DENIS BOUYSSOU AND JEAN-CLAUDE VANSNICK

NONCOMPENSATORY AND GENERALIZED NONCOMPENSATORY PREFERENCE STRUCTURES

1. INTRODUCTION

The aim of this paper is to provide a general study of noncompensatory preference structures. These structures have not been studied very much yet, the attention of most decision theorists being almost exclusively devoted to structures allowing some kind of utility representation. They nevertheless appear frequently in practice both as heuristic approaches to analyse multidimensional evaluations (e.g. disjunctive and lexicographic models, see MacCrimmon, 1973) and as easy to implement methods to perform an aggregation of several attributes for decision-aid (ELECTRE methods, see Roy, 1968, 1971; Roy and Bertier, 1972).

The paper is organized as follows. We present our notations in Section 2. In Section 3 we recall some definitions and propositions about noncompensatory preference structures and introduce the notion of concordance preference structure. In Section 4 we propose a generalization of these notions introducing the idea of discordance. Section 5 provides a brief description of how such preference structures can be used for decision-aid.

2. NOTATIONS AND PRELIMINARY DEFINITIONS

Throughout the paper we will note:
$\mathbb{R}_o^+$ the set of strictly positive real numbers,
$\mathbb{N}=\{0, 1, 2, \dots\}$, $\mathbb{N}_o=\mathbb{N} \setminus \{0\}$,
$\Omega=\{1, 2, \dots, n\}$ with $n\in\mathbb{N}$ and $n\geqslant 2$,
$P(\Omega)$ the set of all subsets of Ω,
S the set of all pairs of disjoint subsets of Ω,
$S=\{(A, B) \mid A, B\in P(\Omega) \text{ and } A\cap B=\emptyset\}$,
$X_1, X_2, \dots, X_n$, n nonempty sets which can be interpreted as n sets of levels defining n attributes in a multi-attribute decision problem,

Theory and Decision **21** (1986) 251–266.

$X = X_{i=1}^{n} X_i$ the Cartesian product of these sets,
$(x_i, (y_j)_{j \neq i})$ the element of X $(y_1, y_2, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$,
$\succ$ an asymmetric binary relation on X which can be interpreted as a (strict) preference relation.
$(X, \succ)$ will be called a Preference Structure (P.S.).

We will classically note $\sim$ the binary relation on X defined by $x \sim y$ iff not $(x \succ y)$ and not $(y \succ x)$ and $\succcurlyeq$ the binary relation on X such that $x \succcurlyeq y$ iff not $(y \succ x)$.

DEFINITION 2.1. For all $i \in \Omega$ we define a binary relation $\succ_i$ on X_i by $x_i \succ_i y_i$ iff $(x_i, (a_j)_{j \neq i}) \succ (y_i, (a_j)_{j \neq i})$ for all $(a_j)_{j \neq i} \in X_{j \neq i} X_j$.
From $\succ_i$ we define $\sim_i$ and $\succcurlyeq_i$ as above.

The asymmetry of $\succ$ obviously implies the asymmetry of each $\succ_i$. The definition of $\succ_i$ does not imply any notion of preferential independence since we have $x_i \succ_i y_i$ only if $(x_i, (a_j)_{j \neq i}) \succ (y_i, (a_j)_{j \neq i})$ for *all* vectors $(a_j)_{j \neq i}$.

DEFINITION 2.2. An attribute X_i is essential iff $x_i \succ_i y_i$ for some x_i, $y_i \in X_i$.

We will assume hereafter that all attributes are essential which will prove unrestrictive for our purposes.

DEFINITION 2.3. For each ordered pair $(x, y) \in X^2$ we will note $P(x, y) = \{i \in \Omega \mid x_i \succ_i y_i\}$.

Thus $P(x, y)$ denotes the set of attributes for which there is a partial preference for x on y. The asymmetry of each $\succ_i$ implies that $P(x, y) \cap P(y, x) = \emptyset$ for all $x, y \in x$.

DEFINITION 2.4. We will note $\rhd$ and $\approx$ the binary relations on $P(\Omega)$ defined respectively by $A \rhd B$ iff $(P(x, y), P(y, x)) = (A, B)$ for some x, $y \in X$ such that $x \succ y$ and $A \approx B$ iff $A \cap B = \emptyset$ and not $(A \rhd B)$ and not $(B \rhd A)$.

It is clear that $A \rhd B$ implies $(A, B) \in S$ and that the following lemma holds for all A, $B \in P(\Omega)$.

LEMMA. 2.1. $A \approx B \Leftrightarrow A \cap B = \emptyset$ *and* $[\forall x, y \in X\colon (P(x, y), P(y, x)) = (A, B) \Rightarrow x \sim y]$.

DEFINITION 2.5. We will note $\gg$ the binary relation on $P(\Omega)$ defined by: $A \gg B$ iff $A \cap B = \emptyset$ and $[\forall x, y \in X: (P(x, y), P(y, x)) = (A, B) \Rightarrow x \succ y]$.

Contrary to $\rhd$, $\gg$ is asymmetric when all attributes are essential. According to Definition 2.5. $\gg$ can be interpreted as a "more important than" relation on $P(\Omega)$.

DEFINITION 2.6. A P.S. $(X, \succ)$ has the properties:

- P_1 super additivity iff $[(A \cup C) \cap (B \cup D) = \emptyset,\ A \rhd B$ and $C \rhd D]$ $\Rightarrow A \cup C \rhd B \cup D$,
- P_2 decisivity iff $[(A, B) \in S$ and $(A, B) \neq (\emptyset, \emptyset)] \Rightarrow$ not $(A \approx B)$,
- P_3 attribute acyclicity iff $\rhd$ has no cycles,
- P_4 attribute transitivity iff $[A \rhd B,\ B \rhd C$ and $A \cap C = \emptyset] \Rightarrow A \rhd C$,
- P_5 double essentiality iff $\forall\, i \in \Omega$, $x_i \succ_i y_i$ and $y_i \succ_i z_i$ for some x_i, y_i, $z_i \in X_i$.

3. NONCOMPENSATORY PREFERENCE STRUCTURES (NPS)

In this section we first recall some definitions and propositions about the notion of noncompensatory preference structures introduced independently by Fishburn (1974, 1975 and 1976) and Plott *et al.* (1975). Two special important cases of NPS are analysed: the classical lexicographic preferences and the new concept of concordance preference structures.

DEFINITION 3.1. A P.S. $(X, \succ)$ is noncompensatory iff $\forall\, x, y, z, w \in X$:
(1) $[(P(x, y), P(y, x)) = (P(z, w), P(w, z))] \Rightarrow (x \succ y \Rightarrow z \succ w)$.
(2) $[P(x, y) \neq \emptyset$ and $P(y, x) = \emptyset] \Rightarrow x \succ y$.

The idea of noncompensation appears clearly in this definition since the global preference of x on y only depends on the subsets of Ω on which there is a partial preference of x on y and of y on x. This definition corresponds to a "regular noncompensatory preference structure" in Fishburn (1976). Condition (2) of Definition 3.1. could be omitted, but from a practical point of view only regular structures are of interest.

It results immediately from Definition 3.1. that if $(X, \succ)$ is a NPS then the attributes are mutually preferentially independent (cf. Keeney and Raiffa, 1976) (which can be shown to be implied by condition (1) alone). We have:

LEMMA 3.1. *A P.S.* $(X, \succ)$ *is a NPS iff:*
(1) $\forall A, B \in P(\Omega), A \rhd B \Rightarrow A \gg B$.
(2) $\forall A \in P(\Omega) \setminus \emptyset, A \gg \emptyset$.
Proof. Left to the reader. ■

Condition (1) of Lemma 3.1. obviously implies that for a NPS $\rhd$ is asymmetric and that $\emptyset \approx \emptyset$.

Lexicographic preference structures are an important particular case of NPS.

DEFINITION 3.2. A P.S. $(X, \succ)$ is lexicographic iff there is a permutation σ on Ω such that $\forall x, y \in X, x \succ y$ iff not $(x_i \sim_i y_i)$ for some $i \in \Omega$ and $x_{\sigma(i)} \succ_{\sigma(i)} y_{\sigma(i)}$ for the smallest i for which not $(X_{\sigma(i)} \sim_{\sigma(i)} y_{\sigma(i)})$.

Fishburn (1976) Theorem 1, proves the following results:

THEOREM 3.1. *If* $(X, \succ)$ *is a NPS, then:*
(a) P_2 and $P_3 \Rightarrow P_4$,
(b) P_1 and P_2 and $P_3 \Leftrightarrow (X, \succ)$ is lexicographic,
(c) P_5 and $\succ$ is a weak order $\Rightarrow$ $(X, \succ)$ is lexicographic.

The notion of NPS also provides a new insight into the idea of concordance which appears in a wide variety of multicriteria decision-aid methods such as ELECTRE. We formalize here this notion using an idea introduced by Vansnick (1986). For another approach to the idea of concordance we refer to Huber (1974 and 1979).

DEFINITION 3.3. A P.S. $(X, \succ)$ is a concordance P.S. of type ρ (CPSρ) with $\rho \geq 1$ a rational number, iff

$$\exists f_1, f_2, \ldots, f_n \in \mathbb{R}_0^+ \text{ such that } \forall x, y \in X$$
$$x \succ y \Leftrightarrow \Sigma_{i \in P(x,y)} f_i > \rho \ \Sigma_{j \in P(y,x)} f_j.$$

Figure 1 gives a graphical interpretation of this definition.

We have the following lemmas:

LEMMA 3.2. $\forall \rho \geq 1$, *a CPSρ is a NPS verifying P_4.*
Proof. Obvious (left to the reader). ■

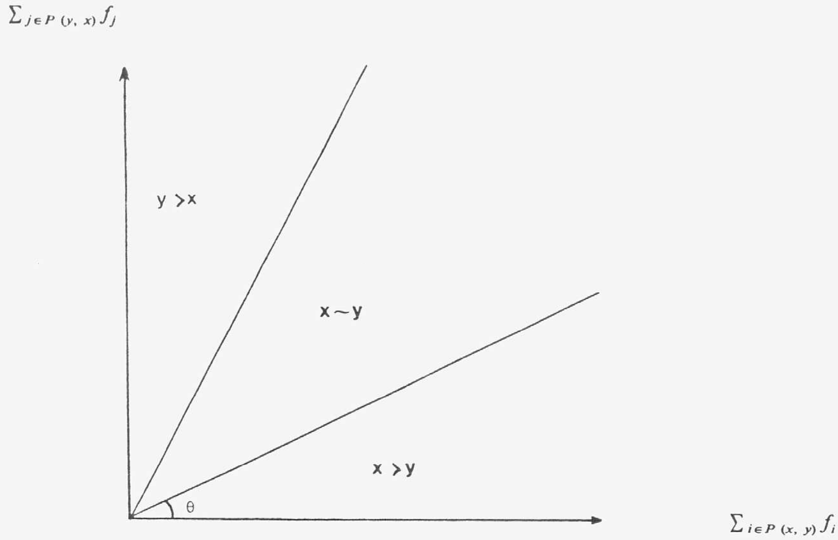

Fig. 1. Graphical interpretation of a CPSρ (here $\rho = 1/\mathrm{tg}\ \theta$).

LEMMA 3.3. *A lexicographic NPS is a CPS*1.
Proof. Follows immediately from taking $f_{\sigma(i)} = 2^{(n+1-i)}$. ■

The following theorem gives necessary and sufficient conditions for the existence of a CPSρ. These conditions are very similar to those appearing in representation theorems of comparative probabilities (see Scott, 1964; Domotor and Stelzer, 1971, Fishburn, 1969; Krantz *et al.*, 1971, chap. 9; Roberts, 1979, chap. 8). This is not surprising because the relation "more important than" between attributes has strong connections with a "more probable than" relation between events. In this theorem, $\forall\ x, y \in X$, $M(x, y)$ will denote the $1 \times n$ matrix $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\forall\ i \in \Omega$, $\alpha_i = 1$ iff $x_i \succ_i y_i$ and $\alpha_i = 0$ otherwise.

THEOREM 3.2. *A P.S.* $(X, \succ)$ *is a CPS*ρ *iff:*
$(X, \succ)$ *is a NPS and*
(3.1) $\quad \forall\ m, k \in \mathbb{N}_0$

$$\sum_{i=1}^{m} M(x^{(i)}, y^{(i)}) + \rho \sum_{0<j<k} M(z^{(j)}, w^{(j)}) \neq$$

$$(3.2) \quad \rho \sum_{i=1}^{m} M(y^{(i)}, x^{(i)}) + \sum_{0<j<k} M(w^{(j)}, z^{(j)})$$

whenever

$x^{(i)}, y^{(i)} \in X$ *and* $x^{(i)} \succ y^{(i)}$ $\forall\ i \in \{1, 2, \ldots, m\}$
$z^{(j)}, w^{(j)} \in X$ *and* $w^{(j)} \sim z^{(j)}$ $\forall\ j \in \mathbb{N}$ *such that* $0<j<k$.

Let us notice that condition (3.2) of Theorem 3.2. contains in fact an infinity of conditions which are rather delicate to interpret as this is the case for the theory of comparative probability. We refer to Vansnick (1986) for a more general presentation of a similar theorem including a threshold.

Proof. Theorem 3.2. can be stated:

$$\exists\ F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \in [\mathbb{R}_0^+]^{n+1} \text{ such that, } \forall\ x, y \in X$$

$x \succ y \Rightarrow M(x, y) \cdot F > \rho\ M(y, x) \cdot F$
$x \sim y \Rightarrow M(x, y) \cdot F \leq \rho\ M(y, x) \cdot F$
iff (3.1) and (3.2).

(a) Necessity

The necessity of (3.1) is obvious. Let $m, k \in \mathbb{N}_0$, $x^{(i)} \succ y^{(i)}$ $\forall\ i \in \{1, \ldots, m\}$ and $w^{(j)} \sim z^{(j)}$ $\forall\ j \in \mathbb{N}$ such that $0<j<k$.

By assumption, there is a $F \in [\mathbb{R}_0^+]^{n+1}$ such that
$M(x^{(i)}, y^{(i)}) \cdot F > \rho\ M(y^{(i)}, x^{(i)}) \cdot F$ $\forall\ i \in \{1, \ldots, m\}$
$\rho\ M(z^{(j)}, w^{(j)}) \cdot F \geqslant M(w^{(j)}, z^{(j)} \cdot F$ $\forall\ j \in \mathbb{N}$ such that $0<j<k$.

After summation, we obtain:

$$[\sum_{i=1}^{m} M(x^{(i)}, y^{(i)}) + \rho \sum_{0<j<k} M(z^{(j)}, w^{(j)})] \cdot F >$$

$$[\rho \sum_{i=1}^{m} M(y^{(i)}, x^{(i)}) + \sum_{0<j<k} M(w^{(j)}, z^{(j)})] \cdot F$$

which implies (3.2).

(b) Sufficiency

Let us observe that it is sufficient to establish that (3.1) and (3.2) imply:

$$\exists\, F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \in \mathbb{R}^{n+1} \text{ such that, } \forall\, x, y \in X:$$

$$(3.3) \qquad \begin{cases} x \succ y \Rightarrow M(x, y)\cdot F > \rho\, M(y, x)\cdot F \text{ and} \\ x \sim y \Rightarrow M(x, y)\cdot F \leq \rho\, M(y, x)\cdot F \end{cases}$$

since (3.1) and (3.3) imply that $f_i > 0\ \forall\, i \in \Omega$. We will show that (3.1) and not (3.3) $\Rightarrow$ not (3.2). Let Y^2 be a set containing one element from each equivalence class of the relation E defined on X^2 by:

$$(x, y)\ E\ (z, w) \text{ iff } P(x, y) = P(z, w) \text{ and } P(y, x) = P(w, z).$$

n being finite, Y^2 contains a finite number of elements.

Let $\{(s^{(i)}, t^{(i)}) \mid i = 1, 2, \ldots, I\}$ be the set of elements in Y^2 such that $s^{(i)} \succ t^{(i)}$.

Let $\{(u^{(j)}, v^{(j)}) \mid j = 1, 2, \ldots, J\}$ be the set of elements in Y^2 such that $u^{(j)} \sim v^{(j)}$.

Given (3.1), not (3.3) is equivalent to: $\nexists\, F \in \mathbb{R}^{n+1}$ such that:

$$M(s^{(i)}, t^{(i)})\cdot F > \rho M(t^{(i)}, s^{(i)})\cdot F\ \forall\, i \in \{1, 2, \ldots, I\}$$

and

$$M(u^{(j)}, v^{(j)})\cdot F \leqslant \rho M(v^{(j)}, u^{(j)})\cdot F\ \forall\, j \in \{1, 2, \ldots, J\}.$$

Each attribute being essential, we have $I \geqslant n > 0$. Therefore, according to Motzkin's transposition theorem (see Vansnick, 1984), there are $\lambda_1, \lambda_2, \ldots, \lambda_I, \mu_1, \mu_2, \ldots, \mu_J \in \mathbb{R}^+$ with $\lambda_i \neq 0$ for at least one $i \in \{1, 2, \ldots, I\}$ such that

$$\sum_{i=1}^{I} \lambda_i\, M(s^{(i)}, t^{(i)}) + \rho \sum_{j=1}^{J} \mu_j\, M(v^{(j)}, u^{(j)}) =$$

$$\rho \sum_{i=1}^{I} \lambda_i\, M(t^{(i)}, s^{(i)}) + \sum_{j=1}^{J} \mu_j\, M(u^{(j)}, v^{(j)}).$$

As the elements of $M(s^{(i)}, t^{(i)})$, $M(t^{(i)}, s^{(i)})$, $M(v^{(j)}, u^{(j)})$ and $M(u^{(j)}, v^{(j)})$ are either 0 or 1 and ρ is a rational number, there exist $\lambda_1^*, \lambda_2^*, \ldots, \lambda_I^*, \mu_1^*, \mu_2^*, \ldots, \mu_J^* \in \mathbb{N}$ such that

$$\sum_{i=1}^{I} \lambda_i^* M(s^{(i)}, t^{(i)}) + \rho \sum_{j=1}^{J} \mu_j^* M(v^{(j)}, u^{(j)}) =$$

$$\rho \sum_{i=1}^{I} \lambda_i^* M(t^{(i)}, s^{(i)}) + \sum_{j=1}^{J} \mu_j^* M(u^{(j)}, v^{(j)})$$

with

$$\sum_{i=1}^{I} \lambda_i^* > 0. \tag{3.4}$$

(3.4) implies that (3.2) does not hold. Indeed, if we let:

(1) $m = \sum_{i=1}^{I} \lambda_i^*,\ k = \sum_{j=1}^{J} \mu_j^* + 1$

(2) for $i = 1, 2, \ldots, I$ and $h = 1, 2, \ldots, m$
$x^{(h)} = s^{(i)}$ and $y^{(h)} = t^{(i)}$ if

$$\sum_{l=0}^{i-1} \lambda_l^* < h \text{ and } h \leqslant \sum_{l=0}^{i} \lambda_l^*$$

(3) for $j = 1, 2, \ldots, J$ and $h' \in \mathbb{N}$ such that $0 < h' < k$
$w^{(h')} = u^{(j)}$ and $z^{(h')} = v^{(j)}$ if

$$\sum_{l=0}^{j-1} \mu_l^* < h' \text{ and } h' \leqslant \sum_{l=0}^{j} \mu_l^*$$

were by convention $\mu_0^* = \lambda_0^* = 0$

we have:

$k, m \in \mathbb{N}_o$
$x^{(h)} \succ y^{(h)}\ \forall\ h \in \{1, 2, \ldots, m\}$
$w^{(h')} \sim z^{(h')}\ \forall\ h' \in \mathbb{N}$ such that $0 < h' < k$

and 3.4. can be written

$$\sum_{h=1}^{m} M(x^{(h)}, y^{(h)}) + \rho \sum_{0<h'<k} M(z^{(h')}, w^{(h')}) =$$

$$\rho \sum_{h=1}^{m} M(y^{(h)}, x^{(h)}) + \sum_{0<h'<k} M(w^{(h')}, z^{(h')})$$

which completes the proof. ∎

4. GENERALIZED NONCOMPENSATORY PREFERENCE STRUCTURES (GNPS)

It is often interesting, from a practical point of view, to weaken the absolute noncompensation of NPS in order to obtain more realistic comparisons (see Roy, 1974; Huber, 1979; and Vansnick, 1986). This is the purpose of the following definition:

DEFINITION 4.1. A P.S. $(X, \succ)$ is a GNPS iff $\forall\, x, y, z, w \in X$:
(1) $[(P(x, y), P(y, x)) = (P(z, w), P(w, z))] \Rightarrow [x \succ y \Rightarrow z \succcurlyeq w]$.
(2) $[P(x, y) \neq \emptyset$ and $P(y, x) = \emptyset] \Rightarrow x \succ y$.

This definition represents a natural generalization of Definition 3.1. allowing to have at the same time $(P(x, y), P(y, x)) = (P(z, w), P(w, z))$, $x \succ y$ and $z \sim w$. The possibility of an absence of preference between z and w aims to encompass the notion of discordance between evaluations (see Roy, 1968; and Roy and Bertier, 1972). In fact, when the difference between the evaluations of z and w becomes important on the attributes belonging to $P(w, z)$ it is unrealistic to suppose $z \succ w$.

Definition 4.1. obviously implies mutual preferential independence in a GNPS. We have:

LEMMA 4.1. *A P.S. $(X, \succ)$ is a GNPS iff*
(1) $\rhd$ is asymmetric.
(2) $\forall\, A \in P(\Omega) \setminus \emptyset$, $A \geqslant \emptyset$.

Proof. Follows immediately from essentiality and Definition 4.1. ∎

It is obvious from the definition that a NPS is also a GNPS. The following definition establishes an interesting link between NPS and GNPS.

DEFINITION 4.2. Let $(X, \succ)$ be a GNPS. We define on X a binary relation $\overset{*}{\succ}$ by $x \overset{*}{\succ} y$ iff $P(x, y) \rhd P(y, x)$.

We have:

LEMMA 4.2. *If $(X, \succ)$ is a GNPS, then $\forall\, x, y, \in X$:*
(1) $x \succ y \Rightarrow x \overset{*}{\succ} y$.
(2) $(X, \overset{*}{\succ})$ *is a NPS.*

Proof. Obvious; left to the reader. ■

Thus, there is a natural way to extend a GNPS into a NPS. For instance let $X=\{x_1, y_1\} \times \{x_2, y_2, z_2\}$. We can represent a P.S. by mean of the following matrix:

	(x_1, x_2)	(x_1, y_2)	(x_1, z_2)	(y_1, x_2)	(y_1, y_2)	(y_1, z_2)
(x_1, x_2)	$\sim$	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
(x_1, y_2)		$\sim$	$\succ$	$\succ$	$\succ$	$\succ$
(x_1, z_2)			$\sim$	$\sim$	$\sim$	$\succ$
(y_1, x_2)				$\sim$	$\succ$	$\succ$
(y_1, y_2)					$\sim$	$\succ$
(y_1, z_2)						$\sim$

It is easily seen that this P.S. is a GNPS with $x_1 \succ_1 y_1$, $x_2 \succ_2 y_2$, $y_2 \succ_2 z_2$, $x_2 \succ_2 z_2$ and $\{1\} \gg \emptyset$, $\{2\} \gg \emptyset$, $\{1, 2] \gg \emptyset$ and $\{1\} \triangleright \{2\}$.

In order to obtain its corresponding NPS, it suffices to add $(x_1, z_2) \succ (y_1, x_2)$ and $(x_1, z_2) \succ (y_1, y_2)$.

Any configuration in the darkened area of the matrix with at least one $\succ$ would have lead to the same associated NPS.

The following considerations allow to specify better the way the discordance effect can work in a GNPS.

DEFINITION 4.3. A GNPS is discordant iff
$\forall\, x, y \in X$ such that $x \overset{*}{\succ} y$:
$[\forall\, j \in P(y, x),\ \exists\, (x_i^*)_{i\neq j}, (y_i^*)_{i\neq j} \in X_{i\neq j}\, X_i$ such that
$((x_i^*)_{i\neq j}, x_j) \succ ((y_i^*)_{i\neq j}, y_j)] \Rightarrow x \succ y$.

In order to be able to interpret this definition, we will use the following:

DEFINITION 4.4. $\forall\, j \in \Omega$, we define a binary relation V_j on X_j by
$x_j\, V_j\, y_j$ iff $\nexists\, (x_i^*)_{i\neq j}, (y_i^*)_{i\neq j} \in X_{i\neq j}\, X_i$ such that
$(y_j, (y_i^*)_{i\neq j}) \succ (x_j, (x_i^*)_{i\neq j})$.

The following lemma establishes the link between these two definitions:

LEMMA 4.3. *In a discordant GNPS*
$x \succ y$ *iff* $x \overset{*}{\succ} y$ *and not* $y_j\ V_j\ x_j$ *for all* $j \in P(y, x)$.

Proof. Follows immediately from Definitions 4.3. and 4.4. ■

Therefore, in a discordant GNPS, the discordance effect is introduced whenever there is an attribute j in $P(y, x)$ for which $y_j\ V_j\ x_j$ which can be interpreted as "y_j is far better than x_j". It should be noticed that this definition implies that each attribute must be considered separately in order to decide for the discordance. Thus, there is no possibility of interaction between the attributes in $P(y, x)$.

We have:

LEMMA 4.4. $\forall\ j \in \Omega$, V_j *is asymmetric.*

Proof. $x_j \succ_j y_j$ implies by definition not $y_j\ V_j\ x_j$. Thus $y_j\ V_j\ x_j$ implies:
– either $y_j \succ_j x_j$ and $x_j\ V_j\ y_j$ is impossible
– or $y_j \sim_j x_j$. The definition of a GNPS and essentially of attribute k thus imply $(y_k, y_j, (y_i^*)_{i \neq k, j}) \succ (x_k, x_j, (y_i^*)_{i \neq k, j})$ for some $(y_i^*)_{i \neq k, j} \in X_{i \neq k, j}\ X_i$. Thus we cannot have $x_j\ V_j\ y_j$. ■

In the case where a weak order underlies each $\succ_i$, some monotonicity conditions on $\succ$, give rise to a semi-order structure for each V_j.

Let us first recall (see for instance Vincke, 1980) that if $\succ_i$ is a semi-order on X_i, the binary relation T_i on X_i defined by:

$$y_i\ T_i\ x_i \text{ iff } \forall\ z_i \in X_i: \begin{cases} z_i \succ_i y_i \Rightarrow z_i \succ_i x_i \\ x_i \succ_i z_i \Rightarrow y_i \succ_i z_i \end{cases}$$

is transitive and strongly connected.

Its asymmetric part T_i^A is thus a weak order and its symmetric part T_i^S an equivalence.

We can now state the following:

THEOREM 4.1. *If a discordant GNPS is such that:*

(4.1) $\succ_j$ *is a semi-order* $\forall\ j \in \Omega$,
$x \succ y \Rightarrow [\forall\ j \in \Omega,\ \forall\ w_j \in X_j$ *such that* $y_j\ T_j\ w_j$:
$x \succ ((y_i)_{i \neq j}\ w_j)]$,

then each V_j is an interval order. Furthermore, if the GNPS also verifies

$$(4.3) \quad x \succ y \Rightarrow [\forall\, j \in \Omega,\ \forall\, z_j \in X_j \text{ such that } z_j\, T_j^A\, x_j: ((x_i)_{i \neq j}, z_j) \succ y],$$

then V_j is a semi-order.

In order to prove Theorem 4.1, we need the following:

LEMMA 4.5. *Under the assumptions* (4.1) *and* (4.2), *we have* $\forall\, j \in \Omega$, $y_j\, V_j\, x_j$ *iff* $y_j\, T_j^A\, z_j$ *for all* $z_j \in S(x_j)$ *where* $S(x_j) = \{y_j \in X_j$ *such that* $\exists\, (x_i^*)_{i \neq j}, (y_i^*)_{i \neq j} \in X_{i \neq j}\, X_i$ *verifying* $((x_i^*)_{i \neq j}, x_j) \succ ((y_i^*)_{i \neq j}, y_j)\}$.

Proof of the lemma. Suppose that $z_j\, T_j\, y_j$ for some $z_j \in S(x_j)$. We have: $((x_i^*)_{i \neq j}, x_j) \succ ((z_i^*)_{i \neq j}, z_j)$ for some $(x_i^*)_{i \neq j}, (z_i^*)_{i \neq j} \in X_{i \neq j}\, X_i$.

(4.2) implies $((x_i^*)_{i \neq j}, x_j) \succ ((z_i^*)_{i \neq j}, y_j)$ which contradicts $y_j\, V_i\, x_i$. The other part of the implication is established as follows:

Suppose that not $y_j\, V_j\, x_j$. Then: $((x_i^*)_{i \neq j}, x_j) \succ ((y_i^*)_{i \neq j}, y_j)$ for some $(x_i^*)_{i \neq j}, (y_i^*)_{i \neq j} \in X_{i \neq j}\, X_i$ and thus $y_j \in S(x_j)$. As not $y_j\, T_j^A\, y_j$, this completes the proof. ■

Proof of Theorem 4.1.

(1) *(4.1) and (4.2)* $\Rightarrow$ V_j interval order. By Lemma 4.4., V_j is asymmetric. Thus, all we have to prove is that, $\forall\, x_j, y_j, z_j, w_j \in X_j$:

$$\left.\begin{array}{l} x_j\, V_j\, y_j \\ \text{and} \\ z_j\, V_j\, w_j \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_j\, V_j\, w_j \\ \text{or} \\ z_j\, V_j\, y_j\,. \end{array}\right.$$

By Lemma 4.5, we have:

$$x_j\, T_j^A\, y'_j \text{ for all } y'_j \in S(y_j)$$
$$z_j\, T_j^A\, w'_j \text{ for all } w'_j \in S(w_j)\,.$$

Suppose now that not $(x_j\, V_j\, w_j)$ so that $\overline{w}_j\, T_j\, x_j$ for some $\overline{w}_j \in S(w_j)$. As $z_j\, T_j^A\, \overline{w}_j$ and $x_j\, T_j^A\, y'_j$ for all $y'_j \in S(y_j)$, we have $z_j\, T_j^A\, y'_j$ for all $y'_j \in S(y_j)$, T_j^A being a weak order. Thus $z_j\, V_j\, y_j$.

(2) *(4.1), (4.2) and (4.3)* $\Rightarrow$ *V_j semi-order.* Given the first part of the theorem, all we have to prove is that $\forall\, x_j, y_j, z_j \in X_j$:

$$\left.\begin{array}{l} x_j\, V_j\, y_j \\ \text{and} \\ y_j\, V_j\, z_j \end{array}\right\} \Rightarrow \forall\, w_j \in X_j \left\{\begin{array}{l} x_j\, V_j\, w_j \\ \text{or} \\ w_j\, V_j\, z_j \end{array}\right.$$

First suppose that $w_j\ T_j\ y_j$. As $y_j\ V_j\ z_j$, $y_j\ T_j^A\ z'_j$ for all $z'_j \in S(z_j)$. Thus $w_j\ T_j^A\ z'_j$ for all $z'_j \in S(z_j)$ *and* $w_j\ V_j\ z_j$.

Suppose now that $y_j\ T_j^A\ w_j$. For any $\overline{w}_j \in S(w_j)$, we have $(w_j, (w_i^*)_{i \neq j}) \succ (\overline{w}_j, (\overline{w}_j^*)_{i \neq j})$ for some $(w_i^*)_{i \neq j}$, $(\overline{w}_i^*)_{i \neq j} \in X_{i \neq j}\ X_i$ and by (4.3) $(y_j, (w_j^*)_{i \neq j})) \succ (\overline{w}_j, (\overline{w}_j^*)_{i \neq j}$ which implies that $\overline{w}_j \in S(y_j)$ for all $\overline{w}_j \in S(w_j)$. We have $x_j\ V_j\ y_j$: therefore $x_j\ T_j^A\ y'_j$ for all $y'_j \in S(y_j)$. As $S(w_j) \subset S(y_j)$, we have $x_j\ V_j\ w_j$. ■

Theorem 4.1. is particularly useful because it generally allows (when there is a numerical representation of the relation V_j) to define for each x_j a "veto threshold".

The reader may be puzzled by the dissymmetry existing between (4.2) and (4.3), (4.3) being slightly weaker than (4.2). Hereafter, we study the consequences of replacing (4.3) by:

$$x \succ y \Rightarrow [\forall\ j \in \Omega,\ \forall\ z_j \in X_j \text{ such that } z_j\ T_j\ x_j\text{: } ((x_i)_{i \neq j}, z_j) \succ y].$$

Let us first observe that, denoting U_i the transitive and strongly connected relation underlying V_i, neither $U_i \subset T_i$ nor $T_i \subset U_i$ is a logical consequence of Theorem 4.1. When V_j is empty, it is obvious that $U_i \subset T_i$ does not hold. In order to prove that we may not have $T_i \subset U_i$ let us consider the following example of GNPS:

$$X = X_1 \times X_2$$
$$X_1 = \{x_1, y_1, z_1\}$$
$$X_2 = \{x_2, y_2\}$$

	(x_1, x_2)	(y_1, x_2)	(z_1, x_2)	(x_1, y_2)	(y_1, y_2)	(z_1, y_2)
(x_1, x_2)	$\sim$	$\succ$	$\succ$	$\succ$	$\succ$	$\succ$
(y_1, x_2)		$\sim$	$\sim$	$\succ$	$\succ$	$\succ$
(z_1, x_2)			$\sim$	$\sim$	$\succ$	$\succ$
(x_1, y_2)				$\sim$	$\succ$	$\succ$
(y_1, y_2)					$\sim$	$\sim$
(z_1, y_2)						$\sim$

We have:

$$x_1 \succ_1 y_1,\ y_1 \sim_1 z_1,\ x_1 \succ_1 z_1$$
$$x_2 \succ_2 y_2$$
$$\{1\} \geqslant \emptyset,\ \{2\} \geqslant \emptyset$$
$$\{1, 2\} \geqslant \emptyset$$
$$\{2\} \rhd \{1\}.$$

It can be verified that this GNPS is discordant, that each $\succ_i$ is a weak-order and that $\succ$ satisfies (4.2), (4.3) but not (4.4) because: $(y_1, x_2) \succ (x_1, y_2)$ and $(z_1, x_2) \sim (x_1, y_2)$, though $z_1\ T_1^S\ y_1$.

In this example, V_2 is empty and we have $x_1\ V_1\ z_1$. We have therefore

$$x_1\ U_1\ z_1,\ x_1\ U_1\ y_1 \text{ and } y_1\ U_1\ z_1$$

but, at the same time

$$x_1\ T_1\ z_1,\ x_1\ T_1\ y_1,\ y_1\ T_1\ z_1 \text{ and } z_1\ T_1\ y_1.$$

We complete our study of discordant GNPS by establishing:

THEOREM 4.2. *In a discordant GNPS verifying* (4.1), (4.2) *and* (4.4), *we have* $T_j \subset U_j\ \forall\ j \in \Omega$.

Remark before proof. T_j and U_j being strongly connected $T_j \subset U_j$ implies $T_j^S \subset U_j^S$ and $U_j^A \subset T_j^A$.

Proof of Theorem 4.2. Suppose $y_j\ T_j\ x_j$ and not $(y_j\ U_j\ x_j)$. Not $(y_j\ U_j\ x_j)$ implies:

either $z_j\ V_j\ y_j$ and not $z_j\ V_j\ x_j$ for some $z_j \in X_j$

or $x_j\ V_j\ w_j$ and not $y_j\ V_j\ w_j$ for some $w_j \in X_j$.

Not $(z_j\ V_j\ x_j) \Rightarrow (x_j, (x_j^*)_{i \neq j}) \succ (z_j, (z_i^*)_{i \neq j})$ for some $(x_i^*)_{i \neq j}, (z_i^*)_{i \neq j} \in X_{i \neq j}\ X_i$.

(4.4) therefore implies:

$$(y_j, (x_i^*)_{i \neq j}) \succ (z_j, (z_i^*)_{i \neq j})$$

which contradicts $z_j\ V_j\ y_j$.

Not $(y_j\ V_j\ w_j) \Rightarrow (w_j, (w_i^*)_{i \neq j}) \succ (y_j, (y_i^*)_{i \neq j})$ for some $(w_i^*)_{i \neq j}, (y_i^*)_{i \neq j} \in X_{i \neq j}\ X_i$

and the application of (4.2) contradicts $x_j\ V_j\ w_j$ ■

5. GNPS AND DECISION-AID

We mentioned in the introduction that the concept of noncompensation has been used in several multi-attribute decision-aid methods. Our purpose in this section is to outline how the theoretical considerations developed above can be helpful in order to design a new method using these concepts (the TACTIC method) and to implement it.

The idea of the TACTIC method is to "build" a global preference relation (see Bouyssou, 1984; and Roy and Bouyssou, 1986 on this notion of construction) given a set of several attributes. Technically, the global preference relation takes the form of a discordant GNPS verifying the conditions (4.1), (4.2) and (4.4), which associated NPS is a CPSρ. Following Vansnick (1986), we will call such P.S. "noncompensatory preference structures with veto". The method first seeks to determine, in agreement with the decision-maker, the semi-orders $\succ_i$ and V_i for each attribute i, with $V_i \subset \succ_i$. This can be done simply by assessing a measurable value function on each attribute and determining two constant thresholds. The method then asks the decision-maker to compare several simple actions in order to obtain inter-criteria information. From this information, it determines simultaneously, following a number of reasonable principles, the 'weights' f_i and the coefficient ρ. Once this information is obtained, the determination of $\succ$ for the complete set of actions is performed easily. For more details on this topic, we refer to Vansnick (1986).

REFERENCES

Bouyssou, D.: 1984, *Approches constructives et descriptives d'aide à la décision: fondements et comparaison*, unpublished doctoral dissertation, Université de Paris-Dauphine, France.

Domotor, Z. and S. Stelzer: 1971, 'Representation of Finetely Additive Semi-Ordered Qualitative Probability Structures', *Journal of Mathematical Psychology* **8**, 145–168.

Fishburn, P. C.: 1969, 'Weak Qualitative Probability on Finite Sets', *The Annals of Mathematical Statistics* **40**, 2118–2126.

Fishburn, P. C.: 1974, 'Lexicographic Orders, Utilities and Decision Rules: A Survey', *Management Science* **20**, 1442–1471.

Fishburn, P. C.: 1975, 'Axioms for Lexicographic Preferences', *The Review of Economic Studies* **42**, 415–419.

Fishburn, P. C.: 1976, 'Noncompensatory Preferences', *Synthese* **33**, 393–403.

Huber, O.: 1974, 'An Axiomatic System for Multidimensional Preferences', *Theory and Decision* **5**, 161–184.

Huber, O: 1979, 'Nontransitive Multidimensional Preferences: Theoretical Analysis of a Model', *Theory and Decision* **10**, 147–165.
Keeney, R. L. and H. Raiffa: 1976, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*, Wiley, New York.
Krantz, D. H., R. D. Luce, P. Suppes, and A. Tversky: 1971, *Foundations of Measurement* Vol. I, Academic Press, New York.
MacCrimmon, K.: 1973, 'An Overview of Multiple Objective Decision Making', in J. L. Cochrane and M. Zeleny (eds.), *Multiple Criteria Decision Making*, University of South Carolina Press.
Plott, C. R., J. T. Little, and R. P. Parks: 1975, 'Individual Choice When Objects Have "Ordinal" Properties', *The Review of Economic Studies* **32**, 403–413.
Roberts, F. S.: 1979, *Measurement Theory with Applications to Decision-Making, Utility and the Social Sciences*, Addison-Wesley, London.
Roy, B.: 1968, 'Classement et choix en présence de points de vue multiples (la méthode ELECTRE)', *RIRO* **8**, 57–75.
Roy, B.: 1971, 'Problems and Methods with Multiple Objective Functions', *Mathematical Programming* **1**, 239–266.
Roy, B.: 1974, 'Critères multiples et modélisation des préférences: l'apport des relations de surclassement', *Revue d'Economie Politique* **84**, 1–44.
Roy, B. and P. Bertier: 1972, 'La méthode ELECTRE II – Une application au média-planning', in M. Ross (ed.), *OR '72*, Proceedings of the IFORS Conference in Dublin, North-Holland, Amsterdam, pp. 291–302.
Roy, B. and D. Bouyssou: 1986, 'Comparison of Two Decision-Aid Models Applied to a Nuclear Plant Sitting Example', *EJOR* **25**, 200–215.
Scott, D.: 1964, 'Measurement Structures and Linear Inequalities', *Journal of Mathematical Psychology* **1**, 233–247.
Vansnick, J. C.: 1986, 'On the Problem of Weights in Multiple Criteria Decision Making', *EJOR* **24**, 288–294.
Vincke, P.: 1980, 'Linear Utility Functions on Semi-Ordered Mixture Space', *Econometrica* **48**, 771–775.

Université de Paris-Dauphine,
F-75775 Paris Cédex 16,
France.