GIACOMO BONANNO

RATIONAL BELIEFS IN EXTENSIVE GAMES

ABSTRACT. Given an extensive game, with every node x and every player i a subset $K_i(x)$ of the set of terminal nodes is associated, and is given the interpretation of player i's knowledge (or information) at node x. A belief of player i is a function that associates with every node x an element of the set $K_i(x)$. A belief system is an n-tuple of beliefs, one for each player. A belief system is rational if it satisfies some natural consistency properties. The main result of the paper is that the notion of rational belief system gives rise to a refinement of the notion of subgame-perfect equilibrium.

Keywords: Information, beliefs, subgame-perfect equilibrium.

1. INTRODUCTION

In an extensive game, information sets capture the notion of what a player knows when it is her turn to move: if two decision nodes x and y belong to the same information set of player i, then player i does not know whether she is making a choice at x or at y. Information sets, however, do not tell us what a player knows at a node that belongs to *another* player. This raises the question of what information players obtain during the play of a game. For example, in general it cannot be the case that every player always observes which player is moving (even though he may not observe what move was made), as the game of Figure 1 shows. When it is player 3's turn to move, he will either have observed that, before him, only player 1 moved or that both players 1 and 2 moved. In the first case he will be able to deduce that he is at node x_2, while in the second case he will know that he is at node x_3. But this contradicts the fact that, according to his information set, player 3 cannot distinguish between nodes x_2 and x_3. The example of Figure 1 also shows that, in general, we cannot think of the play of a game as having a well-defined temporal structure of the following type: the game starts at date $t = 1$ and at every date $t = 2, 3, \ldots$ a move is made until a terminal node is reached. In the above example, if player 3 is asked to move at date $t = 2$ he will know that he is at node x_2,

Theory and Decision **33**: 153–176, 1992.

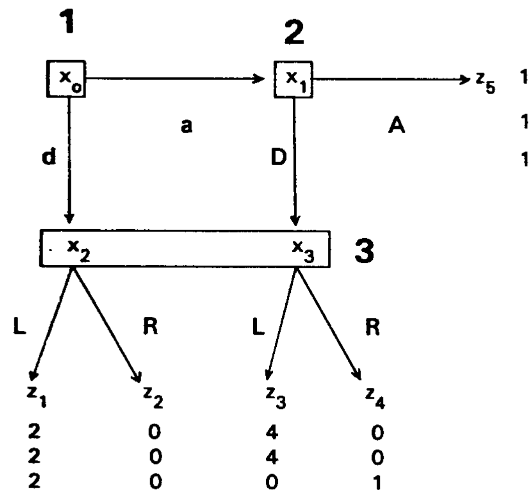

Fig. 1.

while if he is asked to move at date $t = 3$ he will know that he is at node x_3.

Thus the play of an extensive game requires the presence of an outside agent – we shall call her the *umpire* – who provides players with different amounts of information as the play of the game unfolds (for example, in the game of Figure 1, if player 1 chooses a, then player 2 must be informed, while player 3 cannot be informed). In Section 3 we suggest one way of formalizing the information conveyed to each player along every possible play of an extensive game (Section 2 contains some preliminary definitions). With every node x and every player i we associate a subset $K_i(x)$ of the set of terminal nodes, representing what player i knows when node x is reached. The interpretation is that if – when node x is reached – player i's knowledge is given by, say, the set $\{z_1, z_2, z_5, z_8\}$, then player i is informed that the play of the game so far has been such that only terminal nodes z_1, z_2, z_5 and z_8 can be reached.

If x is a decision node that belongs to information set h of player i, we define player i's knowledge at x as the set of all the terminal nodes that can be reached from nodes in h. Thus our definition of knowledge gives an accurate representation of a player's information sets. It seems that, as long as a player's knowledge at her decision nodes is a faithful representation of her information sets, there is a lot of freedom concerning the specification of her knowledge at decision nodes of other players. Thus the definition put forward in this paper is only one of many possibilities.[1] An appealing feature of the definition proposed in this paper is that it provides an intuitive characterization of such notions as perfect recall, perfect information and simultaneity.

In Section 4 we show that our approach offers a new way of thinking about the solutions of a game. We first introduce the notion of *belief system*. We denote by $\beta_i(x)$ player i's belief at node x and we require that $\beta_i(x)$ be an element of the set $K_i(x)$. The interpretation is as follows. Suppose that at node x player i's knowledge is represented by the set $K_i(x) = \{z_1, z_2, z_5, z_8\}$. Then, as explained above, player i knows (is informed) that only terminal nodes z_1, z_2, z_5 and z_8 can be reached. If $\beta_i(x) = z_5$ then player i believes that the final outcome will actually be z_5. (Note that z_5 need not be reachable from x, that is, player i's belief may reflect i's ignorance of the actual play of the game.) Thus a belief of player i is a function that associates with every node x an element of the set $K_i(x)$. A belief system is an n-tuple of beliefs, one for each player. There are some natural consistency properties that one can impose on beliefs. Four simple properties are used in Section 4 to characterize the notion of *rational* belief system. It is then shown that from a (not necessarily rational) belief system β one can extract a pure strategy profile $\sigma = \xi(\beta)$ in a natural way. The main result of this paper is that if β is a *rational* belief system then $\sigma = \xi(\beta)$ is a subgame-perfect equilibrium.[2] While in games of perfect information there is a one-to-one correspondence between (pure-strategy) subgame-perfect equilibria and rational belief systems, in games of *imperfect* information the notion of rational belief system refines that of subgame-perfect equilibrium. We show this by means of an example. Another example is used to show that a rational belief system need not be a sequential equilibrium.[3]

2. PRELIMINARY DEFINITIONS

In this section we define some functions that will be used extensively throughout the paper.

Fix a (finite) extensive game.[4] Let X be the set of *decision* nodes and Z the set of *terminal* nodes. Let $T = X \cup Z$. (In general, we shall denote a decision node by x or y, a terminal node by z and a generic node – decision or terminal – by t.) For every $t \in T$, let

$$\theta(t) \subseteq Z$$

be the set of terminal nodes that can be reached from t. For every $z \in Z$, we set by definition $\theta(z) = \{z\}$. For example, in the game of Figure 1, $\theta(x_1) = \{z_3, z_4, z_5\}$.

If h is an information set, define

$$\theta^*(h) = \bigcup_{x \in h} \theta(x)$$

that is, $\theta^*(h)$ is the set of terminal nodes that can be reached from nodes in h. For example, in the game of Figure 1, $h = \{x_2, x_3\}$ is player 3's information set and we have that $\theta^*(h) = \{z_1, z_2, z_3, z_4\}$.

Recall that a choice c at information set $h = \{x_1, \ldots, x_m\}$ is a set of edges $c = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$ where node y_k is an immediate successor of node x_k $(k = 1, \ldots, m)$. Define

$$\mu(c) = \theta(y_1) \cup \theta(y_2) \cup \cdots \cup \theta(y_m),$$

that is, $\mu(c)$ is the set of terminal nodes that can be reached from nodes in h following the edges that constitute choice c. For example, in the game of Figure 1, $\mu(R) = \{z_2, z_4\}$.

3. THE DEFINITION OF KNOWLEDGE

Fix an extensive game. For every player i and for every (decision or terminal) node t we define a subset $K_i(t)$ of the set Z of terminal nodes, which will be interpreted as 'player i's knowledge at t'. One way of thinking about the proposed definition is as follows. At the root of the tree, denoted by x_0, all players have the same knowledge, namely Z. As the play of the game unfolds and new nodes are reached, an umpire gives (separately) to each player the maximum amount of information that is compatible with that player's information sets, according to the following rules. If z is a terminal node, then every player is informed that the game ended at z. If node x belongs to information set h of player i, then player i is told that her information set h has been reached, but is not told which node in h was reached. If node x does not belong to player i and all the information sets of player i (if any) that are crossed by paths starting at x consist entirely of nodes that are successors of x, then player i is informed that node x has been reached (the justification for this rule is that later on, at any of her information sets, player i will be able to deduce that the play of the game must have gone through node x; hence player i might as well be told at the time when x is reached). When the above condition is not satisfied, player i's knowledge at x either doesn't change (that is, player i is not told anything new) or at most reflects the choice made by player i at the immediate predecessor of x, if that node belonged to player i.

The formal definition is as follows.[5]

(1) Let x_0 be the root of the tree. For every player i set

$$K_i(x_0) = Z .$$

(2) For every $z \in Z$ and for every player i, set

$$K_i(z) = \{z\} .$$

(3) If x is a decision node that belongs to information set h of player i, set

$$K_i(x) = \theta^*(h) .$$

(4) For every decision node x and every player i, let $H_i(x)$ be the set of information sets of player i defined as follows: $h \in H_i(x)$ if and only if h is an information set of player i and there is a node $y \in h$ that is a successor of x. Now, if x is *not* a decision node of player i and either $H_i(x) = \emptyset$ (where $\emptyset$ denotes the empty set) or, for every $h \in H_i(x)$, $\theta^*(h) \subseteq \theta(x)$ (that is, every node in h is a successor of x) then set

$$K_i(x) = \theta(x) .$$

(5) If node x is *not* a decision node of player i and the condition given under (4) is not satisfied (that is, there exists an $h \in H_i(x)$ and a node $y \in h$ such that y is not a successor of x) and x is an immediate successor of decision node t of player i and c is the choice of player i that leads from t to x, then set

$$K_i(x) = \mu(c) .$$

(6) Finally, if x is *not* a decision node of player i and conditions (4) and (5) are not satisfied, set

$$K_i(x) = K_i(\pi_x)$$

where π_x denotes the immediate predecessor of x.

For example, in the game of Figure 2 we have:

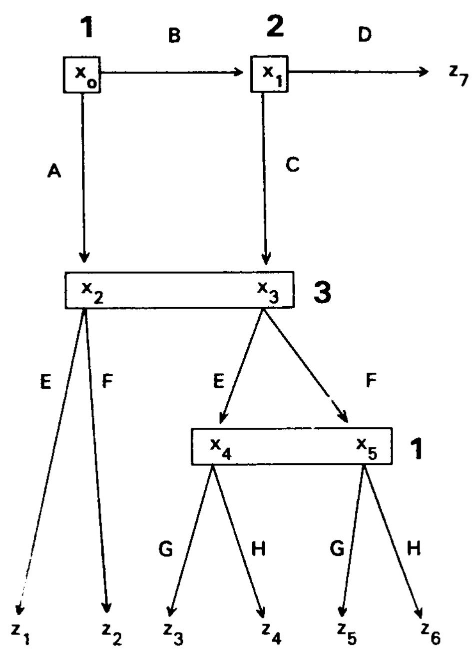

Fig. 2.

By (1): $K_i(x_0) = Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$ for all $i = 1, 2, 3$.

By (2): $K_i(z_j) = \{z_j\}$ for all $i = 1, 2, 3$ and for all $j = 1, \ldots, 7$.

By (3): $K_2(x_1) = \theta^*(\{x_1\}) = \theta(x_1) = \{z_3, z_4, z_5, z_6, z_7\}$.

By (4): $K_1(x_1) = \theta(x_1) = \{z_3, z_4, z_5, z_6, z_7\}$.

By (6): $K_3(x_1) = K_3(x_0) = Z$.

By (4): $K_1(x_2) = K_2(x_2) = \theta(x_2) = \{z_1, z_2\}$.

By (3): $K_3(x_2) = \theta^*(\{x_2, x_3\}) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$.

By (4): $K_1(x_3) = K_2(x_3) = \theta(x_3) = \{z_3, z_4, z_5, z_6\}$.

By (3): $K_3(x_3) = \theta^*(\{x_2, x_3\}) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$.

By (3): $K_1(x_4) = \theta^*(\{x_4, x_5\}) = \{z_3, z_4, z_5, z_6\}$.

By (4): $K_2(x_4) = K_3(x_4) = \theta(x_4) = \{z_3, z_4\}$.

By (3): $K_1(x_5) = \theta^*(\{x_4, x_5\}) = \{z_3, z_4, z_5, z_6\}$.

By (4): $K_2(x_5) = K_3(x_5) = \theta(x_5) = \{z_5, z_6\}$.

Remark 1. It is easy to show (see Bonanno, 1991) that for every node t and for every player i, $\theta(t)$ is a subset of $K_i(t)$. (Recall that $\theta(t)$ denotes the set of terminal nodes that can be reached from node t.)

Remark 2. It is an immediate consequence of point (3) of the above definition that if h is an information set of player i, and x and y are two nodes in h, then $K_i(x) = K_i(y)$. Thus it makes sense to write $K_i(h)$ for player i's knowledge at her information set h.

An appealing feature of the definition of knowledge given above is that it provides an intuitive characterization of such notions as perfect recall, perfect information, simultaneity, etc. For a proof of the following properties see Bonanno (1991).

PROPERTY 1. An extensive game with *perfect recall* satisfies the following property: if node y is a successor of node x, then, for every player i, $K_i(y) \subseteq K_i(x)$. That is, at every node each player knows at least as much as she knew before that node was reached.

The game shown in Figure 3 satisfies the property that if y is a successor of x then $K_i(y) \subseteq K_i(x)$, for every player i, but is not a game with perfect recall. Thus the converse of property 1 is not true.

PROPERTY 2. An extensive game with perfect recall has *perfect information* if and only if at every node all players have the same knowledge, that is, if and only if for every node t and for any two players i and j, $K_i(t) = K_j(t)$.

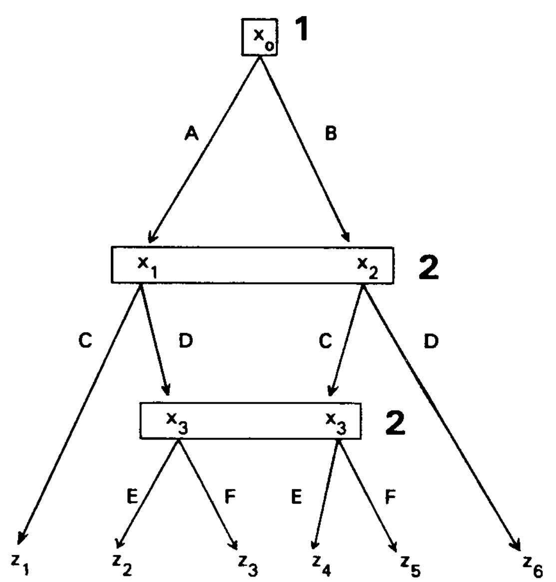

Fig. 3.

Define an extensive game to be simultaneous if every play crosses all the information sets.

PROPERTY 3. An extensive game is *simultaneous* if and only if it satisfies the following property: if x is a decision node of player i, then $K_i(x) = Z$. That is, when a player has to move she knows as much as she did at the root of the tree.

4. BELIEFS

Fix an extensive game. As before, let X be the set of decision nodes, Z the set of terminal nodes, and T the union of X and Z.

DEFINITION. A *belief* of player i is a function

$$\beta_i : T \to Z$$

satisfying the following properties:

(1) $\beta_i(t) \in K_i(t), \quad \forall t \in T,$
(2) If x and y belong to the same information set of player i (so that $K_i(x) = K_i(y)$) then $\beta_i(x) = \beta_i(y)$.

The interpretation is as follows. Consider again the game of Figure 2. At node x_1 we have that $K_1(x_1) = \{z_3, z_4, z_5, z_6, z_7\}$. This means that player 1 knows that only terminal nodes different from z_1 and z_2 can be reached. If $\beta_1(x_1) = z_5$ then player 1 believes that the play of the game will actually end at node z_5 (this obviously implies that player 1 believes that player 2 will take action C and player 3 will take action F and he himself plans to choose G).

Condition (1) in the above definition says that what a player believes must be consistent with what he knows, and condition (2) says that a player cannot have different beliefs at two nodes that belong to one of his information sets, since his knowledge is the same at both nodes. Thus it makes sense to write $\beta_i(h)$ for player i's belief at his information set h.

DEFINITION. A *belief system* is an n-tuple $\beta = (\beta_1, \ldots, \beta_n)$, where, for each player $i = 1, \ldots, n$, β_i is a belief of player i.

DEFINITION. Let β be a belief system. We say that β is *rational* if it satisfies the following properties (which will be discussed immediately below):

(1) *Contraction Consistency*. If x and y are two nodes such that $K_i(x) \supseteq K_i(y)$ and $\beta_i(x) \in K_i(y)$, then $\beta_i(y) = \beta_i(x)$.

(2) *Tree Consistency*. Let h be an information set of player i. Let $x \in h$ be the predecessor of $\beta_i(h)$ and let $\Sigma(x)$ be the set of immediate successors of x. Then

$$\beta_i(y) \in \theta(y) \qquad \forall y \in \Sigma(x)\,.$$

(3) *Individual Rationality*. Let h be an information set of player i. Let $x \in h$ be the predecessor of $\beta_i(h)$ and let $\Sigma(x)$ be the set of immediate successors of x. Then

$$U_i(\beta_i(h)) \geqslant U_i(\beta_i(y)) \qquad \forall y \in \Sigma(x)\,,$$

where $U_i \colon Z \to \Re$ is player i's payoff function. ($\Re$ denotes the set of real numbers.)

(4) *Choice Consistency*. Let x belong to information set h of player i, and let c be the choice at h that precedes $\beta_i(h)$. Then, for every player j, if $\beta_j(x)$ comes after choice d at h, it must be $d = c$.

Property (1) says that, as the knowledge of a player evolves and becomes more refined, the player will not change his belief unless he has to, that is, unless his previous belief is inconsistent with the new information. This is a contraction consistency property which is implied, for example, by Bayesian updating.

The purpose of property (2) is to rule out situations like the one illustrated in Figure 4. There we have that $K_2(h) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ where $h = \{x_1, x_2\}$ is the first information set of player 2, and $K_2(g) = \{z_1, z_2, z_4, z_5\}$ where $g = \{x_3, x_4\}$ is the second information set of player 2.

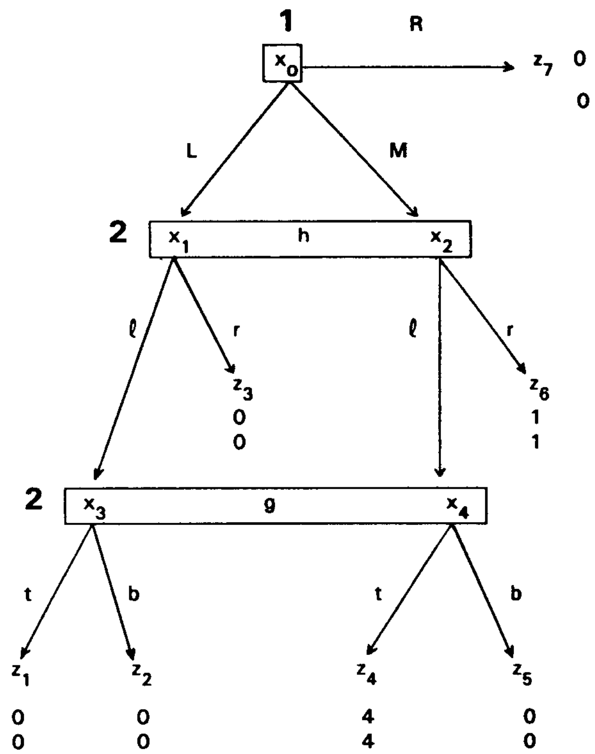

Fig. 4.

Suppose $\beta_2(h) = z_6$ and $\beta_2(g) = z_1$. This belief of player 2 is inconsistent because believing in z_6 at h means believing that node x_2 was reached. Given this belief, if player 2 takes action l, so that the play of the game proceeds to information set g, then node x_4 must be reached, and from x_4 terminal node z_1 cannot be reached. In this example property (2) requires that if $\beta_2(h) = z_6$ then either $\beta_2(g) = z_4$ or $\beta_2(g) = z_5$.

The motivation for property (3) is as follows. If terminal node z represents what player i believes at his information set h (that is, if $z = \beta_i(h)$), then it means that player i believes that he is at that node x in h which lies on the play to z. Suppose that $U_i(z) < U_i(\beta_i(y))$ where y is an immediate successor of x. Then believing in z (at h) is irrational for player i because, instead of making the choice required to reach z, he can – according to his beliefs and by making another choice – move the game to node y where, again according to his beliefs, the game will evolve to an outcome which he prefers to z.[6]

Property (4) says that if player j believes that the play of the game will reach player i's information set h, then player j's belief concerning the *choice* that will be made by i at h must be the same as the choice implied by i's belief at h (although i and j might disagree on the node at which this choice would be made). A justification for this property could be that player j puts himself in the shoes of player i and correctly predicts the choice that player i would make at her information set h.

The properties that define the notion of rational belief system seem to be very natural. Of course, definitions must be judged on the basis of the results that can be obtained from them. The main result of this paper is that the notion of rational belief system gives rise to a refinement of the notion of subgame-perfect equilibrium. In order to prove this result we first need to show how to associate with a (not necessarily rational) belief system $\beta = (\beta_1, \ldots, \beta_n)$ a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$. Let β be a belief system. Let $\sigma = \xi(\beta)$ be the pure-strategy profile obtained as follows. If h is an information set of player i and $z = \beta_i(h)$, let c be the choice at h that precedes z. Set $\sigma_i(h) = c$, that is, c is the choice selected (with probability 1) by player i's strategy at information set h.

The proof of the following proposition is given in Appendix B.

PROPOSITION 1. *Fix an extensive game with perfect recall. Let β be a rational belief system and $\sigma = \xi(\beta)$ the corresponding strategy profile. Then σ is a subgame-perfect equilibrium.*

In Bonanno (1990) it is shown that in games of perfect information there is a one-to-one correspondence between rational belief systems and (pure-strategy) subgame-perfect equilibria. (Note that in a game of perfect information property (2) of the definition of rational belief system (Tree Consistency) is redundant, since, for every player i and for every node t, $K_i(t) = \theta(t)$.) We now show that in games with *imperfect* information the notion of rational belief system refines that of subgame-perfect equilibrium.

In the next example we shall make use of the following lemma (which is proved as Corollary 1 in Appendix B).

LEMMA. *Let β be a rational belief system of a game with perfect recall. Then for every two players i and j, $\beta_i(x_0) = \beta_j(x_0)$. That is, at the root of the tree the beliefs of all the players agree.*

Consider, again, the well-known game of Figure 1 which was first discussed by Selten (1975). This game has two (pure-strategy) Nash equilibria: (a, A, R) and (d, A, L). Both are subgame-perfect since there are no proper subgames. Of these two equilibria only (a, A, R) is sequential (and trembling-hand perfect). We now show that this game has a unique rational belief system β and that $\xi(\beta) = (a, A, R)$.

First of all, by Individual Rationality and Contraction Consistency it cannot be $\beta_3(x_0) = z_2$ or $\beta_3(x_0) = z_3$.[7] By the above lemma, it cannot be $\beta_i(x_0) = z_2$ or $\beta_i(x_0) = z_3$ for any player $i = 1, 2, 3$.

Suppose $\beta_i(x_0) = z_1$ for all $i = 1, 2, 3$. Then, by Contraction Consistency, $\beta_3(x_3) = z_1$ and, by Choice Consistency, $\beta_2(x_3) = \beta_1(x_3) = z_3$. Therefore, by Individual Rationality (for player 2), $\beta_2(x_1) = z_3$. By Choice Consistency and Contraction Consistency, $\beta_1(x_1) = z_3$.[8] Hence by Individual Rationality it cannot be $\beta_1(x_0) = z_1$, a contradiction. Thus $\beta_i(x_0) = z_1$ is ruled out for all $i = 1, 2, 3$.

Similarly, suppose $\beta_i(x_0) = z_4$ for all $i = 1, 2, 3$. Then, by Contraction Consistency $\beta_2(x_1) = z_4$, which violates Individual Rationality for player 2, since $U_2(z_5) > U_2(z_4)$. Thus the only possibility is $\beta_i(x_0) = z_5$ for all $i = 1, 2, 3$. By Contraction Consistency this implies that $\beta_1(x_1) = \beta_2(x_1) = \beta_3(x_1) = z_5$. As noted before, by Individual Rationality it cannot be $\beta_3(x_2) = z_2$ or $\beta_3(x_2) = z_3$. Suppose $\beta_3(x_2) = z_1$. Then by Choice Consistency, since $K_1(x_2) = \{z_1, z_2\}$, it must be $\beta_1(x_2) = z_1$. But this, together with $\beta_1(x_0) = z_5$, contradicts Individual Rationality for player 1. Thus for β to be a rational belief system it must be $\beta_3(x_2) = z_4$, which, by Choice Consistency, implies $\beta_1(x_2) = \beta_2(x_2) = z_2$ and $\beta_1(x_3) = \beta_2(x_3) = z_4$.

Thus we have found a unique rational belief system β given by:

$$\beta_1(x_0) = z_5, \quad \beta_1(x_1) = z_5, \quad \beta_1(x_2) = z_2, \quad \beta_1(x_3) = z_4;$$

$$\beta_2(x_0) = z_5, \quad \beta_2(x_1) = z_5, \quad \beta_2(x_2) = z_2, \quad \beta_2(x_3) = z_4;$$

$$\beta_3(x_0) = z_5, \quad \beta_3(x_1) = z_5, \quad \beta_3(x_2) = z_4, \quad \beta_3(x_3) = z_4.$$

It is easy to check that $\xi(\beta) = (a, A, R)$.

On the basis of the above example, one might wonder if the notion of rational belief system coincides with that of sequential equilibrium. The following example shows that the answer is negative.

Consider the game of Figure 5. [Note that $K_2(x_0) = K_2(x_1) = K_2(x_2) = Z$ and $K_2(\{x_3, x_4, x_5\}) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$, while for every $j = 1, \ldots, 5$, $K_1(x_j) = \theta(x_j)$]. It is easy to check that the following belief system is rational:

$$\beta_1(x_0) = z_8, \quad \beta_1(x_1) = \beta_1(x_4) = z_3,$$

$$\beta_1(x_2) = \beta_1(x_5) = z_5, \quad \beta_1(x_3) = z_1;$$

$$\beta_2(x_0) = \beta_2(x_1) = \beta_2(x_2) = z_8,$$

$$\beta_2(x_3) = \beta_2(x_4) = \beta_2(x_5) = z_5 .$$

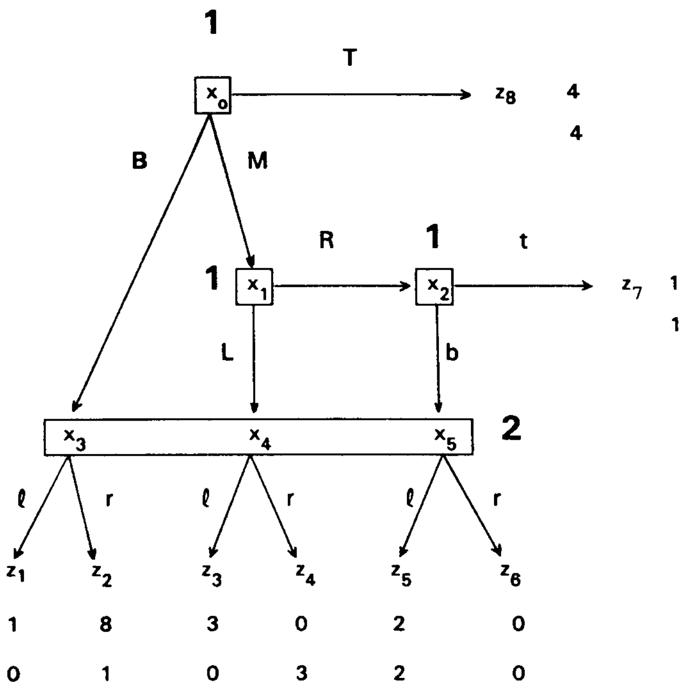

Fig. 5.

It is clear that if $\sigma = \xi(\beta)$ then $\sigma = ((T, L, b), l)$. We will now show that $((T, L, b), l)$ is not a sequential equilibrium. In fact we will prove a stronger claim, namely that *there is no sequential equilibrium that yields outcome* z_8 *with probability* 1.

LEMMA. *In the game of Figure 5 there is no sequential equilibrium where player* 1 *chooses* T *with probability* 1.

Proof. Recall that, for the game of Figure 5, a sequential equilibrium is defined in terms of a pair (σ, ν) where $\sigma = (\sigma_1, \sigma_2)$ is a (behavior) strategy profile and $\nu: \{x_0, x_1, x_2, x_3, x_4, x_5\} \to [0, 1]$ is a function satisfying $\nu(x_0) = \nu(x_1) = \nu(x_2) = 1$ and $\nu(x_3) + \nu(x_4) + \nu(x_5) = 1$. We shall write $\sigma_i(a) = p$ to mean that player i's strategy σ_i assigns probability p to choice a. Now, suppose there is a sequential equilibrium (σ, ν) with $\sigma_1(T) = 1$. Then it must be $\sigma_2(l) > \frac{1}{2}$ (otherwise B would be player 1's unique best choice), which implies that $\sigma_1(b) = 1$ and $\sigma_1(L) = 1$. Thus σ must be of the form $((T, L, b), (p, 1-p))$, where $p > \frac{1}{2}$ is the probability with which player 2 chooses l. Now consider a sequence $\{\sigma^1, \sigma^2, \ldots\}$ of completely mixed strategies whose mth element is given by

$$\sigma^m = ((\underset{T}{1 - a_m - b_m}, \underset{M}{a_m}, \underset{B}{b_m};\ \underset{L}{1 - c_m}, \underset{R}{c_m};\ \underset{b}{1 - d_m}, \underset{t}{d_m});\ (\underset{l}{1 - e_m}, \underset{r}{e_m}))$$

with $a_m, b_m, c_m, d_m \in (0, 1)$ and converging to zero as m tends to infinity, while $e_m \in (0, 1)$ converges to $1 - p$. Let h be the information set of player 2. Now, $\text{Prob}\{h \mid \sigma^m\} = b_m + a_m(1 - c_m d_m)$ and $\text{Prob}\{x_5 \mid \sigma^m\} = a_m c_m(1 - d_m)$. Thus, given σ^m,

$$\text{Prob}\{x_5 \mid h\} = \left(\frac{c_m(1 - d_m)}{\frac{b_m}{a_m} + 1 - c_m d_m} \right)$$

As m goes to infinity, the numerator tends to zero while the denominator either tends to infinity or converges to a number greater than or equal to 1. Hence $\nu(x_5) = 0$, that is, player 2 must attach zero probability to node x_5 if his information set is reached. But $\nu(x_5) = 0$ requires player 2 to play l with probability zero, a contradiction. ■

5. CONCLUSION

For every node t and for every player i we defined a subset $K_i(t)$ of the set of terminal nodes and we interpreted it as the information given to player i when node t is reached. We then defined a belief of a player as a function that associates with every node t an element of $K_i(t)$ and a belief system as an n-tuple of beliefs, one for each player. Natural consistency properties were used to define the notion of rational belief system and it was shown that every rational belief system gives rise to a subgame-perfect equilibrium. Two examples were used to show that the notion of rational belief system refines that of subgame-perfect equilibrium and differs from the notion of sequential equilibrium.

APPENDIX A: DEFINITION OF EXTENSIVE-FORM GAME[9]

A finite game in extensive form without chance moves is a sextuple

$$G = (V, N, P, H, C, U)$$

where the constituents are as follows.

The Game Tree

The game tree V is a pair $(T, \rightarrow)$ where T is a *finite* set of *nodes* and '$\rightarrow$' is a binary relation on T. The interpretation of $x \rightarrow y$ is 'node x immediately precedes node y'.

ASSUMPTION 1. *Asymmetry*. If $x, y \in T$ and $x \rightarrow y$, then not $y \rightarrow x$.

An *edge* is an ordered pair $e = (x, y)$ such that $x \rightarrow y$. We say that e is *incident out of* x and *incident into* y. We shall represent edges as arrows [from x to y if $e = (x, y)$].

For each $x \in T$, let

$$\Sigma(x) = \{y \in T \mid x \rightarrow y\}: \text{ set of immediate successors of } x.$$

$$\Pi(x) = \{y \in T \mid y \rightarrow x\}: \text{ set of immediate predecessors of } x.$$

ASSUMPTION 2. *Root*. There exists a *unique* $x_0 \in T$ such that $\Pi(x_0) = \emptyset$. This element x_0 is called the *root* of the tree.

Thus every node $x \in T \backslash \{x_0\}$ has an immediate predecessor. The following assumption requires uniqueness of predecessors.

ASSUMPTION 3. *Unique predecessor*. For every $x \in T \backslash \{x_0\}$, $\Pi(x)$ is a singleton and its unique element is denoted by π_x.

Let $e_1 = (x_1, y_1)$ and $e_2(x_2, y_2)$ be two edges. We say that e_1 *is adjacent to* e_2 if $y_1 = x_2$. Let $\langle e_1 = (x_1, y_1), \ldots, e_m = (x_m, y_m) \rangle$ be a finite sequence of edges ($m \geqslant 2$). If, for every $k = 1, \ldots, m-1$, e_k is adjacent to e_{k+1}, then we call the sequence a *path from* x_1 *to* y_m.

ASSUMPTION 4. *No cycles*. If $\langle e_1 = (x_1, y_1), \ldots, e_m = (x_m, y_m) \rangle$ is a path from x_1 to y_m, then $x_1 \neq y_m$.

By the finiteness of T and Assumptions 2, 3 and 4, for every $y \in T \backslash \{x_0\}$ there exists a unique path from x_0 to y.

Let $Z = \{x \in T \mid \Sigma(x) = \emptyset\}$. Given the finiteness of T and Assumption 4, the set Z is non-empty. We call Z the set of *terminal nodes* (or end nodes). The set $X = T \backslash Z$ is called the set of *decision nodes*.

A *play* is a path from the root of the tree to a terminal node. As noted above, our assumptions imply that for every $z \in Z$ there is a unique play to it.

The Set of Players

$N = \{1, 2, \ldots, n\}$ is the set of players. Players will be indexed by i.

The Player Partition

The player partition $P = \{P_1, \ldots, P_n\}$ is a partition of the set of decision nodes X. P_i is the *set of decision nodes of player i*. We assume that P_i is non-empty for every $i \in N$.

The Information Partition

The information partition $H = \{H_1, \ldots, H_n\}$ is defined as follows. For every $i \in N$, a subset h of P_i is called *eligible* (as an information set) if:

(i) h is non-empty;
(ii) every play intersects h at most once;[10]
(iii) for every x and y in h, the number of immediate successors of x is equal to the number of immediate successors of y.

H_i is a partition of P_i into eligible subsets. An element of H_i is called an *information set of player i*.

The Choice Partition

The choice partition $C = \{C_1, \ldots, C_n\}$ is defined as follows. For every player i and for every information set $h \in H_i$, let $E(h)$ be the set of edges incident out of nodes in h. We say that a subset c of $E(h)$ is *eligible* (as a choice) if it contains exactly one edge incident out of x, for every $x \in h$. C_i partitions the set of all edges incident out of nodes contained in information sets of player i into eligible subsets. If $h \in H_i$ and c belongs to an element of C_i and also to $E(h)$, then c is called a *choice of player i at information set h*. The set of all choices at $h \in H_i$ is denoted by $C_i(h)$.

The Payoff Function

The *payoff function* U associates with every terminal node $z \in Z$ a vector $U(z) = (U_1(z), \ldots, U_n(z)) \in \Re^n$, called the *payoff vector* at z. The component $U_i(z)$ is called player i's *payoff* at z.

DEFINITION. An extensive game is said to have *perfect recall* if it satisfies the following property: for every player i and any two information sets h and g of player i, if one node $x \in g$ comes after a choice c at h, then every node $y \in g$ comes after this choice.

DEFINITION. An extensive game is said to have *perfect information* if every information set is a singleton. A game which does not have perfect information is called a game of *imperfect information*.

DEFINITION. An extensive game is said to be *simultaneous* if every play of the game crosses all the information sets.

APPENDIX B

In this Appendix we prove Proposition 1. We shall begin with a lemma.

Given a pure-strategy profile σ, for every node x we denote by

$$\zeta(x \mid \sigma)$$

the terminal node reached from x by following σ (if z is a terminal node, we set by definition $\zeta(z \mid \sigma) = z$). Obviously, $\zeta(x \mid \sigma) \in \theta(x)$.

LEMMA 1. *Fix a game with perfect recall. Let β be a belief system that satisfies the properties of Contraction Consistency and Choice Consistency. Then, for every player i and for every node x, if $\beta_i(x) \in \theta(x)$, then $\beta_i(x) = \zeta(x \mid \sigma)$ where $\sigma = \xi(\beta)$.*

Proof. Let β be a belief system that satisfies Contraction Consistency and Choice Consistency and let $\sigma = \xi(\beta)$. Fix an arbitrary node x. If x is a terminal node then $\zeta(x \mid \sigma) = x$ by definition of $\zeta(\cdot)$ and $\beta_i(x) = x$ for every player i, since $K_i(x) = \{x\}$. Suppose, therefore, that x is a decision node. Suppose there is a player i for whom $\beta_i(x) = z' \in \theta(x)$ but $z' \neq z$, where $z = \zeta(x \mid \sigma)$. Let $x_1, x_2, \dots, x_m$ be the sequence of nodes that form the path from x to z (thus $x_1 = x$, $x_m = z$ and, for every $k = 1, \dots, m-1$, x_k is the immediate predecessor of x_{k+1}). Obviously, $\zeta(x_k \mid \sigma) = z$ for all $k = 1, \dots, m$. Let x_k be the node at which the path from x to z and the path from x to z' diverge. (That is, $z, z' \in \theta(x_k)$ and for every immediate successor y of x_k it is not true that both z and z' belong to $\theta(y)$.) Note that x_k could be x itself (in other words, $k = 1$ is a possibility). Clearly, x_k is a decision node. Let h be the information set to which x_k belongs and j be the player to whom h belongs. Two cases are possible: (1) $i = j$, and (2) $i \neq j$.

In case (1), let c be the choice at h that precedes z and c' be the choice at h that precedes z'. By our choice of x_k, $c \neq c'$. We now obtain a contradiction by showing that $\sigma_i(h) = c'$. (If $\sigma_i(h) = c'$ then following σ from x_k one cannot reach z and this contradicts the fact that $z = \zeta(x_k \mid \sigma)$.) By Remark 1 of Section 3, $\theta(x_k) \subseteq K_i(x_k)$. Thus $z' \in K_i(x_k)$, since $z' \in \theta(x_k)$ by the way x_k was chosen. By Property 1 of Section 3, $K_i(x_k) \subseteq K_i(x)$. By Contraction Consistency, it must be $\beta_i(x_k) = z'$. By definition of $\sigma = \xi(\beta)$, it follows that $\sigma_i(h) = c'$.

Now consider case (2), where $j \neq i$. Again, let c be the choice at h that precedes z and c' be the choice at h that precedes z'. By definition of $\sigma = \xi(\beta)$, it must be $\sigma_j(h) = c$. Thus $\beta_j(x_k)$ comes after choice c of player j at h (even though $\beta_j(x_k)$ need not be a successor of x_k) and $\beta_i(x_k)$ comes after choice c' of player j at h, contradicting Choice Consistency.[11] ■

COROLLARY 1. *Fix a game with perfect recall. Let β be a belief system that satisfies the properties of Contraction Consistency and Choice Consistency. Then, for every two players i and j, $\beta_i(x_0) = \beta_j(x_0)$ (where x_0 is the root of the tree).*

Proof. Since, for every player i, $K_i(x_0) = Z = \theta(x_0)$, it follows from Lemma 1 that $\beta_i(x_0) = \zeta(x_0 \mid \sigma)$ for all i. ■

Proof of Proposition 1. (The argument of the proof will be illustrated at the end with the help of Figure 6.) Fix an extensive game G with perfect recall. Let β be a rational belief system and $\sigma = \xi(\beta)$ the corresponding (pure) strategy profile. Suppose σ is not a subgame-perfect equilibrium. Then there exist a decision node x^*, a subgame G' of G with root x^*, a player i and a pure strategy σ_i' of player i such that

$$\text{(B.1)} \qquad U_i(z') > U_i(z^*)$$

where $z^* = \zeta(x^* \mid \sigma)$ is the terminal node reached from x^* by $\sigma = (\sigma_i, \sigma_{-i})$ and $z' = \zeta(x^* \mid \sigma')$ is the terminal node reached from x^* by $\sigma' = (\sigma_i', \sigma_{-i})$ (where σ_{-i} denotes the $(n-1)$-tuple $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$).

Step 1. We first show that $\beta_j(x^*) = z^*$, for all players $j = 1, \ldots, n$. Since x^* is the root of a subgame, for every player j either $H_j(x^*) = \emptyset$

(recall that $H_j(x^*)$ is the set of information sets of player j that have a node which is a successor of x^*) or for every $h \in H_j(x^*)$, $\theta^*(h) \subseteq \theta(x^*)$. Thus $K_j(x^*) = \theta(x^*)$. By Lemma 1, $\beta_j(x^*) = \zeta(x^* \mid \sigma) = z^*$ for every player j.

Step 2. Let y be the node at which the path from x^* to z^* and the path from x^* to z' diverge. (That is, $z^*, z' \in \theta(y)$ and if w is an immediate successor of y then it is not the case that both z^* and z' belong to $\theta(w)$.) Note that $y = x^*$ is a possibility. Then y is a decision node of player i. By Remark 1 (cf. Section 3), $\theta(y) \subseteq K_i(y)$. Thus $z^* \in K_i(y)$. By Property 1 (cf. Section 3), $K_i(y) \subseteq K_i(x^*)$. Thus, by Contraction Consistency, $\beta_i(y) = \beta_i(x^*) = z^*$.

Step 3. Let y_1 be the immediate successor of y on the path from y to z' and let $z_1 = \beta_1(y_1)$. Then, by Tree Consistency, $z_1 \in \theta(y_1)$ (hence $z_1 \neq z^*$). Thus, by Lemma 1, $z_1 = \zeta(y_1 \mid \sigma)$. By Individual Rationality, since $\beta_i(y) = z^*$ and y_1 is an immediate successor of y,

(B.2) $\quad U_i(z_1) \leq U_i(z^*)$

If $z_1 = z'$ the proof is complete, since (B.2) contradicts our supposition that $U_i(z') > U_i(z^*)$. If $z_1 \neq z^*$, proceed to Step 4.

Step 4. Let y_2 be the node at which the path from y_1 to z_1 and the path from y_1 to z' diverge ($y_2 = y_1$ is a possibility). Then y_2 is a decision node of player i. By Remark 1 (cf. Section 3), $\theta(y_2) \subseteq K_i(y_2)$ and, therefore, since $z_1 \in \theta(y_2)$, it follows that $z_1 \in K_i(y_2)$. By Contraction Consistency (since, by Property 1 (cf. Section 3), $K_i(y_2) \subseteq K_i(y_1)$), $\beta_i(y_2) = z_1$.

Step 5. Let y_3 be the immediate successor of y_2 on the path from y_2 to z' and let $z_3 = \beta_i(y_3)$. By Tree Consistency, $z_3 \in \theta(y_3)$. By Lemma 1, $z_3 = \zeta(y_3 \mid \sigma)$. By Individual Rationality,

(B.3) $\quad U_i(z_3) \leq U_i(z_1)$

and, using (B.2) and (B.3),

(B.4) $\quad U_i(z_3) \leq U_i(z^*)$

Step 6. If $z_3 = z'$ the proof is complete, since (B.4) contradicts

(B.1). If $z_3 \neq z'$, by proceeding the same way – since the number of nodes (and hence of information sets) is finite – we must reach a node y_r such that:

(i) the path from x^* to z' and the path from x^* to $z_r \equiv \beta_i(y_r)$ diverge at y_r (and, therefore, y_r is a decision node of player i; furthermore, $z_r = \zeta(y_r \mid \sigma)$);

(ii) if $\hat{y}$ is the immediate successor of y_r on the path from y_r to z',

$$\zeta(\hat{y} \mid \sigma) = \zeta(\hat{y} \mid \sigma')$$

(either because none of the nodes on the path from $\hat{y}$ to z' are decision nodes of player i, or because σ_i and σ_i' coincide at the information sets of player i crossed by this path);

(iii) $U_i(z_r) \leq U_i(z^*)$ (by the same argument that led to (B.4)).

Then, by Tree Consistency, $\beta_i(\hat{y}) \in \theta(\hat{y})$, so that, by Lemma 1, $\beta_i(\hat{y}) = \zeta(\hat{y} \mid \sigma)$. By (ii), above, $\zeta(\hat{y} \mid \sigma) = \zeta(\hat{y} \mid \sigma') = z'$. (Recall that $z' = \zeta(x^* \mid \sigma')$; hence, since $\hat{y}$ is on the path from x^* to z', $\zeta(\hat{y} \mid \sigma') = z'$.) By Individual Rationality, $U_i(z') \leq U_i(z_r)$ and by (iii), above, $U_i(z_r) \leq U_i(z^*)$. Thus

$$U_i(z') \leq U_i(z^*) ,$$

contradicting our supposition that $U_i(z') > U_i(z^*)$. ■

Figure 6 illustrates the argument of the proof. In Figure 6 continuous double edges denote the strategy profile σ, while dashed double edges denote σ_i'. In this case we have: $i = 1$, $y = x^*$, $y_1 = x_1$, $y_2 = y_r = x_3$, $\hat{y} = x_4$.

ACKNOWLEDGEMENTS

I am grateful to Yossi Greenberg for thought-provoking discussions and stimulating comments and to an anonymous referee for very helpful and constructive comments.

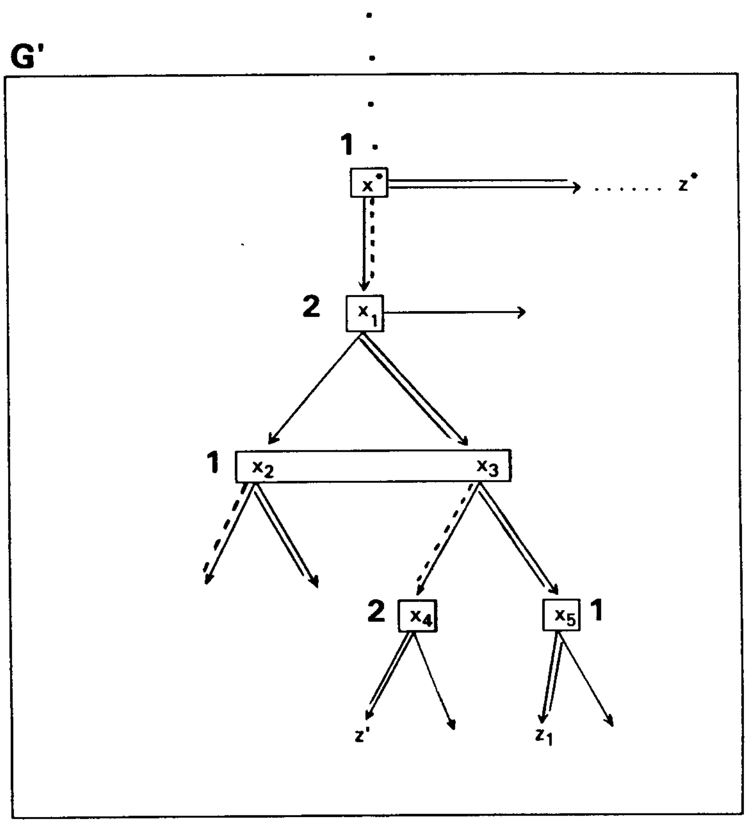

Fig. 6.

NOTES

[1] An alternative definition, which is coarser than the one used in this paper, is given in Bonanno (1991).
[2] The notion of subgame-perfect equilibrium was introduced by Selten (1965, 1973) and is a generalization of the backward induction argument suggested by Zermelo (1912).
[3] The notion of sequential equilibrium was introduced by Kreps and Wilson (1982).
[4] For simplicity we shall restrict ourselves to games without chance moves, although not necessarily with perfect recall. The definition of extensive game without chance moves is given in Appendix A, together with other definitions used in this paper.
[5] Points (1)–(5) define a function $K: N \times T \rightarrow 2^Z$, where N is the set of players and 2^Z denotes the set of subsets of Z. Given a pair (i, t), where i is a player and t is a node, we use the notation $K_i(t)$ instead of $K(i, t)$.

[6] Property (3) is somewhat reminiscent of the notion of 'internal stability' discussed by Greenberg (1990).

[7] Suppose $\beta_3(x_0) = z_2$. Then, since $z_2 \in K_3(x_2)$, by Contraction Consistency we must have $\beta_3(x_2) = z_2$. But z_1 is an immediate successor of x_2 and $\beta_3(z_1) = z_1$, since $K_3(z_1) = \{z_1\}$. Since $U_3(z_1) = 2 > U_3(z_2) = 0$, Individual Rationality would be violated for player 3. The argument for $\beta_3(x_0) = z_3$ is similar.

[8] By Choice Consistency, since $\beta_2(x_1) = z_3$, it must be either $\beta_1(x_1) = z_3$ or $\beta_1(x_1) = z_4$. The latter, however, would require, by Contraction Consistency, that $\beta_1(x_3) = z_4$, contradicting what we have already established, namely that $\beta_1(x_3) = z_3$.

[9] Our definition is along the lines of Selten's (1975).

[10] That is, if $\langle e_1 = (x_1, y_1), \ldots, e_m = (x_m, y_m)\rangle$ is a play (thus $x_1 = x_0$ and $y_m \in Z$) then the following conditions are satisfied:

(a) if $x_k \in h$ for some $k = 1, \ldots, m$, then, for all $j = 1, \ldots, m$, $y_j \not\subseteq h$ and, for all $j \neq k$, $x_j \not\subseteq h$;
(b) if $y_k \in h$ for some $k = 1, \ldots, m-1$, then, for all $j = 1, \ldots, m$, $x_j \not\subseteq h$, and, for all $j \neq k$, $y_j \not\subseteq h$.

[11] It is easy to show that if either Contraction Consistency or Choice Consistency is not satisfied then Lemma 1 is not true.

REFERENCES

Bonanno, G.: 1990, 'Knowledge and beliefs in extensive games', Department of Economics Working Paper No. 361, University of California, Davis.

Bonanno, G.: 1991, 'Players' information in extensive games', to appear in: *Mathematical Social Sciences*.

Greenberg, J.: 1990, *The Theory of Social Situations*, Cambridge University Press, Cambridge.

Kreps, D. and Wilson, R.: 1982, 'Sequential equilibria', *Econometrica*, **50**, 863–894.

Selten, R.: 1965, 'Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrägetragheit', *Zeitschrift für die gesamte Staatswissenschaft*, **12**, 301–324 and 667–689.

Selten, R.: 1973, 'A simple model of imperfect competition where 4 are few and 6 are many', *International Journal of Game Theory*, **2**, 141–201.

Selten, R.: 1975, 'Re-examination of the perfectness concept for equilibrium points in extensive games', *International Journal of Game Theory*, **4**, 25–55.

Zermelo, E.: 1912, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels', *Proceedings, Fifth International Congress of Mathematicians*, Vol. 2, pp. 501–504.

Department of Economics,
University of California,
Davis, CA 95616, U.S.A.