Analytical initial-*g*uess-free solution to Kepler's transcendental equation using Boubaker Polynomials Expansion Scheme *BPES*

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An analytical initial-guess-free solution to the Kepler problem is proposed. The resolution protocol allows, oppositely to initial-guess methods, the determination of the real root of Kepler's equation without any first guess.

The presented resolution protocol provides an analytical supply to works aiming to understand some of the new techniques of celestial mechanics.

Keywords: Kepler's equation; Mean anomaly; BPES; Analytical solution.

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Introduction

The present study deals with a famous historical problem: Kepler's equation [1-10]. This well-known equation has been encountered in the context of a classical bound two-body problem, where one of two targeted bodies follows an elliptical trajectory or orbit (Fig. 1). Despite having many possible parameterizations, the major part of studies used the set of geometrical angles called anomalies.

The first formulations of the Kepler's equation, as described [1-3] in the beginning of the XVIIth, traduces the mathematical links between these anomalies. Given the fundamental importance and frequent application of Kepler's equation, numerous mathematicians and scientists have developed, during three centuries, different methods for solving it numerically. Those methods involved typically initial-values protocols [2-10].

In this paper, an analytical investigation of the Kepler's equation roots, for given mean anomaly values, is presented. The main advantages of the proposed method are the non-necessity of any first guess, and the mathematical properties of the solutions expression which are continuous and infinitely differentiable.

Kepler's equation

In the Keplerian two-body problem, a body *A* follows an elliptical orbit whose a focus *F* is identified to the second body *B*. The instantaneous location of the body A is parameterized by two main parameters (Fig.1):

- ✓ The eccentric anomaly E : the angle between the pericentre direction and the auxiliary point *P* (deduced from the true position of the body *A*), measured from the centre of the elliptic trajectory.

- ✓ The true anomaly θ: the angle between the pericentre direction and the instantaneous location of the body A, as seen from the main focus F of the elliptic trajectory.

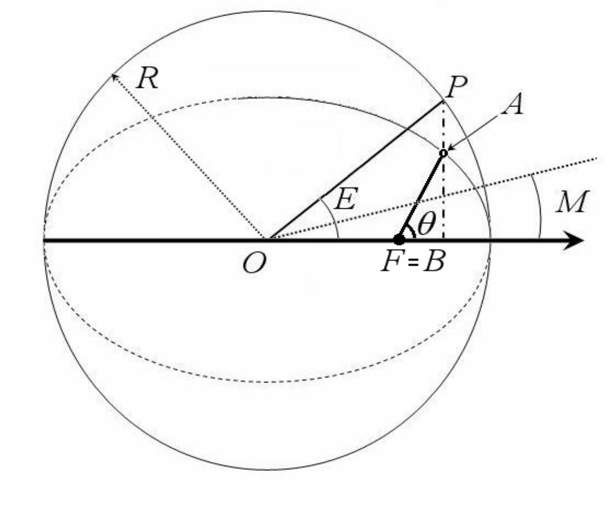

Figure 1: Kepler problem parameterization scheme.

It has been demonstrated [2-6] that the eccentric anomaly, as well as the true anomaly, does not increase uniformly with time.

For this purpose, and according to Kepler's second law (area law), the mean anomaly M was defined as the O-summit angle (Fig.1) which is proportional to the area swept by the vector $\overrightarrow{FA}$ line since the last cross with the pericentre direction. This mean anomaly, which increases uniformly from 0 to 2π radians during each orbit, does not have any simple interpretation as a geometric angle, it could be considered as time measured in radians.

Kepler's equation relates the eccentric anomaly E as an intrinsic parameter to the already defined mean anomaly:

$$E - e \times \sin E = M \quad (1)$$

where e is the elliptic orbital eccentricity ($0 \leq e < 1$).

As this equation is transcendental, its solutions had to be found either empirically or iteratively.

Resolution historical glossary

In the XVIIth, attempts to solve (Eq. 1) were not numerous. The earliest published solutions were proposed, more than one century after Kepler, by J. Wallis [3] and Newton [4], using the prolate cycloid geometrical properties.

The end of the XIXth brought several solutions of Kepler's equation by graphical-iterative methods as proposed by P. Horrebow [5], H. Simpson *et al.* [6] and L. Euler [7].

In the last two centuries, the searches of a real root to Eq. (1), for a given value of *M*, have been the subject of J. Machin [8], N. Fergola [9], P. Clowell [10] and J. De Pacassi [11] developed studies. It is recognized that, except the results published by Siewart *et al.* [12], and L. Gergely *et al.* [13-15], the proposed solutions [2-11,16-30] were generally numerical and approximated.

BPES-related solution

In this study, an analytical solution is proposed to the standardized system (Eq. 2):

$$\begin{cases} E - e \times \sin E = M_0; & 0 \leq e < 1 \\ 0 < E < \dfrac{\pi}{2} \quad \text{and} & 0 \leq M_0 < \dfrac{\pi}{2} \end{cases} \quad (2)$$

where M_0 is a given value of mean anomaly.

The resolution algorithm is essentially based on the Boubaker polynomials expansion scheme (BPES)[31-49].

This scheme is an analytical tool that was used in several applied physics, numerical analysis and mathematics investigations. During the last decade, O. B. Awojoyogbe *et al.* [36] developed many solutions to human blood flow system using the BPES. S. Slama *et al.* [39-42] proposed also several BPES-related models of dynamic processes. Many other investigations used the BPES in order to solve semiconductors binary compounds characterization [43,44], multimode heat transfer [45,46] or biophysical [47] problems.

The resolution algorithm is applied to system (Eq. 2), by assuming the relations:

$$\begin{cases} \sin(x) = 1 + \dfrac{1}{2N_0} \displaystyle\sum_{q=1}^{N_0} \mu_q . B_{4q}(\hat{x}) \\ x = 1 + \dfrac{1}{2N_0} \displaystyle\sum_{q=1}^{N_0} \mu'_q \; B_{4q}(\hat{x}) \\ \hat{x} = 2 \dfrac{\beta_q}{\pi} x \end{cases} \tag{3}$$

where B_{4q} is the *4q*-order Boubaker polynomial, β_q is the minimal positive root [34,49] of B_{4q}, $\mu_q\big|_{q=1..N_0}, \mu'_q\big|_{q=1..N_0}$ are unknown coefficients, and N_0 is a given integer.

This expansion presents primly the advantage of verifying the main boundary conditions:

$$\begin{cases} \sin(x)\big|_{x=0} = x\big|_{x=0} = 1 + \dfrac{1}{2N_0}\displaystyle\sum_{q=1}^{N_0} \mu_q . B_{4q}(0) = 0 \\ \sin(x)\big|_{x=\frac{\pi}{2}} = 1 + \dfrac{1}{2N_0}\displaystyle\sum_{q=1}^{N_0} \mu_q . B_{4q}(\beta_q) = 1 \end{cases} \tag{4}$$

since β_q is a root of B_{4q}.

By introducing the expression (3) in the system (2), one obtains:

$$1 + \frac{1}{2N_0}\sum_{q=1}^{N_0} \mu_q' B_{4q}(\hat{x}) - e \times \left(1 + \frac{1}{2N_0}\sum_{q=1}^{N_0} \mu_q . B_{4q}(\hat{x}) \right) = M_0 \tag{5}$$

then, by differentiating:

$$\frac{1}{2N_0}\sum_{q=1}^{N_0} \mu_q' \frac{dB_{4q}(\hat{x})}{d\hat{x}} - e \times \left(\frac{1}{2N_0}\sum_{q=1}^{N_0} \mu_q . \frac{dB_{4q}(\hat{x})}{d\hat{x}} \right) = 0 \tag{6}$$

At this step, the integral Θ_{N_0} is defined on [0;π/2] :

$$\Theta_{N_0} = \int_0^{\frac{\pi}{2}} \left(\sum_{q=1}^{N_0} \left(\mu_q' - e \times \mu_q \right) \frac{dB_{4q}(\hat{x})}{d\hat{x}} \right) d\hat{x} = \sum_{q=1}^{N_0} \left(\mu_q' - e \times \mu_q \right) \left(B_{4q}(\beta_q) - B_{4q}(0) \right) \tag{7}$$

Since $B_{4q}(\alpha_q) = 0$ and $B_{4q}(0) = -2$, one has:

$$\Theta_{N_0} = -2\sum_{q=1}^{N_0} \left(\mu_q' - e \times \mu_q \right) \tag{8}$$

Consequently, a physically acceptable solution to Eq. (6) is the set of coefficients $(\mu_q)_{M_0}\big|_{q=1..N_0}$, $(\mu'_q)_{M_0}\big|_{q=1..N_0}$ that minimize the integral Θ_{N_0} under the given boundary conditions. Finally, the problem is reduced to the $2N_0$ variables system:

$$\begin{cases} \min\left[\Theta_{N_0} = -2\sum_{q=1}^{N_0}\left((\mu'_q)_{M_0} - e\times(\mu_q)_{M_0}\right)\right] \\ 1+\dfrac{1}{2N_0}\sum_{q=1}^{N_0}(\mu'_q)_{M_0}\, B_{4q}(\hat{x}) - e\times\left(1+\dfrac{1}{2N_0}\sum_{q=1}^{N_0}(\mu_q)_{M_0}\, B_{4q}(\hat{x})\right) = M_0 \\ \sum_{q=1}^{N_0}(\mu_q)_{M_0} = N_0; \sum_{q=1}^{N_0}(\mu'_q)_{M_0} = N_0 \end{cases} \quad (9)$$

Since the solution of the system (9) depends on M_0, it is noted E_{M_0}, and consequently :

$$E_{M_0} = \int_0^{\frac{\pi}{2}} \arcsin\left(\frac{1}{2N_0}\left(\sum_{q=1}^{N_0}(\mu_q)_{M_0}\times B_{4q}(\hat{x})\right)-1\right)d\hat{x} \qquad (10)$$

Application to Mercury orbital

A BPES-related solution to the equation (1) is proposed in the case of the system Sun-Mercury orbital [12]. This orbital is known to have the highest eccentricity (e≈0.2056) of all the solar system planets for the smallest planet size, as mentioned by R. J. Buenker [49]. The solution is presented in Figure 2 along with the solutions E_{H-G} proposed around 1640 by J. Horrocks [50] and revised (Eq. 11) three centuries later by S. Gaythorpe [51].

$$E_{H-G} = \arctan\left(\frac{\sin M}{\cos M - e}\right) \tag{11}$$

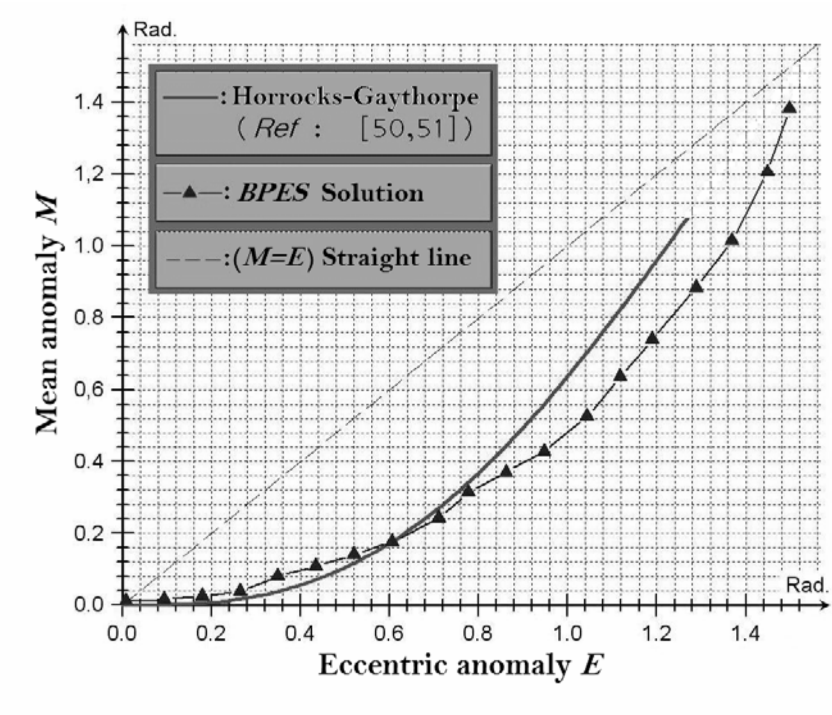

Figure 2: PBES solution to Kepler's problem
(along with precedent solutions: ref [50,51])

It can be noticed that the BPES-related solution is not very far from the precedent referred expression [50,51]. The behaviours of the two curves are approximately the same at the critical points $E = 0$ and $E = \pi/2$. Nevertheless, the quadratic error f_e, as established by R Ivanov *et al.* [52], and calculated using (Eq. 12) was of about 9.9% for the values of $E \in [0.1, 1.3]$.

$$f_e = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{E_{BPES}^{i} - E_{G-H}^{i}}{E_{BPES}^{i}}\right)^2} \tag{12}$$

where is the number of recorded points, $E_{G-H}^{i}\Big|_{i=1\ldots n}$ are the values yielded by S. Gaythorpe [51], and $E_{BPES}^{i}\Big|_{i=1\ldots n}$ are the values yielded by the actual study.

In reference to the recent works of J.M. Danby *et al.* [17,18], an iterative solution yielded, for the same problem, the accurate value ($E = 1.402738$) for a mean anomaly ($M = 1.2$). In Fig. 2, it is obvious that the PBES solution gives a closer value ($E \approx 1.4$) than the precedent model ($E \approx 1.3$).

Conclusion

This work presents an attempt to find an analytical solution to the well-known Kepler's problem [1-10]. Since its establishment in 1609 by J. Kepler [1-2], the problem of finding the real-valued root of Kepler's equation was continuously analysed and discussed during three centuries, nevertheless, the few proposed analytical solutions were relatively complicated and difficult to apply elsewhere. As shown in Fig. 2, the proposed solution is in good agreement with some precedent ones [50,51].

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