The Existence of Numbers (Or: What is the Status of Arithmetic?)
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I begin with a personal confession. Philosophical discussions of existence have always bored me. When they occur, my eyes glaze over and my attention falters. Basically ontological questions often seem best decided by banging on the table--rocks exist, fairies do not. Argument can appear long-winded and miss the point. Sometimes a quick distinction resolves any apparent difficulty. Does a falling tree in an earless forest make noise, ie does the noise exist? Well, if noise means that an ear must be there to hear it, then the answer to the question is evidently "no." But if noise means that, if there were (counterfactually) someone there, then he would hear it, then just as obviously, the answer becomes "yes."

Nonetheless, I warn the reader who partakes of my philistine tendencies, that this posting would normally send shivers down my spine, since it is a (long) discussion of the existence of mathematical objects.

Begin by noting, that *no matter what*, there are meanings of "exist" where 2 does not exist (take exist = "is an irrational number"), and where 2 does exist (take exist = "is a natural number"). So, it is not merely the answer to the ontological question which is important, but the exact, precise content of the question itself. Alas this is often not made explicit, but one way of testing this is to ask what the answer actually implies. Presumably no one who asserts that 2 exists is saying that, when he dies, he will find the number 2 waiting for him in Heaven ready to engage him in conversation (about Plato?), or that the number 2 is a ghost who overturns books in the study. On the other hand, presumably those who assert that 2 does not exist, are not asserting that 2 + 2 no longer equals 4. In brief, just because the answers are superficially different, doesn't mean people are actually disagreeing, and one must be wary of supposing that this is so.

One of the problems here is that much of ontological discussions turns on metaphor. Eg consider the statement "mathematical objects are just like chairs." Now of course in at least one sense, this is so: all mathematical objects (whatever, or even if, they may be) and all chairs are objects. And in another, it is not: mathematical objects, unlike many material things, cannot be touched by your fingers. Sometimes analogies are explained by another analogy, which hardly helps. "Mathematical objects are just like chairs because they can be perceived." But of course, if this assertion is to be true, the perception mentioned is presumably not the normal feel-see-hear type. My fingers are never likely to touch the number 2. It has to be something else, but, even without looking at the matter in detail, *of course* there is *some* meaning of "perceived" *somewhere* which will make this statement true. Whether mathematical objects are "external" to us plays on a similar analogy, and so likewise the question is ambiguous. It should be obvious that no amount of metaphor, however detailed, actually advances the discussion, because at no point is any of it made concrete, and there are alternate interpretations or mappings of all the relevant terms which lead to different truth values being assigned to each statement. Again just because some people are affirming and others are denying the likeness of mathematical objects with chairs, or our perception of mathematical objects, or that mathematical objects are external to us, doesn't mean they really disagree. And again, how many are really asserting that we can talk to the number 2, or that 2 + 2 doesn't equal 4?

So one must be careful of the content of ontological assertions. This is particularly important in assertions about fictional characters, such as Oliver Twist. If someone were to say "Oliver Twist exists," then he is probably making one of two claims. He might be saying that "Oliver Twist exists as a fictional character," in which case the assertion is true. Or he might be asserting that "Oliver Twist is or was alive and living somewhere in London", which is evidently false.

Let us return to arithmetic, by classifying its truths according to their ontological presumption. Some make none (call these type I), some assume that specific numbers exists (type II), and some (type III) that numbers exist ad infinitum.

An example of type I would be the Commutative Law of Addition, in the form, "$\forall x \forall y \forall z (x + y = z \Rightarrow y + x = z)$". While numbers are mentioned, there is no ontological pre-supposition. The Commutative Law (in this form) assumes the situation where there are numbers x, y, z such that $x + y = z$, and then concludes that these same numbers are such that $y + x = z$. It does not, clearly, assert that there are such numbers, and it would still be true, vacuously, if one or all did not exist.

An example of type II would be "2 + 2 = 4". There is a reference to the numbers 2 and 4; and, like "The King of France is bald," if either 2 or 4 didn't exist, then apparently "2 + 2 = 4" would not be true.

An example of type III is the Chinese Remainder Theorem. This is proved by taking a set of numbers and constructing their product, which is a much bigger number. In order to go through, the natural number sequence must continue indefinitely, and there must always be a next biggest number.

In *Arithmetic without the Successor Axiom* there is presented a system, in second-order logic with arithmetical (predicative) comprehension, with the following axioms:

(F1) Uniqueness. $\forall P \forall n \forall m \, (Mn, P \mathbin{\&} Mm, P \Rightarrow n = m)$
(F2) Zero. $\forall P (M0, P \Leftrightarrow \forall x \neg Px)$
(F3) Successoring.

$$\forall P \forall Q \forall a \forall n \forall m \, (Nn \mathbin{\&} Sn, m \mathbin{\&} \neg Pa \mathbin{\&} \forall x \, (Qx \Leftrightarrow Px \lor x = a) \Rightarrow (Mm, Q \Leftrightarrow Mn, P))$$

(F4) Induction. From: $\phi(0) \mathbin{\&} \forall n \forall m (Nn \mathbin{\&} Sn, m \mathbin{\&} \phi(n) \Rightarrow \phi(m))$
Conclude: $\forall n \, (Nn \Rightarrow \phi(n))$
(F5) N0
(F6) *Ad infinitum.*

$$\forall n \forall P \forall a \, (Nn \mathbin{\&} Mn, P \mathbin{\&} \neg Pa \Rightarrow \exists m \, (Nm \mathbin{\&} Mm, \{x : Px \lor x = a\}))$$

Here: Mn, P means "P numbers n" (M is a third-order predicate), Nn means "n is a finite number", and Sn, m means "m succeeds n (in the natural number sequence).

Remark that (F1) through (F6) can prove the Peano Axioms.

Now (F6) is evidently the axiom by which assertions of Type III must be proved. (F2) and (F5) are axioms of Type II. And (F1), (F3), and (F4) are axioms of Type I. Thus it would seem that we should look at the system (F1), (F3), and (F4), to discern what arithmetic truths are of Type I.

Unfortunately, the system {F1,F3,F4} would, I believe, have difficulty proving anything substantial. For while the axioms {F1,F2,F3,F4} are a strong system, F2 plays a crucial role in the derivations.

Nonetheless, it is possible to modify the axioms to make F2's ontology disappear, as follows. Use $zero(z)$ to abbreviate $\forall P(Mz,P \Leftrightarrow \forall x \neg Px)$.

And consider:

(G1) Uniqueness. $\forall P \forall n \forall m\ (Mn,P\ \&\ Mm,P \Rightarrow n = m)$
(G2) Zero. $\forall P \forall n\ (Mn,P\ \&\ \neg\, zero(n) \Rightarrow \exists x\, Px)$
(G3) Successoring:

$$\forall P \forall Q \forall a \forall n \forall m\ (Nn\ \&\ Sn,m\ \&\ \neg\, Pa\ \&\ \forall x\ (Qx \Leftrightarrow Px \lor x = a) \Rightarrow (Mm,Q \Leftrightarrow Mn,P))$$

(G4) Induction.
From: $\forall z\ (zero(z) \Rightarrow \phi(z))\ \&\ \forall n \forall m\ (Nn\ \&\ Sn,m\ \&\ \phi(n) \Rightarrow \phi(m))$
Conclude: $\forall n\ (Nn \Rightarrow \phi(n))$

This G system is now of Type I. Importantly, it has substantially the same deductive power as {F1,F2,F3,F4}, since the following lemma goes through:

L1. Suppose $\exists n\ Nn$. Then $\exists z\ (zero(z)\ \&\ Nz)$.
Proof:
Prove $\forall n(Nn \Rightarrow \exists z\ (zero(z)\ \&\ Nz))$ by induction, with $\phi(n)$ as $(Nn \Rightarrow \exists z\ (zero(z)\ \&\ Nz))$.
Remark: In fact, by (G1), the z is also unique.

Now, {F1,F2,F3,F4}, so the G system, can define addition, multiplication, and exponentiation, and prove "downward" arithmetic theorems. A "downward" theorem is one which, given a natural number, only needs smaller natural numbers to exist. The Euclidean Algorithm and Unique Prime Factorization are both "downward". The Commutative, Associative, and Distributive Laws of Addition are not "downward," but can be readily modified to be so. For instance, the Commutative Law of Addition asserts that $\forall x \forall y\ (Nx\ \&\ Ny \Rightarrow (x + y) = (y + x))$. This is "upward", since given two numbers x and y, a new, larger number--namely $(x + y)$--needs to exist. However, one can modify it as, $\forall x \forall y \forall z\ (x + y = z \Rightarrow y + x = z)$, as it was written above, to make it "downward."

Remark that Fermat's Last Theorem is also "downward", but it would be an open question (at least, a question which I cannot answer!) whether it can be proven in the F or G systems.

Type I assertions, not making any ontological pre-suppositions, evidently do not turn on what one means by "exists." Without arguing for the point, it seems reasonable to say that Type I assertions are analytic and necessary. That is, the theorems in the G system are all analytic.

In order to prove Type II or Type III assertions, additional axioms must be added to the G system. E.g., one can posit the existence of specific numbers, which are "Type II", as follows:

$$\exists a\ (zero(a)\ \&\ Na)$$
$$\exists a \exists b\ (zero(a)\ \&\ Na\ \&\ Sa,b\ \&\ Nb)$$
$$\exists a \exists b \exists c\ (zero(a)\ \&\ Na\ \&\ Sa,b\ \&\ Nb\ \&\ Sb,c\ \&\ Nc)$$

and so on (with the idea that at some point one stops, and this series does not continue ad infinitum). With these axioms added to the G system (call these the G+ systems), one can prove statements like "2 + 2 = 4", in addition to the "downward" theorems. Although there is evidently no way to prove this, it does seem that this is "practical" arithmetic, in that the "real" world does not require more. Nonetheless, ontological assumptions are evidently being made, and it is important to be clear about them.

My interest in the existence of mathematical objects turns on the following issue. For me, an object (mathematical or not) exists if it can be referred to. 2 exists because "2" refers successfully. This may seem to have small value, by merely transforming the original question into another, namely what can be referred to, which may seem equally opaque. It may also be thought of as putting the cart before the horse. One might say that it is not that the King of France does not exist because "The King of France" does not refer; rather it is that "the King of France" does not refer, because there is no King of France.

Nonetheless, this sense of "existence" is useful because it is as extensive as possible. If an object exists, under whatever sense, then it can be referred to (we have just done it!), so it exists in our sense. Perhaps someone may object that "Oliver Twist does not exist" is true, and so we can refer to what does not exist. But evidently "Oliver Twist" here is not referring to a human being living or having lived in London; if it were, then indeed the assertion would not be true (in the same way that "The King of France is a man" or "The King of France exists" is untrue). Instead it is referring to the fictional character, which may be referred to and so does exist (as a fictional character). However, "Oliver Twist does not exist" is not asserting of the fictional character that he does not exist (as a fictional character). It is claiming that there is no living human being corresponding to the fictional character Oliver Twist, which is of course true.

Now the reasoning about 2 goes as follows. 2 exists, because "2" refers successfully. "2" refers successfully, eg in "2 is a number." And 2 is a number, because my hands are 2 (in number).

This kind of reasoning is in fact unexceptional. "Red" refers successfully, because red is a color. And red is a color, because a Stop sign is red (in color). Red exists, just like 2. Generally, any property or characteristic can be referred to, so it exists, again in some minimal sense.

The content of "2 exists" then relies on assertions of the type "My hands are 2 (in number)." Only if someone were to deny the possibility of the truth of these assertions, would I accept that he could rightfully deny the existence of 2.

Remark that Type II truths seem reasonably classifiable as analytic. If one agrees that "2 + 2 = 4" is a proposition, then "2" and "4" must be concepts, and so 2 and 4 "exist," at least under some meaning of the term.

This leaves the status of Type III statements. Where does our idea that numbers continue *ad infinitum* come from? What is the justification for axiom F6? F6, it should be noted again, is not really needed for "practical" arithmetic; it is, however, convenient.

Without too much reflection, it seems apparent that the basis comes from our notion of counting. If I know how to count, then I believe I can always count one more. Now this would appear to give F6 some kind of analytic status as well, making it intrinsic to the notion of "counting" and thence the notion of "finite number". Against this, surely it is my experience of "counting," and indeed a sort of induction, which gives me this confidence. So the judgment appears to be synthetic.

Now of course all concepts are ultimately based on experiences. To know that "All bachelors are unmarried," I have had to learn the concepts of "bachelor" and "unmarried," i.e. I have had to have some experiences. Indeed, my learning has probably been of an inductive sort.

Nonetheless, there are a couple of important differences. My induction to arrive at a notion of "unmarried" is one of associating a property to things; it does not purport to materialize new things. This, on the other hand, is the effect of the ad infinitum axiom. Secondly, if my induction about "unmarried" is incorrect, there can be an example in the external world which can put me aright. On the other hand, there can be no example which will show me my induction on counting is incorrect. To expropriate Popper, it is an unfalsifiable induction.

The ad infinitum axiom is akin to the parallel postulate in geometry. Recall that the parallel postulate was, since the time of the ancient Greeks, considered more suspect than the others, because it did not involve some kind of immediate intuition, but talked of an extension of a line ad infinitum. So with the arithmetical ad infinitum axiom.

In brief, arithmetic theorems which require the ad infinitum axiom and are of Type III, are synthetic, and not analytic. They can be admitted as "true" only if we decide it is convenient, only if the price of admitting more entities than are strictly necessary is paid back by producing greater simplicity somewhere else. There will never be, because there can never be, a definite answer. The existence of numbers ad infinitum is, will always be, and must be provisional.

The status of arithmetic, then, is not that its truths are analytic, as Frege would have had it. They are not all synthetic, either, as Mill would have had it. Some arithmetic truths are analytic, and some are synthetic. That this extends to the rest of mathematics, in particular analysis, goes without saying; the question is only which theorems are which.