Problem Solving as Theorizing: A New Model for School Mathematics

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The current paradigm in school mathematics is dominated by the drive to solve problems. Whether they come from a textbook, a teacher, or a standardized exam, the problems take the same form. Each problem is written in such a way as to evoke certain algorithms that the student is expected to have learned that can be used to get to a solution. Crafting such problems is an element of mathematics education methods courses. While in research and practice there is often talk of different approaches to problems and promoting creative thinking, there is one unambiguous answer to each problem in school math, which can be confirmed by the back of the book or by the exam key. As students progress through the grades, they are expected to be able to solve increasingly more problems, all the while developing the elusive skill called "problem-solving" that can be applied to thinking across different contexts.

I argue that the problem-solving paradigm in mathematics education engenders rigid and uncritical thinking habits that are ultimately harmful to democratic education. I propose turning to naturalist philosophies of mathematics to revisit the idea of the mathematics problem, to rethink what it is to think mathematically, and suggests that a more open understanding of the nature of math problems can found richer mathematics pedagogy. Following a short orientation with respect to pertinent traditions in education, I will make three movements. The first will document the emergence of the problem-solving skills paradigm in school mathematics. The second will contrast the vision of mathematics promoted by this current paradigm with the practice of pure mathematicians. Finally, I will argue that meaningful change in mathematics pedagogy, toward the kinds of teaching advocated by the democratic tradition, might find its roots in the professional practice of mathematics.

Two aspects of the problem-solving skills model of school math make it problematic for democratic education. The first is that it implicitly figures teaching as the distribution of these problem-solving skills, and learning as the acquisition of these skills. Within this paradigm the only salient questions about justice are about what constitutes fair distribution, and what fair distribution might mean for the deep diversity of our schools and students. As Robbie McClintock has shown, conceiving of education essentially as distribution and justice as fairness severely limits the possibilities for schooling.[1]

Social justice has an epistemic component as well.[2] This is to say there are aspects of social justice that are not reducible to the fair distribution of physical and cognitive resources. There is a long and varied humanist tradition of educational thinkers who argue that a functioning multicultural democracy requires an educated (though not necessarily schooled) citizenry that can transcend antiquated and oppressive social orders and institutions.[3] This tradition maintains that progressive democracy requires citizens who think well in order to disrupt persistent oppression. More recently, scholars of entrenched oppression have argued that reversing persistently unjust social arrangements requires thinking practices that disrupt traditional epistemology.[4] Teaching children that good thinking is merely a matter of efficiently and uncritically applying memorized processes cannot create this sort of engagement. This is the second problem with the model: the problem-solving skills paradigm reproduces knowledge habits and patterns that maintain unjust social arrangements.

The dangers of teaching by traditional methods and rote learning have been extolled for generations by humanist educators. Scholars and educators alike have warned that teaching students to think automatically, superficially, and uncritically is antithetical to the needs of deep democracy. These seem to be the unavoidable characteristics of mathematics though. After all, algebra doesn't change, it is said, and the object of math class is to learn the same arithmetic and algorithms that students have learned for decades. The fixed and unchanging nature of mathematics knowledge does not seem to allow for the constructivist and multicultural methods prominent in other disciplines.[5] Because mathematics knowledge is established as the paragon of objective truth, there is no space for student-created knowledge or disrupting established norms. Reform efforts that prioritize democratic education and social justice can be generally understood as either business as usual in disguise, or as something other than mathematics. The effect is that socially responsive mathematics pedagogy and rigorous mathematics pedagogy come to be conflicting ideals, when they should not be. Revisiting what it is to do mathematics provides a way out and a way forward toward a richer and more responsible pedagogy.

Problem-Solving Skills

The idea that mathematics can be conceptualized as a more general mental activity of solving problems resonates with the ideas informing the dominant political tides of around the turn of the twenty-first century. The 1983 *A Nation at Risk* report had set in motion a series of policy decisions and movements toward national standardization of policy and curriculum, and it had been influential in framing public education as a primarily economic endeavor.[6] The report linked the success of schools, particularly in mathematics and science, to the nation's position as a global superpower. It named a crisis of education, warning that if the schools continued to be ineffective, the workforce would suffer, rendering the United States unable to compete with emerging world powers. The report was most influential not in direct policy recommendations, but in framing the educational policy discourse around a singular problem. Whereas traditionally schools had a number of functions (including integrating recent immigrants, providing for social mobility, creating an informed and democratic populace), after the report, public discourse about schools was limited primarily to their economic functions and their ability to produce a skilled workforce.

Within mathematics education, there had long been tension between traditionalists and constructivists.[7] The former largely embrace a Platonic realist view of mathematics, advocating a return to basics and the need for students to learn an established canon of information. Constructivist reformers, for the most part, embrace constructivist versions of mathematics, and emphasize student-centered learning and growth over content knowledge.[8] Educators found a third approach in the work of George Polya,

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which allowed them to circumvent this conflict and avoid joining one side or the other.[9] He gave the option of a value-free version of mathematics — one dedicated not to knowledge of material but to processing power. This value-free approach fit well with the wider social calls for focusing exclusively on producing a technically literate workforce for an increasingly technical economy.

As a result, Polya's methods were widely embraced by the mathematics education community in the 1980s, one author stating that "for mathematics education and for the world of problem solving, it marked a line of demarcation between two eras, problem solving before and after Polya."[10] In 1980, the National Council of Teachers of Mathematics (NCTM) declared that problem solving should be the basic skill of mathematics, on which school mathematics should be focused.[11] Curriculum specialists took up this directive, publishing a host of studies and resources for teaching problem-solving skills.[12] In the 1990s, the NCTM went even further, placing *all* mathematics education in the context of problem solving. The most recent iteration comes with the 2010 publishing of the Common Core State Standards, which lay out eight "process strands," or norms of mathematical thinking.[13] The first is, explicitly, problem solving, and the remaining seven implicitly figure problem solving as the primary activity in question.[14]

The core commitments of the Problem-Solving Skills[15] paradigm are:

- A focus on "knowledge *how*" over "knowledge *that*."
- Themes and processes are broken down into measurable and articulable subprocesses.
- The unchanging nature of conceptual units — an indication of a good tool is that it does not need maintenance or upgrading.
- A focus on justifying and articulating steps to show mastery and understanding of the subprocesses.
- Learning happens by observing and modeling, with a focus on practice.
- Additive model of acquisition implies that it is the learner's task to accumulate increasingly more skills.
- Universal assessability, that is, there are transparent and universally accepted norms for executing procedures.
- Transferability and generalizability.

If we turn to the recent history of mathematics, we will see that this emerging paradigm of school mathematics does not resonate with the practice of the professional mathematician. Specifically, mathematicians do not understand problems to be pre-given, fixed, or transparent. Rather, they are opportunities to think differently about what one knows or feels certain. I will now proceed by first recounting the most widely publicized mathematical result of the twentieth century, and giving the standard interpretation, within the problem-solving paradigm. I will then rely on recent trends in philosophy of mathematics to problematize the standard interpretation and offer an alternative account of mathematics.

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Fermat's Last Theorem

Fermat's Last Theorem appears as one of many marginal notations in Pierre Fermat's translated copy of Diophantus' *Arithmetica*.[16] Following Fermat's death in 1665, his notes in this book were incorporated into a new translation, many of which were unproven theorems and conjectures, and most of which have been shown to be correct. Next to the Pythagorean Theorem, Fermat had made a note that (as expressed in modern notation) there are no such integers x, y, z, n such that $x^n + y^n = z^n$, when $n > 2$. That is to say, the Pythagorean Theorem has no analog for powers higher than two. Fermat famously wrote that he had "a truly marvelous demonstration of this proposition which this margin is too narrow to contain."[17] The Pythagorean Theorem has been proven in dozens of different ways and accepted since some of the earliest recorded mathematics. But while many of the most accomplished mathematicians from the centuries since this comment was published have tried to fashion a proof for it, and tens of thousands of dollars offered in prize money for finding a proof, it remained unproven until 1995. Howard Whitley Eves notes that Fermat's Last Theorem has the dubious distinction of being "the mathematical problem for which the greatest number of incorrect proofs have been published."[18] Fermat's Last Theorem captures the imagination because it seems so easy to understand. On its surface, it appears that it can be either proven or disproved with basic mathematics and cleverness. The British mathematician Andres Wiles became interested in the problem at age 10 for this very reason. It caught his attention for its seeming simplicity and he did not let it go until he became famous for working out a proof, years later, in 1995.[19] As Reuben Hersch and Vera John-Steiner note, the proof brought together "virtually all of the breakthroughs in the 20th century number theory and incorporated them in one mighty proof."[20] He created new techniques and combined the existing ones in innovative ways, opening up possibilities for approaching a wealth of other problems. Wiles first submitted the proof for peer review in 1994, but the reviewer found a fundamental error, sending Wiles back for revision. By 1995, the proof was complete and was brought to the public.

The standard account of problem solving is that there exist open questions in mathematics and mathematicians work steadily on answering them. Once solutions are found, the questions are no longer live, and the answers provide ground for the solutions to further problems. Mathematics is inherently individualized work, the community functioning only to safeguard against errors. Superficially, Wiles's trajectory fits into this narrative nicely. Fermat's Last Theorem was a problem that had existed for generations; Wiles took it up in solitude; and worked until he came across a solution to be shared with the wider community. Problem solving in school math is designed to follow this standard account — as students learn increasingly complex algorithms, they are given more difficult problems to practice on, with no intention of, or reason, to return to earlier problems.

This account, however, renders invisible a number of essential aspects of mathematics. Notably, the standard account misses the fallibility of proofs, the role of exploration and so-called "false steps," the role of invention, the possibility and necessity of changing the question, and the role of the mathematical community.

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These elements have been articulated by naturalist philosophers of mathematics including, Ludwig Wittgenstein, Imre Lakatos, and Philip Kitcher.[21] On the whole, mathematics is not the sort of unimpeachable, objective, unchanging knowledge it is made out to be.

The first point is that mathematical knowledge is not flawless. One of Wiles's errors was caught in 1994, sending him back to revise, but there is no guarantee that there are no other inconsistencies lying in wait, either in Wiles's own work or in the many theorems he invoked in crafting his proof. This complaint is often brushed off as a technicality, but the recorded history of mathematics has plenty of errors that have been overlooked for just this reason. A number of articles in the past several years have been dedicated to errors in mathematics,[22] and in the most extreme cases, errors have been identified in work that had gone unquestioned for centuries and even millennia. On the standard account, mistakes and inconsistencies are lamented as inevitable outcomes of human fallibility. The role of community and discussion is to guard against such mistakes and to aid in heightened vigilance. As Wiles makes clear, however, in reflecting on his process, the inconsistency was generative for him and ended up being integral to his final product.[23] Far from being an anomaly, Lakatos argues, the discursive process of working through inconsistencies is characteristic of mathematical practice.[24]

But not only is mathematical knowledge flawed at times, Kitcher shows that it also evolves.[25] Rather than growing by simple addition, new work in mathematics changes the concepts that were there before. Our predecessors generally agreed that there is no number that when multiplied by itself equals -1. That is to say in modern notation that $\sqrt{-1}$ does not exist or that there is no x such that $x^2 = -1$. Mathematicians today would disagree, showing that the unit i is such a number. It has been defined so that $i^2 = -1$. With the creation of i, the reference potential of the word "number" has been expanded and we now wish to say that there is no *real number* x such that $x^2 = -1$. We do not tend to say, though, that the ones who came before us were wrong, like we might say that they were wrong about the sun going around the Earth. We do not say that they held false beliefs because given the concepts available at the time, before the development of the complex numbers; there was no such number. As new concepts are introduced, the reference potential of existing terms is expanded and thus, though at times incrementally, their meanings are changed.

In the case of more complex proofs, at times new concepts are introduced, but their primary work is to connect existing ideas to one another. The proposition in question, when proven, is oriented in respect to an existing framework of accepted propositions. We are thus enabled to see it in a new light and have the chance to see things that might otherwise be obscured or go unnoticed. Wiles's proof of Fermat's Last Theorem, for example, rids us of the notion that it is a simple property of elementary mathematics. Even if we do not understand Wiles's proof, we have been shown that the problem is not what it seemed to be. Accepting the proof requires shifting the way we see it or a shift in our understanding of it. This is the crux of a transformational theory of mathematics — that mathematical change is characterized by transformation of existing concepts rather than simple addition.

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Questions change in different ways. Some established questions are simply answered — mathematicians craft an appropriately thorough and targeted response that fills the gap highlighted by the question. Questions can also cease to be meaningful. In cases where presuppositions of a question are shown to be false, or are incompatible with newer classifications, mathematicians simply leave questions behind. Most interestingly, as language evolves, new questions emerge. Commonly, as the reference potential of terms expand, new categories of questions are presented which are analogous to the old questions. With the development of complex numbers, for example, a whole host of questions presents itself, such as: In what ways do complex numbers behave like real numbers and how are they different? Can complex numbers have logarithms? Can they be factored?

In Wiles's description of his progress toward proving Fermat's Last Theorem, we can see this sort of mathematical change. The mathematician explains that he was initially inspired to take up Fermat's Last Theorem by a number of developments by his contemporaries that put a solution within reach. The first was the Shimura-Tanayama conjecture, which states that every elliptic curve is modular. This conjecture established a method of transformation between the two types of curves, enabling mathematicians to translate between the two structures. The second development was Gerhard Frey's contrapositive, which states that if there exists a solution to $x^n + y^n = z^n$, when $n > 2$, it would create an elliptic curve that is not modular, thereby disproving Shimura-Tanayama. Frey's proposal (for which he initially gave a plausibility argument, but was later proven) meant that if Shimura-Tanayama was correct, then so too was Fermat. The question of proving Fermat's Last Theorem was thus changed: the problem was no longer to prove the nonexistence of a solution to the equation, but to perfect the Shimura-Tanayama transformation method.

Another, subtler, mode of mathematical change figures prominently. Wiles describes one of his most important breakthroughs as essentially changing the question at hand. He was able to convert modular forms into a different representation (Galois representations) thereby making them easier to catalog and count. Redefining the question in this way ultimately allowed him to answer it.

A Different Mathematics Paradigm

In light of this developing picture of mathematics as discursive and transformational rather than inert and additive, it becomes clear that the problem-solving skills paradigm in school mathematics gets mathematics wrong. Solving mathematics problems entails changing and interpreting them. In math class, we spend most of our time answering perfectly stated questions. Terminology is clear, no information is extra, and none omitted. The work of solving school math problems is as much recognizing and responding to the intentions of the problem's author as it is performing calculations. There is always a right answer and a best way to do it, in advance of the student's work.

When we are limited to the process-oriented problem-solving skills paradigm, we miss:

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- The dispositional and habit aspects of thinking mathematically. We want students to consistently use number, pattern, and structure to interpret and engage with the world both inside and outside of school and a simple skills framework cannot do this.
- The place of judgment in mathematical thinking. The ability to judge appropriate criteria is context-dependent and relies on subject- and problem-specific details.
- Change and adaptation. Skills do not require adaptation or change, and in skill-learning we are not asked to reconsider what we have already mastered. Learning new skills is a process of simple accrual.
- Mistakes and exploration. The only thing valued by skills-learning is mastery. Mistakes are valued only insofar as they can reveal wrong thinking that can then be corrected.
- A commitment to the truth. Subject matter is seen as fodder for practicing skills, so we move between real situations, hypothetical situations, and hybrids thereof in order to set a course to be navigated.
- Imagination.

The problem-solving skills paradigm derives from the general ideals of humanist education. The intention is to conduct schooling in ways that support students' quality thinking. We want them to be able to leverage centuries of human thought and achievement to make sense of the world, for their own fulfillment and well being as well as for their vocation and the betterment of society. This impulse goes wrong when we reduce good thinking to a set of processes or mindless routines that can be universally applied to the world. The remedy is to refrain from making this conceptual leap from skilled thinking to the possibility of isolatable training on the sub-components of more complex processes.[26]

In order to do that, a different paradigm is needed; one that provides exemplars of skilled thinking, that establishes a terrain on which meaningful evaluations can be made, questions can be asked, and interpretations of data can be made. Mathematics itself can provide this paradigm, provided we interpret it differently. I propose that essential to an improved paradigm for mathematics education is a more nuanced and accurate understanding of the nature of mathematical problems and what it means to solve them. For all of the philosophers mentioned here, the mathematics problem has played a central role, and indeed, working problems is central to the practice of mathematics. There are different ways to understand the role of the problem — either as supporting the development of theory, or being supported by it. While the standard interpretation and the problem-solving skills paradigm assume the latter — that mathematical knowledge is developed for the sake of solving ever more complex problems — the history of mathematics reveals, however, that the former is more often the case. Mathematics problems function primarily as impetus to challenge what we know and to expose limitations and contradictions in what we take for granted.

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The mathematician is not in the business of constant forward motion, gradually adding on to his established knowledge, amassing an ever more powerful problem-solving arsenal. He is rather in an iterative process of seeking out quandaries or contradictions, and adjusting his prior understandings to alleviate them. The actual practice of theoretical mathematics exemplifies the thinking habits characteristic of good citizenship. This means that democratic mathematics pedagogy need not be one that is less rigorous or overlaid with competing social concerns. The pedagogy that enables responsible agency is the one that is the most mathematically authentic.

Pointing out that there are differences between school math and professional mathematics does not necessarily imply that the two should be brought closer together. There is a latent pedagogical question of whether students need a base of content knowledge before they can begin to do "real" mathematics, and it is beyond the scope here to explore particular materials of methods that might embody the ideals I promote. I am encouraged by developments in other disciplines, science especially, that have turned the focus of learning away from a rote learning of information and toward a practice based pedagogy. Mathematics has been late to the game, so to speak, and has not yet seen the same progress. Just as science education down to the earliest grades has become about learning and participating in the scientific process, rather than only learning about the work done by others, so too can mathematics education be reimagined to be about practicing genuine mathematics. Before that can happen though, we need to be clear about what the practice of mathematics really is.

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2. For a full discussion of epistemic injustice, see Miranda Fricker, *Epistemic Injustice: Power and the Ethics of Knowing* (Oxford: Oxford University Press, 2009).

3. For example, see John Dewey, *Democracy and Education: An Introduction to the Philosophy of Education* (New York: Free Press, 1997); Paulo Freire, *Pedagogy of the Oppressed*, 30th Anniversary ed. (New York, NY: Continuum, 2000); and Maxine Greene, *The Dialectic of Freedom* (New York: Teachers College Press, 1988).

4. For example, see George Yancy, *Black Bodies, White Gazes: The Continuing Significance of Race* (Lanham, MD: Rowman and Littlefield, 2008); Eduardo Bonilla-Silva, *Racism Without Racists: Color-Blind Racism and the Persistence of Racial Inequality in the United States* (New York: Rowman and Littlefield, 2003); Barbara Applebaum, *Being White, Being Good: White Complicity, White Moral Responsibility, and Social Justice Pedagogy* (Lanham, MD: Lexington Books, 2010); Cris Mayo, "Certain Privilege : Rethinking White Agency," in *Philosophy of Education 2004*, ed. Chris Higgins (Urbana, IL: Philosophy of Education Society, 2005), 308–16; and Sara Ahmed, "A Phenomenology of Whiteness," *Feminist Theory* 8, no. 2 (2007): 149–68.

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10. Alan H Schoenfeld, "Polya, Problem Solving, and Education," *Mathematics Magazine* 60, no. 5 (1987): 283–91.

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11. The Board of Directors, National Council of Teachers of Mathematics, "An Agenda for Action: Recommendations for School Mathematics of the 1980s," Reston, VA: The National Council of Teachers of Mathematics, (1980), http://www.nctm.org/standards/content.aspx?id=17278.

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13. Common Core State Standards Initiative, "Common Core State Standards for Mathematics," Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers (2010), http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf.

14. Ibid., 6–8, from the section "Standards for Mathematical Practice": "1. Make sense of problems and persevere in solving them.... 2. Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problem situations.... 5. Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem)."

15. This list is analogous to, and influenced by, Steven Johnson's criteria for framing critical thinking as a skill from Steven Johnson, "Teaching Thinking Skills," in *Teaching Thinking Skills*, 2nd ed., ed. Steven Johnson, Harvey Siegel, and Christopher Winch (New York: Continuum, 2010), 1–50.

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17. Singh, *Fermat's Enigma*, 62.

18. Eves, *An Introduction to the History of Mathematics*, 293.

19. Reuben Hersh and Vera John-Steiner, *Loving and Hating Mathematics: Challenging the Myths of Mathematical Life* (Princeton, NJ: Princeton University Press, 2011).

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23. Singh, *Fermat's Enigma*, 274–276.

24. Lakatos, *Proofs and Refutations*.

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