A note on the undecidability of the reachability problem for o-minimal dynamical systems

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In this paper we prove that the reachability problem is BSS undecidable for o-minimal dynamical systems.

1 Introduction

Nowadays transition systems are universally recognised as a mathematical model of real life systems. A major question in the study of transitions systems is to know whether a system can reach some set of final states (from an initial set of states). This question is known as the *reachability problem*. It is natural to classify transition systems into subclasses depending on whether the reachability problem is decidable or not for this subclass. A lot of work has already been done in this direction and some decidable subclasses have emerged and remain an active domain of research. Examples of such classes are finite automata [HU79], Petri nets [Kos82], and more recently timed automata [AD94].

A classical way to prove that the reachability problem is decidable for a subclass of "infinite" transition systems is to prove that there exists a finite *bisimulation* that can be effectively constructed for each system of this subclass[1]. This technique has been fruitfully used in [AD94, Hen95].

O-minimal structures [PS86, KPS86, PS88, vdD98] enjoy very nice finiteness properties. Regarding the above discussion it seemed interesting to define "o-minimal transition systems". This was done in [LPS00][2] where the author introduced the notion of *o-minimal hybrid systems*. These hybrid systems allow a rich continuous dynamics but have strong conditions on the discrete transitions implying that the study of o-minimal hybrid systems reduces to the study of *o-minimal dynamical system*. The main result of [LPS00] is the existence of *finite bisimulations* for o-minimal hybrid systems. This result was also proved in [Dav99] using a more topological approach to bisimulations.

In [BMRT04] (see also [BM05]) we extended the definition of *o-minimal dynamical systems* by defining the dynamics as a function (see Definition 2.5 in this paper) allowing "non-determinism" in the continuous behaviour. However in order to prove existence of *finite bisimulations* we needed to include extra assumptions on the determinism of the behaviour of the o-minimal dynamical systems. Moreover we investigated conditions under which the finite bisimulation is effective. In particular we noticed that the decidability of the existential theory of the structure $\mathcal{M}$ (in which the o-minimal dynamical system

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[1] Let us mention that there exist subclasses of transition systems with decidable reachability problem and no finite bisimulation, this is the case for Petri nets.
[2] Let us notice that [LPS00] appeared as a preprint in 1998.

is defined) is a crucial assumption to obtain decidability of the reachability problem and effectiveness of the bisimulation construction. Let us also mention that the *word encoding technique*, introduced in [BMRT04] has been used in order to compute the size of the finite bisimulation in the interesting case of *pfaffian hybrid systems* (see [KV04, KV05]).

The decidability of the reachability problem for general o-minimal dynamical systems *i.e.* "non-deterministic" ones, remained open in [BM05]. Although we had the suspicion it should be undecidable, we did not manage to prove it at that time. However we recently[3] realised that the Turing undecidability of this problem could easily be derivated from results in [AMP95], for three dimensional systems. In this paper we prove that this problem is already *Turing* undecidable for two dimensional systems. In addition, we prove its undecidability in the *BSS* model of computation.

The rest of the paper is organised as follows. In Section 2 we briefly recall essential definitions. Section 3 contains the proof of the Turing-undecidability of the reachability problem of o-minimal dynamical systems by reduction to the *halting problem* for *two-counter machines.* In Section 4 we prove that the reachability problem of o-minimal dynamical systems is BSS-undecidable using that the *Cantor set* is not BSS-recursive.

2 Preliminaries

In this section we just recall essential definitions in order to understand the undecidability proof. More details, explanations and examples can be found in [BMRT04, BM05, Bri].

2.1 O-minimality and definability

Let $\mathcal{M}$ be a structure. In this paper when we say that some subset, function is definable, we mean it is first-order definable (possibly with parameters) in the sense of the structure $\mathcal{M}$. A general reference for first-order logic is [Hod93]. All the notions related to o-minimality can be found in [PS86, KPS86, PS88], see also [vdD98] for a nice overview. We start with the definition of an o-minimal structure:

Definition 2.1 An extension of an ordered structure $\mathcal{M} = \langle M, <, ...\rangle$ is *o-minimal* if every definable subset of M is a finite union of points and open intervals (possibly unbounded).

In other words the definable subsets of M are the simplest possible: the ones which are definable with parameters in $\langle M, <\rangle$. This assumption implies that definable subsets of M^n (in the sense of $\mathcal{M}$) admit very nice structure theorems (like *Cell decomposition*). The following are examples of o-minimal structures.

Example 2.2 The field of reals $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$, the group of rationals $\langle \mathbb{Q}, <, +, 0\rangle$, the field of reals with exponential function, the field of reals expanded by restricted pfaffian functions and the exponential function.

2.2 Bisimulation and dynamical systems

Definition 2.3 A *transition system* $T = (Q, \Sigma, \rightarrow)$ consists of Q a set of states, Σ a finite alphabet of events, and $\rightarrow \subseteq Q \times \Sigma \times Q$ a transition relation.

A transition (q_1, a, q_2) belonging to $\rightarrow$ is denoted by $q_1 \xrightarrow{a} q_2$. A transition system is said to be finite if Q is finite. If the alphabet of events is reduced to a singleton, we will denote the transition system by $(Q, \rightarrow)$ and omit the event.

Definition 2.4 Given a transition system $T = (Q, \Sigma, \rightarrow)$, a *finite path in T* is a finite sequence of transitions $q_0\ q_1\ q_2\ \cdots\ q_n$ such that for all $i = 1, ..., n$ there exists $a_i \in \Sigma$ such that $q_{i-1} \xrightarrow{a_i} q_i$. We denote it as follows:

$$\rho = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots \xrightarrow{a_n} q_n.$$

Definition 2.5 A *dynamical system* is a pair $(\mathcal{M}, \gamma)$ where:

[3] This was in fact suggested by a referee when reviewing a previous version of this paper.

- $\mathcal{M} = \langle M, < \rangle$ is a totally ordered structure,
- $\gamma : M^{k_1} \times M \to M^{k_2}$ is a function definable in $\mathcal{M}$, for some fixed $k_1,\ k_2 \in \mathbb{N}$.

The function γ is called the *dynamics* of the dynamical system. More generally, we can consider the case where γ is defined on subsets of $\mathcal{M}$, that is $\gamma : V_1 \times V \to V_2$ with $V_1 \subseteq M^{k_1}$, $V \subseteq M$ and $V_2 \subseteq M^{k_2}$.

Classically, when M is the field of the reals, we see M as the time, $M^{k_1} \times M$ as the space-time, M^{k_2} as the (output) space and M^{k_1} as the input space. We keep this terminology in the more general context of a structure $\mathcal{M}$.

Definition 2.6 An *o-minimal dynamical system* is a dynamical system where $\mathcal{M} = \langle M, <, ... \rangle$ is an o-minimal structure.

Definition 2.7 Given $(\mathcal{M}, \gamma)$ a dynamical system, we define a *transition system* $T_\gamma = (Q, \to_\gamma)$ *associated with the dynamical system* by:

- the set Q of states is M^{k_2};
- the transition relation $y_1 \to_\gamma y_2$ is given by:

$$\exists x \in M^{k_1},\ \exists t_1, t_2 \in M,\ \big(t_1 \leqslant t_2 \text{ and } \gamma(x, t_1) = y_1 \text{ and } \gamma(x, t_2) = y_2\big)$$

Let us notice that the transition system T_γ allows some *non-determinism.* Indeed one can easily imagine dynamical systems where there exists $x_1,\ x_2 \in V_1$ and $t_1,\ t_2 \in M$ with $t_1 < t_2$ such that $\gamma(x_1, t_1) = \gamma(x_2, t_1)$ and $\gamma(x_1, t_2) \neq \gamma(x_2, t_2)$ (see [Bri, Example 2.22]). This is crucial in the proof of the undecidability results.

Definition 2.8 Reachability Problem Given $T = (Q, \Sigma, \to)$ a transition system, $Init \subseteq Q$ and $Fin \subseteq Q$ two subsets of states, is there a *finite path of transitions* ρ from $Init$ to Fin?

We say that *the reachability problem is Turing-decidable for a subclass* $\mathbf{C}$ *of transition systems* if there exists a Turing algorithm that decides whether there is a finite path in T from $Init$ to Fin, given any $T = (Q, \Sigma, \to)$ in $\mathbf{C}$, $Init \subseteq Q$ and $Fin \subseteq Q$. If such a Turing algorithm does not exists, we say that *the reachability problem is Turing-undecidable for the subclass* $\mathbf{C}$. The same kind of definition holds for BSS-undecidability.

3 Turing-undecidability result

In [AMP95] the authors have proved that the reachability problem is Turing-undecidable for *piecewise-constant derivative (PCD) systems.* The result holds for three dimensional PCD systems. One can easily be convinced that any PCD systems can be seen as o-minimal dynamical systems whose underlying o-minimal structure is the expansion of a group. In [AMP95] the authors also proved that the reachability problem is decidable for two dimensional PCD systems. We give here a short proof that the reachability problem is already undecidable for two-dimensional[5] o-minimal dynamical systems definable in $\langle \mathbb{Q}, \leqslant, +, 0, 1 \rangle$. This shows in some sense that o-minimal dynamical systems are more powerful than PCD systems.

Theorem 3.1 *The reachability problem is Turing-undecidable for (o-minimal) dynamical systems definable in* $\langle \mathbb{Q}, \leqslant, +, 0, 1 \rangle$.

Proof. Given a two-counter machine $\mathbf{M}$, we construct an o-minimal dynamical system $(\mathcal{M}, \gamma)$ that mimics $\mathbf{M}$. Moreover we define two subsets $Init \subseteq V_2$ and $Fin \subseteq V_2$ such that $\mathbf{M}$ halts (from the initial configuration) if and only if Fin is reachable from $Init$. It will follow that the reachability problem for o-minimal dynamical system is undecidable.

We here consider the classical model of two-counter machine (see [Min67]). A two-counter machine consists in a finite list of instructions and two counters denoted by c_1 and c_2. The different types of labeled instructions are given in Table 1. We assume the finite list of instructions of $\mathbf{M}$ is labeled by

zero test	k : if $c_i = 0$ then goto k' else goto k''
increment	k : $c_i := c_i + 1$ (goto $k+1$)
decrement	k : $c_i := c_i - 1$ (goto $k+1$)
stop	k : STOP

Table 1 The possible instructions of a two-counter machine.

the set $\{0, ..., s\} \subseteq \mathbb{N}$. A *configuration* of the machine $\mathbf{M}$ is given by a triple (k, c_1, c_2) which represents the (label of the) current instruction of $\mathbf{M}$ and two counter values, thus $(k, c_1, c_2) \in \{0, 1, ..., s\} \times \mathbb{N}^2$. Without loss of generality we can make the following assumptions about the two-counter machines $\mathbf{M}$. The first instruction of $\mathbf{M}$ is labeled by 0 and the *stop instruction* is labeled by s. The *initial configuration* of $\mathbf{M}$ is $(0, 0, 0)$. We also assume that there is a zero test before each *decrementation instruction* such that the counter value is not modified each time it is equal to zero.

We now define an o-minimal dynamical system $(\mathcal{M}, \gamma)$ that simulates $\mathbf{M}$, where $\mathcal{M} = \langle \mathbb{Q}, \leqslant, +, 0, 1 \rangle$ is an ordered group. We define $\gamma : V_1 \times [0, 1] \to V_2$ where $V_1 = V_2 = [0, s+1[^2$. The configuration (k, c_1, c_2) of the machine $\mathbf{M}$ will be encoded by the couple of V_2 given by

$$\left(k + \frac{1}{2^{c_1+1}}, k + \frac{1}{2^{c_2+1}} \right).$$

Let us now define the two subsets $Init$ and Fin: $Init = \{(0.5, 0.5)\}$ and $Fin = [s, s+1[^2$. The idea is that $Init$ corresponds to the initial configuration of $\mathbf{M}$ and Fin corresponds to the configurations of $\mathbf{M}$ labeled with the *stop instruction*.

We now need to explain how to encode the different instructions of $\mathbf{M}$ through the definition of a dynamics γ. We will define $\gamma_k : [k, k+1[\to V_2$ for $k = 0, ..., s$. Let us start with the *zero test instruction* for counter c_1. The encoding for counter c_2 is similar. When k is a *zero test instruction* for counter c_1, we have that

$$\gamma_k(x_1, x_2, t) = \begin{cases} (x_1, x_2) & \text{if } t = 0 \\ \big(x_1 + (k' - k), x_2 + (k' - k)\big) & \text{if } t > 0 \ \wedge \ x_1 = k + 0.5 \\ \big(x_1 + (k'' - k), x_2 + (k'' - k)\big) & \text{if } t > 0 \ \wedge \ x_1 \neq k + 0.5 \end{cases}$$

Let us recall that since $\langle \mathbb{Q}, \leqslant, +, 0 \rangle$ is a divisible torsion-free ordered group[6], for any $x \in \mathbb{Q}$ there exists a unique $y \in \mathbb{Q}$ satisfying $y + y = x$. We denote this y by $\frac{x}{2}$. We now explain the encoding for an instruction of incrementation of c_1. When k is an instruction for incrementing c_1 we define

$$\gamma_k(x_1, x_2, t) = \begin{cases} (x_1, x_2) & \text{if } t = 0 \\ \big(\frac{x_1 - k}{2} + (k+1), x_2 + 1\big) & \text{if } t > 0 \end{cases}$$

We now explain the encoding for an instruction of decrementation of c_1. When k is an instruction for decrementing c_1, we define

$$\gamma_k(x_1, x_2, t) = \begin{cases} (x_1, x_2) & \text{if } t = 0 \\ \big(2(x_1 - k) + (k+1), x_2 + 1\big) & \text{if } t > 0 \end{cases}$$

The same kind of constructions is used in order to increment and decrement c_2. From the above encoding, one can easily be convinced that Fin is reachable from $Init$ if and only if $\mathbf{M}$ halts. □

Corollary 3.2 *The reachability problem is Turing-undecidable for o-minimal dynamical systems.*

Remark 3.3 Let us notice that the encoding of the proof uses two dimensional dynamical systems. The decidability of the reachability problem thus remains open for one dimensional systems. One could also wonder what happens if the language only consists of $\{\leqslant\}$.

[5] In the case of o-minimal dynamical systems, the dimension of the system is the o-minimal dimension of the (output) space V_2.

[6] Let us notice that this is the case of any o-minimal group, see [PS86].

4 BSS-undecidability result

We do not recall the basis of BSS complexity but refer to [BSS89]. Through this section we denote by $\mathcal{C}$ the fractal known as the *Cantor set*. From Proposition 1 of [BSS89] one can easily deduce the following result.

Lemma 4.1 *The complement of the Cantor set is BSS recursively enumerable but it is not BSS recursive.*

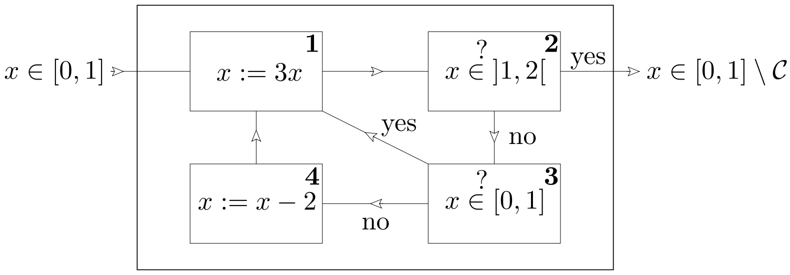

Fig. 1 A BSS machine that halts on the complement of the Cantor set.

Theorem 4.2 *The reachability problem is BSS-undecidable for (o-minimal) dynamical systems definable in* $\langle \mathbb{R}, \leqslant, +, \cdot, 0, 1\rangle$.

P r o o f. Let us consider the o-minimal dynamical system $(\mathcal{M}, \gamma)$ where $\mathcal{M} = \langle \mathbb{R}, \leqslant, +, \cdot, 0, 1\rangle$ and $\gamma : [0,1] \times \{1,2,3,4,5\} \times M \to M \times \{1,2,3,4,5\}$. Given $k \in \{1, ..., 5\}$ we denote by γ_k the function $\gamma(\cdot, k, \cdot) : [0,1] \times \mathbb{R} \to \mathbb{R}$. In order to define γ we will define γ_k for $k = 1, ..., 5$.

$$\gamma_1(x,t) = \begin{cases} (x,1) & \text{if } t = 0 \\ (3x,2) & \text{if } t > 0 \end{cases} \quad ; \quad \gamma_2(x,t) = \begin{cases} (x,2) & \text{if } t = 0 \\ (x,5) & \text{if } t > 0 \ \wedge \ x \in]1,2[\\ (x,3) & \text{if } t > 0 \ \wedge \ x \notin]1,2[\end{cases}$$

$$\gamma_3(x,t) = \begin{cases} (x,3) & \text{if } t = 0 \\ (x,1) & \text{if } t > 0 \ \wedge \ x \in [0,1] \\ (x,4) & \text{if } t > 0 \ \wedge \ x \notin [0,1] \end{cases} \quad ; \quad \gamma_4(x,t) = \begin{cases} (x,4) & \text{if } t = 0 \\ (x-2,1) & \text{if } t > 0 \end{cases}$$

and $\gamma_5(x,t) = (x,5)$.

Due to the definition of γ we have that $(\mathcal{M}, \gamma)$ mimics the BSS machine of Figure 1. Let Fin be the set $\{(y,5) \mid y \in \mathbb{R}\} \subseteq V_2$. Let us suppose, for a contradiction, that given any $(x,1) \in \mathbb{R}^2$ we can decide if we can reach Fin from $(x,1)$. Hence we could decide if a given point of $[0,1]$ belongs to the complement of the Cantor Set, contradicting Lemma 4.1. □

Corollary 4.3 *The reachability problem is BSS-undecidable for o-minimal dynamical systems.*

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References

[AD94] R. Alur and D.L. Dill, *A theory of timed automata*, Theoretical Computer Science **126** (1994), no. 2, 183–235.

[AMP95] Eugene Asarin, Oded Maler, and Amir Pnueli, *Reachability analysis of dynamical systems having piecewise-constant derivatives*, Theoret. Comput. Sci. **138** (1995), no. 1, 35–65, Special issue on hybrid systems.

[BM05] T. Brihaye and C. Michaux, *On the expressiveness and decidability of o-minimal hybrid systems*, Journal of complexity **21** (2005), no. 4, 447–478.

[BMRT04] T. Brihaye, C. Michaux, C. Rivière, and C. Troestler, *On o-minimal hybrid systems*, HSCC'2004 (Rajeev Alur and George J. Pappas, eds.), Lecture Notes in Computer Science, vol. 2993, Springer, 2004, pp. 219–233.

[Bri] T. Brihaye, *Words and bisimulation of dynamical systems*, to appear in the Journal of Automata, Languages and Combinatorics (23 pages).

[BSS89] L. Blum, M. Shub, and S. Smale, *On a theory of computation and complexity over the real numbers: NP- completeness, recursive functions and universal machines.*, Bull. Am. Math. Soc., New Ser. **21** (1989), no. 1, 1–46.

[Dav99] J.M. Davoren, *Topologies, continuity and bisimulations*, Theor. Inform. Appl. **33** (1999), no. 4-5, 357–381, Fixed points in computer science (Brno, 1998).

[Hen95] T.A. Henzinger, *Hybrid automata with finite bisimulations*, ICALP'95: Automata, Languages, and Programming, Lecture Notes in Computer Science, vol. 944, Springer-Verlag, 1995, pp. 324–335.

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[HU79] John E. Hopcroft and Jeffrey D. Ullman, *Introduction to automata theory, languages, and computation*, Addison-Wesley Publishing Co., Reading, Mass., 1979, Addison-Wesley Series in Computer Science.

[Kos82] S. Rao Kosaraju, *Decidability of reachability in vector addition systems (preliminary version)*, STOC'82: Proceedings of the fourteenth annual ACM symposium on Theory of computing (New York, NY, USA), ACM Press, 1982, pp. 267–281.

[KPS86] Julia F. Knight, Anand Pillay, and Charles Steinhorn, *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 593–605.

[KV04] M. Korovina and N. Vorobjov, *Pfaffian hybrid systems*, CSL 2004 (Jerzy Marcinkowski and Andrzej Tarlecki, eds.), Lecture Notes in Computer Science, vol. 3210, Springer-Verlag, 2004, pp. 430–441.

[KV05] ———, *Upper and lower bounds on sizes of finite bisimulations of pfaffian hybrid systems*, preprint 2005.

[LPS00] G. Lafferriere, G. J. Pappas, and S. Sastry, *O-minimal hybrid systems*, Math. Control Signals Systems **13** (2000), no. 1, 1–21.

[Min67] Marvin L. Minsky, *Computation: finite and infinite machines*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1967, Prentice-Hall Series in Automatic Computation.

[PS86] Anand Pillay and Charles Steinhorn, *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 565–592.

[PS88] ———, *Definable sets in ordered structures. III*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 469–476.

[vdD98] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.