VARIETIES OF FINITISM

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§1 Introduction

Two of the central problems in the philosophy of mathematics are whether there are numbers (at all) and if there are numbers how many of them there are.

This essay does not deal with the first question. Here we are concerned with the problem of infinity. Infinity/the infinite is beset with (supposed) paradoxes of the infinitely large and the infinitely small. We will not deal with these paradoxes here, they can be dealt with (cf. Oppy 2006).

One may have *epistemological* problems with the infinite. These usually centre on the question how we can know about the properties of infinite structures or of objects (like numbers) infinitely away from our (epistemic) point of view. Such concerns may lead one either to change the *logic* one is working with when considering such an – supposedly, epistemically almost inaccessible – realm. So intuitionists gave away on *tertium non datur* in general (i.e. apart from constructively accessible structures). If one does not want to change logic one can restrict oneself to *parts* of standard set theory and mathematics by restricting oneself to their *recursive* parts (i.e. concern oneself only with those infinite subsets x of infinite sets, like ω, where the membership in x is recursive) thus working in *recursive mathematics* (cf. McCarty 1985). We are not concerned with such epistemological considerations here.

We are rather concerned with *ontological* scruples one may have about the infinite. Even if one otherwise is a staunch realist one may have serious doubts whether there *is* anything infinite. As far as we know the universe is finite, at least the number of elementary particles in it may well be finite, and whether one has to assume an infinity of space-time-points beyond the elementary particles may raise just more ontological scruples. The infinity of (non-relativistic) time – always running on, or even having always been running – might have been already a difficult idea to grasp, but a completed infinity (a collection of infinitely many entities) is even harder to grasp. And such doubts are aggravated for a nominalist or anyone denying the existence of abstract entities, since with them more option for infinities may seem available. The question of finitism has to be kept apart from that of nominalism. For a nominalist even one abstract entity is one too much. For a finitist nothing is gained if the infinities are not postulated in some abstract realm but as non-abstract entities (e.g. as a continuum of space-time points, these being of an ontological category *sui generis*). Finitists will find it easier, though, to be nominalists.

A mixture of epistemological and ontological concerns may be *methodological* problems. If one denies the existence of completed totalities, but sees the infinite as being always *in the making* (typically in time or in construction), where this process can never stop, one is forced to disallow (like Poincare) impredicative definitions over such totalities, since at different stages of the construction of the infinite such definitions may pick out different objects.

We are concerned on the one hand with problems that are linked with the traditional distinction between the potential and the actual infinite (cf. Moore 1993). On the other hand we are concerned with the question whether we need an infinity of natural numbers at all to have arithmetic (or even more advanced mathematical theories). Linked to these concerns are the topic of schematic representation and the idea of the indefinite, as to be distinguished from the infinite.

Some may accept countable infinities ω but reject uncountable infinities (like $\aleph_1$). This requires to change standard set theory, not with respect to the Axiom of Infinity, but, at least, with respect to both the Axiom of Powersets and the the Axiom of Replacement to block deriving the larger cardinalities. We are concerned here with blocking or avoiding even the countable infinity of ω.

§2 Potential and Actual Infinities

The idea of the potential infinite seems to spring from the idea of continuously or repeatedly doing the same thing – going on "for ever and ever". The paradigm example is counting: starting "1,2, 3, 4" we are easily led to "and so on (and on)". On the other hand is it not clear that our idea of continuing *indefinitely* with a process is the same as or implies the idea of continuing *infinitely* long. Even early Greek mathematics operated with the idea of having *unbounded* quantities. Again, however, working with the assumption of unboundedness may simply mean that we never hit on a limit and need not mean an infinite quantity. The idea of the potential infinite typically (e.g. by strict intuitionists like Brouwer or already in Aristotle) is set up as a contrast to the actual infinite: The infinite is considered as that which cannot be traversed (and thus cannot be collected having finished such a traversing).

The idea of the actual infinite seems to spring from the idea – championed by Cantor – that any such process as imagined as potentially infinite has to have a domain on which it works or which is created by it. If the former then one should easily talk about this infinite domain, thus proceeding by this *Domain Principle* from a process (or process talk) to a domain (or domain talk). If the latter, then this nevertheless invites *collecting the results of the infinite process*, especially so since the paradigm examples of potentially infinite progressions are built by applications of functions, which should take no time at all, being a mere matter of logic or mathematics. Traditional arguments (like those of Aristotle) claim the infinite to be untraversable, since they understand such a traversing as being a temporal process and time cannot be completed. This restriction does not apply to a logical process (i.e. an iterated application of a function). Frege argued that there is the singleton of every *x*, so that if there is one *x* (say the empty set, ∅) there are infinitely many sets. Thus there is *a totality* which *is* infinite. On the other hand does a collected infinity seem to be determinate (so to say be enclosed within the set) whereas the infinite should resist being thus restrained or conclusively collected – isn't this what its being unbounded – even by bounds of sets and membership – means? Once we have sets we seem to be able to unite or enlarge them. Thus the idea of collecting infinities provides room for the further idea of infinities of different size

In Naïve Set Theory an infinite set is guaranteed by an instance of Naïve Comprehension. Within standard set theory ZFC enough numbers are provided by the Axiom of Infinity, which explicitly states as an axiom the condition that would be used in Naïve Comprehension. One of its versions is:

> (Axiom of Infinity)
>
> There is a set such that ∅ is an element of that set, and for each element of that set the singleton of that element is in the set.
>
> $(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \supset \{y\} \in x))$

This formulation ensures that the proposed set is infinite, since the "process" of having another singleton in it never ends. In this image of putting singletons into this set the construction is looking ahead. The image of putting singletons into this set, of course, is an image inspired by the idea of potential infinity whereas the axiom itself is a paradigm of the *Domain Principle*.

It is explicitly stated that we assume to deal with infinite amounts. In combination with the Axiom of Powersets and Cantor's Theorem (therewith derivable) the infinites of ZFC become larger and larger in cardinality. Ordinal numbers of any rank are reached with the Axiom of Replacement. What is unlimited now in ZFC is the process (i.e. ontological hierarchy) of having ever larger infinities (cardinal numbers). The totality of all this cannot be a set in ZFC, but may be added in stronger set theories, which than have their own even larger totalities. [Cantor considered such ultimate totalities (like the collection of all ordinals) to be *inconsistent totalities*, too big to treated as a domain. Nevertheless he described them as *absolute infinities*.]

The logicists (especially Russell), who wanted to reduce mathematics and set theory to logic, had their problems with the Axiom of Infinity, since it clearly makes an *ontological* claim beyond the logical notions involved. Russell justifies the axiom as being necessary for real number theory, although *not* for arithmetic (cf. Russell 1919). Russell also explicitly sees the axiom as an insurance for never conceding "n = n + 1" [cf. §6 below].

Even though intuitionism criticizes using standard logical principles (like *tertium non datur*) or non-constructive axioms (like the Axiom of Choice) when dealing with infinities, standard intuitionistic set theory contains the standard form of the Axiom of Infinity, with the quantifiers being understood intuitionistically. Very large cardinalities are blocked, since Cantor's Theorem is not constructive, but countable infinities are present in the usual versions of intuitionistic or constructive mathematics. That infinities are considered as not being completed is mirrored in the invalidity of "$\neg(\forall x)F(x) \supset (\exists x)\neg F(x)$" in intuitionistic logic: even if the assumption that all *x* we have are *F* has to be rejected that does not mean that we can give some *specific x* such that this *x* having *F* can be rejected (this *x* may be "too far" away for our constructions).

That we need an Axiom of Infinity for a theory of natural numbers and standard arithmetic turns out, however, to be an artificiality of (systems like) ZFC. One can have standard arithmetic without infinity.

§3 Arithmetic with Virtuality instead of Infinity

One way to have standard arithmetic without infinity in an almost standard set theory is Quine's theory of *virtual classes*. [1] In *Set Theory and its Logic* (Quine 1963) Quine tries to set out the common core of different conceptions of sets, i.e. he tries to develop as much set theory as possible with as little axiomatic assumptions as possible before introducing the axioms that set, say, ZFC and NF or NBG apart. One of his main tools in this enterprise is his theory of virtual classes. "Virtual classes" are set expressions built by curly brackets and set abstraction (like: $\{x \mid x > y \wedge x \neq z\}$) that occur on the right hand side of "$\in$". These set expressions thus are used to built statements like: $w \in \{x \mid F(x)\}$. Since the language under consideration allows for statements like "$x \in y$" these set expressions function as singular terms syntactically on a par with variables that

[1] Quine speaks of "classes" but this is used synonymously to "sets". We speak of "sets" here and mean by "classes" so-called "proper classes".

can be interpreted as having some set as value. The crucial point about their *virtuality* is that they cannot be quantified over (in that position to the right of "∈"). Since Quine follows the methodological maxim that only those entities are admitted into a theory that are quantified over, these set expressions do not stand for or denote entities. They are short hand for statements in which conversion has occurred, i.e. "$w \in \{x \mid F(x)\}$" is short (depending on the length of "F", of course) for "F(w)". They are *virtual* also in the sense that some such expression might be quantified over later, so we do not know when we see such an expression whether it never *materializes* into a set later on. They *can* be quantified over indirectly in expressions like: $(\exists y)(y = \{x \mid F(x)\} \wedge y \in z)$. One can thus introduce existential commitments piecemeal.

Quine starts with a definition and an axiom for "=" ensuring extensionality of sets and a pair of weak axioms (providing the existence of ∅ and of pair sets, {x, y} for all x and y), which given the framework of virtual classes provide the finite sets (only).[2]

These finite sets, however, are sufficient for standard arithmetic! *Each* natural number can be constructed as a finite set, say the set of its predecessors (the predecessor relation being the converse of the usual successor relation). For some purposes of arithmetic we need to talk about the set of all natural numbers however. ZFC introduces the set of natural numbers for this purpose by the axiom of infinity. This need for infinity can be circumvented. The decisive idea is to use a virtual class instead of the Axiom of Infinity. The Axiom of Infinity uses the successor operation and so "looks forward" towards infinity. One may also use the converse of the successor operation and "look backwards" instead. We take ∅ as representing 0, as usual. The successor function is modelled by the function giving for any *x* the unit set {x}. Let us denote the predecessor function by "ϕ" and the closure of a function *f* with respect to a set *x* by "f^*x". Now we can define that some number *x* is smaller or equal than a number *y* by:

$$(\leq) \quad x \leq y \overset{\text{def}}{=} (\forall z)(y \in z \wedge \phi^*z \subseteq z \supset x \in z)$$

i.e. x is smaller than y if x is present in all sets which contain y and are closed under the predecessor function. We can now define ℕ by

(ℕ) ℕ is short for "$\{x \mid \varnothing \leq x\}$"

Nothing demands that ℕ is more than virtual! Note than the quantified in (≤) need only to range over finite sets.

[2] This framework contains First Order Logic (with a ι-operator) and the usual set theoretic constructions like unions and cuts. It contains the identification of objects with their unit sets! It also can express the *existence* of a set *x* by "$x \in V$" with *V* being the *universal set* $\{x \mid x = x\}$, which may itself be merely virtual, however! [*V* only contains existents, since the "x" left to "|" carries ontological commitment.] Proper classes are thus excluded from the theory. Existential formula are needed, since by virtuality not every singular term refers, and the usual quantification rules have to be restricted to existing objects.

The finite sets can be identified at this point as sets that contain some number as largest element and are closed with respect to the predecessor relation. A further axiom – a finite version of the Axiom of Replacement – is added:

(FR) The range of a function applied to a finite set *exists*.

This again yields only further finite sets. By this axiom mathematical induction can be derived as the scheme

(I) $F(\emptyset) \wedge (\forall x)(F(x) \supset F(\{x\})) \wedge y \in \mathbb{N} \supset F(y)$

Given the finite version of replacement, induction and the thus available notions of iteration and ancestral the well known arithmetical operations and (Peano) axioms for addition, multiplication and exponentiation can be derived (Quine 1963: §16).

Arithmetic can thus be done without infinity, it seems. No *explicit* commitment to infinity has to be introduced in the corresponding core set theory.

Quine's theory, however, gives way to ever larger infinites when the need for real numbers arises, *supposing* that there is a need for real numbers. Rational and real numbers are introduced as sets of sets of natural numbers. For these definitions to work (i.e. get beyond the empty set) one has to ensure that for arbitrary subsets of $\mathbb{N}$ their union *exists*, and this is an existential commitment to infinity. One *such* axiom of infinity then is: $(\forall x)(x \subset \mathbb{N} \supset x \in V)$.

Another problem is – even if not the idea of *mere* virtuality itself – the presupposed and non explicit meta-theory. The quantifier in (≤) has to range over the set of *all* finite sets, and this set, of course, is a non-finite set. The meta-theory laying down the truth conditions for the quantifiers in this set theory has to use an infinite domain.

§4 Finitism I: The Idea of the Indefinite

Finitism comes in at least three versions.

On the one hand one may try to develop a *formalism* that does not commit one to the actual infinite by some *axiom*.

Hilbert's finitism (Hilbert 1925) was not directed against the idea of the ever larger infinities of "Cantor's paradise", but was inspired by the idea of secure foundations for talk of infinity, where in these foundations (considered as meta-logic) the notion of the infinite was not to be presupposed. What is interesting even for the critic of infinities in Hilbert's finitism is his method of trying to work around the commitment to actual infinities (cf. George/Velleman 2002: 147-72; Shapiro 2000: 158-65).

Hilbert wants to justify set theory and Cantor's theory of transfinite numbers by using only *finitary arithmetic* in the meta-theory. Finitary arithmetic includes equations and their truth functional combinations. Also sentences with *bounded* quantifiers, like "($\forall$x<120)", are admissible. Any combination of such sentences is effectively decidable (by dealing with finitely many specific numbers and their properties). Now, to include some generality (i.e. to be able to make general statements like the commutativity of addition) Hilbert introduces *schematic* letters: *a, b….* One can thus express

(CA) $a + b = b + a$

Hilbert considers a statement like (CA) to be finitary! The idea is: Which ever specific numerals we choose to replace "a" and "b" the corresponding statement will be an acceptable (decidable) finitary statement.

Whereas in standard logic one typically reasons

(UG) For some arbitrary x: F(x)
…
G(x)
Thus: $(\forall x)(F(x) \supset G(x))$ since x was arbitrarily chosen.

the finitary reasoning is different. (UG)-like reasoning infers to *the totality* of the domain. Finitary reasoning rather argues:

(FG) The following proof *scheme* is valid *for any instance*:
F(a)
…
G(a)
Thus: $F(a) \supset G(a)$

Nothing is supposed about a totality of objects. It is rather provided a scheme to turn the assumption "F(a)" for any given or thought of individual term "a" into a proof of "G(a)". One might express this as the dialogical challenge "Once you name the object, I will provide the proof that it is well-behaved as well."

Employing the Wittgensteinian distinction between *saying* and *showing* one can understand the distinction between (UG) and (FG) as having (UG) *saying* what (FG) only *shows*, where, of course, Wittgenstein, who after 1929 took a position close to the use of schemata by Hilbert or Skolem (cf. Marion 1998), would add that what (FG) shows cannot be *said* at all, since there are *no* completed infinities.

Since no totality of objects to be quantified over is presupposed once need not assume that there is such a totality as the (infinite) set of natural numbers. One just claims that theorems can be proven for any number expression that someone comes up with. Since this is a general claim about forms/schemata of theorems – what else should one wish for in arithmetic?

Apart from the difficulties with Hilbert's larger program (like the incompleteness theorems, especially Gödel's Second Incompleteness Theorem) finitism in this sense has its own difficulties. What to think, for example, about the negation of schemata like (CA)? One might think they are equivalent to sentences or schemata with an *unbounded* existential quantifier. Hilbert thus sees them as "transfinite propositions", i.e. as illegitimate in finitism.[3]

Nevertheless one may take up the idea of schematic representation.

If the idea of schematic indefinite understanding was viable, it would have far reaching consequence well beyond the philosophy of mathematics or set theory. So, for example, one might think that a schematic presentation of defining truth within the standard semantic hierarchy could be given somewhat like:

(T) $[\text{"F(a)" is true-in-}L_n \equiv F(a)]_{n+1}$

where "F( )", "a" and – especially! – "n" are *schematic*, and the bracketed sentence being thought of as place at level L_{n+1}. One would so – supposedly – understand that truth can be defined for some (level of) language at the next higher meta-language (level). One might then go on and argue that we thus understand what it means to define semantic concepts by a semantic hierarchy by understanding schemata of this sort. The case for the inexpressibility of such hierarchies would seem to break down – one of the mayor arguments for dialetheism (cf. Priest 2006: 18-25).

Again, however, one wonders where to place such schemata. They cannot be somewhere within the hierarchy. First because then the usual regained antinomies reoccur. Second because by *being schematic* they cannot be on any non-schematic level at all. Where are they then? Do we have some additional faculty of schematic abstraction? That needed some explanation. And, in any case, such an explanation would not solve but aggravate the problem that once we were able to express what this faculty comes to and how it works we are placed again within the semantic hierarchy, thus facing the regained antinomies. The ineffability problem raises its head again.

Further on, leaving semantics to the side, one may ask in general what understanding a schema comes to. Is it not just to understand that some schematic representation is true/well-formed/valid for *all* its specifications? In understanding the schema we seem to have access to the *domain* of its instances

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[3] And even excluding negations of schemata like (CA) seems to leave one with *primitive recursive arithmetic* (cf. also Tait 1981). Remember that the *primitive* recursive function do not include μ-minimization. Primitive recursion comes down to bounded quantification. To have the usual means of logic available Hilbert allows these non-finite formula in, but considers them as „ideal" (i.e. devoid of respectable finitist content). Formulas are only considered in their inferential role. The formulas themselves can then be taken as the new (material) objects of this reasoning. Thus finitism gives way to *formalism*.

(respectively the domain that these instances are talking about). In this case we seem to have a strong intuition in favour of some domain principle.

§5 Finitism II: The Axioms of Zillions

The second version of Finitism takes up the idea of the indefinite, but also tries to combine it with the intuition that there is *some* domain the instances come from. The essential property of this domain, of course, has to be its finitude.

The formalism (cf. Lavine 1994: 267-308) proceeds by changing all quantifiers into bounded quantifiers. The bounds are given by large collections Ω_i which work like indefinite ordinals ("i" representing some rational number[4]). One calls them "zillions".

A formula is *regular* if it is bounded and whenever Ω_i occurs within the scope of Ω_j then j < i. (Domains reachable relative to some other indefinite domain have to be *larger*. They need *not* contain all the elements of the source domain.)

φ is a finitization of φ' if we get φ' by dropping all the bounds on the quantifiers in φ. "$\Omega_i(a)$" says that Ω_i is large enough to contain *a*. The axiom schemata of finitization then are:

(1) $\Omega_i(a)$

(2) $(\forall x_1 \dots x_n \in \Omega_i)\ \Omega_j(f(x_1 \dots x_n))$ for $i < j$

(3) $(\forall x \in \Omega_i)\ \Omega_j(x)$ for $i < j$

(4) $(\forall x \in \Omega_i)((\forall y \in \Omega_j)\varphi \equiv (\forall y \in \Omega_k)\varphi)$ for $i < j$, $i < k$, φ regular

To be added are relativized axiom schemata for identity (like: $(\forall x,y \in \Omega_j)(x = y \supset y = x)$). Axiom 2 says that if *f* is a function with n arguments available in Ω_i then the range of *f* is any set Ω_j indefinitely large with respect to Ω_i. Axiom 3 expresses the inclusion relation between extended indefinite domains. Axiom 4 says that the Ω are indiscernible: One can exchange any of them for a larger one without changing the truth values of formulas (cf. Mycielski 1986).

Having finitized a theory *T* in this fashion gives Fin(T). Fin(T) may have infinite models if *T* has, but it does also have finite models of indefinite size. This is so because we commit ourselves only to the instances of (bounded) formulas as we need them. If we make use of only a finite number of instances of the axiom schemata in some theory Fin(T), then we can have a model the universe of

[4] Rational numbers are chosen only for convenience: One uses a rational to insert some Ω between others, e.g. Ω_3, Ω_4, now we insert $\Omega_{7/2}$ for some indefinite realm between the two. A renumbering would do as well.

which is a *subset* of the universe of *T* and Fin(T) and *T agree* on the interpretation of the shared language (i.e. predicates, constants, functions) – provided that there is a model of *T*. Thus given a finitization of arithmetic PA, Fin(PA), we consider only as many numbers as we are interested in, and have then a finite model that agrees with PA on the interpretation of the arithmetic operations (for these numbers).

A finitization of the Axiom of Infinity yields the Axiom of Zillions:

(AZ) $(\exists x \in \Omega_0)(\emptyset \in x \wedge (\forall y \in \Omega_1)(y \in x \supset \{y\} \in x))$

This axiom says that there is come indefinitely large collection Ω_0 which contains a set *x* that contains the empty set, and from Ω_0 a larger indefinite collection Ω_1 can be reached such that if some of its objects are in the set *x*, then their singleton is in *x* as well. If infinite sets should be excluded one has to focus on a case where there is some number $n \in \Omega_1$ such that $\{n\} \in x$, but where this successor then is not in Ω_1 itself, $\{n\} \notin \Omega_1$. So *x* and Ω_1 can have some greatest member which breaks the progression to infinity. (Remember that Ω_1 need not – and in this case here must not – include all the members of Ω_0.)

The decisive result relating finitizations to their base theories is (Mycielski 1986):

(MT) If φ' is a regular relativization of φ: $\vdash_T \varphi$ iff $\vdash_{Fin(T)} \varphi'$

This immediately entails that *T* is *consistent* iff Fin(T) is.[5]

The finite versions of our theories thus deliver *in some sense* the same results as the original versions. The only difference is that every result carries a restriction rider on it.

One wonders how the Ω-construction is distinct from ω if it allows for a function that takes up the work of the successor function. Are the successors of Ω_0 at most in Ω_1? Do we not need then a new successor function for each Ω_i? It seems that either we have no general successor function or are simply back to ω or just schemata. In fact to have finite models one *has to* use the *rule* that one is only committed to the used instances of schemata. The procession of Ωs then may come to a halt. No closure conditions are forced on the arithmetic operations!

The main philosophical question about the theory of zillions may centre on the very idea of some indefinite boundary. The approach uses the idea of some boundary, but does not make the boundary *explicit*. This runs against any realistic intuitions concerning numbers or objects in general: Even if the domain

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5 This does not provide us with an easy road to the consistency of ZFC, since proving that *every* subset of Fin(ZFC) is consistent may be well beyond us by the Ωs being indefinitely large. One cannot simply check them – but cf. strict finitism below!

is indefinitely large it certainly has to *have* some size (cardinality). Even if we cannot count up to that size, we can introduce a name for it. Let us call it "#" And then we have to modify either our mathematics or our logic since the successor operation either cannot be applied to # or yields: # = # + 1.

Further on, the positive claim that we can have a finite model for some Fin(T) presupposes that there is a model of *T*. Then, however, the commitments to infinity that come with standard arithmetic and, say, ZFC are in no way avoided, at least not in the meta-theory.

If the Ωs mentioned in the Axiom of Zillions can be superseded by larger Ω_j there seems to be a clear sense in which the numbers in the set *x* cannot be *all* the numbers. And if we restart counting within some other Ω_j the union of the generated number sets will be a number set *y* such that there are *gaps* in the progression of numbers (otherwise *x* would be the only number set). This seems a strange consequence: We have then numbers without successor and numbers without predecessor within the same number set. This deviates clearly from our ordinary concept of number.

Even if there is some largest number, and the models of zillions allow for there to be a largest number, there can never be a theorem that explicitly states that there is a largest number or gives a name to it.

It is here that we have to turn to strict finitism.

§6 Finitism III: Strict Finitism in Arithmetic

The third version of finitism is *strict finitism*. Strict finitism really assumes that there is *no* infinity of natural numbers. This, of course, is a controversial and non-standard approach.

Strict finitism can be presented in a way similar to the finitude of zillions taking up Hilbert's idea of the indefinitely large. This version of strict finitism may proceed as follows, adding arithmetic axioms to First Order Logic:

The axioms of arithmetic do not imply infinity by themselves. One can keep the following axioms, with "s( )" being the successor function (cf. Mycielski 1981):

(A1) $\neg(\exists x)(s(x) = 0)$

(A3) $(\forall x)(x + 0 = 0)$

(A4) $(\forall x,y)(x + s(y) = s(x + y))$

(A5) $(\forall x)(x \bullet 0 = 0)$

(A6) $(\forall x,y)(x \bullet s(y) = (x \bullet y) + x)$

and the schema of induction

(IS) $F(0), (\forall x)(F(x) \supset F(s(x))) \vdash (\forall x)F(x)$

what has to be changed is the axiom of successor functionality:

(A2') $(\forall x,y)(x \neq s(x) \wedge y \neq s(y) \supset (s(x) = s(y) \supset x = y))$

These axioms allow for models in which there is some number # such that $(\forall x<\#)(x \neq s(x))$, because $(\forall x<\#)(s(x) = x + 1)$, and $s(\#) = \#$ (i.e. # being something like the largest number beyond which there is no further number). These axioms allow also for infinite models, however, just because elements of $\mathbb{N}$ make the antecedent of (A2') always true. A strict finitistic arithmetic is reached (cf. van Bendegem 2003, 2006) by adding the axiom:

(A7) $(\exists x)(x = \# \wedge s^{\#}s((0)) = \#)$

for the constant "#" and "$s^{\#}$" being the $\#^{th}$ application of the successor function (thus giving here: $s(\#) = \#$), and rewriting (A2') as:

(A2'') $(\forall x,y)(x \neq \# \wedge y \neq \# \supset (s(x) = s(y) \supset x = y))$

is the unique largest number in this system, as one can now prove: $(\forall x)(x = s(x) \supset x = \#)$. Quantifiers can be eliminated in favour of conjunctions or disjunctions up to having #-many sub-formulas.

By this the resulting system of finite arithmetic has remarkable properties:

(SFAT) *Strictly Finite Arithmetic Meta-Theorem*

The system (A1), (A2''), (A3)-(A7), (IS) is:

(i) decidable

(ii) categorical

(iii) consistent

(iv) deduction-complete

(v) Löwenheim-Skolem immune (i.e. theorems of this type do not apply)

Proof (Outline):

The resulting system is *decidable* by being finite: The sentences of the language (of arithmetic) have recursive truth conditions and in case of quantification only finitely many cases have to be considered. It is *categorical*, since on the one hand finite systems are not compact, and thus do not yield non-standard models for arithmetic (cf. Gurevich 1984), and on the other hand the system has to be categorical by having *only* models of size #, required by (A7). It is *consistent* (i.e. does not contain φ and $\neg\varphi$ for all φ of the language) by having finite *models* with domains of size #. And because quantifiers are eliminable and propositional reasoning is deduction-complete one derives all arithmetically valid φ of the language, thus the system is *deduction-complete*. By finitude and categoricity Löwenheim-Skolem theorems cannot apply. ■

Strict finite arithmetic thus has some of those admirable properties that the limitative theorems exclude for standard arithmetic. Further on, categoricity, which otherwise may be seen as the main reason to have second order systems in mathematics (cf. Shapiro 1991), is achieved within a first order language.

This version of strict finitism contains the idea of the indefinite in that the largest number # is kept indefinite. On the one hand this seems clear, since it is doubtful how we could ever come to recognize what the largest number is. On the other hand this makes using detachment on the assumption "$5 \neq \#$" looking like making use of some *indefinite formula* (i.e. a formula the truth of which seems to be indefinite itself for some numeral *n* large enough). One may rather bite the bullet and reason with some fixed but large enough[6] largest number like 2^{1000}. Or one may consider a *range* of systems each of which assumes some specific largest number [see below]. Further on it might be seen as an advantage if one need not change the axioms or arithmetics (in comparison to standard PA). Changing the underlying logic seems, of course, the stronger deviation than changing the axioms, but if one has already changed the underlying logic for some other reasons it may be an advantage to save standard arithmetic axioms.[7]

Strict finitism can also be presented containing the explicit denial of infinity or the explicit statement of the existence of a largest number *in combination* with the more or the less standard arithmetical axioms. This combination then is inconsistent and strict finitism of this kind has to be embedded within a paraconsistent logic. One arrives at *inconsistent mathematics*, here: inconsistent arithmetic.

Inconsistent arithmetics that are finite may have any finite size you like. They contain one largest number. Since we do not know which number really is the largest we may assume that one of these arithmetics is true, although we do not know which. Which one it is is not that important, since all these arithmetics have common properties:

Let *n* be some natural number, then let N_n be a set of arithmetic sentences. Let N be standard arithmetic, as usual (i.e. the set of all standard arithmetic truths). These sets N_n then have the following properties (cf. Priest 1994, 1997):

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6 Present physics is compatible with a discrete picture of the universe where the basic building blocks are space-time regions having the minimal size of Planck-length (10^{-35}m) and the minimal temporal duration of Planck-time (10^{-43}s), thus comprising $10^{-148}m^3s$. Given present theories the universe is roughly $10^{78}m^3$ large and 10^{18}s old. Thus the number of basic objects being 10^{244}, well below 2^{1000}. Of course nothing forbids choosing $2^{1000000}$ if needed.

7 So if one for some other reason has adopted a paraconsistent logic this step is (further) *vindicated* by finding another field of useful application of paraconsistency. Providing the logical framework for strict finitism thus strengthens the case for strong paraconsistency.

(IAT) *Inconsistent Arithmetic Meta-Theorem*

(i) $N \subset N_n$.

(ii) N_n is inconsistent.

(iii) $\varphi \in N_n$ for a (negated) equation φ concerning numbers < n *if and only if* $\varphi \in N$.

(iv) N_n is decidable.

(v) N_n is representable in N_n (thus we have a N_n truth predicate).

(vi) For the proof predicate B( ) of N_n every instance of B("φ") $\supset \varphi$ is in N_n.

(vii) If φ is not a theorem of N_n $\neg$B("φ")$\in N_n$.

(viii) For the Gödel sentence G for N_n $G \in N_n$ and $\neg G \in N_n$.

(ix) N_n is finitely axiomatizable.

An inconsistent arithmetic N_n thus has quite remarkable properties:

- by (i) we have that N_n is *negation-complete*, since N is.
- by (ii) and (viii) we have, of course, that it is *inconsistent*.
- by (iv) it has all the nice properties that N does not have, *although* N_n is complete!
- by (v) we can in the language of arithmetic define a *truth predicate* for that very same language.
- by (vi) N_n has an ordinary proof predicate.
- by (vii) in conjunction with (iii) we have not only that N_n is not trivial (by excluding some the equations that are excluded by N), but that this non-triviality can be established within N_n itself.

A key element is that *any* finite theory is decidable, (iv): just check the finitely many objects for having the property in question. How do we get (IAT)?

Proof (Outline):

• A theory with less numbers than N can have less counterexamples to a given arithmetic sentence. Thus it contains at most more sentences (as true). This holds in general (called *Collapsing Lemma*). Therefore (i). So we do not lose any of the power of N by switching to N_n.

• Since N is negation complete *adding* any sentence (as true) means adding a sentence for which the negation is already in N. Thus the resulting theory contains for at least one φ, φ and $\neg\varphi$. Thus (ii). This means that the logic of these arithmetic theories has to be a paraconsistent logic.

• Representability of truth is a consequence of (iv) and (i). The same holds for the representability of the proof predicate, (vi). Once the proof predicate is representable in the decidable theory N_n we can represent non-provability, and thus have (vii) and finally (viii). Whereas *N* is not axiomatizable at all, (ix) is another trivial consequence of finitude.

• (iii) is the most interesting property and results from the way the domain of a corresponding model is constructed [see below]. ■

The properties of inconsistent arithmetics by this even exceed those of strictly finite arithmetics which use First Order Logic. Since these arithmetics are (already) inconsistent neither *Tarski's Theorem* nor *Gödel's Second Incompleteness Theorem* have any force. For finite inconsistent theories truth and provability *cannot* come apart. That is the reason that the Gödel sentence behaves like a liar sentence in these theories. Provability can be as inconsistent as truth. Although checking whether some number codes a proof is primitive recursive it can be the case that an inconsistent arithmetic N_n claims that some number *is* and *is not* a number of a proof (of φ). The positive claim typically is provided by going through an effective computation of proof coding. The negative claim then is provided by *another* proof of the claim that the number is not the number of a proof of φ (typically because one can prove the generalisation that *no number* codes a proof of φ, for example in the case of the Gödel sentence *G*).

If one considers the systems N_n as a series one may consider standard arithmetic PA as the limit of this series (i.e. as generated by taking the transfinite union of all the systems of the series, as ZFC would have it). Once one takes this limit all the admirable properties stated in (FAT) and (IAT) are lost, and PA thus may be considered to be "quite unnatural" (van Bendegem 2003) compared to the series of finite systems.

A model of a theory N_n can be constructed as a filtering of an ordinary arithmetic model. In general one can reduce the cardinality of some domain by substituting for the objects equivalence classes given some equivalence relation (i.e. instead of objects o_1, o_2 ... we have $[o_1]$, $[o_2]$...). The equivalence classes provide then the substitute objects. Since the objects within the equivalence class are equivalent in the sense of interest in the given context the predicates still apply (now to the substitute object).

The trick in case of N_n is to chose the filtering which puts every number < n into its equivalence class, *and nothing else*; and puts *all* numbers ≥ n into n's equivalence class. The numerals are taken to refer to such equivalence classes.

An identity statement "m = n" is true iff $(\exists x,y)(x\in[m]\wedge y\in[n]\wedge x=y)$. For functions f: $f(m) = n$ iff $(\exists x,y)(x\in[m]\wedge y\in[n]\wedge f(x)=y)$. As a result of this for $x < n$ the standard equations are true (of [x]), while in case of $y \geq n$ *everything* that could be said of such a y is true of [n]. So we have immediately n = n (by identity) and n = n + 1 (since for y = n + 1 in *N* this is true). For x < n the usual functional values/equations obtain, for $y \geq n$ *everything* that could be reached by a function applied to $y \in [n]$ can be reached from [n]. With respect to x < n the standard arithmetical truths are preserved (so-called *Collapsing Lemma* as noted in (IAT) clause (iii)).

The *domain* of a theory N_n so is of cardinality n. The denotation of "n" now is *an inconsistent object* of N_n.

A simpler model, which avoids the usage of equivalence classes, may be given by having all numerals preceding "n" having their ordinary denotation and mapping all numerals starting with "n" to n. If for the moment we picture the successor function by arrows we can picture the structure of a model of N_n thus:

$$0 \rightarrow 1 \rightarrow \dots \rightarrow \underset{\circlearrowleft}{n}$$

Such models are called *heap models*.

The logic modelling N_n has to be paraconsistent. And it has to have restrictions on standard first order reasoning as well: identity elimination, substitution of identicals (=E), *cannot hold* for *n* of N_n if triviality is to be avoided, consider:

1. $n - n = n - n$	Theorem (for any number)
2. $n = n + 1$	Assuming n to be the largest number
3. $n - n = n - (n+1)$	(=E) 1, 2
4. $n - n = 0$	Theorem
5. $0 = n - (n+1)$	(=E) 4, 3
6. $0 = (n+1) - n$	Commutativity, 5
7. $((n+1)-n)\times(b-a)/((n+1)-n)=((n+1)-n)\times(b-a)/((n+1)-n)$	Theorem
8. $(0\times(b-a)/((n+1)-n) = ((n+1)-n)\times(b-a)/((n+1)-n)$	(=E) 7, 6
9. $0\times(b-a)/((n+1)-n) = 0$	Theorem
10. $0 = ((n+1)-n)\times(b-a)/((n+1)-n)$	(=E) 8, 9
11. $((n+1)-n)\times(b-a)/((n+1)-n) = b - a$	Theorem
12. $0 = b - a$	(=E) 10, 11
… 18. $a = b$	by some Theorems for "+", "–" ■

A logic with unrestricted (=E) so yields a=b for *any* numbers!

Mortensen (1995) chooses RM3# (an arithmetic based on the Relevant Logic RM3) as basic system and finitizes it by substituting for a number n the number n *modulo* some m. Thus the domain becomes {0, 1, 2, ... m-1}. The resulting arithmetic $RM3^m$ (where the "m" indicates the cardinality of the domain) is complete, non-trivial and decidable.

$RM3^m$ is *axiomatisable* by adding to RM3# the axioms:

$\vdash 0 = m$

and all instances of the following axiom scheme for $n \in \{0, 1, ... m-1\}$:

$\vdash (0 = n \leftrightarrow 0 = 1)$.

The approach "*modulo* some m" has *at least* the same deviant results than the heap models mentioned before: In $RM3^5$ we have $4 + 2 = 6$ (since $RM3^5$ is complete, i.e. has all theorems of *N*) and $4 \times 6 = 4$ (since "6" denotes 1). And the approach "*modulo* some m" has these deviant sentences for some *known* numbers! Heap models may, therefore, be preferred in the light of the recapture problem. In general it is a good idea to consider finite systems with a largest number way beyond any physical magnitude we might encounter (say 2^{1000}). One avoids entertaining too many abnormal equations by this.

Arithmetic is constructed thus as a finite theory. One can generalize the steps of this procedure to apply it to other mathematical theories.

Van Bendegem (1993) distinguishes the following steps:

1. Take any first-order theory *T* with finitely many predicates. Let *M* be a model of *T*.
2. Reformulate the semantics of *T* in a paraconsistent fashion (i.e. the mapping to truth values and overlapping extensions of P^+ and P^-).
3. If the models of *M* are infinite, define an equivalence relation *R* over the domain *D* of *M* such that *D*/*R* is finite.
4. The model *M/R* is a finite paraconsistent model of the given first-order theory *T* such that *validity* is at least preserved.

The restriction to theories with finitely many predicates is no real restriction in any field of applied mathematics or formal linguistics, since no physical device (be it human or machine) can store a non-enumerable list of basic predicates.

One may to express the strength of this paraconsistent procedure may be called the *Paraconsistent Downward Löwenheim/Skolem-Theorem*. The standard *Löwenheim/Skolem-Theorem* is one of the limitative or negative meta-theorems of standard arithmetic and First Order Logic. It says that any theory presented in First Order Logic has a *denumerable* model. This is strange, since there are first order representations not only of real number theory (the real numbers being *presented* there as uncountable), but of set theory itself. Thus the denumerable models are deviant models (usually Herbrand models of self-representation), but they cannot be excluded.[8] Given the general procedure to finitize an existing mathematical first order theory using paraconsistent semantics, there is a paraconsistent strengthened version of the *Löwenheim/Skolem-Theorem*:

(PDLST) Any mathematical theory presented in first order logic has a *finite* paraconsistent model.

Ideally strict finitism agrees with standard arithmetic as much as possible. Similarly to the problem of *classical recapture* in paraconsistent logics in general (i.e. the problem of keeping as much as possible of standard logic when reasoning about a consistent premise set) paraconsistent arithmetic should contain Peano Arithmetic. Thus property (iii) in the characterisation of the arithmetics N_n is crucial.

This advantage of combining the fruits of standard arithmetic and standard logic with an adherence to strict finitism can be strengthened when one uses an Adaptive Logic to embed strict finite arithmetic.

Adaptive logics (cf. Batens 2000) employ standard logic in consistent context and with respect to consistent objects and use a paraconsistent logic for the inconsistent cases. They are adaptive in that one proceeds on the assumption that one deals with a consistent case only on explicit information that the context is inconsistent some supposed consequences have to be retracted. Practically this works by adding to natural deduction style derivation a further column in which one notes the consistency or normality assumptions/presuppositions that have to be made when employing some critical rules of inference. For example, the paraconsistent logic LP makes – as do paraconsistent logics typically – Disjunctive Syllogism invalid; since LP, further on, uses the standard material conditional this means that *Modus Ponens* is not valid in general; but it is valid on the assumption that the antecedent φ of the conditional $\varphi \supset \phi$ used in an instance of *Modus Ponens* is a consistent statement. Thus noting the assumption $^\circ\varphi$ ("°" expressing the consistency of a formula) in the extra column of a derivation one can employ *Modus Ponens*, but once it turns out by the internal dynamics of drawing further consequences that φ was not consistent after all, the derived line and all lines dependent on it have to be retracted. We use such an adaptive version of LP here (often called "minimally inconsistent LP", cf. Priest 2006: 221-30). Standard or non-standard quantificational rules are added.[9] We have to deal however with the failure of substitution of identicals for inconsistent objects. Identity elimination, (=E), has to be restricted to consistent

8 They can be excluded in some second order semantics for second order set and number theory, see (Shapiro 1991).

9 Since paraconsistency is not a matter of quantificational theory one may add standard quantificational rules or one's preferred non-standard account (like Free Logic quantification).

objects. We define a *consistency predicate* "K( )" for objects (as a logical constant, of course) to do this:

(DK) $K(a) \overset{\text{def}}{=} \neg(\exists P)(P(a) \wedge \neg P(a))$

Since we do not use a second order system here, we may employ (DK) in that way that we note $\neg K(a)$ in some line of a derivation if for the object named "a" we could have a line with an instance of the schema: $P(a) \wedge \neg P(a)$. Identity Elimination then takes the form:

n.<m>	$P(a)$	...	Γ
o.<k>	$a = e$	...	Λ
p.<m,k>	$P(e)$	(=E) n,o	$\Gamma \cup \Lambda \cup \{K(e)\}$

where the column on the right takes down the sets of normality/consistency assumptions (or other presuppositions, cf. Bremer 2005: 224-36). The principal inconsistent object we are concerned with here is, of course, #. For # we have "$\# = \# + 1$" by (A7) and we have, if we use the standard axioms of arithmetic, "$\# \neq \# + 1$" as well. Thus using (=E) with # is not possible and thus back-propagating of "$\# = \# + 1$" to "$1 = 0$" is blocked (compare the use of (=E) in the proof of a generic "$a = b$" above).

We can now use the standard axiom of successor functionality:

(A2) $(\forall x,y)(s(x) = s(y) \supset x = y))$

To make the assumption of strict finitism *explicit* in our proof theory we assume (A7) with "#" denoting the unique largest number (say 2^{1000}).

As an example of a proof in this system (consisting of LPQ_m, the identity rules (=E) and $(\forall x)(x=x)$, (A7) and PA) we prove (with "n", "m", "l" denoting specific numbers):

(T1) $(\forall x,y,z)(x = y \supset x + z = y + z)$

1.<1>	$n = m$	Assumption	$\varnothing$
2.<>	$(\forall x)(x = x)$	(=I)	$\varnothing$
3.<>	$n + l = n + l$	($\forall$E), 2	$\varnothing$
4.<1>	$n + l = m + l$	(=E), 1, 3	$\{K(m)\}$
5.<>	$n = m \supset n + l = m + l$	($\supset$I), $\underline{1}$, 4	$\{K(m)\}$
6.<>	$(\forall x,y,z)(x = y \supset x + z = y + z)$	($\forall$I), 5	$\{K(m)\}$

The right column tells us that we derived this theorem on the assumption that "m" denotes a consistent object (i.e. does not denote #). So for all consistent numbers the theorem holds. If "m" turns out to denote an inconsistent object the whole proof (lines 3 – 5) is retracted. (We can prevent this by using the numerals "1", "2", "3" instead of "n", "m", "l".)

Vermeir (Vermeir 1999) uses another paraconsistent lower limit logic in his version of adaptive paraconsistent arithmetic. The logics CLuN and CLuNs are often used in adaptive approaches. CLuN adds a *non-compositional* negation to the positive part of standard propositional logic. CLuNs adds properties of negation (like dualities, DeMorgan-laws, double negation elimination) not available in CLuN. Only atomic negations (literals) are independent of positive sentences in CluNs. Since CLuNs contains a bottom particle and does not restrict (=E) one cannot have (A7) with standard (A2) without trivialization. Thus a non-standard axiomatization of arithmetic is needed. In these systems either one has to *prove* that some number is a standard number or inference rules like *Modus Tollens* are blocked *generally* in key axioms. The systems use two types of conditionals. For these reasons the adaptive system used above may be preferred.

The central – and obvious – problem with strict finitism of the inconsistent variety is the existence of an inconsistent object: the greatest number #. What an inconsistent object is supposed *to be* is quite not clear. If # is a physical object (like a structural universal or a trope) one wonders how a physical objects could ever be inconsistent (i.e. manage to have and not have a property). If numbers are seen as structures (of cardinality), which are where the heaps are they characterize (with respect to cardinality), it seems to be beyond imagination how a structure can the *thus* and *not thus* at the same time. If, on the other hand, numbers are considered to be abstract objects, it may be better conceived how they just may have inconsistent properties. But when one assumes abstract entities in the first place, why not then have as much of them as one wishes, namely infinitely many. Our inclination towards finitude is based on our conception of a – supposedly – finite universe. There is no reason, it seems, why an abstract realm should have limitations of size. Some (like Priest 2005), therefore, have proposed a theory of non-being (*noneism*) according to which numbers *neither* are *nor* are not – a theory well beyond the comprehension of many.

If the largest number is large enough one might avoid any problems with numbering any sentences that we can ever encounter (using some standard coding for Gödel numbers). Coding sentences with numbers *close* to the largest number seems to raise the problem that we run out of numbers to code such sentences.

A similar problem connected to the finitude of the set of numbers seems to be the consequences this should have for notions like countability and higher cardinalities provided by *Cantor's Theorem*. It seems that the powerset of the set of numbers up to # cannot be countable, since there are no numbers available to count it.

In fact *there are not* any numbers around that are used in this counting, but nevertheless the set can be paired with numerical expressions. The decisive shortcoming of these numerical expressions just is that most of them refer to #. Paraconsistent finitism keeps the standard theorems and thus the standard concepts like countability. Gödel numbering provides us with coding numerical expressions. Now, however, we often cannot take their standard realistic *reading* for granted (i.e. a reading functional with respect to denotation).[10] Even this problem can be circumvented if one works with a series of finite systems. Then somewhere in the series a system N_m follows N_n, able to functionally code the consistent subsystem and any part of the inconsistent subsystem of N_n we like.

These reservations point to the task to reconsider set theory as well once one starts with paraconsistency. Dialetheism has the option to use Naïve Comprehension with all its power (like making special axioms of infinity or choice superfluous) – and all the inconsistent objects that come with this. This may be left to another occasion.

These reservations may also lead one on to incorporate more *epistemological* considerations into strict finitism. So far strict finitism was presented as an ontological theory (i.e. concerning the *denotation* of arithmetical expressions). In this restriction to finitely many numbers we have made use of the problematic assumption of having enough *numerals* around. In fact if the universe is finite and one does not take numerals to be abstract entities there can be only finitely many numerals. If numerals are objects used to label other objects and labelling is functional not all objects *can be labelled* (cf. van Bendegem 1999). Given this, however, not everything can be counted, and it may well be beyond the given representational resources to Gödel number all of (finite) arithmetic within (finite) arithmetic. The corresponding positive powers of self-description and representability of truth and provability (as proclaimed in (IAT) as clauses (v) – (ix)) are lost, it seems. On the other hand nothing else should be expected in such a (naturalistic) theory of arithmetics. There are (simply by the finitude of human attention span, life span and brain size) limits to the length of human expressible numerals. In some sense (some) logical properties of numbers are thus *inexpressible*. This inexpressibility, however, is not one of the mysterious kind we find in Wittgenstein's *Tractatus* or in the logical inexpressibility of semantic hierarchies, but an inexpressibility that results itself from human finitude.

[10] Remember that numerals are just devices of convenience to substitute for expressions like: 0''', 0'''', 0''''''' – etc. A finitist thus has not to assume some pre-given infinite set of numerals (or variables).

§7 (*Strict*) *Finitism Beyond Arithmetic*

Looking beyond arithmetic might be considered necessary, since it is often claimed that the sciences are committed at least to analysis if not to complex numbers.

Mortensen (Mortensen 1995) set outs an inconsistent analysis, but it uses infinitely large hyperreals and infinitesimals. Mycielski's original aim (cf. Mycielski 1981) was to develop a finitistic version of analysis. This analysis makes use of indefinitely large numbers and indefinitely fine agreement between ratios or reals. It thus is not a strictly finitistic analysis, but a finitistic analysis following the approach of the axioms of zillions.

Just by the generality of the procedures of finitization one can generate some Fin(T) of zillions for any theory *T*. So one could have a finitized version of analysis as well. Limits, for example, may be modelled as indefinite extensions (cf. Lavine 1994: 279-84).

The same general remark applies to van Bendegem's procedure of paraconsistent finitization. The difference being that van Bendegem's procedure yields a *strictly* finitistic theory.

With respect to the rationals $\mathbb{Q}$ constructing them from the finitely many natural numbers yields a limited density. One may introduce rationals – as usual – as ordered pairs of natural numbers:

$$\mathbb{Q} = \{<n,m> \mid n \in \mathbb{N} \wedge m \in \mathbb{N} \wedge m \neq 0\}$$

the set $\mathbb{N}$ being available as the closure of the successor function (going into the loop at the largest number #). To identify 2/4 with 1/2 one may introduce an equivalence relation "∽" for elements *x* and *y* of $\mathbb{Q}$ by having $x = <x',x''> \backsim y = <y',y''>$ (with x', x'', y', y'' all being elements of $\mathbb{N}$) iff $x' \bullet y'' = y' \bullet x''$. The corresponding equivalence classes [x] for $x \in \mathbb{Q}$ then being the reduced rational numbers.

As a fact about multiplication in strict finite arithmetic we note:

(LM) $(\forall x,y)(\exists z)((\exists w)(w \leq \# \wedge x \bullet y = w \wedge z = w) \vee z = \#)$

Thus the smallest unit of rational discreteness (by this limit of multiplication) is 1/#. We can then define and postulate a *limited density* as a property of strictly finite rationals $\mathbb{Q}$:

(LD) $(\forall x,y)(y < x \wedge (\exists n \in \mathbb{N})(x - y \geq 2n \bullet 1/\#) \supset (\exists w)(x < w \wedge w < y))$

The standard proof of the density of the rationals does not go through in our adaptive system, since it uses (∀x,y)(∃z)(y•z > x), which is not true only (but also false) with respect to substituting # for "x".

Since the largest number is to be chosen as being beyond physical or psychological applicability the smallest ratio will be without any applicability and thus supposedly small enough. (Given 2^{1000} as # the smallest ratio 2^{-1000} is way below the Planck-length 10^{-35}, whatever the units may be.)

Finitism as here introduced has no use for real numbers, especially given the picture of a discrete universe. It is, of course, an option for a finitist – as for anyone else – to be a mathematical fictionalist about ℝ if this was needed.

§8 Conclusion

We have considered here several versions of finitism or conceptions that try to work around postulating sets of infinite size.

Restricting oneself to the so-called potential infinite seems to rest either on temporal readings of infinity (or infinite series) or on anti-realistic background assumptions. Both these motivations may be considered problematic.

Quine's virtual set theory points out where strong assumptions of infinity enter into number theory, but is implicitly committed to infinity anyway.

The approaches centring on the indefinitely large and the use of schemata would provide a work-around to circumvent usage of actual infinities if we had a clear understanding of how schemata work and where to draw the conceptual line between the indefinitely large and the infinite. Neither of this seems to be clear enough.

Versions of strict finitism in contrast provide a clear picture of a (realistic) finite number theory. One can recapture standard arithmetic without being committed to actual infinities. The major problem of them is their usage of a paraconsistent logic with an accompanying theory of inconsistent objects. If we are, however, already using a paraconsistent approach for other reasons (in semantics, epistemology or set theory), we get finitism for free. This strengthens the case for paraconsistency.

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