Simone Bova

Lewis Dichotomies in Many-Valued Logics

Abstract. In 1979, H. Lewis shows that the computational complexity of the Boolean satisfiability problem dichotomizes, depending on the Boolean operations available to formulate instances: intractable (NP-complete) if negation of implication is definable, and tractable (in P) otherwise [17]. Recently, an investigation in the same spirit has been extended to nonclassical propositional logics, modal logics in particular [18, 19]. In this note, we pursue this line in the realm of many-valued propositional logics, and obtain complexity classifications for the parameterized satisfiability problem of two pertinent samples, Kleene and Gödel logics.

Keywords: Parameterized satisfiability, complexity dichotomy, many-valued logics.

1. Introduction

A fundamental problem in classical propositional logic and theoretical computer science is the Boolean satisfiability problem, SAT: Given a propositional formula over a complete basis of logical connectives, does there exist a Boolean assignment of the variables that makes the formula true? In the early Seventies, S. Cook and L. Levin show (independently) that SAT is NP-complete [5, 16]. Nevertheless, restricting instances to certain incomplete bases (say, monotone bases) the satisfiability problem becomes easy. In 1979, H. Lewis shows that the satisfiability problem is computationally hard if the allowed basis, even if incomplete, defines negation of implication, $x \wedge \neg y$, and polynomial-time tractable otherwise: an entirely nonobvious dichotomic classification, since if P$\neq$NP, there are infinitely many complexity degrees between P and NP [13]. In this spirit, during the last decade the complexity of several other problems on propositional formulae and Boolean circuits, for instance the circuit value problem [23] and the propositional entailment problem [20], has been systematically classified, for every possible parameterizing set of logical connectives (or Boolean operations).

The aforementioned complexity classifications rely on *Post lattice* (compare Figure 1), the lattice of clones of Boolean operations titanically established by E. Post in the Twenties [22]. Recently, the possibility of applying Post lattice to the complexity classification of parameterized satisfiability

Presented by **?**; *Received* September 30, 2010

problems for propositional *nonclassical* logics has been successfully explored, in particular for *modal* propositional logics, leading to a neat trichotomy classification [18]. In this note, we explore the same possibility in the realm of nonclassical propositional *many-valued* logics.

The work is motivated by the following observation. Let $\mathbf{A} = (A, F)$ be an algebra on signature σ, where $A = \{0, 1, \dots\}$, and let $a \in A$; we think at F as a many-valued language, and at a as a designated value. Given a term t on σ, consider the problem whether or not there is an assignment $\mathbf{a}$ of the variables of t in A such that the term operation defined by t evaluates to a at $\mathbf{a}$. We are interested in a systematic classification of the complexity of this problem, for every set F of operations on A; if $A = \{0, 1\}$ and $a = 1$, this is the problem classified by H. Lewis. Unfortunately, if $|A| > 2$, a classification relying on the lattice of clones of operations on A is unfeasible. In fact, there is evidence that an explicit description of the lattice of clones on a set of more than two elements is unreachable; for instance, the lattice of clones on the three-element set does not satisfy any nontrivial lattice identity [4]. Nevertheless, if F carries a propositional semantics, and the problem is intended as a propositional satisfiability problem, it is typically assumed that F preserves $\{0, 1\}$ and a is equal to 1. We insist that, upon reading 0 and 1 as the classical truthvalues, the mentioned additional conditions are natural: a many-valued logic is obviously (and typically) required to behave classically on classical cases.

Interestingly, this additional property of many-valued logics allows for a reduction to Post lattice, which we explore in this note with respect to two historically established and inherently diverse samples of many-valued logics, namely *Kleene* and *Gödel* logics. The former is a three-valued logic whose logical deduction manages *undetermined* propositions (or, partial truth assignments to the propositional variables). The latter is a *fuzzy* logic in the hierarchy of triangular norms logics, intended to support logical deduction on *vague* propositions. For a comprehensive discussion, including philosophical foundations and artificial intelligence applications justifying interest in computational complexity issues, we refer the reader to [25, 24].

The paper is organized as follows. In the rest of this section, we collect the background on clone theory and Post lattice which we need to establish our complexity classifications. In Section 2, we recall the parameterized version of the Boolean satisfiability problem and sketch the proof by Lewis in the language of Post lattice. In Section 3, we introduce the parameterized version of the many-valued satisfiability problem, and then prove the main results: a complete complexity classification of the satisfiability problem parameterized by Kleene and Gödel operations (Section 3.2 and Section 3.1).

We conclude this note with a partial result and open problem on *deMorgan* operations, a clone of that includes Kleene operations. [1]

1.1. Background

We collect the algebraic notions and facts that we will use in establishing our complexity results. [2] Let A be a nonempty set, and let n be a natural number.

We let $\mathbf{O}_A^n = A^{A^n}$ denote the set of n-ary operations on A, and we let $\mathbf{O}_A = \bigcup_{1\leq n} \mathbf{O}_A^n$ denote the set of finitary operations on A. A *clone* on A is a subset C of $\mathbf{O}_A$ that contains the projection operations (that is, for all $1 \leq n$ and $1 \leq i \leq n$, the operation $\pi_i^n(x_1, \ldots, x_n) = x_i$ is in C) and is closed under compositions (that is, if g is an m-ary operation in C, and $f_1, \ldots, f_m$ are n-ary operations in C, then the n-ary operation $f(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is in C). The set of all clones on A, in symbols Cl_A, ordered by inclusion, is a bounded algebraic lattice, whose meet and join operations are respectively the intersection of clones, and the closure (under compositions) of unions of clones. If $F \subseteq \mathbf{O}_A$, we let $[F]$ denote the *(clone) closure* of F, that is, the smallest clone on A that contains F; if C is a clone, we let $C_n = C \cap \mathbf{O}_A^n$ denote the n-ary fragment of C.

If R is a relation on A, we let $\mathrm{ar}(R)$ denote the arity of R. [3] We let $\mathbf{R}_A^n = 2^{A^n}$ denote the set of n-ary relations on A, and we let $\mathbf{R}_A = \bigcup_{1\leq n} \mathbf{R}_A^n$, denote the set of finitary relations on A. A *coclone* on A is a subset $S \subseteq \mathbf{R}_A$ that contains the diagonal relation (that is, $\{(a, a) \mid a \in A\}$ is in S), and is closed under Cartesian products (if $R_1, R_2 \in S$, then

$$\{(a_1, \ldots, a_{\mathrm{ar}(R_1)}, b_1, \ldots, b_{\mathrm{ar}(R_2)}) \mid (a_1, \ldots, a_{\mathrm{ar}(R_1)}) \in R_1, (b_1, \ldots, b_{\mathrm{ar}(R_2)}) \in R_2\}$$

is in S), identification of coordinates (if R is in S, then for all $1 \leq i < j \leq$

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[1] Several computations have been assisted by the general algebra software tool developed by Freese and Kiss (`www.uacalc.org`).

[2] For further background in clone theory and computational complexity, we refer the reader to [15] and [21] respectively.

[3] Let $S = \{(a_{1,1}, \ldots, a_{1,n}), (a_{2,1}, \ldots, a_{2,n}), \ldots, (a_{k,1}, \ldots, a_{k,n})\}$ be an n-ary relation on A. In the sequel, we often display S by laying its tuples columnwise, as follows:

$$\begin{array}{cccc} a_{1,1} & a_{2,1} & \cdots & a_{k,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{k,n} \end{array}.$$

$\mathrm{ar}(R)$,

$$\{(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_{\mathrm{ar}(R)}) \mid (a_1, \ldots, a_{\mathrm{ar}(R)}) \in R, a_i = a_j\}$$

is in S), and projection of coordinates (if R is in S, then for all $1 \leq i \leq \mathrm{ar}(R)$,

$$\{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{\mathrm{ar}(R)}) \mid (\exists a_i \in A)\ (a_1, \ldots, a_i, \ldots, a_{\mathrm{ar}(R)}) \in R\}$$

is in S). The set of all coclones on A, in symbols Co_A, ordered by inclusion, is a bounded algebraic lattice, whose meet and join operations are respectively the intersection of coclones, and the closure (under Cartesian products, identifications of coordinates, and projections of coordinates) of unions of coclones. If $S \subseteq \mathbf{R}_A$, we let $[S]$ denote the *(coclone) closure* of S, that is, the smallest coclone on A that contains S.

Let $R \in \mathbf{R}_A^k$ and let $f \in \mathbf{O}_A^n$. Then, we say that f *preserves* R if R is a subalgebra of the k-th power of the algebra (A, f), that is,

$$(a_{1,1}, \ldots, a_{1,k}), \ldots, (a_{n,1}, \ldots, a_{n,k}) \in R$$

implies

$$(f(a_{1,1}, \ldots, a_{n,1}), \ldots, f(a_{1,k}, \ldots, a_{n,k})) \in R.$$

For $S \subseteq \mathbf{R}_A$, we let $\mathrm{Pol}(S)$ denote the set of all finitary operations on A that preserve each relation in S (called, the *polymorphisms* of S), and for $F \subseteq \mathbf{O}_A$, we let $\mathrm{Inv}(F)$ denote the set of all finitary relations on A that are preserved by each operation in F (called, the *invariants* of F). It is easy to check that $\mathrm{Pol}(S)$ is a clone, and that $\mathrm{Inv}(F)$ is a coclone. Hence, Pol maps Co_A to Cl_A, and Inv maps Cl_A to Co_A. Moreover, Geiger [8] and Bodnarchuk et al. [26] show that,

$$\mathrm{Inv}(F) = \mathrm{Inv}([F]) \text{ and } \mathrm{Pol}(\mathrm{Inv}(F)) = [F], \tag{1}$$

and

$$\mathrm{Pol}(S) = \mathrm{Pol}([S]) \text{ and } \mathrm{Inv}(\mathrm{Pol}(S)) = [S]; \tag{2}$$

for a proof, [15, Theorem 2.6.2, Theorem 2.9.1, and Theorem 2.9.2].

Equations (1) and (2) imply that Cl_A and Co_A are lattice antiisomorphic via Pol (or Inv), as clearly Pol and Inv are antitone (by definition, if $S, S' \subseteq \mathbf{R}_A$ are such that $S \subseteq S'$, then $\mathrm{Pol}(S') \subseteq \mathrm{Pol}(S)$, and if $F, F' \subseteq \mathbf{O}_A$ are such that $F \subseteq F'$, then $\mathrm{Inv}(F') \subseteq \mathrm{Inv}(F)$).

As a consequence, if $F \subseteq \mathbf{O}_A$, then there exists $S \subseteq \mathbf{R}_A$, unique up to coclone closure, such that $[F] = \mathrm{Pol}(S)$. Indeed, pick any subset S of $\mathbf{R}_A$

such that $[S] = \mathrm{Inv}(F)$. By (1) and (2), $[F] = \mathrm{Pol}(\mathrm{Inv}(F)) = \mathrm{Pol}([S]) = \mathrm{Pol}(S)$. If $[F] = \mathrm{Pol}(S)$, we call S a *relational presentation* of $[F]$.

We conclude this section collecting a number of facts on Post lattice for future use. We adopt the terminology, notation, and display in [7]. Compare Figure 1.

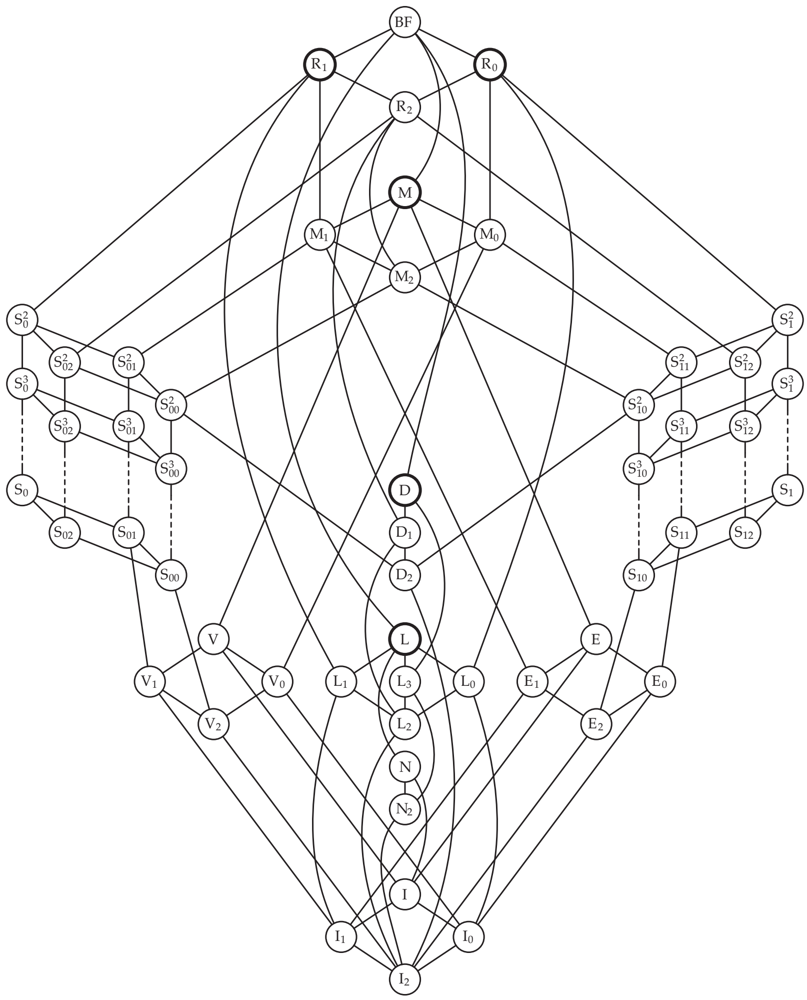

Figure 1. Boolean clones, or Post lattice [7].

An operation $f \in \mathbf{O}_{\{0,1\}}$ is said a *Boolean* operation. In the sequel, we describe Boolean operations as term operations over a basic package of Boolean operations including $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$,

$x \oplus y = x + y \mod 2$, $\neg x = 1 - x$, $\bot = 0$, $\top = 1$ (customarily, we regard $\{0, 1\}$ as an ordered ring). The following presentation of the clone of all Boolean operations is easy to check,

$$\mathbf{O}_{\{0,1\}} = [\{x \wedge y, \neg x\}] = \mathrm{Pol}\begin{pmatrix} 0 & 1 \end{pmatrix} \leftrightharpoons \mathrm{Pol}(\mathcal{B}).$$

A Boolean operation f is said 1-*reproducing* if $f(1, \ldots, 1) = 1$; *selfdual* if $f(a_1, \ldots, a_n) = 1 - f(1 - a_1, \ldots, 1 - a_n)$; *monotone* if $f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n)$ if $a_i \leq b_i$ for $i = 1, \ldots, n$; *affine* if it has a term definition of the form $(c_0 \oplus (c_1 \wedge x_1) \oplus \cdots \oplus (c_n \wedge x_n))$ for some constants $c_0, \ldots, c_n \in \{\bot, \top\}$. It is easy to check that the previous properties are preserved under composition, therefore 1-reproducing, selfdual, monotone, affine Boolean operations form clones, respectively denoted by R_1, D, M, L; both operational and relational presentations are known [22, 6]:

$$\begin{aligned}
R_1 &= [\{x \vee y, x \oplus y \oplus \top\}] = \mathrm{Pol}\begin{pmatrix} 1 \end{pmatrix} \leftrightharpoons \mathrm{Pol}(\mathcal{R}_1), \\
D &= [\{(x \wedge \neg y) \vee (x \wedge \neg z) \vee (\neg y \wedge \neg z)\}] = \mathrm{Pol}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightharpoons \mathrm{Pol}(\mathcal{D}), \\
M &= [\{x \wedge y, x \vee y, \bot, \top\}] = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \leftrightharpoons \mathrm{Pol}(\mathcal{M}), \\
L &= [\{x \oplus y, \top\}] = \mathrm{Pol}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \leftrightharpoons \mathrm{Pol}(\mathcal{L}).
\end{aligned}$$

2. Lewis Dichotomy

In this section, we introduce the *parameterized* version of the Boolean satisfiability problem, and we sketch the proof of Lewis dichotomy.

Let σ be a (finite, algebraic) signature, and let $\mathbf{B} = (\{0, 1\}, F)$ be an algebra on σ. If F is any subset of $\mathbf{O}_{\{0,1\}}$, we define the Boolean satisfiability problem, parameterized by $\mathbf{B}$, as follows:

Problem SAT($\mathbf{B}$).

Instance A term $t(x_1, \ldots, x_n)$ over σ.

Question Does there exist an assignment $(a_1, \ldots, a_n) \in \{0, 1\}^n$ such that $t^{\mathbf{B}}(a_1, \ldots, a_n) = 1$?

If $F = \{\wedge, \neg\}$, then SAT($\mathbf{B}$) is the classical SAT problem, NP-complete by [5, 16].

In [17], Lewis classifies the computational complexity of the Boolean satisfiability problem for all possible parameterizing algebras $\mathbf{B}$, showing that the problem dichotomizes, either NP-complete or polynomial-time tractable.

THEOREM 1 (Lewis Dichotomy). *Let* $\mathbf{B} = (\{0,1\}, F)$ *be an algebra. Then,* $\mathrm{SAT}(\mathbf{B})$ *is NP-complete (under logspace many-one reductions) if*

$$x \wedge \neg y = \neg(x \to y) = \begin{array}{c|cc} b & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array} \in [F],$$

and in P otherwise.

PROOF. [4] For NP-completeness, assume $b \in [F]$. Since $[b] \leftrightharpoons S_1$, we have $S_1 \subseteq [F]$. Inspection of Post lattice shows that $\{b, 1\}$ generates all Boolean operations, indeed $[1] \leftrightharpoons I_1$ and $S_1 \vee I_1 = \mathbf{O}_{\{0,1\}}$. Let $\mathbf{B}' = (\{0,1\}, F') = (\{0,1\}, \{x \wedge \neg y, \top\})$. Then, $\mathrm{SAT}(\mathbf{B}')$ is NP-complete: containment is clear, and for hardness, it is possible to reduce SAT to $\mathrm{SAT}(\mathbf{B}')$ because $[F'] = \mathbf{O}_{\{0,1\}}$ warrants that the term definitions of $x \wedge y$ and $\neg x$ on F' contain, respectively, only one occurrence of each of the variables x and y, and only one occurrence of the variable x [17, Lemma 1 and Lemma 2]. The reduction of $\mathrm{SAT}(\mathbf{B}')$ to $\mathrm{SAT}(\mathbf{B})$, given a term $t(x_1, \dots, x_n)$ on signature $x \wedge \neg y, \top$, returns the term $t'(x_1, \dots, x_n, x_{n+1}) = x_{n+1} \wedge_\sigma t[\top/x_{n+1}]$, where x_{n+1} is a fresh variable, $t[\top/x_{n+1}]$ denotes the result of replacing all occurrences of $\top$ in t with x_{n+1}, and $\wedge_\sigma$ denotes the term definition of the operation $\wedge$ on the signature of $\mathbf{B}$; note that $S_1 \subseteq [F]$ implies $\wedge \in [F]$, hence this definition exists. Clearly, if $t(a_1, \dots, a_n) = 1$ in $\mathbf{B}'$, then by sending in addition x_{n+1} to 1, $t'(a_1, \dots, a_n, 1) = 1$ in $\mathbf{B}$; and conversely, if $t'(a_1, \dots, a_n, a_{n+1}) = 1$ in $\mathbf{B}$, then $a_{n+1} = 1$, and then $t(a_1, \dots, a_n) = 1$ in $\mathbf{B}'$. So, $\mathrm{SAT}(\mathbf{B})$ is NP-hard. As $\mathrm{SAT}(\mathbf{B})$ is in NP for every algebra $\mathbf{B}$, we conclude that $\mathrm{SAT}(\mathbf{B})$ is NP-complete.

For tractability, if $b \notin [F]$, then $S_1 \not\subseteq [F]$, and inspection of Post lattice yields the following cases. If $F \subseteq R_1$, then t is 1-reproducing, therefore, every t is a "Yes" instance to $\mathrm{SAT}(\mathbf{B})$, and the problem is trivial (solvable in constant time). If $F \subseteq D$, then t is selfdual, then $t(0, \dots, 0) = 1$ or $t(1, \dots, 1) = 1$; therefore, again, every t is a "Yes" instance to $\mathrm{SAT}(\mathbf{B})$. If $F \subseteq M$, then t is monotone, then, t is a "Yes" instance if and only if $t(1, \dots, 1) = 1$; thus the problem reduces to a single evaluation of t, and this problem is in P; indeed, for every $F \subseteq \mathbf{O}_{\{0,1\}}$, the problem is in P

[4] We follow [23, Theorem 3.24].

[14]. If $F \subseteq L$, then t is affine, so that t is a "Yes" instance iff $c_i = \top$ for some $i = 0, \dots, n$. Now, $c_0 = \top$ iff $t(0, \dots, 0) = 1$ and $c_i = \top$ iff $c_0 \oplus t(0, \dots, 1, \dots, 0) = 1$ for $i = 1, \dots, n$. Again, the problem reduces to at most $n + 1$ evaluations of t, and this problem is in P. ■

Lewis dichotomy allows for an explicit description of tractable cases through Post lattice, both in operational and relational terms, and yields a decision procedure that, given any algebra $\mathbf{B} = (\{0,1\}, F)$ of finite signature, establishes whether or not $\mathrm{SAT}(\mathbf{B})$ is in P.

3. Many-Valued Dichotomies

Let $\mathbf{L} = (A, F)$ be an algebra of signature σ such that $\{0,1\} \subseteq A$ and

$$F \subseteq \mathrm{Pol}(\mathcal{B}); \tag{3}$$

in words, for any $f \in F$, the restriction of f to $\{0,1\}$ ranges over $\{0,1\}$, a defensibly natural condition for a many-valued language with an intended propositional semantics. Notice that (3) implies $[F] \subseteq \mathrm{Pol}(\mathcal{B})$. [5] By universal algebraic facts, $[F]_n$ is the universe of $\mathbf{F}_{HSP(\mathbf{L})}(n)$, the free n-generated algebra in the variety generated by $\mathbf{L}$, or the Lindenbaum-Tarski algebra (of the n-variate fragment) of the intended many-valued propositional language.

The *parameterized many-valued satisfiability problem*, corresponding to $[F]$, is defined as follows. Let $\mathbf{A} = (A, F')$ be an algebra of signature σ such that $F' \subseteq [F]$.

Problem $\mathrm{SAT}(\mathbf{A})$

Instance A term $t(x_1, \dots, x_n)$ on σ.

Question Does there exists $(a_1, \dots, a_n) \in A^n$ such that $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$?

This problem includes parameterized Boolean satisfiability as a special case, and in this more general setting, inspired by the complexity classification given in Lewis dichotomy, the natural question arises: How is the complexity of $\mathrm{SAT}(\mathbf{A})$ affected by the parameterizing algebra $\mathbf{A}$?

As mentioned in the introduction, a direct approach through the lattice of clones on A, along the lines of Theorem 1, is unfeasible if $|A| > 2$. Interestingly, condition (3) allows for a reduction to Post lattice. Exploiting

[5] Typically, and in particular in all cases under consideration, a many-valued language F with logical inspiration satisfies the stronger condition, that the restriction of its operations to $\{0,1\}$ forms a complete Boolean basis, that is, $[\{f|_{\{0,1\}} \mid f \in F\}] = \mathbf{O}_{\{0,1\}}$.

this observation, we obtain complete complexity classifications, analogous to Lewis dichotomies, for the parameterized satisfiability problem of two established many-valued logics: Kleene logic (Section 3.1) and Gödel logic (Section 3.2).

3.1. Kleene Operations

In this section, we classify the complexity of the satisfiability problem parameterized by Kleene operations, defined as follows.

Let $\mathbf{K} = (\{0,2,1\}, K)$ be the algebra defined by $K = \{\wedge, \neg, \top\}$, where $\top = 1$, $\neg 0 = 1, \neg 2 = 2, \neg 1 = 0$, and $\wedge$ is the meet on the chain $0 < 2 < 1$,

$$\begin{array}{c|ccc} \wedge & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 1 \end{array}.$$

The algebra $\mathbf{K}$ has the following intended three-valued propositional semantics: 1 and 0 stand for "true" and "false" respectively, and 2 stands for "undetermined" [3, 2]. A *Kleene operation* is a term operation on $\mathbf{K}$.

Upon adjoining term operations $x \vee y = \neg(\neg x \wedge \neg y)$ and $\bot = \neg\top$, and expanding the signature accordingly, the algebra $\mathbf{K}$ generates the variety of *Kleene algebras*: bounded distributive lattices with an involution, $\neg\neg x = x$, satisfying identically the deMorgan and Kleene equations, $\neg(x \vee y) = \neg x \wedge \neg y$ and $x \wedge \neg x \leq y \vee \neg y$ respectively [12]. By universal algebraic facts, $[K]_n$ forms the universe of the free n-generated Kleene algebra, $\mathbf{F}_{HSP(\mathbf{K})}(n)$, that can be interpreted as the Lindenbaum-Tarski algebra (of the n-variate fragment) of the above three-valued propositional language. The clone $[K]$ has a nice relational presentation [2]:

$$[K] = \mathrm{Pol}\left(\begin{array}{cc} 0 & 1 \end{array}, \begin{array}{ccccc} 0 & 2 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{array} \right) \eqqcolon \mathrm{Pol}(\mathcal{B}, \mathcal{K}),$$

which clearly implies that Kleene operations satisfy condition (3).

If $\mathbf{A} = (\{0,2,1\}, F)$ is any algebra, over a signature σ, whose fundamental operations are Kleene operations,

$$F \subseteq [K],$$

we obtain the following dichotomy classification of the parameterized satisfiability problem.

THEOREM 2. *Let* $\mathbf{A} = (\{0,2,1\}, F)$ *be such that* $F \subseteq [K]$. *Then,* $\mathrm{SAT}(\mathbf{A})$ *is NP-complete if*

$$\begin{array}{c|ccc} k_1 & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{array} \in [F] \ \text{or} \ \begin{array}{c|ccc} k_2 & 0 & 2 & 1 \\ \hline 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{array} \in [F],$$

and in P otherwise.

REMARK 3. *The statement implies that* $[k_1]$ *and* $[k_2]$ *are incomparable in the lattice of clones on* $\{0,2,1\}$.

PROOF OF THEOREM 2. Since every $f \in F$ is a Kleene operation, in particular, f preserves $\mathcal{B}$, that is, $f|_{\{0,1\}}$ ranges over $\{0,1\}$, and then, $\mathbf{A}' = (\{0,1\}, \{f|_{\{0,1\}} \mid f \in F\})$ is an algebra, which we display on the same signature of $\mathbf{A}$, say σ. We claim that

$$\mathrm{SAT}(\mathbf{A}) = \mathrm{SAT}(\mathbf{A}'), \tag{4}$$

that is, the set of satisfiable terms on $\mathbf{A}$ is equal to the set of satisfiable terms on $\mathbf{A}'$. Indeed, let t be any term on σ. If $(a_1, \dots, a_n) \in \{0,2,1\}^n$ is such that $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$, then pick any $(a'_1, \dots, a'_n) \in \{0,1\}^n$ such that $a'_i = a_i$ whenever $a_i \in \{0,1\}$. Notice that $(a_i, a'_i) \in \mathcal{K}$ for all $i = 1, \dots, n$, therefore, $(t^{\mathbf{A}}(a_1, \dots, a_n), t^{\mathbf{A}}(a'_1, \dots, a'_n)) \in \mathcal{K}$ by preservation of $\mathcal{K}$; since $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$, it follows that $t^{\mathbf{A}}(a'_1, \dots, a'_n) = 1$. By construction, the restriction of $t^{\mathbf{A}}$ to $\{0,1\}$ is equal to $t^{\mathbf{A}'}$, then $t^{\mathbf{A}'}(a'_1, \dots, a'_n) = 1$. Conversely, suppose that $(a_1, \dots, a_n) \in \{0,1\}^n$ is such that $t^{\mathbf{A}'}(a_1, \dots, a_n) = 1$. Since the restriction of $t^{\mathbf{A}}$ to $\{0,1\}$ is equal to $t^{\mathbf{A}'}$, it follows that $t^{\mathbf{A}}(a_1, \dots, a_n) = 1$, and the claim is proved. We now prove the dichotomy.

For intractability, suppose $k_1 \in [F]$; the case $k_2 \in [F]$ is similar. As the operation k_1 is in the clone generated by F, and the operation b in Theorem 1 is the restriction of k_1 to $\{0,1\}$, it follows that b is in the clone generated by $\{f|_{\{0,1\}} \mid f \in F\}$. Then, by Lewis dichotomy, $\mathrm{SAT}(\mathbf{A}')$ is NP-complete, and by (4), $\mathrm{SAT}(\mathbf{A})$ is NP-complete.

For tractability, suppose that neither k_1 nor k_2 are in $[F]$. By direct computation, there are exactly 4 binary operations in $[K]$ whose restriction to $\{0,1\}$ is the operation b in Lewis dichotomy; in addition to the operations k_1 and k_2 in the statement, we have:

$$\begin{array}{c|ccc} k_3 & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array} \ \text{and} \ \begin{array}{c|ccc} k_4 & 0 & 2 & 1 \\ \hline 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}.$$

Since k_1 is term definable over k_3,

$$k_1(x,y) = k_3(x, k_3(x, k_3(x,y))),$$

and k_2 is term definable over k_4,

$$k_2(x,y) = k_4(k_4(x,y), k_4(y,x)),$$

it follows that neither k_3 nor k_4 are in $[F]$. Then, there is no binary operation in $[F]$ whose restriction to $\{0,1\}$ is b. Thus, by Lewis dichotomy, $\mathrm{SAT}(\mathbf{A}')$ is in P, and by (4), $\mathrm{SAT}(\mathbf{A})$ is in P. ∎

A tangible description of tractable sets of Kleene operations follows directly, using the relational presentation of the Kleene clone, and the relevant Boolean clones discussed in the introduction. We remark that an operational presentation of tractable cases is likely nontrivial, since the obvious reduction to Post lattice fails (compare Example 6).

Corollary 4. *Let $\mathbf{A} = (\{0,2,1\}, F)$ be such that $F \subseteq [K]$. Then,* $\mathrm{SAT}(\mathbf{A})$ *is in P if and only if F is contained in* $\mathrm{Pol}(\mathcal{R}_1, \mathcal{B}, \mathcal{K})$*, or in* $\mathrm{Pol}(\mathcal{M}, \mathcal{B}, \mathcal{K})$*, or in* $\mathrm{Pol}(\mathcal{D}, \mathcal{B}, \mathcal{K})$*, or in* $\mathrm{Pol}(\mathcal{L}, \mathcal{B}, \mathcal{K})$*.*

Remark 5. *The statement implies that, given a finite set of operations in $\mathbf{O}_{\{0,1,2\}}$, it is decidable whether or not they induce a tractable Kleene satisfiability problem, by checking finitely many preservation conditions.*

Proof of Corollary 4. If $\mathrm{SAT}(\mathbf{A})$ is not in P, then by Theorem 2, either k_1 or k_2 is in $[F]$, say w.l.o.g., $k_1 \in [F]$. A direct check reveals that k_1 preserves neither $\mathcal{R}_1$, nor $\mathcal{M}$, nor $\mathcal{D}$, nor $\mathcal{L}$. Indeed, $k_1(1,1) = 0$, with $(1) \in \mathcal{R}_1$ but $(0) \notin \mathcal{R}_1$; $k_1(1,0) = 1$ and $k_1(1,1) = 0$, with $(1,1), (0,1) \in \mathcal{M}$ but $(1,0) \notin \mathcal{M}$; $k_1(0,0) = 0$ and $k_1(1,1) = 0$, with $(0,1) \in \mathcal{D}$ but $(0,0) \notin \mathcal{D}$; $k_1(0,0) = 0$, $k_1(0,1) = 0$, $k_1(1,0) = 1$ and $k_1(1,1) = 0$ with $(0,0,1,1), (0,1,0,1) \in \mathcal{L}$ but $(0,0,1,0) \notin \mathcal{L}$. By (1), $\mathrm{Inv}(F) = \mathrm{Inv}([F])$, and we are done.

Conversely, if $\mathrm{SAT}(\mathbf{A})$ is in P, then neither k_1 nor k_2 is in $[F]$, and along the lines of Theorem 2, b is not in $[F|_{\{0,1\}}]$. Since $[b] = S_1$, $S_1 \not\subseteq [F|_{\{0,1\}}]$ and by inspection of Post lattice, $[F|_{\{0,1\}}]$ is contained in either $R_1 = \mathrm{Pol}(\mathcal{R}_1)$, $L = \mathrm{Pol}(\mathcal{L})$, $M = \mathrm{Pol}(\mathcal{M})$, or $D = \mathrm{Pol}(\mathcal{D})$. For instance, suppose $[F|_{\{0,1\}}]$ is contained in $\mathrm{Pol}(\mathcal{R}_1)$. By hypothesis, $F \subseteq \mathrm{Pol}(\mathcal{B}, \mathcal{K})$, and then, any $f \in F$ is a Kleene operation whose restriction to $\{0,1\}$ is 1-reproducing. But $\mathrm{Pol}(\mathcal{R}_1, \mathcal{B}, \mathcal{K})$ is the largest set of Kleene operations whose Boolean restrictions are 1-reproducing, hence it must contain F. The other cases are similar. ∎

EXAMPLE 6. *We know that* $R_1 = [\{x \vee y, x \oplus y \oplus \top\}]$. *However the clone of Kleene operations whose Boolean restrictions are* 1*-reproducing is not generated by the Kleene operations whose Boolean restrictions are equal to* $x \vee y$ *or to* $x \oplus y \oplus \top$, *as the following counting shows.*

There are five Kleene operations $g_1, \ldots, g_5$ *whose Boolean restriction is equal to* $x \vee y$ *or to* $x \oplus y \oplus \top$. *Now, the clone of Kleene operations generated by* $\{g_1, \ldots, g_5\}$ *contains only* 25 *binary Kleene operations; but there are* 42 *binary Kleene operations having a* 1*-reproducing Boolean restriction.*

3.2. Gödel Operations

In this section, we classify the complexity of the satisfiability problem parameterized by Gödel operations, which we now define.

Let $\mathbf{G} = ([0,1], G)$ be the algebra defined by $G = \{\wedge, \rightarrow, \neg, \bot\}$, where $\bot = 0$, $x \wedge y = \min\{x, y\}$,

$$x \rightarrow y = \begin{cases} 1 & x \leq y \\ y & \text{otherwise} \end{cases},$$

and $\neg x = x \rightarrow \bot$. A *Gödel operation* is a term operation on $\mathbf{G}$.

For $m \geq 1$ integer, let $[0,1]_m = \{0, 1/m, 2/m, \ldots, 1\}$. It is easy to check that

$$\mathbf{G}_m = ([0,1]_m, G_m),$$

where $G_m = \{f|_{[0,1]_m} \mid f \in G\}$, is a subalgebra of $\mathbf{G}$. We refer the reader to [10] for a discussion of the intended fuzzy propositional semantics of the algebra $\mathbf{G}$ in the general framework of triangular norm based logics.

The algebra $\mathbf{G}$ generates the variety of Gödel algebras, namely, commutative bounded integral divisible prelinear idempotent residuated lattices [11]. Therefore, $[G]_n$ forms the universe of the free n-generated Gödel algebra $\mathbf{F}_{HSP(\mathbf{G})}(n)$, that can be interpreted as the Lindenbaum-Tarski algebra (of the n-variate fragment) of the fuzzy propositional logic mentioned above.

For future use, we prepare a relational presentation of the clone of Gödel operations.[6]

THEOREM 7. *For* $m \geq 1$, *let* $\mathcal{G}_m = \{S \mid S$ *subuniverse of* $\mathbf{G}_m$ *or* $\mathbf{G}_m^2\}$. *Then,*

$$[G] = \mathrm{Pol}\left(\bigcup_{m \geq 1} \mathcal{G}_m\right) \leftrightharpoons \mathrm{Pol}(\mathcal{G}).$$

[6] To our knowledge, this is the first relational presentation of Gödel operations.

Proof. If $f \in [G]$, then f is a term operation of $\mathbf{G}$, thus f preserves subuniverses of $\mathbf{G}$ and powers of subuniverses of $\mathbf{G}$, therefore f preserves $\mathcal{G}$.

Now suppose $f \notin [G]$. Let n be the arity of f, so that $f\colon [0,1]^n \to [0,1]$ and f is not in the n-ary fragment $[G]_n$ of $[G]$. Since G_{n+1} is a subuniverse of $[0,1]$, the restriction of any n-ary term operation g of $\mathbf{G}$ to $[0,1]_{n+1}$ is a term operation of $\mathbf{G}_{n+1}$, or equivalently, is contained in the n-ary fragment of the clone $[G_{n+1}]$. The following statement from [9] is here reported as a fact.

Fact 8. *The map $h(g) = g|_{[0,1]_{n+1}}$ is a bijection between $[G]_n$ and $[G_{n+1}]_n$.*

Example 9. *The bijection h between $[G]_1$ and $[G_2]_1 \subseteq \{0, 1/2, 1\}^{\{0,1/2,1\}}$ is $\perp \mapsto (0,0,0)$, $x \mapsto (0,1/2,1)$, $\neg x \mapsto (1,0,0)$, $x \vee \neg x \mapsto (1,1/2,1)$, $\neg\neg x \mapsto (0,1,1)$, $\top \mapsto (1,1,1)$.*

By Fact 8, $f|_{[0,1]_{n+1}} \notin [G_{n+1}]_n$, otherwise $h^{-1}(f|_{[0,1]_{n+1}}) = f \in [G]_n$.

Claim 10. *If $f|_{[0,1]_{n+1}} \notin [G_{n+1}]_n$, then $f|_{[0,1]_{n+1}} \notin \mathrm{Pol}\,(\mathcal{G}_{n+1})$.*

Proof of Claim 10. Let for short f' denote $f|_{[0,1]_{n+1}}$, and suppose that $f' \notin [G_{n+1}]_n$. We preliminarily collect some background terminology and a nontrivial fact from [1].

Let $\mathbf{a} = (a_1, \dots, a_n) \in [0,1]^n_{n+1}$. The ordered partition $(B_1, \dots, B_m)$ of $\{\perp, 1, \dots, n, \top\}$ is said *induced* by $\mathbf{a}$ if: $\perp \in B_1$; $\top \in B_m$; for all $1 \leq k \leq m$, we have $i, j \in B_k$ iff $a_i = a_j$; for all $1 \leq k < m$, $i \in B_k$ and $j \in B_{k+1}$ iff $a_i < a_j$. For $1 \leq k \leq m$, if $\{v\} = \{a_i \mid i \in B_k\}$, we call v the *value* of B_k; we stipulate that the values of B_1 and B_k are 0 and 1 respectively.

Example 11. *The point $(1, 2/3) \in [0,1]^2_3$ induces over $\{\perp, 1, 2, \top\}$ the ordered partition $(\{\perp\}, \{2\}, \{1, \top\})$, whose blocks have values $(0, 2/3, 1)$. The point $(2/3, 1/3) \in [0,1]^2_3$ induces the partition $(\{\perp\}, \{2\}, \{1\}, \{\top\})$, whose blocks have values $(0, 1/3, 2/3, 1)$. The two partitions share the first two blocks.*

Fact 12. *$f' \notin [G_{n+1}]_n$ iff: either (Case 1) $f'(a_1, \dots, a_n) \notin \{a_1, \dots, a_n\} \cup \{0,1\}$ for some $(a_1, \dots, a_n) \in [0,1]^n_{n+1}$; or (Case 2) there exist points $\mathbf{a}$ and $\mathbf{b}$ in $[0,1]^n_{n+1}$ that respectively induce partitions $(A_1, \dots, A_i, \dots, A_j)$ and $(B_1, \dots, B_i, \dots, B_k)$ sharing the first i blocks ($i \leq j, k$), with v_t the value of A_t ($1 \leq t \leq j$) and w_s the value of B_s ($1 \leq s \leq k$), such that the following holds: $f'(\mathbf{a}) = v_r$ and $f'(\mathbf{b}) = w_s$ with either $r \neq s \leq i$, or $r \leq i < s$, or $s \leq i < r$.*

In light of Fact 12, we distinguish two cases. Case 1: In this case, f does not preserve the subuniverse $\{0, a_1, \ldots, a_n, 1\}$ of $\mathbf{G}_{n+1}$, which settles the first case. Case 2: In this case, f' does not preserve the subuniverse R of $\mathbf{G}^2_{n+1}$ given by

$$\{(0,0),(v_1,w_1),\ldots,(v_i,w_i)\} \cup \{v_{i+1},\ldots,v_{j-1},1\} \times \{w_{i+1},\ldots,w_{k-1},1\},$$

as we now check. First notice that R is actually a subuniverse of $\mathbf{G}^2_{n+1}$. Now, $(a_1,b_1),\ldots,(a_n,b_n) \in R$ because for all $1 \leq l \leq n$ the following holds by construction: If (a_l,b_l) is such that l is in a shared block, say $l \in A_p = B_p$ with $p \leq i$, then $(a_l,b_l) = (v_p,w_p) \in R$; otherwise, if (a_l,b_l) is such that l is not in a shared block, say $l \in A_p$ and $l \in B_q$ with $i < p,q$, then $(a_l,b_l) = (v_p,w_q) \in R$. But, $(f'(a_1,\ldots,a_n), f'(b_1,\ldots,b_n)) = (v_r,w_s) \notin R$ by the hypothesis on r and s and the construction of R. Then, f' does not preserve R. This settles the second case, and the claim. ■

Since $\mathcal{G}_{n+1} \subseteq \mathcal{G}$, the claim implies that $f \notin \mathrm{Pol}(\mathcal{G})$. ■

In particular, $\mathcal{B}$ is the universe of $\mathbf{G}_1$, so that $\mathcal{B} \in \mathcal{G}$, and then, any set of Gödel operations satisfies condition (3).

If $\mathbf{A} = ([0,1], F)$ is any algebra, over a signature σ, whose fundamental operations are Gödel operations,

$$F \subseteq [G],$$

we obtain the following dichotomy classification of the parameterized satisfiability problem.

THEOREM 13. *Let $\mathbf{A} = ([0,1], F)$ be such that $F \subseteq [G]$. Then,* $\mathrm{SAT}(\mathbf{A})$ *is NP-complete if*

$$(x \wedge \neg y)^{\mathbf{G}} \in [F] \text{ or } (\neg(x \rightarrow y))^{\mathbf{G}} \in [F]$$

and in P otherwise.

REMARK 14. *The statement implies that $[(x \wedge \neg y)^{\mathbf{G}}]$ and $[(\neg(x \rightarrow y))^{\mathbf{G}}]$ are incomparable in the lattice of clones on $[0,1]$.*

LEMMA 15. *Let $\mathbf{A} = ([0,1]_3, F)$ with $F \subseteq [G_3]$. Then,* $\mathrm{SAT}(\mathbf{A})$ *is NP-complete if*

$$\begin{array}{c|cccc} g_1 & 0 & 1/3 & 2/3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array} \in [F] \text{ or } \begin{array}{c|cccc} g_2 & 0 & 1/3 & 2/3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array} \in [F],$$

and in P otherwise.

REMARK 16. *The statement implies that* $[g_1]$ *and* $[g_2]$ *are incomparable in the lattice of clones on* $[0,1]_3$.

PROOF OF LEMMA 15. Since every $f \in F$ is a Gödel operation, in particular, f preserves $\mathcal{B}$, that is, $f|_{\{0,1\}}$ ranges over $\{0,1\}$, and then, $\mathbf{A}' = (\{0,1\}, \{f|_{\{0,1\}} \mid f \in F\})$ is an algebra, which we display on the same signature of $\mathbf{A}$, say σ. We claim that

$$\mathrm{SAT}(\mathbf{A}) = \mathrm{SAT}(\mathbf{A}'), \tag{5}$$

that is, the set of satisfiable terms on $\mathbf{A}$ is equal to the set of satisfiable terms on $\mathbf{A}'$. Indeed, let t be any term on σ. If $(a_1, \ldots, a_n) \in [0,1]_3^n$ is such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$, then pick any $(a'_1, \ldots, a'_n) \in \{0,1\}^n$ such that $a'_i = 0$ if $a_i = 0$ and $a'_i = 1$ otherwise. By direct inspection, $R = \{(0,0), (1/3,1), (2/3,1), (1,1)\}$ is a subuniverse of $\mathbf{G}_3^2$, so that $t^{\mathbf{A}}$ preserves R. Clearly, $(a_1, a'_1), \ldots, (a_n, a'_n) \in R$ therefore,

$$(t^{\mathbf{A}}(a_1, \ldots, a_n), t^{\mathbf{A}}(a'_1, \ldots, a'_n)) = (1, t^{\mathbf{A}}(a'_1, \ldots, a'_n)) \in R$$

by preservation, so $t^{\mathbf{A}}(a'_1, \ldots, a'_n) = 1$. Since the restriction of $t^{\mathbf{A}}$ to $\{0,1\}$ is equal to $t^{\mathbf{A}'}$, we have that $t^{\mathbf{A}'}(a'_1, \ldots, a'_n) = 1$. The converse is clear, and the claim is proved. We now prove the dichotomy.

For intractability, suppose $g_1 \in [F]$; the case $g_2 \in [F]$ is similar. As the operation g_1 is in the clone generated by F, and the operation b in Theorem 1 is the restriction of g_1 to $\{0,1\}$, it follows that b is in the clone generated by $\{f|_{\{0,1\}} \mid f \in F\}$. Then, by Lewis dichotomy, $\mathrm{SAT}(\mathbf{A}')$ is NP-complete, and by (5), $\mathrm{SAT}(\mathbf{A})$ is NP-complete.

For tractability, suppose that neither g_1 nor g_2 are in $[F]$. By direct computation, g_1 and g_2 are the only binary operations in $[G_3]$ whose restriction to $\{0,1\}$ equals the operation b in Theorem 1. Then, there is no binary operation in $[F] \subseteq [G_3]$ whose restriction to $\{0,1\}$ is b. Thus, by Lewis dichotomy, $\mathrm{SAT}(\mathbf{A}')$ is in P, and by (4), $\mathrm{SAT}(\mathbf{A})$ is in P.

The lemma is settled. ∎

PROOF OF THEOREM 13. Along the lines above, noticing that $[0,1]_3$ is a subuniverse of $[0,1]$, define $\mathbf{A}' = ([0,1], F')$ with $F' = \{f|_{[0,1]_3} \mid f \in F\}$. We show that $\mathrm{SAT}(\mathbf{A}) = \mathrm{SAT}(\mathbf{A}')$. If $\mathbf{a} = (a_1, \ldots, a_n) \in [0,1]^n$ is such that $t^{\mathbf{A}}(\mathbf{a}) = 1$, then pick $\mathbf{a}' = (a'_1, \ldots, a'_n) \in [0,1]_3^n$ such that $a'_i = a_i$ if $a_i = 0$ and $a'_i = 1$ otherwise. Now, $t^{\mathbf{A}} \in [G]$ implies that $t^{\mathbf{A}}$ preserves the subuniverse $R = \{(0,0), (a,1) \mid 0 < a\}$ of $\mathbf{G}^2$. We have $(a_1, a'_1), \ldots, (a_n, a'_n) \in R$ and then, by preservation, $(t^{\mathbf{A}}(\mathbf{a}), t^{\mathbf{A}}(\mathbf{a}')) = (1, t^{\mathbf{A}}(\mathbf{a}')) \in R$, therefore, $t^{\mathbf{A}}(\mathbf{a}') = 1$. But then, $t^{\mathbf{A}'}(\mathbf{a}') = 1$, since by definition $t^{\mathbf{A}'}$ is equal to the restriction of $t^{\mathbf{A}}$ to $[0,1]_3$. The converse is clear. We now prove the dichotomy.

For intractability, suppose that $(x \wedge \neg y)^{\mathbf{G}} \in [F]$. Since $g_1 \in [F']$, by Lemma 15, $\mathrm{SAT}(\mathbf{A}')$ is NP-complete, and then $\mathrm{SAT}(\mathbf{A})$ is NP-complete. The case $(\neg(x \to y))^{\mathbf{G}} \in [F]$ is similar. For tractability, if neither $(x \wedge \neg y)^{\mathbf{G}}$ nor $(\neg(x \to y))^{\mathbf{G}}$ is in $[F]$, then neither g_1 nor g_2 are in $[F']$, then by Lemma 15, $\mathrm{SAT}(\mathbf{A}')$ is in P. So, $\mathrm{SAT}(\mathbf{A})$ is in P. The theorem is settled. ■

We parallel the Kleene case and state an explicit characterization of tractable sets of Gödel operations, exploiting the relational presentation of the Gödel clone in Theorem 7. We insist that, as in the Kleene case, an operational presentation of the same characterization seems nontrivial (compare Example 19).

COROLLARY 17. *Let* $\mathbf{A} = ([0,1], F)$ *be such that* $F \subseteq [G]$. *Then,* $\mathrm{SAT}(\mathbf{A})$ *is in P if and only if* F *is contained in* $\mathrm{Pol}(\mathcal{R}_1, \mathcal{G})$, *or in* $\mathrm{Pol}(\mathcal{M}, \mathcal{G})$, *or in* $\mathrm{Pol}(\mathcal{D}, \mathcal{G})$, *or in* $\mathrm{Pol}(\mathcal{L}, \mathcal{G})$.

REMARK 18. *The statement implies that, given a (finite encoding of a) finite set* F *of operations in* $\mathbf{O}_{[0,1]}$, *it is decidable whether or not they induce a tractable Gödel satisfiability problem. In fact, it is sufficient to check finitely many preservation conditions, as along the lines of Theorem 7, if an* n*-ary operation* $f \in F$ *preserves* $\mathcal{G}_{n+1}$, *then it preserves* $\mathcal{G}$.

PROOF OF COROLLARY 17. Along the lines of Corollary 4. ■

EXAMPLE 19. *We know that* $L = [\{x \oplus y, \top\}]$. *However the clone of Gödel operations whose Boolean restrictions are linear is not generated by the Gödel operations whose Boolean restrictions are equal to* $x \oplus y$ *or to* $\top$, *as the following counting shows.*

There is one unary Gödel operation whose Boolean restriction is equal to $\top$, *and there are four binary Gödel operations whose Boolean restriction is equal to* $x \oplus y$. *Now, the clone of Gödel operations generated by these five operations contains only* 36 *binary Gödel operations; but there are* 136 *binary Gödel operations having a linear Boolean restriction.*

In view of the fuzzy propositional nature of the intended semantics of $\mathbf{G}$, the question about the complexity of the *graded* satisfiability problem over an algebra $\mathbf{A} = ([0,1], F)$ with $F \subseteq [G]$ is pertinent: Fix $0 < \epsilon \leq 1$. Given a term $t(x_1, \ldots, x_n)$, does there exist an assignment $(a_1, \ldots, a_n) \in$

$[0,1]^{\{x_1,\dots,x_n\}}$ such that $t(a_1,\dots,a_n) > \epsilon$ holds in $\mathbf{A}$? A direct preservation argument shows that this problem is equivalent (in complexity) to the satisfiability problem parameterized by Gödel operations. [7]

4. Conclusion

In this note, widening the dichotomy by Lewis on Boolean operations, we show that the satisfiability problem parameterized by certain sets of operations over the three-element set and over the unit interval, Kleene and Gödel operations, dichotomizes with respect to computational complexity. Interestingly, even in the absence of a description of the lattice of clones over the three-element set or the unit interval, the dichotomy presented is *effective* in that, if the parameterizing set of Kleene or Gödel operations is finite, it is possible to decide whether or not a given problem $\mathrm{SAT}(\mathbf{A})$ is tractable through a reduction to Post lattice.

An intriguing case study, which we leave open, is the characterization of the complexity of satisfiability parameterized by *deMorgan* operations or *m-valued Łukasiewicz* operations: here, the technique used in this note does not apply directly. We recall that the clone of deMorgan operations $[M] \subseteq \mathbf{O}_{\{0,1,2,3\}}$, is defined by

$$[M] = \mathrm{Pol}\begin{pmatrix} 0 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 3 & 1 & 2 & 2 & 2 & 0 & 1 \\ 0 & 2 & 3 & 1 & 0 & 1 & 3 & 3 & 3 \end{pmatrix} \eqqcolon \mathrm{Pol}(\mathcal{S}, \mathcal{M}');$$

this is the clone of term operations over the algebra $\mathbf{M} = (\{0,2,3,1\}, M)$ with $M = \{\wedge, \neg, \top\}$ where $\top = 1$, $\neg 0 = 1$, $\neg 2 = 2$, $\neg 3 = 3$, $\neg 1 = 0$, and $\wedge$ is the meet operation on the diamond $0 < 2 < 1$, $0 < 3 < 1$, $2 \parallel 3$. In turn, $\mathbf{M}$ generates the variety of bounded distributive lattices with an involution satisfying deMorgan equation [12]. For $m \geq 1$ integer, the clone of m-valued Łukasiewicz operations $[L_m] \subseteq \mathbf{O}_{\{0,1/m,2/m,\dots,1\}}$, defined by

$$[L_m] = \mathrm{Pol}(\{dk/m \mid 0 \leq k \leq m/d\})_{1 \leq d | m},$$

forms the universe of the free algebra in the variety of m-valued Łukasiewicz algebras, or, the variety generated by the algebra $\mathbf{L} = (\{0, 1/m, \dots, 1\}, L)$

[7] In fact, there exists $(a_1,\dots,a_n) \in [0,1]^n$ such that $t^{\mathbf{A}}(a_1,\dots,a_n) > \epsilon$ if and only if there exists $(a_1,\dots,a_n) \in [0,1]^n$ such that $t^{\mathbf{A}}(a_1,\dots,a_n) = 1$. Sufficiency is obvious. For necessity, notice that $R = \{(a,0),(b,1) \mid a \leq \epsilon < b\}$ is a subuniverse of $\mathbf{G}^2$. If $(a_1,\dots,a_n) \in [0,1]^n$ is such that $t^{\mathbf{A}}(a_1,\dots,a_n) > \epsilon$, then pick $(a'_1,\dots,a'_n) \in \{0,1\}^n$ such that $a'_i = 0$ if $a_i \leq \epsilon$ and $a'_i = 1$ if $\epsilon < a_i$, then $(a_1,a'_1),\dots,(a_n,a'_n) \in R$ and $t^{\mathbf{A}}(a_1,\dots,a_n) > \epsilon$ imply $t^{\mathbf{A}}(a'_1,\dots,a'_n) = 1$.

with $L = \{\odot, \to, \perp\}$ where $\perp = 0$, $x \odot y = \max\{0, x + y - 1\}$, $x \to y = \min\{1, y + 1 - x\}$.

We conclude reporting a partial result on deMorgan operations.

Proposition 20. *Let* $\mathbf{A} = (\{0, 2, 3, 1\}, F)$ *such that* $F \subseteq [M]$. *Then,* $\mathrm{SAT}(\mathbf{A})$ *is NP-complete if* $[m] \subseteq [F]$, *where* $m|_{\{0,1\}} = b$, *and is in P if* $F|_{\{0,1\}} \subseteq \mathrm{Pol}(\mathcal{R}_1), \mathrm{Pol}(\mathcal{D})$.

Proof. By direct computation there are nine binary deMorgan operations whose Boolean restriction equals the operation b in Theorem 1,

$$\begin{array}{c|cccc} m_1 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_2 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_3 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 \\ 1 & 1 & 2 & 3 & 0 \end{array},$$

$$\begin{array}{c|cccc} m_4 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 3 & 0 \\ 3 & 3 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_5 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_6 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array},$$

$$\begin{array}{c|cccc} m_7 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 2 \\ 3 & 3 & 0 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_8 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 3 & 2 \\ 3 & 3 & 2 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array}, \quad \begin{array}{c|cccc} m_9 & 0 & 2 & 3 & 1 \\ \hline 0 & 0 & 2 & 3 & 0 \\ 2 & 2 & 2 & 0 & 2 \\ 3 & 3 & 0 & 3 & 3 \\ 1 & 1 & 2 & 3 & 0 \end{array}.$$

In fact, m_7 is term definable over m_1, and m_3 is term definable over m_7; m_8 is definable over m_2, m_4 and m_9 are definable over m_8, m_5 and m_6 are definable over m_9; and no other pair of operations among $m_1, \ldots, m_8$ is related by term definability, in particular, m_1 does not define m_2 and viceversa. Notice that m_1 and m_2 preserve the relation $R = \{(0, 0), (1, 1), (0, 2), (0, 3), (1, 2), (1, 3)\}$, therefore $m_3, \ldots, m_9$ preserve R.

Since every $f \in F$ is a deMorgan operation, in particular, f preserves $\mathcal{S}$ and then $\mathcal{B}$, so that $\mathbf{A}' = (\{0, 1\}, \{f|_{\{0,1\}} \mid f \in F\})$ is an algebra on the same signature of $\mathbf{A}$, say σ. We claim that $\mathrm{SAT}(\mathbf{A}) = \mathrm{SAT}(\mathbf{A}')$. Indeed, let t be any term on σ. If $(a_1, \ldots, a_n) \in \{0, 2, 3, 1\}^n$ is such that $t^{\mathbf{A}}(a_1, \ldots, a_n) = 1$, then pick any $(a'_1, \ldots, a'_n) \in \{0, 1\}^n$ such that $a'_i = a_i$ if $a_i \in \{0, 1\}$, and $a'_i \in \{0, 1\}$ if $a_i \in \{2, 3\}$. Notice that $(a'_i, a_i) \in R$ for all $i = 1, \ldots, n$, therefore, $(t^{\mathbf{A}}(a'_1, \ldots, a'_n), t^{\mathbf{A}}(a_1, \ldots, a_n)) = (t^{\mathbf{A}}(a'_1, \ldots, a'_n), 1) \in R$, since any term operation on $\mathbf{A}$ preserves R. Then, $t^{\mathbf{A}}(a'_1, \ldots, a'_n) = 1$. Since the restriction of $t^{\mathbf{A}}$ to $\{0,1\}$ is equal to $t^{\mathbf{A}'}$, we have $t^{\mathbf{A}'}(a'_1, \ldots, a'_n) = 1$. The converse is easy.

For the hardness part, if $[m_i] \subseteq [F]$ for some $i = 1, \ldots, 9$, then $b \in [F|_{\{0,1\}}]$ and $\mathrm{SAT}(\mathbf{A}')$ is NP-complete by Theorem 1; by the claim, $\mathrm{SAT}(\mathbf{A})$ is NP-complete.

The tractability part is trivial (every instance is a "Yes" instance). ■

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Simone Bova
Department of Mathematics
Vanderbilt University
Nashville TN 37240, USA
simone.bova@vanderbilt.edu