Looking for the Right Thing at the Right Place: Phase Transition in an Agent Model with Heterogeneous Spatial Resources

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Received August 16, 2004; revised December 6, 2004; accepted December 6, 2004

We introduce a simple model describing an agent traveling within a medium containing resources randomly distributed in space. The agent, initially located at some origin, performs a single step toward a target of his choice that minimizes an economy function. We use an economy function that depends on the weight of the target and its distance from the origin. The cost and length statistics of the displacements are analyzed. We find a critical point for a particular resource weight distribution, such that the average cost vanishes (or becomes very small in practice). This problem can describe an economical agent looking for a firm with the best price or largest product choice, a tourist in search of an interesting and nearby city to visit, or an animal foraging in a complex landscape. *Complexity 10: 52–55, 2005*

Key Words: deterministic walks, phase transitions, agent models, foraging

1. INTRODUCTION

Although the theory of random walks [1], including random walks in disordered media [2], stands on firm foundations, the field of deterministic walks is much less understood [3]. Deterministic walk problems have many applications in physics as well as in some fields outside physics such as sociology, in animal behavior [4], and in economics, where, for example, the assumption that customers choose where to buy according to a convenience criterion [5] determines the location sites of stores in order to maximize their markets. The well-known Traveling Salesman Problem is a good illustration of the mathematical challenges brought by such problems. Even simpler models, involving walkers visiting fixed location points according to some local deterministic displacement rules (see, e.g., the "tourist problem" [6, 7]) become very difficult to analyze formally as soon as spatial disorder is present.

Here we introduce a simple problem that can be solved exactly and that provides interesting insights regarding the outcome of complex spatial distributions of resources on the trajectories of an agent. Consider a random distribution of localized, heterogeneous targets (representing resources, for instance) and consider an agent, located at some point, who wants to reach a target sufficiently large and not too far away. The question asked is: What is the most convenient target to choose? This problem can have many implications.

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DOI 10.1002/cplx.20068

For instance, the complex spatial distribution of cities and the wide variations in their size have shaped transportation networks and the movements of travelers in order to lower costs [8]. At an other scale, fruit-eating or herbivorous animals foraging in their environment are faced with similar situations [9]: maximizing food intake at least effort. Such problem is not simple: most ecological systems (for instance, tropical forests) contain resources heterogeneous in size and distributed spatially in a disordered way [10]. Similarly, a tourist visiting cities of varying interest would have to take similar decisions [6, 7]. Complex movement patterns can appear as a consequence of these distributions, but little is known about their properties.

We are interested in characterizing the statistics of displacements that arise from a minimal cost decision on such "landscapes" of randomly distributed resources, as well as the distribution of the cost that results from this decision. The problem studied in this article is a single step process and there resides its simplicity. However, the model exhibits nontrivial features that could persist in more complicated situations. In particular, we show that for realistic choices of the economy function upon which the decision is taken, if the weight (or size, or importance, etc.) of any resource is random and follows an inverse power-law probability distribution function, then the system can fall in a "critical" state. At this particular point, the statistics of the displacements have very large fluctuations and the mean cost of a displacement vanishes.

In the following section, we briefly present the model and a derivation of its solution. We then discuss the main results in Section 3.

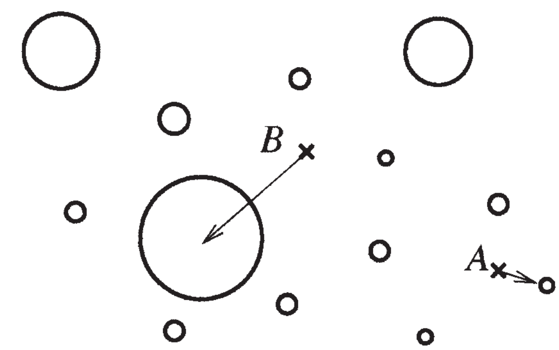
2. THE MODEL

In general terms, we consider an agent at the origin of an infinite d-dimensional system ($d = 2$ for practical purposes) in which targets are placed randomly with uniform number density ρ. Each target is point-like for simplicity, and offers a weight k. The weights k are independent identically distributed random variables with probability distribution $\phi(k)$. We assume that the agent has complete knowledge of the locations of each target as well as the values of k offered by them. Further, we assume that the agent decides which target to visit according to a criterion based on the optimization (minimizing) of an economy function (cost) $E(l_i, k_i)$. In general, $E(l_i, k_i)$ is a decreasing function of the value k_i offered by the ith target and a growing function of the distance l_i to that target. (see Figure 1). For instance, $E(l_i, k_i) = l_i / k_i$.

Under these conditions, we are interested in finding the probability that the agent performs a sojourn of length l to the optimal target. We denote this probability distribution as $P(l)$, and we denote by $q(C)$ the distribution of the cost of the optimal sojourn.

We begin by presenting the general formalism for calculating these quantities. We denote by C the minimum of $E(l_i, k_i)$ over the system, corresponding to a sojourn of length l^* toward a target of weight k^*. Inverting $C = E(l^*, k^*)$, we can write $k^* = F(l^*, C)$; similarly we can invert to obtain $l^* = G(C, k^*)$. This implies that for all other targets, $k_i < F(l_i, C)$. Given the assumptions of independence, the probability for this to be the case can be calculated by noting that every region dV around position $\boldsymbol{r}$ in the system must be either empty or occupied by a target with a value satisfying the above inequality. That is

FIGURE 1



Schematic representation of the model. An agent (A, B…) initially standing at some arbitrary location in a random and heterogeneous environment moves toward the nearest/biggest target.

$$Q(C) = \prod_{\bar{r}} [(1 - \rho dV) + \mathrm{Prob}(k < F(r, C))\rho dV]$$
$$= \exp\left\{-\Omega(d)\rho \int_0^\infty [1 - \mathrm{Prob}(k < F(r, C))] r^{d-1} dr\right\}, \quad (1)$$

where $\Omega(d)$ is the d-dimensional solid angle and $\mathrm{Prob}(k < F(r, C)) = \int_0^{F(r, C)} \phi(k)\, dk$. Thus, $Q(C)$ is the probability that all values of $E(l, k)$ in the system are larger than C. The distribution $q(C)$ of the minimum cost is $q(C) = -dQ/dC$.

Next we require the probability to find the optimum target at a distance $l^* = l$. Now, among all the targets with fixed value k, the one that represents the lowest cost is, of course, the nearest. Thus we first obtain the distribution of distances to the nearest target with value k, which we denote by $p_k(l)$. Since the density of targets with value k is merely $\rho\phi(k)$, the required probability is

$$P_k(l) = -\frac{d}{dl}\exp\{-\rho\phi(k)\Omega(d)l^d\}$$
$$= \rho\phi(k)\Omega(d)dl^{d-1}\exp\{-\rho\phi(k)\Omega(d)l^d\}. \quad (2)$$

From this expression, the distribution of the distance to the optimal target can be calculated directly as

$$P(l) = \sum_{k=1}^{\infty} P_k(l) \prod_{m \neq k} \left[\int_{G(E(l,k),m)}^{\infty} P_m(\lambda) d\lambda \right]$$

$$= \rho\Omega(d) d l^{d-1} \sum_{k=1}^{\infty} \phi(k)$$

$$\times \exp\left\{ -\rho\Omega(d) \sum_{m=0}^{\infty} \phi(m)[G(E(l, k), m)]^d \right\}. \quad (3)$$

We are now in position to analyze the statistical properties for specific cases. In what follows we specialize the above expressions to the case in which $\phi(k)$ is an inverse power-law distribution: the medium is composed of many small targets and fewer large ones broadly distributed. This is a realistic description of tree-size distributions in forests [10], for instance. To simplify, let us consider $\phi(k)$ concentrated on the positive integers ($k = 1, 2\ldots$):

$$\phi(k) = \frac{k^{-\alpha}}{\zeta(\alpha)} \sum_{n=1}^{\infty} \delta(k - n) \quad (4)$$

with α a resource exponent larger than 1. The normalization constant is the Riemann zeta function, $\zeta(\alpha) = \Sigma_{i=1}^{\infty} i^{-\alpha}$. This choice allows us to evaluate the effect on the resulting distributions of the convergence or divergence of the moments of $\phi(k)$. More importantly, we will restrict the economy function to the simple case $E(l, k) = l/k^{\gamma}$ with $\gamma > 0$. Then, if the optimal sojourn has $E(l, k) = C$, we have $k = F(l, C) = [l/C]^{1/\gamma}$ and $l = G(C, k) = Ck^{\gamma}$.

The first expression we need to evaluate is the integral appearing in $Q(C)$, namely:

$$\int_0^{\infty} [1 - \mathrm{Prob}(k < [r/C]^{1/\gamma})] r^{d-1} dr = \gamma C^d \int_0^{\infty} [1 - \mathrm{Prob}(k < \lambda)] \lambda^{\gamma d - 1} d\lambda$$

$$= \frac{C^d}{d} \frac{\zeta(\alpha - \gamma d)}{\zeta(\alpha)}. \quad (5)$$

which, substituted into Eq. (1) yields

$$Q(C) = \exp\left\{ -\frac{\Omega(d)\rho\zeta(\alpha - d\gamma)}{\zeta(\alpha) d} C^d \right\}. \quad (6)$$

We will return to a more thorough discussion of this form later. We now turn to the evaluation of $P(l)$ for this case. First we consider the awkward sum appearing in the exponential in Eq. (3):

$$\sum_{m=0}^{\infty} \phi(m)[G(E(l, k), m)]^d = \frac{1}{\zeta(\alpha)} \frac{l^d}{k^{d\gamma}} \sum_{m=0}^{\infty} m^{-(\alpha - d\gamma)}$$

$$= \frac{\zeta(\alpha - d\gamma)}{\zeta(\alpha)} \frac{l^d}{k^{d\gamma}}. \quad (7)$$

Thus, the distribution of step lengths is given by

$$P(l) = \rho \frac{\Omega(d) d l^{d-1}}{\zeta(\alpha)} \sum_{k=1}^{\infty} k^{-\alpha} \exp\left\{ -\rho\Omega(d) \frac{\zeta(\alpha - d\gamma)}{\zeta(\alpha)} \frac{l^d}{k^{d\gamma}} \right\}. \quad (8)$$

To get a better understanding of the behavior of this function, we focus on its large l behavior. In this limit we can convert the sum into an integral, which yields

$$P(l) \sim l^{-[(\alpha-1)/\gamma - d + 1]}$$

$$\times \frac{\rho d\Omega(d)}{\zeta(\alpha)} \int_0^{\infty} x^{-\alpha} \exp\left\{ -\rho\Omega(d) \frac{\zeta(\alpha - d\gamma)}{\zeta(\alpha)} x^{-d\gamma} \right\} dx. \quad (9)$$

The above continuous expression holds if the Riemann zeta functions are finite, i.e., if $\alpha > \gamma d + 1$.

3. DISCUSSION

Expression (6) shows that the distribution of the cost of the optimal move is always narrow (C does not fluctuate much if the agent starts from different origins), whereas the step length distribution (9) can decay slowly with l. Therefore, a same typical cost can be associated with markedly different travel lengths. An interesting phenomenon occurs when the resource exponent α reaches (from above) the critical value

$$\alpha_c = d\gamma + 1. \quad (10)$$

The function $\zeta(\alpha - d\gamma)$ in Eq. (6) diverges at that value. The cost distribution function for $\alpha < \alpha_c$ is therefore simply the delta-function, $q(C) = \delta(C)$ [recall that $Q(C) = \int_C^{\infty} dx\, q(x)$]. This situation is paradoxical at first sight: as C is a positive quantity by definition, how can the result be $C = 0$! The explanation can be formulated as follows. At low values of α ($<\alpha_c$), the density of large targets is relatively high and they strongly dominate the statistics. For instance, if one chooses $\gamma = 1$ and $d = 2$, then $\alpha_c = 3$ and the average $\langle k^2 \rangle = \infty$ below that transition value. Let us consider a finite system, with number density ρ and N targets. The largest target has a typical weight $k_{max} \sim N^{1/(1-\alpha)}$, that increases with N. If $\alpha < \alpha_c$, the result tells that the typical cost $C = l/k^{\gamma}$ goes to zero at large N (keeping ρ constant), meaning that, in average k^{γ} increases with N faster than l. Thus, for low α values, the sojourn of minimal cost can traverse a macroscopic fraction of the system, as ever bigger targets can be found at increasing distances. The situation is somehow similar to a condensation phenomenon in physics: here, the cost "condensates" toward the value zero. On the other hand, if $\alpha > \alpha_c$, large targets are sufficiently scarce so that the typical cost remains finite and independent of the system size for large systems.

The average cost $\langle C \rangle$ of a step is easily deduced from (6). For α close to α_c ($\alpha > \alpha_c$), one obtains

$$\langle C \rangle \sim (\alpha - \alpha_c)^{1/d}. \tag{11}$$

Considering $\langle C \rangle$ as the order parameter of the transition (α being the control parameter), one concludes that the transition is of second order, with an order parameter exponent equal to $1/d$.

We then analyze the length distribution $P(l)$ in the case $\alpha > \alpha_c$. In this range $P(l)$ is given by expression (9) and decays algebraically with l:

$$P(l) \sim l^{-\mu}. \tag{12}$$

As $\alpha \to \alpha_c$, one obtains $\mu \to 1^+$: at the critical resource distribution, the length of the steps suffers large fluctuations [right at α_c normalization of P in any finite system is ensured by the existence of a k_{max} and the associated exponential cutoff; see Eq. (8)]. Away from criticality, $P(l)$ decays faster than $1/l$ at large l, but its second moment $\langle l^2 \rangle$ still remains infinite in the interval $\alpha_c < \alpha < \alpha_1$, where $\alpha_1 = (d + 2)\gamma + 1$. In that interval, the distribution of l is thus asymptotically equivalent to a Lévy distribution. For values of the resource exponent larger than α_1 , the steps have a finite mean and finite fluctuations around it. In other words, the k's of the targets are seen as essentially homogeneous by the agent.

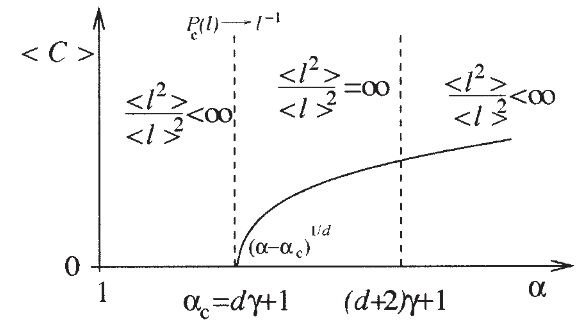
We have not analyzed in detail $P(l)$ for $\alpha < \alpha_c$. However, one can show from Eq. (8) that below criticality, the sojourn length distribution is characterized by a length ξ_0 that is of the order of the linear size of the system, $(N/\rho)^{1/d}$. Numerical simulations of the model (not shown) confirm this property. The existence of this length scale, although large, makes the fluctuation ratio $\langle l^2 \rangle / \langle l \rangle^2$ finite and independent of N, unlike the case $\alpha_c < \alpha < \alpha_1$.

Figure 2 summarizes the main results in a "phase diagram" as the resource distribution exponent α is varied.

4. CONCLUSION

We have introduced a simple model where an agent, located in a medium containing a complex spatial distribution of resources, chooses a target than minimizes an economy function and travels to it. Namely, the agent has to choose a target large enough and located not too far away. The model exhibits a nontrivial behavior: for a particular size distribution of resources, the lengths of the displacements have very large fluctuations and the typical cost of a move vanishes (if the system is infinite). This property is analogous to a critical phenomenon occurring at a phase transition. The critical resource distribution is an inverse power-law with a well-defined exponent: the number density of large targets must be high enough, but not too high. Too many large targets kill fluctuations as well as the scaling behavior of the step length distribution. Too few large targets have the same effects. This problem could have interesting applications in the study of animal foraging.

FIGURE 2



Mean cost of a displacement as a function of the resource exponent α. The corresponding statistics of the length l are also indicated.

ACKNOWLEDGMENTS

This work was supported by the Consejo Nacional de Ciencia y Tecnología (CONACYT, Mexico) Grant number 40867-F and by DGAPA-UNAM through grant IN100803.

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