M_0 \prec N$ be models and A be a set. We say that $tp(A/N)$ does not fork over M_0, written $A \underset{M_0}{\perp} N$, iff for all small $a \in A$ and all small $N^- \prec N$, we have that $tp(a/N^-)$ is realized in M_0.*

Thus a type does not fork over a base model iff all small approximations to it, both in length and domain, are realized in the base model. This definition is a relative of the finite satisfiability–also known as coheir–notion of forking that is extensively studied in stable theories. It is an AEC version of the non-forking defined in Makkai and Shelah [MaSh285] for categorical $L_{\kappa,\omega}$ theories when κ is strongly compact.

We now list the properties that, under our ideal conditions, our nonforking notion will have. These properties can be thought of as axiomatizing an independence relation. The ones listed below are commonly considered and are similar to the properties that characterize nonforking in first order, stable theories, although this list is most inspired by [MaSh285]. However, many of these properties have been changed because we require the bottom and right inputs to be models. This is similar to good λ-frames, which appear in [Sh:h].II, althought we don't require the parameter set A to be a singleton and we allow the sets and models to be of any size.

The properties we introduce are heavily parameterized. The interesting and hard to prove properties–Existence, Uniqueness, and Symmetry–are each given with three parameters: λ, μ, and χ. These parameters allows us to conduct a fine analysis of exactly what assumptions are required to derive these properties. The order of these parameters is designed to be as uniformized as possible: the λ refers to the size of the left object, μ refers to the size of the middle object, and χ refers to the size of the right object. If we write a property without parameters, then we mean that property for all possible parameters.

Definition 3.3. *Fix an AEC K. Let $\overset{*}{\perp}$ be a ternary relation on models and sets so that $A \underset{M_0}{\overset{*}{\perp}} N$ implies that A is a subset of the monster model and $M_0 \prec N$ are both models. We say that $\overset{*}{\perp}$ is an* independence relation *iff it satisfies all of the following properties for all cardinals referring to sets and all cardinals that are at least κ when the cardinal refers to a model.*

(I) **Invariance**
Let $f \in Aut\ \mathfrak{C}$ be an isomorphism. Then $A \underset{M_0}{\overset{*}{\perp}} N$ implies $f(A) \underset{f(M_0)}{\overset{*}{\perp}} f(N)$.

(M) **Monotonicity**
If $A \underset{M_0}{\overset{*}{\perp}} N$ and $A' \subset A$ and $M_0 \prec M_0' \prec N' \prec N$,, then $A' \underset{M_0'}{\overset{*}{\perp}} N'$.

(T) **Transitivity**
If $A \underset{M_0'}{\overset{*}{\perp}} N$ and $M_0' \underset{M_0}{\overset{*}{\perp}} N$ with $M_0 \prec M_0'$, then $A \underset{M_0}{\overset{*}{\perp}} N$.

$(C)_{<\kappa}$ **Continuity**
(a) *If for all small $A' \subset A$ and small $N' \prec N$, there is $M_0' \prec M_0$ and $N' \prec N^* \prec N$ so $M_0' \prec N^*$ and $A' \underset{M_0'}{\overset{*}{\perp}} N^*$, then $A \underset{M_0}{\overset{*}{\perp}} N$.*
(b) *If $\langle A_i, M_0^i \mid i < \kappa \rangle$ are filtrations of A and M_0 and $A_i \underset{M_0^i}{\overset{*}{\perp}} N$ for all $i < \kappa$, then $A \underset{M_0}{\overset{*}{\perp}} N$.*

$(E)_{(\lambda,\mu,\chi)}$ (a) **Existence**
Let A be a set and M_0 be a model sizes λ and μ, respectively. Then $A \underset{M_0}{\overset{*}{\perp}} M_0$.
(b) **Extension**
Let A be a set and M_0 and N be models of sizes λ, μ, and χ, respectively, so that $M_0 \prec N$ and $A \underset{M_0}{\overset{*}{\perp}} N$. If $N^+ \succ N$ of size χ, then there is A' so $A' \underset{M_0}{\overset{*}{\perp}} N^+$ and $tp(A'/N) = tp(A/N)$.

$(S)_{(\lambda,\mu,\chi)}$ **Symmetry**
Let A_1 be a set, M_0 be a model, and A_2 be a set of sizes λ, μ, and χ, respectively, so that there is a model M_2 with $M_0 \prec M_2$ and $A_2 \subset |M_2|$ such that $A_1 \overset{*}{\underset{M_0}{\perp}} M_2$. Then there is a model $M_1 \succ M_0$ that contains A_1 so that $A_2 \overset{*}{\underset{M_0}{\perp}} M_1$.

$(U)_{(\lambda,\mu,\chi)}$ **Uniqueness**
Let A and A' be sets and $M_0 \prec N$ be models of sizes λ, λ, μ, and χ, respectively. If $tp(A/M_0) = tp(A'/M_0)$ and $A \overset{*}{\underset{M_0}{\perp}} N$ and $A' \overset{*}{\underset{M_0}{\perp}} N$, then $tp(A/N) = tp(A'/N)$.

These axioms and their names are primarily drawn from [MaSh285]. Since our list of requirement is subset of the ones for good λ-frames, the notion introduced here is a much more general notion than good λ-frame; see Definition II.2.1 in [Sh:h]. A large difference is that good λ-frames specify a distinguished set of types (called "basic types") that are a generalization of regular types in superstable theories.

Notice that the Existence property implies that $M_0 \in K_{\geq\kappa}$ is κ-saturated. However, this is not a serious restriction in comparison with Shelah's results on good frames. In all examples where Shelah has isolated good λ-frame, the "base model" M_0 is saturated and almost always the class is λ-categorical (e.g. [Sh:h].II.3.4 and .II.3.7 and [Sh:h].V.4.10). Theorem 5.4 below shows that categoricity in some $\lambda = \lambda^{<\kappa}$ implies that all sufficiently large models are κ saturated. In [JrSh875], Jarden and Shelah studied "weak λ-frames," and our notion is also more general than that notion.

The monotonicity and invariance properties are actually necessary to justify our formulation of nonforking as based on types. Without them, whether or not a type doesn't fork over a base model could depend on the specific realization of the chosen type. Since these properties are clearly satisfied by our definition of nonforking, this is not an issue.

The axioms $(E)_{(\lambda,\mu,\chi)}$ combines two notions. The first is Existence: that a type does not fork over its domain. This is similar to the consequence of simplicity in first order theories that a type does not fork over the algebraic closure of its domain. As mentioned above, in this context, existence is equivalent to every model being κ saturated. In the first order case, where finite satisfiability is the proper analogue of our non-forking, existence is an easy consequence of the elementary substructure relation. In [MaSh285], this holds for $< \kappa$ satisfiability, their nonforking, because types are formulas from $L_{\kappa,\kappa}$ and, due to categoricity, the strong substructure relation is equivalent to $\prec_{L_{\kappa,\kappa}}$.

The second notion is the extension of nonforking types. In first order theories (and in [MaSh285]), this follows from compactness but is more difficult in a general

AEC. We have separated these notions for clarity and consistency with other sources, but could combine them in the following statement:

> Let A be a set and M_0 and N be models of sizes λ, μ, and χ, respectively, so that $M_0 \prec N$. Then there is some A' so that $tp(A'/M_0) = tp(A/M_0)$ and $A' \overset{*}{\underset{M_0}{\perp}} N$.

As an alternative to assuming (E), and thus assuming all models are κ saturated, we could simply work with the definition and manipulate the nonforking relationships that occur. This is the strategy in Section 6. In such a situation, κ saturated models, which will exist in $\lambda^{<\kappa}$, will satisfy the existence axiom.

The relative complexity of the symmetry property is necessitated by the fact that the right side object is required to be a model that contains the base. If the left side object already satisfied this, then there is a simpler statement.

Proposition 3.4. *If $(S)_{(\lambda,\mu,\chi)}$ holds, then so does the following*

$(S^*)_{(\lambda,\mu,\chi)}$ *Let M, M_0, and N be models of size λ, μ, and χ, respectively so that $M_0 \prec N$ and $M_0 \prec M$. Then $M \overset{*}{\underset{M_0}{\perp}} N$ iff $N \overset{*}{\underset{M_0}{\perp}} M$.*

In first order stability theory, many of the key dividing lines depend on the local character $\kappa(T)$, which is the smallest cardinal so that any type doesn't fork over some subset of its of domain of size less than $\kappa(T)$. The value of this cardinal can be smaller than the size of the theory, e.g. in an uncountable, superstable theory. However, since types and nonforking occur only over models, the smallest value the corresponding cardinal could take would be $LS(K)^+$. This is too coarse for many situations. Instead, we follow [ShVi635], [Sh:h].II, and [GV06b] by defining a local character cardinal based on the length of a resolution of the base rather than the size of cardinals. As different requirements appear in different places, we give two definition of local character: one with no additional requirement, as in [Sh:h].II, and one requiring that successor models be universal, as in [ShVi635] and [GV06b].

Definition 3.5. $\kappa_\alpha(\overset{*}{\perp}) = \min\{\lambda \in REG \cup \{\infty\}$: *for all $\mu = cf\,\mu \geq \lambda$ and all increasing, continuous chains $\langle M_i : i < \mu\rangle$ and all sets A of size less than α, there is some $i_0 < \mu$ so $A \overset{*}{\underset{M_{i_0}}{\perp}} \cup_{i<\mu} M_i\}$*

$\kappa^*_\alpha(\overset{*}{\perp}) = \min\{\lambda \in REG \cup \{\infty\}$: *for all $\mu = cf\,\mu \geq \lambda$ and all increasing, continuous chains$\langle M_i : i < \mu\rangle$ with M_{i+1} universal over M_i and κ saturated and all sets A of size less than α, there is some $i_0 < \mu$ so $A \overset{*}{\underset{M_{i_0}}{\perp}} \cup_{i<\mu} M_i\}$*

In either case, if we omit α, then we mean $\alpha = \omega$.

In Section 6, we return to these properties and examine natural conditions that imply that $\kappa^*(\perp) = \omega$.

Although we do not use it in this paper, we explain the changes that must be made if we don't work inside of a monster model but still assume amalgamation. In that case, the definition of the type of A over N must be augmented by a model contain both; that is, some $\widehat{M} \in K$ so $A \subset |\widehat{M}|$ and $N \prec M$. We denote this type $tp(A/N, \widehat{M})$. Similarly, we must add this fourth input to the nonforking relation that contains all other parameters. Then $A \underset{M_0}{\overset{\widehat{M}}{\perp}} N$ iff $M_0 \prec N \prec \widehat{M}$ and $A \subset |\widehat{M}|$ and all of the small approximations to the type of A over N *as computed in* $\widehat{M}$. The properties are expanded similarly with added monotonicity for changing the ambient model $\widehat{M}$ and the allowance that new models that are found by properties such as existence or symmetry might exist in a larger big model $\widehat{N}$. All theorems proved in this paper about nonforking only require amalgamation, although some of the results referenced make use of the full power of the monster model.

We end this section with an easy exercise in the definition of nonforking that says that A and N must be disjoint outside of M_0.

Proposition 3.6. *If we have* $A \underset{M_0}{\perp} N$, *then* $A \cap |N| \subset |M_0|$.

Proof: Let $x \in A \cap |N|$. Since N is a model, we can find a small $N^- \prec N$ that contains x. Then, by the definition of nonforking, $tp(x/N^-)$ must be realized in M_0. But since $x \in |N^-|$, this type is algebraic so the only thing that can realize it is x. Thus, $x \in |M_0|$. †

4. Connecting Existence, Symmetry and Uniqueness

In this section, we investigate what AEC properties cause the axioms of our independence relation to hold. The relations are summarized in the proposition below.

Proposition 4.1. *Suppose that K doesn't have the weak κ-order property and is $(<\kappa, \lambda+\chi)$-type short for θ-sized domains and $(<\kappa, \theta)$-tame for $<\kappa$ length types. Then*

1. $(E)_{(\chi,\theta,\lambda)}$ *implies* $(S)_{(\lambda,\theta,\chi)}$.
2. $(S)_{(<\kappa,\theta,<\kappa)}$ *implies* $(U)_{(\lambda,\theta,\chi)}$.

This proposition and the lemma used to prove it below rely on an order property.

Definition 4.2. *K has the* weak κ-order property *iff there are lengths $\alpha, \beta < \kappa$, a model $M \in K_{<\kappa}$, and types $p \neq q \in S^{\alpha+\beta}(M)$ such that there are sequences $\langle a_i \in {}^\alpha\mathfrak{C} : i < \kappa\rangle$ and $\langle b_i \in {}^\beta\mathfrak{C} : i < \kappa\rangle$ such that, for all $i, j < \kappa$,*

$$i \leq j \implies tp(a_i b_j/M) = p$$
$$i > j \implies tp(a_i b_j/M) = q$$

This order property is a generalization of the first order version to our context of Galois types and infinite sequences. This is one of many order properties proposed for the AEC context (we introduce another one in Section 6) and is similar to 1-stability that is studied by Shelah in [Sh1019] in the context of $L_{\theta,\theta}$ theories where θ is strongly compact. The adjective 'weak' is in comparison to the $(<\kappa,\kappa)$-order property in Shelah [Sh394]. The key difference is that [Sh394] requires the existence of ordered sequences of *any* length, while we only require a sequence of length κ. We discuss the implications of the weak κ-order property property in the next section. For now, we use it to prove the following result, similar to one in [MaSh285].

Lemma 4.3. *Suppose K is an AEC that is $(<\kappa,\theta)$-tame for $<\kappa$ length types and doesn't have the weak κ-order property. Let $M_0 \prec M, N$ such that $\|M_0\| = \theta$ and let $a, b, a' \in \mathfrak{C}$ such that $\ell(a) = \ell(a') < \kappa$, $\ell(b) < \kappa$, $b \in N$, and $a' \in M$. If*

$$tp(a/M_0) = tp(a'/M_0) \text{ and } a \underset{M_0}{\perp} N, \text{ and } b \underset{M_0}{\perp} M$$

then $tp(ab/M_0) = tp(a'b/M_0)$.

Proof: Assume for contradiction that $tp(ab/M_0) \neq tp(a'b/M_0)$. We will build sequences that witness the weak κ-order property. By tameness, there is some $M^- \prec M_0$ of size $<\kappa$ such that $tp(ab/M^-) \neq tp(a'b/M^-)$. Set $p = tp(ab/M^-)$ and $q = tp(a'b/M^-)$. We will construct two sequences $\langle a_i \in {}^{\ell(a)}M_0 : i < \kappa\rangle$ and $\langle b_i \in {}^{\ell(b)}M_0 : i < \kappa\rangle$ by induction. We will have, for all $i < \kappa$

(1) $a_i b \vDash p$;
(2) $a_i b_j \vDash q$ for all $j < i$;
(3) $ab_i \vDash q$; and
(4) $a_i b_j \vDash p$ for all $j \geq i$.

Note that, since $b_i \in M_0$, (3) is equivalent to $a' b_i \vDash q$.

This is enough: (2) and (4) are the properties necessary to witness the weak κ-order property.

Construction: Let $i < \kappa$ and suppose that we have constructed the sequence for all $j < i$. Let $N^+ \prec N$ of size $<\kappa$ contain b, M^-, and $\{b_j : j < i\}$ Because $a' \underset{M_0}{\perp} N$, there is some $a_i \in M_0$ that realizes $tp(a/N^+)$. This is witnessed by $f \in Aut_{N^+}\mathfrak{C}$ with $f(a) = a_i$.

Claim: (1) and (2) hold.

f fixes M^- and b, so it witnesses that $tp(ab/M^-) = tp(a_i b/M^-)$. Similarly, it fixes b_j for $j < i$, so it witnesses $q = tp(ab_j/M^-) = tp(a_i b_j/M^-)$. $\dagger_{Claim}$ Similarly, pick $M^+ \prec M$ of size $<\kappa$ to contain M^-, a', and $\{a_j : j \leq i\}$. Because $b \underset{M_0}{\perp} M'$, there is $b_i \in M_0$ that realizes $tp(b/M^+)$. As above, (3) and (4) hold. $\dagger$

Now we are ready to prove our theorems regarding when the properties of $\perp$ hold. The first four properties always hold from the definition of nonforking.

Theorem 4.4. *If K is an AEC with $LS(K) < \kappa \leq \lambda$, then $\mathop{\smile}\limits^{\vert}$ satisfies (I), (M), (T), and $(C)_{<\kappa}$.*

To get the other properties, we have to rely on some degree of tameness, type shortness, no weak order property, and the property (E).

Proof of Proposition 4.1:

(1) Suppose $(E)_{(\chi,\theta,\lambda)}$ holds. Let $A_2 \mathop{\smile}\limits^{\vert}_{M_0} M_1$ and $A_1 \subset |M_1|$ with $|A_2| = \lambda$, $\|M_0\| = \theta$, and $|A_1| = \chi$. Let M_2 contain A_2 and M_0 be of size λ. By $(E)_{(\chi,\theta,\lambda)}$, there is some A_1' such that $tp(A_1/M_0) = tp(A_1'/M_0)$ and $A_1' \mathop{\smile}\limits^{\vert}_{M_0} M_2$. It will be enough to show that $tp(A_1A_2/M_0) = tp(A_1'A_2/M_0)$. By $(< \kappa, \lambda + \chi)$-type shortness over θ-sized domains, it is enough to show that, for all $a_2 \in A_2$ and corresponding $a_1 \in A_1$ and $a_1' \in A_1'$ of length $< \kappa$, we have $tp(a_1a_2/M_0) = tp(a_1'a_2/M_0)$. By (M), we have that $a_1' \mathop{\smile}\limits^{\vert}_{M_0} M_2$ and $a_2 \mathop{\smile}\limits^{\vert}_{M_0} M_1$, so this follows by Lemma 4.3 above.

Now that we have shown the type equality, let $f \in Aut_{M_0}\mathfrak{C}$ such that $f(A_1A_2) = A_1'A_2$. Applying f to $A_1' \mathop{\smile}\limits^{\vert}_{M_0} M_2$, we get that $A_1 \mathop{\smile}\limits^{\vert}_{M_0} f(M_2)$ and $A_2 = f(A_2) \subset f(M_2)$, as desired.

(2) Suppose $(S)_{(<\kappa,\theta,<\kappa)}$. Let A and A' be sets of size λ and $M_0 \prec N_0$ of size θ and χ, respectively, so that

$$tp(A/M_0) = tp(A'/M_0) \text{ and } A \mathop{\smile}\limits^{\vert}_{M_0} N \text{ and } A' \mathop{\smile}\limits^{\vert}_{M_0} N$$

As above, it is enough to show that $tp(AN/M_0) = tp(A'N/M_0)$. By type shortness, it is enough to show this for every $n \in N$ and corresponding $a \in A$ and $a' \in A'$ of lengths less than κ. By (M), we know that $a \mathop{\smile}\limits^{\vert}_{M_0} N$ and $a' \mathop{\smile}\limits^{\vert}_{M_0} N$. By applying $(S)_{(<\kappa,\theta,<\kappa)}$ to the former, there is $N_a \succ M_0$ containing a such that $n \mathop{\smile}\limits^{\vert}_{M_0} N_a$. As above, Lemma 4.3 gives us the desired conclusion. †

5. The main theorem

We now state the ideal conditions under which our nonforking works.

Theorem 5.1. *Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ such that*

1. *K is fully $< \kappa$-tame;*
2. *K is fully $< \kappa$-type short;*
3. *K doesn't have the weak κ-order property; and*

(4) $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies* (E)

then $\mathop{\smile\!\!\!\!\!\!|}$ *is an independence relation.*

Proof: First, by Theorem 4.4, $\mathop{\smile\!\!\!\!\!\!|}$ always satisfies (I), (M), (T), and $(C)_{<\kappa}$. Second, (E) is part of the hypothesis. Third, by the other parts of the hypothesis, we can use Proposition 4.1. Let χ, θ, and λ be cardinals. We know that $(E)_{(\chi,\theta,\lambda)}$ holds, so $(S)_{(\lambda,\theta,\chi)}$ holds. From this, we also know that $(S)_{(<\kappa,\theta,<\kappa)}$ holds. Thus, $(U)_{(\lambda,\theta,\chi)}$ holds. So $\mathop{\smile\!\!\!\!\!\!|}$ is an independence relation. †

In the following sections, we will assume the hypotheses of the above theorem and use $\mathop{\smile\!\!\!\!\!\!|}$ as an independence relation. First, we discuss these hypotheses and provide some examples that satisfy them.

<u>"amalgamation, joint embedding, and no maximal models"</u>
These are a common set of assumptions when working with AECs that appear often in the literature; see [Sh394], [GV06a], and [GVV] for examples. Readers interested in work on AECs *without* these assumptions are encouraged to see [Sh576] or Shelah's work on good λ-frames in [Sh:h] and [JrSh875].

<u>"fully $<\kappa$-tame" and "fully $<\kappa$-type short"</u>
As discussed in [Bonc], these assumptions say that Galois types are equivalent to their small approximations. Without this equivalence, there is no reason to think that our nonforking, which is defined in terms of small approximations, would say anything useful about an AEC.

On the other hand, we argue that these properties will occur naturally in any setting with a notion of independence or stability theory. This is observed in the introduction to [GV06a]. Additionally, the following proposition says that the existence of a nonforking-like relation that satisfies stability-like assumptions implies tameness and some stability.

Proposition 5.2. *If there is a nonforking-like relation* $\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}$ *that satisfies* (U)*,* (M)*, and* $\kappa_\alpha(\overset{*}{\mathop{\smile\!\!\!\!\!\!|}}) < \infty$*, then* K *is* $(<\mu,\mu)$ *tame for less than* α *length types for all regular* $\mu \geq \kappa_\alpha(\overset{*}{\mathop{\smile\!\!\!\!\!\!|}})$*.*

Proof: Let $p \neq q \in S^{<\alpha}(M)$ so their restriction to any smaller submodel is equal and let $\langle M_i \in K_{<\mu} : i < \mu\rangle$ be a resolution of M. By the local character, there are i_p and i_q such that p does not fork over M_{i_p} and q does not fork over M_{i_q}. By (M), both of the types don't fork over $M_{i_p+i_q}$ and, by assumption, $p \restriction M_{i_p+i_q} = q \restriction M_{i_p+i_q}$. Thus, by (U), we have $p = q$. †

The results of [Bonc].§3 allow us to get a similar result for type shortness.

The arguments of [MaSh285].4.14 and Theorem **??** show that this is enough to derive stability-like bounds on the number of Galois types.

<u>"no weak κ order property"</u>
In first order model theory, the order property and its relatives (the tree order

property, etc) are well-studied as the nonstructure side of dividing lines. In broader contexts such as ours, much less is known. Still, there are some results, such as Shelah [Sh:e].III, which shows that a strong order property, akin to getting any desired order of a certain size in an EM model, implies many models. Note that he does not explicitly work inside an AEC, but his proofs and definitions are sufficiently general and syntax free to apply here.

Ideally, the weak κ-order property could be shown to imply non-structure for an AEC. While this is not currently known in general, we have two special cases where many models follows by combinatorial arguments and the work of Shelah.

First, if we suppose that κ is inaccessible, then we can use Shelah's work to show that there are the maximum number of models in every size above κ. We will show that, given any linear order, there is an EM model with the order property for that order. This implies [Sh:e]'s notion of "weakly skeleton-like", which then implies many models by [Sh:e].III.24.

Proposition 5.3. *Let κ be inaccessible and suppose K has the weak κ-order property. Then, for all linear orders I, there is EM model M^*, small $N \prec M^*$, $p \neq q \in S(M)$, and $\langle a_i, b_i \in M^* : i \in I\rangle$ such that, for all $i, j \in I$,*

$$i \leq j \implies tp(a_i b_j/M) = p$$
$$i > j \implies tp(a_i b_j/M) = q$$

Thus, for all $\chi > \kappa$, K_χ has 2^χ nonisomorphic models.

We sketch the proof and refer the reader to [Sh:e] for more details.

Proof Outline: Let $p \neq q \in S(N)$ and $\langle a_i, b_i : i < \kappa\rangle$ witness the weak order property. Since K has no maximal models, we may assume that this occurs inside an EM model (see [Bond] for details). In particular, there is some Φ proper for linear orders so $N \prec EM(\kappa, \Phi) \upharpoonright L$ that contains $\langle a_i, b_i : i < \kappa\rangle$, $L(\Phi)$ contains Skolem functions, and κ is indiscernible in $EM(\kappa, \Phi) \upharpoonright L$. Recall that, for $X \subset EM(\kappa, \Phi)$, we have $\text{Contents}(X) := \cap\{I \subset \kappa : X \subset |EM(I, \Phi)|\}$. By inaccessibility, we can thin out $\{\text{Contents}(a_i b_i) : i < \kappa\}$ to $\{\text{Contents}(a_i b_i) : i \in J\}$ that is a head-tail Δ system of size κ and are all generated by the same term and have the same quantifier free type in κ. Since κ is regular and $\text{Contents}(N)$ is of size $< \kappa$, we may further assume the non-root portion of this Δ system is above $\sup \text{Contents}(N)$.

By the definition of EM models, we can put in any linear order into $EM(\cdot, \Phi) \upharpoonright L$ and get a model in K. Thus, we can take the blocks that generate each $a_i b_i$ with $i \in J$ and arrange them in any order desired. In particular, we can arrange them such that they appear in the order given by I. Then, the order indiscernibility implies that the order property holds as desired.

We have shown the hypothesis of [Sh:e].III.24 and the final part of our hypothesis is that theorem's conclusion. †

We can also make use of these results without large cardinals. To do so, we 'forget' some of the tameness and type shortness our class has to get a slightly weaker relation. Suppose K is $< \kappa'$ tame and type short. Let λ be regular such that $\lambda^{\kappa'} = \lambda > \kappa'$. By the definitions, K is also $< \lambda$ tame and type short, so take λ to be our fixed cardinal κ. In this case, the ordered sequence constructed in the proof of Lemma 4.3 is actually of size $< \kappa'$. This situation allows us to repeat the above proof and construct 2^κ non-isomorphic models of size κ. Many other cardinal arithmetic set-ups suffice for many models.

$\underline{(E)}$

This has already been discussed after Definition 3.3. Here we show that Existence, the simplicity style assumption that is equivalent to every models being κ saturated, follows from categoricity in a cardinal with favorable cardinal arithmetic.

Theorem 5.4. *Suppose K is an AEC satisfying the amalgamation property. If K is categorical in a cardinal λ satisfying $\lambda = \lambda^{<\kappa}$, then every member of $K_{\geq \chi}$ is κ-saturated, where $\chi = min\{\lambda, sup_{\mu<\kappa}(\beth_{(2^\mu)^+})\}$.*

Proof: First, note that by using the AP and the assumption $\lambda = \lambda^{<\kappa}$ we can construct a κ-saturated member of K_λ. Since this class is categorical, all members of K_λ are κ-saturated.

The easy case is when $\lambda < \chi$: Suppose $M \in K$ is not κ-saturated and $\|M\| > \lambda$. Then there is some small $M^- \prec M$ and $p \in S(M^-)$ such that p is not realized in M. Then let $N \prec M$ be any substructure of size λ containing M^-. Then N doesn't realize p, which contradicts its κ saturation.

For the hard part, suppose $M \in K$ is not κ saturated and $\|M\| \geq \sup_{\mu<\kappa}(\beth_{(2^\mu)^+})$. There is some small $M^- \prec M$ and $p \in S(M^-)$ such that p is not realized in M. We define a new class $(K^+, \prec^+)$ that depends on K, p and M^- as follows:

$L(K^+) := L(K) \cup \{c_m : m \in |M^-|\}$ by

$$\begin{aligned} K^+ \quad &= \quad \{N : N \text{ is an } L(K^+) \text{ structure st } N \upharpoonright L(K) \in K, \text{ there exists} \\ &\qquad h : M^- \to N \upharpoonright L(K) \text{ a } K\text{-embedding such that } h(m) = (c_m)^N \\ &\qquad \text{for all } m \in M^- \text{ and } N \upharpoonright L(K) \text{ omits } h(p)\}. \\ N_1 \prec^+ N_2 \quad &\iff \quad N_1 \upharpoonright L(K) \prec N_2 \upharpoonright L(K) \text{ and } N_1 \subset_{L(K^+)} N_2. \end{aligned}$$

This is clearly an AEC with $LS(K^+) = \|M^-\| + LS(K) < \kappa$ and $\langle M, m\rangle_{m\in|M^-|} \in K^+$.

By Shelah's presentation Theorem K^+ is a $PC_{\mu,2^\mu}$ for $\mu := LS(K^+)$. By Thoerems VII.5.5(2) and VII.5.5(6) of [Sh:c] the Hanf number of K^+ is $\leq \beth_{(2^\mu)^+} \leq \chi$.

Thus, K^+ has arbitrarily large models. In particular, there exists $N^+ \in K^+_\lambda$. Then $N^+ \upharpoonright L(K) \in K_\lambda$ is not κ-saturated as it omits its copy of p. †

Remark 5.5. *While for the rest of the results we assume that K satisfies Hypotheses 3.1, in the proof of Theorem 5.4 we use only the amalgamation property and also avoid any use of tameness or type shortness.*

Before continuing, we also identify a few contexts which are known to satisfy this hypothesis, especially (1), (2), and (3) of Theorem 5.1.

- **First order theories** Since types are syntactic and over sets, they are $< \aleph_0$ tame and $< \aleph_0$ type short and (4) follows by compactness. Additionally, when (3) holds, the theory is stable so coheirs are equivalent to non-forking. While we don't claim to have discovered anything new about first-order theories, formally speaking our framework apply to K_T where T is a superstable first-order theory and K_T is the class of $|T|^+$-saturated models (our κ is $|T|^+$).
- **Large cardinals** Boney [Bonc] proves that (1),(2), and Extension hold for any AEC K that are essentially below a strongly compact cardinal κ (this holds, for instance, if $LS(K) < \kappa$). Slightly weaker (but still useful) versions of (1) and (2) also hold if κ is measurable or weakly compact. See Section 8 for more.
- **Homogeneous model theory** The homogeneity of the monster model ensures that the types are tame and type short. Hyttinen and Shelah [HySh629]
- **Zilber's pseudoexponentiation** See page 190 in Baldwin's book [Bal09].

6. Getting Local Character

Local character is a very important property for identifying dividing lines. In the first order context, some of the main classes of theories–superstable, strictly stable, strictly simple, and unsimple–can be identified by the value of $\kappa(T)$. By finding values for $\kappa_\alpha(\mathop{\smile\!\!\!\!\!\!\!\;\;\,|}\,)$ under different hypotheses, we get candidates for dividing lines in AECs.

Readers familiar with first order stability theory will recall that there is a notion of an heir of a type that is the dual notion to coheir, which our nonforking is based on. Heir is equivalent to to the notion of coheir under the assumption of no order; see [Pil83].1 and .2 as a reference. We develop an AEC version of heir and explore its relation with nonforking. We further show that there is an order property that implies their equivalence. This equivalence allows us to adapt an argument of [ShVi635] to calculate $\kappa(\mathop{\smile\!\!\!\!\!\!\!\;\;\,|}\,)$ from categoricity. In this discussion, we only assume the properties of nonforking that follow immediately from the definition, like those in Theorem 4.4, and explicitly state any other assumptions. In particular, note that Theorem 6.6 doesn't assume (E), the failure of the weak κ-order property, or tameness or type shortness.

Recall that 'small' refers to objects of size $< \kappa$.

Definition 6.1. *We say that $p \in S^I(N)$ is an* heir *over $M \prec N$ iff for all small $I_0 \subset I$, $M^- \prec M$, and $M^- \prec N^- \prec N$ (with M^- possibly being empty), there is some $f : N^- \to_{M^-} M$ such that $f(p^{I_0} \upharpoonright N^-) \leq p$; that is, $f(p)^{I_0} \upharpoonright f(N^-) = p^{I_0} \upharpoonright f(N^-)$. We also refer to this by saying p is a heir of $p \upharpoonright M$.*

$$\begin{tikzcd} M \arrow[r] & N \\ M^- \arrow[u] \arrow[r] & N^- \arrow[u] \arrow[ul, "f"'] \end{tikzcd}$$

At first glance, this property seems very different from the first order version of heir. However, if we follow the remark after Theorem 5.1, we can think of restrictions of p as formulas and small models as parameters. Then, M^- is a parameter from M, N^- is a parameter from N, $f(N^-)$ is the parameter from M that corresponds to N^- (notice that it fixes M^-), and $f(p \upharpoonright N^-) \leq p$ witnesses that it the original formula $p \upharpoonright N^-$ is still in p with a parameter from M.

If we restrict ourselves to models, then the notions of heir over and nonforking (coheir over) are dual with no additional assumptions.

Proposition 6.2. *Suppose $M_0 \prec M, N$. Then $tp(M/N)$ does not fork over M_0 iff $tp(N/M)$is an heir over M_0.*

Proof: First, suppose that $M \underset{M_0}{\downarrow} N$ and let $a \in |N|$ be of length $< \kappa$. Let $M_0^- \prec M_0$ and $M^- \prec M$ both be of size $< \kappa$ such that $M_0^- \prec M^-$. Find $N^- \prec N$ of size $< \kappa$ containing M_0^- and a. By the definition of nonforking, $tp(M^-/N^-)$ is realized in M_0. This means that there is $g \in Aut_{M^-}\mathfrak{C}$ such that $g(M^-) \prec M_0$. Set $f = g \upharpoonright M^-$. Then $f : M^- \to_{M_0^-} M_0$ such that $f(tp(a/M^-)) = tp(a/f(M^-))$. Since a, M_0^-, and M^- were arbitrary, $tp(N/M)$ is an heir over M_0.

Second, suppose that $tp(N/M)$ is an heir over M_0. Let $b \in M$ and $N^- \prec N$ both be of size $< \kappa$. Since M is a model, we may expand b to a model $M^- \prec M$ of size $< \kappa$. Then, if we can realize $tp(M^-/N^-)$ in M_0, we can find a realization of $tp(b/N^-)$ there as well. By assumption, there is some $f : M^- \to M_0$ such that $tp(f(N^-)/f(M^-)) = tp(N^-/f(M^-))$. This type equality means that there is some $g \in Aut_{f(M^-)}\mathfrak{C}$ such that $g(f(N^-)) = N^-$. Thus, $g \circ f$ is in $Aut_{N^-}\mathfrak{C}$ and sends M^- to $f(M^-) \prec M_0$. Thus, $tp(M^-/N^-) = tp(f(M^-)/N^-)$ and is realized in M_0, as desired. †

This proposition was proven just from the definitions, without assuming any tameness or type shortness. If we assume even the weak symmetry (S^*), then we have that nonforking and heiring are equivalent for models. Assuming full symmetry (S) is enough to get the full implication in one direction.

Theorem 6.3. *Suppose $\downarrow$ satisfies (S). If $p \in S(N)$ and $M \prec N$, then p does not fork over M holds implies p is an heir over M.*

Proof: Suppose $p \in S(N)$ does not fork over M. Then, given A that realizes p, we have $A \underset{M}{\perp} N$. By (S), we can find $M^+ \succ M$ containing A such $N \underset{M}{\perp} M^+$. By Proposition 6.2, we then have $tp(M^+/N)$ is an heir over M. By motonicity, $p = tp(A/N)$ is an heir over M. †

However, for the other direction, this does not suffice. It would be possible to completely redevelop the stability theory of the previous sections for the notion of heiring, but this would not help us understand the real connection between nonforking and heiring. Instead, we draw a parallel to the first order case. There, the equivalence of heir and coheir uses the order property, as does the first order version of Lemma 4.3 above. Following this, we introduce a new order property, order$_2$, that characterizes the relationship between nonforking and heiring. We refer to order$_2$ as "an order property" because, like Definition 4.2, it is witnessed by a sequence whose order is semantically definable inside of the AEC.

Definition 6.4. *We say that an AEC K has the* (λ, α)-order$_2$ property *iff there are parameters b and $\langle b_i : i < \alpha \rangle$ and models $\langle N_i \in K : i < \alpha \rangle$ such that $\ell(b_i) + \|N_i\| < \lambda$ and, for all $i, j < \alpha$, we have*

$$i \leq j \text{ iff } b_j \models tp(b/N_i)$$

We now prove that no order$_2$ property means that heiring implies nonforking. This follows the first order version as presented in [Pil83].2.2.

Theorem 6.5. *Let K be an AEC and $M \prec N$ be models such that M is κ saturated. If there is $p \in S(N)$ that is a heir over M and also forks over M, then K has the (κ, κ)-order$_2$ property, and it is witnessed in M.*

Proof: Suppose that $b \models p$. Since $\neg(b \underset{M}{\perp} N)$, there is some $N^- \prec N$ such that $tp(b/N^-)$ is not realized in M. We are going to construct two sequences $\langle b_i \in |M| : i < \kappa \rangle$ and increasing $\langle N_i^- \prec M : i < \kappa \rangle$ that witness the (κ, κ)-order$_2$ property.
Suppose that we have our sequences defined for all $j < i$ for some fixed $i < \kappa$. Set small $M_i^+ \prec M$ to contain all $\{N_j^-, b_j : j < i\}$ and $N_i^+ \prec N$ to contain M_i^+ and N^-, both of size $< \kappa$; if $i = 0$, then we just take $M_i^+ = \emptyset$ and $N_i^+ = N^-$. Since $tp(b/N)$ is a heir over M, we can find some $f_i : N_i^+ \to_{M_i^+} M$ such that $tp(f_i(b)/f_i(N_i^+)) = tp(b/f_i(N_i^+))$. Set $N_i^- = f_i(N_i^+)$ and extend f_i to an automorphism f_i^+ of $\mathfrak{C}$. By the κ saturation of M, there is $b_i \in |M|$ that realizes $tp(b/N_i^-)$.
Now we want to show that these exhibit the order$_2$ property:
$\mathbf{i \leq j}$: By construction, $N_i^- \prec N_j^-$, so, in particular, $tp(b/N_i^-) \leq tp(b/N_j^-)$. Also, $b_j \models tp(b/N_j^-)$, so we have $b_j \models tp(b/N_i^-)$.

i > j: Suppose $b_j \vDash tp(b/N_i^-)$. This means

$$\begin{array}{rcll} b_j & \vDash & tp(b/N_i^-) & \\ b_j & \vDash & tp(f_i^+(b)/N_i^-) & \text{by the definition of heir} \\ (f_i^+)^{-1}(b_j) & \vDash & tp(b/N_i^+) & \\ b_j & \vDash & tp(b/N_i^+) & b_j \in |M_i^+| \\ b_j & \vDash & tp(b/N^-) & N^- \prec N_i^+ \end{array}$$

which contradicts our assumption that $tp(b/N^-)$ is not realized in M.
So $\langle b_i, N_i^- : i < \kappa\rangle$ witnesses the (κ, κ)-order$_2$ property. †

Now that we have established an equivalence between nonforking and being an heir, we aim to derive local character. For this, we use heavily the proof [ShVi635].2.2.1, which shows that, under certain assumptions, the universal local character cardinal for non-splitting is ω. Examining the proof, much of the work is done by basic independence properties–namely (I), (M), and (T)–and the other assumptions on K–namely categoricity, amalgamation, and EM models, which follow from no maximal models. Only in case (c), defined below, do they need the exact definition of their independence relation (non μ-splitting) and GCH. In this case, we can use the definition of heir to complete the proof.

Theorem 6.6. *Suppose that K has no (κ, κ)-order$_2$ property, is categorica lin some $\lambda \geq \kappa$, and is stable in κ. Then $\kappa_\omega^*(\downfree) = \omega$. That is, if*

(1) *$\langle M_i \in K_\mu : i \leq \alpha\rangle$ is increasing and continuous;*
(2) *each M_{i+1} is universal over M_i and κ saturated;*
(3) *$cf\, \alpha = \alpha < \mu^+ \leq \lambda$; and*
(4) *$p \in S^{<\omega}(M_\alpha)$*

then, for some $i < \alpha$, p does not fork over M_i

Proof: Deny and set $M = M_\alpha$. As in [ShVi635], we consider the three following cases:

(a) for all $i < \alpha$, $p \upharpoonright M_i$ does not fork over M_0;
(b) (a) is impossible and for all $i < \alpha$, $p \upharpoonright M_{2i+1}$ forks over M_{2i} and M_{2i+2} does not fork over M_{2i+1}
(c) (a) and (b) are impossible and $\alpha = \mu \geq \kappa$ and for all $i < \alpha$, $p \upharpoonright M_{i+1}$ forks over M_i

Shelah and Villaveces first show that, using only (M), (I), and (T), one of these three cases must hold. Then, cases (a) and (b) are eliminated using categoricity and EM models, both of which are part of the assumptions. Thus, we can assume that we are in case (c).

Then, by Theorem 6.5 and the assumption of no (κ, κ)-order$_2$ property, we know that $p \upharpoonright M_{i+1}$ is not a heir over M_i for all $i < \alpha$. Find the minimum σ such that $2^\sigma > \kappa$. Then $\sigma \leq \kappa$ and $2^{<\sigma} \leq \kappa$. We are going to contradict stability in κ by finding 2^σ many types over a model of size $2^{<\sigma}$.

Step 1: We define $\langle M^i \prec N^i \mid i < \alpha\rangle$ as follows: for each $i < \alpha$, since $p \restriction M_{i+1}$ is not a heir over M_i, there exists some $M^i \prec N^i \in K_{<\kappa}$ such that $M^i \prec M_i$ and $N^i \prec M_{i+1}$ and for any $h : N^i \to_{M^i} M_i$, we have $h(p \restriction N^i) \neq p \restriction h(N^i)$.

Now define $\langle \hat{M}_i \prec \hat{N}_i \in K_{<\kappa} \mid i < \alpha\rangle$ increasing and continuous and $g_i : \hat{N}_i \to_{\hat{M}_i} \hat{M}_{i+1}$ by setting $\hat{M}_0 = M^0$ and $\hat{N}_0 = N^0$ and taking unions at limits. If we have $\hat{M}_i$ and $\hat{N}_i$ defined, then we can use the saturation of $M_{i+1} \succ \hat{M}_i$ to find some $g_i : \hat{N}_i \to_{\hat{M}_i} M_{i+1}$. Then pick $\hat{M}_{i+1} \prec M_{i+1}$ to contain $g_i(\hat{N}_i)$ and M^{i+1} and $\hat{N}_{i+1} \prec M_{i+2}$ to contain N^{i+1} and $\hat{M}_{i+1}$.

Now that we have finished this construction, notice that $g_i \restriction N^i : N^i \to_{M^i} \hat{M}_{i+1} \prec M_{i+1}$, so $g_i(p \restriction N^i) \neq p \restriction g_i(N^i)$. Since inequality of types always transfers up, we have $g_i(p \restriction \hat{N}_i) \neq p \restriction g_i(\hat{N}_i)$.

Step 2: First, we relabel elements as standard. We change:

$$\begin{tikzcd}
\dots \arrow[r] & \hat{N}_{2i} \arrow[r] \arrow[dr, "g_{2i}"] & \hat{N}_{2i+1} \arrow[r] \arrow[dr, "g_{2i+1}"] & \hat{N}_{2i+2} \arrow[r] & \dots \\
\dots \arrow[r] & \hat{M}_{2i} \arrow[r] \arrow[u] & \hat{M}_{2i+1} \arrow[r] \arrow[u] & \hat{M}_{2i+2} \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

to

$$\begin{tikzcd}
\dots \arrow[r] & N_{2i} \arrow[r] \arrow[dr, "g_{2i}"] & N_{2i+1} \arrow[r, "f_{2i+1}"] \arrow[dr] & N_{2i+2} \arrow[r] & \dots \\
\dots \arrow[r] & M_{2i} \arrow[r] \arrow[u] & M_{2i+1} \arrow[r] \arrow[u] & M_{2i+2} \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

by setting $f_{2i+1} = g^{-1}_{2i+1}$; $M_{2i+2} = g^{-1}_{2i+1}(\hat{M}_{2i+2})$, $M_{2i+1} = \hat{M}_{2i+1}$; $N_{2i+2} = g^{-1}_{2i+1}(\hat{N}_{2i+2})$, and $N_{2i+1} = \hat{N}_{2i+1}$. We do this so we have identities where we want them and embeddings (the f_{2k+1}'s) on paths we ignore. Now define, for each $i < \alpha$, $N_i \prec N_i^1, N_i^2 \prec N_{i+1}$ all in $K_{<\kappa}$ with $\langle N_i \mid i < \alpha\rangle$ increasing and continuous and $h_i : N_i^1 \to_{N_i} N_i^2$ such that $h_i(p \restriction N_i^1) \neq p \restriction h_i(N_i^1)$. This is done by setting $N_i = \hat{M}_{2i}$, $N_i^1 = \hat{N}_{2i}$, $N_i^2 = \hat{M}_{2i+1}$, and $h_i = g_{2i}$.

Step 3: We now construct a tree of types of height κ that will all be different at the top. For each $\eta \in {}^{\leq\sigma}2$, we are going to construct

- $\widehat{h}_\eta \in Aut\ \mathfrak{C}$;
- increasing, continuous $N_\eta \in K_{<\kappa}$ such that $\widehat{h}_\eta(N_{\ell(\eta)}) = N_\eta$; and
- increasing $p_\eta \in S(N_\eta)$.

We work by induction on the length of η.

- When $\eta = 0$, set $\widehat{h}_\emptyset = \mathrm{id}_{\mathfrak{C}}$, $N_\emptyset = N_0$, and $p_\emptyset = p \restriction M_0$.

- When η is the successor of ν, set $\widehat{h}_{\nu^\frown 0} = \widehat{h}_\nu$ and $\widehat{h}_{\nu^\frown 1} = \widehat{h}_\nu \circ h_{\ell(\nu)}$. Then set $N_{\nu^\frown i} = \widehat{h}_{\nu^\frown i}(N_{\ell(\nu+1)})$, as required, and $p_\eta = \widehat{h}_\eta(p \restriction N_\ell(\eta))$.
- When η is a limit, set $N_\eta = \cup_{\alpha<\ell(\eta)} N_{\eta\restriction\alpha}$. Then we have that $\langle \widehat{h}_{\eta\restriction\alpha} \restriction N_{\eta\restriction\alpha} : \alpha < \ell(\eta)\rangle$ is an increasing sequence, so set $\widehat{h}_\eta$ to be any automorphism of $\mathfrak{C}$ extending their union and $p_\eta = \widehat{h}_\eta(p \restriction N_{\ell(\eta)})$.

Once we have completed this construction, note that our choice of σ guarantees that there is some M^* of size κ such that $N_\eta \prec M^*$ for all $\eta \in {}^{<\sigma}2$. Thus, $N_\eta \prec M^*$ for all $\eta \in {}^\sigma 2$. Then, for each $\eta \in {}^\sigma 2$, we can extend p_η to some $p^*_\eta \in S(M^*)$. Once we prove the following claim, we will have contradicted stability in κ since $2^\sigma > \kappa$.

Claim: If $\eta \neq \eta' \in {}^\sigma 2$, then $p^*_\eta \neq p^*_{\eta'}$.

Set $\nu = \eta \cap \eta'$ and $i = \ell(\nu)$. WLOG, $\nu^\frown 0 \subset \eta$ and $\nu^\frown 1 \subset \eta'$. From their construction, we know the following things about these types:

- $p_{\nu^\frown 0} = \widehat{h}_{\nu^\frown 0}(p \restriction N_{i+1}) = \widehat{h}_\nu(p \restriction N_{i+1}) \leq p_\eta$;
- $p_{\nu^\frown 1} = \widehat{h}_{\nu^\frown 1}(p \restriction N_{i+1}) = \widehat{h}_\nu(h_i(p \restriction N_{i+1})) \leq p_{\eta'}$;
- $p \restriction N_i^2 \leq p \restriction N_{i+1}$;
- $h_i(p) \restriction N_i^2 \leq h_i(p) \restriction N_{i+1}$; and
- $p \restriction N_i^2 \neq h_i(p) \restriction N_i^2$.

Since inequality of types transfers upwards, this is enough. The bottom three lines imply that $p \restriction N_{i+1} \neq h_i(p) \restriction N_{i+1}$. Since the first two lines show that the same map $\widehat{h}_\nu$ maps the lefthand-side as a subtype of p_η and the righthand-side as a suptype of $p_{\eta'}$, this finishes the claim and the proof. †

This construction could not go further than κ many steps because the definition of heir requires all of the models and tuples involved to be of size $< \kappa$. Thus, we need to know that stability fails at κ. If we knew that nonforking and nonsplitting were the same, instead of just nonforking and heiring, then we would have a more general argument. The connection between these two notions of independence and other is explored more in [BGKV].

Once we have the universal local character, we can get results on the uniqueness of limit models. Limit models (introduced as brimmed in [Sh600], see the definition below) have been suggested as a substitute for saturated models and the question of uniqueness of limit models has been suggested as a dividing line for AECs; see Shelah [Sh576]. Shelah and Villaveces [ShVi635] claimed uniqueness of limit models from categoricity. However, in 1998, VanDieren discovered a gap in the proof. In 2000, Shelah admitted that he is unable to fix that gap. VanDieren [Van06] [Van13] proved uniqueness of limit models from categoricity with weaker assumptions than we have here, namely instead of full amalgamation it was assumed that only unions of limit models are amalgamation bases. A follow up is in Grossberg, VanDieren, and Villaveces [GVV].

Definition 6.7. (1) *Let $M \in K_\lambda$ and $\alpha < \lambda^+$ be a limit ordinal. N is (λ, α)-limit over M iff there is a resolution of N $\langle N_i \in K_\lambda : i < \alpha \rangle$ so $N_0 = M$ and M_{i+1} is universal over M_i.*

(2) *K has unique limit models in λ iff if $M, N_1, N_2 \in K_\lambda$ and $\alpha_1, \alpha_2 < \lambda^+$ so that N_ℓ is (λ, α_ℓ)-limit over M, then $N_1 \cong_M N_2$.*

It is an easy exercise to show that (2) holds if $cf\alpha_1 = cf\alpha_2$. While many of the above papers prove the uniqueness of limit models in different contexts, the most relevant for our context is the proof that is outlined in [Sh:h].II.4 and detailed in Boney [Bonb]. There, Shelah's frames are used to create a matrix of models to show that limit models are isomorphic. Inspecting the proof, the only property used that is not a part of an independence property is a stronger continuity restricted to universal chains. This follows from universal local character.

Theorem 6.8 ([Bonb].8.2)**.** *If $\perp$ is an independence relation so $\kappa^*_\omega(\perp) = \delta$, then any two limit models of length at least δ are isomorphic. Thus, if $\kappa^*_\omega(\perp) = \omega$, then K has unique limit models.*

Remark 6.9. *It is unclear if this theorem is indeed an improvement of [Van06], [Van13] and [GVV]. In Theorem 6.8 full amalgamation is used. Notice that the proof of Theorem 6.8 uses that $\perp$ is symmetric. The other approaches for uniquness of limit models don't use symmetry.*

A proof for the original uniqueness statement from [ShVi635] is still not known.

Corollary 6.10. *Suppose there is some $\kappa > LS(K)$ so that*

(1) *K is fully $< \kappa$-tame;*
(2) *K is fully $< \kappa$-type short;*
(3) *K doesn't have the weak κ-order property or the (κ, κ)-order$_2$ property;*
(4) *$\perp$ satisfies (E); and*
(5) *it is categorical in some $\lambda > \kappa$*

Then K has a unique limit model in each size in $[\kappa, \lambda)$. Moreover, if λ is a successor, then K has unique limit models in each size above κ.

Proof: The first part follows from Theorems 5.1, 6.5, 6.6, and 6.8. The moreover follows from the categoricity transfer of [GV06a]. †

Note that the uniqueness of limit models as stated does not follow trivially from categoricity because it requires that the isomorphism fixes the base.

7. The U-Rank

Independence relations and ranks go hand in hand in first order theories: in the appropriate contexts, splitting is equivalent to an increase of the two-rank [Gro1X].6.4.4, non-weak minimality to an increase of the Deg [Sh31].4.2, forking to an increase in the local rank [Sh:c].Theorem III.4.1.

Here we develop a U-rank for our forking and show that, under suitable conditions, it behaves as desired. The U-rank was first introduced by Lascar [Las75] for first order theories and first applied to AECs by [Sh394]. They have also been studied by Hyttinen, Kesala, and Lessman in various nonelementary contexts; see [Les00], [Les03], [HyLe02], and [HyKe06].

For this section, we add the hypotheses of the main theorem so that $\smile$ will be an independence relation. Indeed, the results of this section do not use our specific definition of nonforking, but just that it satisfies the axioms of an independence relation given in Definition 3.3.

Hypothesis 7.1. *Suppose that there is some $\kappa > LS(K)$ such that*

(1) *K is fully $< \kappa$-tame;*
(2) *K is fully $< \kappa$-type short;*
(3) *K doesn't have the weak κ-order property; and*
(4) *$\smile$ satisfies (E).*

Definition 7.2. *We define U with domain a type and range an ordinal or ∞ by, for any $p \in S(M)$*

(1) *$U(p) \geq 0$;*
(2) *$U(p) \geq \alpha$ limit iff $U(p) \geq \beta$ for all $\beta < \alpha$;*
(3) *$U(p) \geq \beta + 1$ iff there is $M' \succ M$ with $\|M'\| = \|M\|$ and $p' \in S(M')$ such that p' is a forking extension of p and $U(p') \geq \beta$;*
(4) *$U(p) = \alpha$ iff $U(p) \geq \alpha$ and $\neg(U(p) \geq \alpha + 1)$; and*
(5) *$U(p) = \infty$ iff $U(p) \geq \alpha$ for every α.*

First we prove a few standard rank properties. The first several results are true without the clause about the sizes of the model, but this is necessary later when we give a condition for the finiteness of the rank for Lemma 7.8.

Lemma 7.3 (Monotonicity). *If $M \prec N$, $p \in S(M)$, $q \in S(N)$, and $p \leq q$, then $U(q) \leq U(p)$.*

Proof: We prove by induction on α that $p \leq q$ implies that $U(q) \geq \alpha$ implies $U(p) \geq \alpha$. For limit α, this is clear, so assume $\alpha = \beta + 1$ and $U(q) \geq \beta + 1$. Then there is a $N' \succ N$ and $q^+ \in S(N')$ that is a forking extension of q and $IU(q^+) \geq \beta$. By (M), it is also a forking extension of p. Then $U(p) \geq \alpha$ as desired. †

Lemma 7.4 (Invariance). *If $f \in Aut\mathfrak{C}$ and $p \in S(M)$, then $U(p) = U(f(p))$.*

Proof: Clear. †

Proposition 7.5 (Ultrametric). *The U rank satisfies the ultrametric property; that is, if we have $M \prec N_i$, $p \in S(M)$ and distinct $\langle q_i \in S(N_i) \mid i < \alpha \rangle$ are such that $a \models p$ iff there is an $i_0 < \alpha$ such that $a \models q_{i_0}$, then we have $U(p) = \max_{i<\alpha} U(q_i)$.*

Note that, as always, we assume α is well below the size of the monster model.

Proof: We know that $p \leq q_i$ for all $i < \alpha$, so, by Lemma 7.3, we have $\max_{i<\alpha} U(q_i) \leq U(p)$. Since we have a monster model, we can find some $N^* \in K$ that contains all N_i. By (E), we can find some $p^+ \in S(N^*)$ such that p^+ is a non-forking extension of p. Now, let $a \models p^+$. Since $p \leq p^+$, $a \models p$. Since '$p(\mathfrak{C}) = \cup_{i<\alpha} q_i(\mathfrak{C})$,' there is some $i_0 < \alpha$ such that $a \models q_{i_0}$. But then $a \underset{M}{\perp} N^+$ implies $a \underset{M}{\perp} N_{i_0}$ by (M), so $tp(a/N_{i_0}) = q_{i_0}$ does not fork over M. Then

$$U(p) = U(q_{i_0}) = \max_{i<\alpha} U(q_i)$$

†

We want to show that same rank extensions correspond exactly to non-forking when the U-rank is ordinal valued. One direction is clear from the definition. For the other, we generalize first order proofs to the AEC context; this proof follows the one in [Pil83]. First, we prove the following lemma.

Lemma 7.6. *Let $N_0 \prec N_1 \prec \bar{N}_1$, $N_0 \prec \bar{N}_0 \prec \bar{N}_1$, and $N_0 \prec N_2$ be models with some $c \in \bar{N}_0$. If*

$$N_1 \underset{N_0}{\perp} \bar{N} \text{ and } N_2 \underset{\bar{N}_0}{\perp} \bar{N}_1$$

then there is some N_3 extending N_1 and N_2 such that

$$c \underset{N_2}{\perp} N_3$$

Proof: We can use (S) twice on $N_2 \underset{\bar{N}_0}{\perp} \bar{N}_1$ to find $\bar{N}_2$ extending N_2 and $\bar{N}$ such that $\bar{N}_2 \underset{\bar{N}}{\perp} \bar{N}_1$. This contains c, so (M) implies that $N_2 c \underset{\bar{N}}{\perp} \bar{N}_1$. By applying (S) to the other nonforking from our hypothesis, we know $\bar{N} \underset{N_0}{\perp} N_1$. By (T), this means that $N_2 c \underset{N_0}{\perp} N_1$.

Applying (S) to this, there is some N_3' extending N_2 and containing c such that $N_1 \underset{N_0}{\perp} N_3'$. By (M), we have that $N_1 \underset{N_2}{\perp} N_3'$. Applying (S) one final time, we can find an N_3 extending N_1 and N_2 such that $c \underset{N_2}{\perp} N_3$. †

Theorem 7.7. *Let $p \in S(M_0)$ and $q \in S(M_1)$ such that $p \leq q$ and $U(p), U(q) < \infty$. Then*

$$U(p) = U(q) \iff q \text{ is a nonforking extension of } p$$

Proof: By definition, $U(p) = U(q)$ implies q does not fork over M_0. For the other direction, we show by induction on α that, for any q that is a nonforking extension of some p, $U(p) \geq \alpha$ implies $U(q) \geq \alpha$.

If α is 0 or limit, this is straight from the definition.

Suppose that $U(p) \geq \alpha + 1$. Then, there are $M_2 \succ M_0$ and $p_1 \in S(M_2)$ such that p_1 is a forking extension of p and $U(p_1) \geq \alpha$.

Claim: We may pick M_2 and p_1 such that there is a M_3 extending M_1 and M_2 and $q_1 \in S(M_3)$ so

- $q_1 \geq q, p_1$; and
- q_1 does not fork over M_2.

Once we prove this claim, we will be done.

Assume for contradiction that q_1 does not fork over M_1. By [BGKV].6.9, a right version of transitivity also holds of our nonforking:

$$\text{if } A \underset{M_0}{\perp} M_1 \text{ and } A \underset{M_1}{\perp} M_2 \text{ with } M_0 \prec M_1 \prec M_2, \text{ then } A \underset{M_0}{\perp} M_2$$

Thus, q_1 would also not fork over M_1. By (M), this would imply that p_1 does not fork over M_0, a contradiction. Thus, q_1 is a forking extension of q of U rank at least α. Thus, $U(q) \geq \alpha + 1$.

To prove the claim, let d realize q and d' realize p_1. Since both of these types extend p, there is some $f \in Aut_{M_0}\mathfrak{C}$ such that $f(d') = d$. Set $M_2' = f(M_2)$. We know that $d \underset{M_0}{\perp} M_1$, so by (S), there is some $\bar{M}_0 \succ M_0$ that contains d so $M_1 \underset{M_0}{\perp} \bar{M}_0$. Pick $\bar{M}_1 \prec \mathfrak{C}$ that contains $\bar{M}_0$ and M_1. By (E), there is some M_2'' so that $tp(M_2'/\bar{M}_0) = tp(M_2''/\bar{M}_0)$ and $M_2'' \underset{\bar{M}_0}{\perp} \bar{M}_1$. Let $g \in Aut_{\bar{M}_0}\mathfrak{C}$ such that $g(M_2') = M_2''$; note that this fixes d.

We may now apply our lemma. This means there is some M_3 that extends M_2'' and M_1 such that $d \underset{M_2''}{\perp} M_3$. Now this proves our claim with M_2'' and $tp(d/M_2'') = g(f(p_1))$ and witnesses M_3 and $q_1 = tp(d/M_3)$. †

We now give a condition for the U rank to be ordinal valued, as in [Sh394].5. First, note that clause about the model sizes in the definition of U gives a bound for the rank.

Lemma 7.8 (Ordinal Bound)**.** *If $M \in K_\mu$ and $p \in S(M)$, then $U(p) > (2^\mu)^+$ implies $U(p) = \infty$.*

Theorem 7.9 (Superstability)**.** *Let $M \in K_\mu$ and $p \in S(M)$. Then the following are equivalent:*

(1) $U(p) = \infty$.
(2) *There is an increasing sequence of types $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and p_{n+1} is a forking extension of p_n for all $n < \omega$.*

Proof: First, suppose $U(p) = \infty$ and set p_0. We will construct our sequence by induction such that $U(p_n) = \infty$. Then $U(p_n) > (2^\mu)^+ + 1$, so there is a forking extension p_{n+1} with the same sized domain and $U(p_{n+1}) > (2^\mu)^+$. But then $U(p_{n+1}) = \infty$ and out induction can continue.

Second, suppose we have such a sequence $\langle p_n : n < \omega \rangle$ and we will show, by induction on α, the $U(p_n) \geq \alpha$ for all $n < \omega$. The 0 and limit stages are clear. At stage $\alpha + 1$, p_{n+1} is a forking extension of p_n with rank at least α. Thus, $U(p_n) \geq \alpha + 1$. †

Ranks in a tame AEC have also been explored by Lieberman [Lie13]. Under a tameness assumption, he introduces a series of ranks that emulate Morley Rank.

Definition 7.10 ([Lie13].3.1). *Let $\lambda \geq \kappa$, where K is κ-tame. For $M \in K_\lambda$ and $p \in S(M)$, we define $R^\lambda(p)$ inductively by*

- *$R^\lambda[p] \geq 0$;*
- *$R^\lambda[p] \geq \alpha$ for limit α iff $R^\lambda[p] \geq \beta$ for all $\beta < \alpha$; and*
- *$R^\lambda[p] \geq \beta + 1$ iff there is $M' \succ M$ and $\langle p_i \in S(M') : i < \lambda^+ \rangle$ such that $p \leq p_i$ and $R^\lambda[p_i] \geq \beta$ for all $i < \lambda^+$.*

If $\|M\| > \lambda$ and $p \in S(M)$, then

$$R^\lambda[p] = \min\{R^\lambda[p \restriction N] : N \prec M, \|N\| = \lambda\}$$

Our U-rank dominates these Morley Ranks at least for domains of size λ. Thus, the finiteness of the U-rank, which follows from local character, implies us that an AEC is totally transcendental and that the stability transfer results of [Lie13].§5 apply.

Theorem 7.11. *Let $M \in K_\lambda$, $p \in S(M)$, and $\lambda \geq \kappa$. Then $U(p) \geq RM^\lambda(p)$.*

Proof: We prove, simultaneously for all types, that $RM^\lambda(p) \geq \alpha$ implies $U(p) \geq \alpha$ for all α by induction. For $\alpha = 0$ or limit, this is easy.

Suppose $RM^\lambda(p) \geq \alpha + 1$. Let M' and $\langle p_i \in S(M') : i < \lambda^+ \rangle$ witness this. p has a unique nonforking extension to M', call it p^*. Thus, almost all of the p_i fork over M; let p_{i_0} be one of them. Then, $p_{i_0} \neq p^*$, so there is some $M_0 \prec M'$ of size $< \kappa$ such that $p_{i_0} \restriction M_0 \neq p^* \restriction M_0$. Let $M'' \prec M'$ contain M and M_0 such that $\|M\| = \|M''\|$ and $p' = p_{i_0} \restriction M''$. Then

- p' extends p;
- p' is a forking extension of p because it differs from the nonforking extension, $p^* \restriction M_{i_0}$; and
- $RM^\lambda(p') \geq RM^\lambda(p_{i_0})$ by [Lie13].3.3. So $RM^\lambda(p') \geq \alpha$. By induction, this means $U(p') \geq \alpha$.

So $U(p) \geq \alpha + 1$, as desired. †

8. Large cardinals revisited

In this section, we discuss the behavior of non-forking in the presence of large cardinals. We return to just assuming Hypothesis 3.1, that K satisfies amalgamation, joint embedding, and no maximal models.

Recall that κ is strongly compact iff every κ complete filter can be extended to a κ complete ultrafilter; see [Jec06].20 for a reference. Boney [Bonc] proved that the tameness and type shortness hypotheses of Theorem 5.1 follow from κ being a strongly compact cardinal.

Theorem 8.1 ([Bonc].4.5)). *If κ is strongly compact and K is an AEC with $LS(K) < \kappa$, then K is fully $< \kappa$ tame and fully $< \kappa$ type short.*

A similar result holds for AECs axiomatized in $L_{\kappa,\omega}$. However, [MaSh285] deals with this case more fully, so we focus on $LS(K) < \kappa$. In fact, extending the results of [MaSh285] to general AECs via the methods of [Bonc] was the motivation for this paper. The key tool of [Bonc] is a Łoś' Theorem for AECs (see [Bonc].4.3 and .4.7) that says that such AECs are closed under ultraproducts, that ultraproducts of embeddings is an embedding of the ultraproducts, and more.

We now detail a construction that will be used often in the following proof. This construction and the proof of the following theorem draw inspiration from [MaSh285]. Suppose that $M \prec N$ and U is a κ complete ultrafilter over I. Then Łoś' Theorem for AECs states that the canonical ultrapower embedding $h : N \to \Pi N/U$ that takes n to the constant function $[i \mapsto n]_U$ is a K-embedding. We can expand h to some h^+ that is an $L(K)$ isomorphism with range $\Pi N/U$ and set $N^U := (h^+)^{-1}[\Pi N/U]$. This is a copy of the ultraproduct that actually contains N. Similarly, we can set $M^U := (h^+)^{-1}[\Pi N/U]$. The following claim is key.

Claim: $M^U \underset{M}{\perp} N$.

Proof: Let small $N^- \prec N$ and $a \in M^U$. Then $h^+(a) = [f]_U$ for some $[f]_U \in \Pi M/U$. Denote $tp(a/N^-)$ by p. Then, by Łoś' Theorem, version 2, we have

$$\begin{array}{rcl} a & \vDash & p \\ h^+(a) = [f]_U & \vDash & h^+(p) = h(p) \\ X & := & \{i \in I : f(i) \vDash p\} \in U \end{array}$$

Since $[f]_U \in \Pi M/U$, there is some $i_0 \in X$ such that $f(i_0) \in M$. Then $f(i_0) \vDash p$ as desired. †

We now show that non-forking is very well behaved in the presence of a strongly compact cardinal. Note that the second part says that the local character property holds very strongly if the type does not fork over its domain and the third part improves on Theorem 5.4 by showing that categoricity implies an analogue of superstability instead of just an analogue of simplicity.

Theorem 8.2. *Suppose κ is strongly compact and K is an AEC such that $LS(K) < \kappa$. Then*

(1) *$\perp$ satisfies Extension.*
(2) *If $M = \cup_{i<\alpha} M_i$, $p \in S^\chi(M)$ for (possibly finite) $\chi < cf\,\alpha$, and p does not fork over M, then there is some $i_0 < \alpha$ such that p does not fork over M_{i_0}*

(3) *If K is categorical in some $\lambda = \lambda^{<\kappa}$, then $\mathop{\perp}$ is an independence relation with $\kappa_\alpha(\mathop{\perp}) \leq \omega + |\alpha|$.*

Proof:

(1) Suppose that $A \mathop{\perp}\limits_{M_0} N$ and let $N^+ \succ N$. In particular, this means that $A \mathop{\perp}\limits_{N} N$ and every $< \kappa$ approximation to $tp(A/N)$ is realized in N. We can use this to construct U as in [Bonc] such that $h(tp(A/N))$ is realized in $\Pi N/U$. That means that $tp(A/N)$ is realized in N^U. Call this realization A'. By the above claim, $N^U \mathop{\perp}\limits_{N} N^+$. By (M), this implies $A' \mathop{\perp}\limits_{N} N^+$. Since $tp(A/N) = tp(A'/N)$, $A' \mathop{\perp}\limits_{M_0} N$ by invariance. Thus, by (T), $A' \mathop{\perp}\limits_{M_0} N^+$, as desired.

(2) We break into cases based on the cofinality of α.

If cf $\alpha < \kappa$, then, as before, we can use the fact that p does not fork over M to find a κ complete ultrafilter U on I such that p is realized in M^U. Since cf $\alpha < \kappa$ and U is κ complete, we have that $M^U = \cup_{i<\alpha} M_i^U$. Let $A \in M^U$ realize p. Since A if of size χ and $\chi <$ cf α, there is some $i_0 < \alpha$ such that $A \in M_{i_0}^U$. Thus, by the claim,

$$M_{i_0}^U \mathop{\perp}\limits_{M_{i_0}} M$$

$$A \mathop{\perp}\limits_{M_{i_0}} M$$

Thus, $p = tp(A/M)$ does not fork over M_{i_0}.

Now suppose that cf $\alpha \geq \kappa$. For contradiction, suppose that p forks over M_i for all $i < \alpha$. We now build an increasing and continuous sequence of ordinals $\langle i_j : j < \chi^+ + \aleph_0 \rangle$ by induction. Let $i_0 < \alpha$ be arbitrary. Given i_j, we know that p forks over M_{i_j}. By the definition, there is a small $M^- \prec M$ and small $I_0 \subset \chi$ such that $p^{I_0} \upharpoonright M^-$ is not realized in M_{i_j}. Since cf $\alpha \geq \kappa$, there is some $i_{j+1} > i_j$ such that $M^- \prec M_{i_{j+1}}$. Then $p \upharpoonright M_{i_{j+1}}$ forks over M_{i_j}. Set $M^* = \cup_{j<\chi^+} M_{i_j}$. Then, by Monotonicity, $p \upharpoonright M^*$ forks over M_{i_j} for all $j < \chi^+$. Since $\chi^+ < \kappa$, this contradicts the first part.

(3) Note that the results of [Bonc] say that this categoricity assumption also implies that $K_{\geq\kappa}$ has amalgamation, joint embedding, and no maximal models, which significantly weakens the reliance on or eliminates the need for Hypothesis 3.1.

From inaccessibility, we know that $\sup_{\mu<\kappa}(\beth_{(2^\mu)^+}) = \kappa$, so Existence holds by Theorem 5.4. Then Extension holds by the first part, so (E) holds. Theorem **??** tells us that K is $< \kappa$ tame and type short. Finally, as outlined in the discussion after Theorem 5.1, the weak κ order property with κ inaccessible implies many models in all cardinals above κ, which is contradicted by categoricity in λ. †

Additionally, with the full strength of a strongly compact cardinal, we can reprove much or all of [MaSh285].§4 in an AEC context. One complication is that Definition [MaSh285].4.23 defines weakly orthogonal types by having an element in the nonforking relation where we require a model. However, this definition has already been generalized at [Sh:h].III.6.

[Bonc] also proves weaker of Theorem **??** from assumptions of measurable or weakly compact cardinals. These in turn could be used to produce weaker versions of Theorem 8.2. However, [MaSh285] is not the only time independence relations have been studied in infinitary contexts with large cardinals. Kolman and Shelah [KoSh362] and Shelah [Sh472] investigate the consequences of categoricity in $L_{\kappa,\omega}$ when κ is measurable. In [KoSh362], they use heavily 'suitable operations,' by which they mean taking κ complete ultralimits. The denote such an ultralimit of M by $Op(M)$ and the canonical embedding by $f_{Op} : M \to Op(M)$. In [Sh472], Shelah introduces the following independence relation.

Definition 8.3 ([Sh472].1.5)**.** *Let K be essentially below κ measurable. Define a 4-place relation ${}^S\!\perp$ by $M_1 {}^S\!\underset{M_0}{\overset{M_3}{\perp}} M_2$ iff there is an ultralimit operation Op with embedding f_{Op} and $h : M_3 \to Op(M_1)$ such that the following commutes*

$$\begin{tikzcd}
M_1 \arrow[rr, "f_{Op}"] \arrow[dr] & & Op(M_1) \\
& M_3 \arrow[ur, "h"] & \\
& M_2 \arrow[u] \arrow[dr, "h"] & \\
M_0 \arrow[uuu] \arrow[ur] \arrow[rr, "f_{Op}"] & & Op(M_0) \arrow[uuu]
\end{tikzcd}$$

In these conditions, this notion turns out to be dual to our non-forking. Thus, by Proposition 6.2, it is equivalent to heir over.

Theorem 8.4. *Let K be an AEC essentially below κ measurable and let $M_0 \prec M_1, M_2 \in K$. Then*

$$M_1 \underset{M_0}{\perp} M_2 \iff \exists M_3 \text{ so } M_2 {}^S\!\underset{M_0}{\overset{M_3}{\perp}} M_1$$

Proof: First, suppose that $M_1 \underset{M_0}{\perp} M_2$. Then we can find a κ complete ultrafilter U such that M_0^U realizes $tp(M_1/M_2)$.Then $M_0^U \underset{M_0}{\perp} M_2$. Thus, there is some $f \in$

$Aut_{M_2}\mathfrak{C}$ such that $f(M_1) \prec M_0^U$. Set $M_3 = f^{-1}[M_2^U]$. Then we have the following commuting diagram:

$$\begin{tikzcd}
M_2 \arrow[rr] \arrow[dr] & & M_2^U \arrow[r, "h^+"] & \Pi M_2/U \\
& M_3 \arrow[ur, "f"] & & \\
& M_1 \arrow[u] \arrow[dr, "f"] & & \\
M_0 \arrow[uuu] \arrow[ur] \arrow[rr] & & M_0^U \arrow[uuu] \arrow[r, "h^+"] & \Pi M_0/U \arrow[uuu]
\end{tikzcd}$$

Collapsing this diagram gives

$$\begin{tikzcd}
M_2 \arrow[rr, "h"] \arrow[dr] & & \Pi M_2/U \\
& M_3 \arrow[ur, "h \circ f"] & \\
& M_1 \arrow[u] \arrow[dr, "h \circ f"] & \\
M_0 \arrow[uuu] \arrow[ur] \arrow[rr, "h"] & & \Pi M_0/U \arrow[uuu]
\end{tikzcd}$$

Note that an ultraproduct is a suitable ultralimit operation, and the ultrapower embedding is its corresponding embedding. Thus $M_2{}^S \underset{M_0}{\overset{M_3}{\smile}} M_1$.

Second, suppose that there is an M_3 such that $M_2{}^S \underset{M_0}{\overset{M_3}{\smile}} M_1$. The claim above generalized to ultralimits implies $f_{Op}^{-1}(Op(M_0)) \underset{M_0}{\smile} M_2$. We have that $h : M_1 \to Op(M_0)$, so by Monotonicity $f_{Op}^{-1}(h(M_1)) \underset{M_0}{\smile} M_2$. By the diagram, $f_{Op}^{-1} \circ h$ fixes M_2, we have that $tp(f_{Op}^{-1}(h(M_1))/M_2) = tp(M_1/M_2)$. By Invariance, this means that $M_1 \underset{M_0}{\smile} M_2$. †

9. Future work

As always, new answers lead to new questions.

Based on the results that we have, a further investigation of type shortness would be useful. Because it was only defined recently, there has been no study of type shortness outside of these two papers. A starting place would be to look at known examples of AECs and determine whether or not they are type short. The relationship between tameness and type shortness explored in [Bonc] suggests that the examples of [HaSh323], [BK09], and [BlSh862] would be good places to look for the failure of type shortness, while the list of tame AECs given in [GV06a] would be good candidates to prove type short.

In addition to type shortness, the above results require that the AEC be tame for long types, not just for types of length 1. Unfortunately, tameness for 1-types is the property that is typically studied. Thus, it would be interesting to see if there is some transfer theorem that shows, given $\beta < \alpha$, if tameness for β-types implies tameness for α-types or if there is some counterexample. A partial transfer theorem has recently been obtained by Boney and Vasey [BoVa] by using Shelah's good λ-frames.

A natural question to ask following the introduction of a strong independence relation in this contexts is if it is the *only* such relation, akin to first order results of Lascar for superstable theories [Las76], Harnik and Harrington for stable theories [HH84], and Kim and Pillay for simple theories [KP97]. This has been explored by Boney, Grossberg, Kolesnikov, and Vasey in [BGKV] with a positive answer:

Theorem 9.1 ([BGKV].7.1)**.** *Under the hypotheses of Theorem 5.1, $\perp$ is the only independence relation on $K_{\geq\kappa}$. In particular, if $\overset{*}{\perp}$ satisfies (I), (M), (E), and (U), then $\overset{*}{\perp} = \perp$.*

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