Dynamical *vs.* variational symmetries: understanding Noether's first theorem

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Abstract

It is argued that awareness of the distinction between dynamical and variational symmetries is crucial to understanding the significance of Noether's 1918 work. Special attention is paid, by way of a number of striking examples, to Noether's first theorem which establishes a correlation between dynamical symmetries and conservation principles.

1 Introduction

Emmy Noether's continuing fame in physics is attributable to a paper she published in 1918 on a problem in the calculus of variations,[1] and in particular to the systematic treatment therein of the connection between symmetries of a particular kind (the so-called rigid, or global symmetries) and conservation principles for systems described by Lagrangian dynamics. This treatment was not novel in kind, nor was it the main focus of the 1918 paper, but it was the most systematic and general to date. We shall not dwell here on the rest of Noether's paper, except to say that it had to do with symmetries of a more general (local) nature, and in particular with the role of general covariance (diffeomorphism invariance) within Einstein's general theory of relativity in clarifying the confusing status of 'conservation' principles in the theory.[2] Since this main section of the paper contained at least one significant further theorem, it is not uncommon to refer to the better-known theorem involving global symmetries as Noether's first theorem. Its applications in physics are legion, from particle and field theory to crack mechanics and fluid dynamics.[3]

But to use the term 'symmetry' in the context of Noether's 1918 paper is not innocent. What Noether was actually concerned with were the consequences of the existence of any groups of infinitesimal transformations of the dependent and independent variables under which the given action of the system is invariant. Noether herself never used the word 'symmetry', although the just-mentioned condition is occasionally referred to in the subsequent literature as the existence of a *variational symmetry*.[4] The real question for the physicist is what this has to do with a *dynamical symmetry*, defined to be a group of transformations which map solutions of the equations of motion—the Euler-Lagrange equations of the system—into solutions. In re-addressing this question in various of its facets, we hope in this note to further clarify the significance of Noether's results. Although the considerations raised here are relevant to all parts of her 1918 paper, we shall concentrate on Noether's first theorem and the nature of the connection between dynamical symmetries and conservation principles.

2 Noether's variational problem

Suppose we have a system of fields ϕ_i $(i = 1, \ldots, N)$ defined on four-dimensional space-time, whose dynamical behaviour can be obtained from

[1] See [1]. An English translation is found in [2].

[2] For a discussion of the full content of Noether's paper, see e.g. [3-7]. An extensive historical study of Noether's work and the literature it has spawned is found in [8].

[3] For examples of the last two applications see [9] and [10] respectively.

[4] See e.g. [5].

the Lagrangian density[5] $\mathcal{L}$ by appropriate applications of Hamilton's principle.[6] Consider a specific group of transformations of the dependent and independent variables which depend smoothly on a number of arbitrary constant parameters ω_k[7]

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu + \cdots \tag{1}$$
$$\phi_i(x) \to \phi'_i(x') = \phi_i(x) + \delta\phi_i + \cdots \tag{2}$$

where $\mu = 0, \ldots, 3$ and $\delta x^\mu, \delta\phi_i$ signify the terms linear in the ω_k.[8] The question that Noether posed in 1918 was this. What are the consequences of the claim that the *infinitesimal* transformations

$$x'^\mu = x^\mu + \delta x^\mu \tag{3}$$
$$\phi'_i(x') = \phi_i(x) + \delta\phi_i \tag{4}$$

leave the action $S = \int_R \mathcal{L} d^4x$ invariant?[9] The variation in the action is defined as:

$$\delta S = S[\phi'_i, \partial_\mu\phi'_i, x'^\mu] - S[\phi_i, \partial_\mu\phi_i, x^\mu] \tag{5}$$
$$= \int_{R'} \mathcal{L}(\phi'_i, \partial_\mu\phi'_i, x'^\mu) d^4x' - \int_R \mathcal{L}(\phi_i, \partial_\mu\phi_i, x^\mu) d^4x \tag{6}$$

So Noether was concerned with the consequences of the claim that for the specific transformations (3), (4) $\delta S = 0$. It has since become commonplace (indeed it was anticipated by Noether herself) to weaken this condition to 'quasi-invariance'[10], i.e. invariance up to a surface term (equivalently, the

[5] We follow the usual procedure of considering Lagrangians that depend on the fields and their first derivatives, but the generalization to higher order derivatives is straightforward. For a general treatment of Noether's theorems which allows for dependency on first and second derivatives, see e.g. [3] and particularly [4].

[6] Note that we do not initially assume that *all* the fields ϕ_i are subject to Hamilton's principle.

[7] In the case of Noether's 'second' theorem, the transformations are allowed to depend on arbitrary functions on space-time, as in the case of diffeomorphisms in general relativity or local gauge transformations in electrodynamics.

[8] It can always be arranged that the identity transformations correspond to zero values of the ω_k.

[9] Noether's problem must not be conflated with the more familiar stationarity principle of Hamilton which concerns only infinitesimal transformations of the *dependent* variables ϕ_i, and where the transformations are *arbitrary* except that they vanish on the boundary of the region of integration associated with the action.

[10] Recall that two Lagrangian densities that differ by a total divergence yield the same equations of motion in response to the same applications of Hamilton's principle. This fact was noted by Noether in her 1918 paper, but fuller recognition of its consequences later led her to suggest a generalization of her invariance condition to Bessel-Hagen, who was involved in applying Noether' s first theorem to the case of Maxwellian electrodynamics following a suggestion of Klein. Bessel-Hagen [11] accordingly analysed the case in which the action is only quasi-invariant under the transformations (1), i.e. invariant up to

space-time integral of a total divergence[11]):

$$\delta S = \int_R d_\mu(\Lambda^\mu) d^4x. \tag{7}$$

Any group of transformations of the dependent and independent variables under which the action is quasi-invariant in this sense will be called a *Noether group* relative to the Lagrangian density $\mathcal{L}$. If we now apply the calculus of variations to (7) and consider only the first order contribution to δS, then it can be shown that the following expression must hold:

$$\int_R \sum_i \left(E_i^{\mathcal{L}} \delta_0 \phi_i\right) d^4x = -\int_R \sum_i d_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta_0 \phi_i + \mathcal{L} \delta x^\mu - \Lambda^\mu\right) d^4x \tag{8}$$

where d_μ signifies the "total" derivative relative to x^μ and where $\phi_{i,\mu}$ means the partial derivative $\partial\phi_i/\partial x^\mu$; the Einstein summation convention is being used for Greek indices. The term $\delta_0\phi_i$ denotes the Lie drag (or 'form' variation)

$$\delta_0 \phi_i = \phi_i'(x) - \phi_i(x). \tag{9}$$

Comparing with (2) we see that:[12]

$$\delta\phi_i = \delta_0\phi_i + (\partial_\mu \phi_i)\delta x^\mu \tag{10}$$

to first order. Finally, the $E_i^{\mathcal{L}}$ in (8) is the Euler expression associated with the field ϕ_i:

$$E_i^{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \phi_i} - d_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}}\right). \tag{11}$$

The reader will recall that if Hamilton's principle can be applied in relation to the field ϕ_i, then the Euler-Lagrange equation of motion

$$E_i^{\mathcal{L}} = 0 \tag{12}$$

must hold.[13]

a surface term. This generalization allowed for application of a Noether-type theorem to the case of Galilean covariance under coordinate boosts in standard classical particle mechanics. But perhaps more importantly for Noether, it also covered the case of the Lagrangian density introduced by Einstein in his 1916 treatment of general relativity based on an action principle—which unlike Hilbert's Lagrangian density is not a scalar density. (For a brief discussion of the Einstein Lagrangian density, see [12].)

[11] It is also common to find the claim in the literature that the divergence term involved, if any, must have no higher derivatives of the dependent variables than are found in the Lagrangian density (see for instance [13])). We believe this claim to be mistaken; indeed the 1916 Einstein action mentioned earlier provides a counterexample. For further discussion see [7] and particularly the Appendix in [14].

[12] The particularly important property of $\delta_0\phi$ is that it commutes with ∂_μ, the ordinary derivative. The variation $\delta\phi$, on the other hand, does not.

[13] Note that it is not necessarily the case that given (12) the field ϕ_k itself becomes 'dynamical'; for some $\mathcal{L}$ it may be that (12) fixes the dynamical behaviour of a different field ϕ_l, $l \neq k$.

Now since the region of integration in (8) is arbitrary, we thus arrive at the following solution to the Noether variational problem:

Noether expression

$$\sum_i E_i^{\mathcal{L}} \delta_0 \phi_i = -\sum_i d_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \delta_0 \phi_i + \mathcal{L} \delta x^\mu - \Lambda^\mu \right) \tag{13}$$

This differs from Noether's 1918 expression (restricted to the case of first-order Lagrangians in 4-dimensional space-time) only by the presence of the term involving Λ^μ on the right-hand side.

3 Noether's first theorem

We can express the variations δx^μ in (1) as

$$\delta x^\mu = \sum_k \omega_k \eta_k^\mu(x). \tag{14}$$

It follows from (10) that the the form variations of the fields are also linear in the ω_k, so we have

$$\delta_0 \phi_i = \sum_k \omega_k \xi_{ki}(x). \tag{15}$$

We stress that ξ_{ki} and η_k^μ are in general coordinate dependent, and may depend on both the fields and their derivatives.[14]

Similarly, we re-express the Λ^μ term appearing in (9) as an expansion:

$$\Lambda^\mu = \sum_k \omega_k \zeta_k^\mu(x) \tag{16}$$

On substituting the new expressions for δx^μ, $\delta_0 \phi_i$ and Λ^μ into the Noether Expression (13), since the parameters ω_k are independent of the coordinates both sides of the equation are linear in these parameters, and we can compare coefficients in each case, leading to the ρ identities associated with

Noether's first theorem

[14] It has become common in the literature to distinguish between Noether transformations that are 'geometrical', or 'point' transformations, and the 'generalized' or 'velocity-dependent' variety. In the former, the transformations (1), (2) may depend on the dependent variables ϕ_i but not their derivatives, whereas in the latter this restriction is lifted. Contrary to what is sometimes claimed, Noether in 1918 explicitly derived her theorems for generalized transformations, and not just geometrical ones. For a useful discussion of the origins of this confusion, see [5], pp. 286 and particularly 374–5. Up to the 1970s scores of papers were published either rederiving the version of Noether's first theorem (see below) limited to point transformations, or attempting to generalize it, only to reprove the original result or special cases of it; see [5], p. 282.

If a continuous (Lie) group of transformations depending on ρ constant parameters ω_k $(k = 1, 2, ..., \rho)$ is a Noether group with respect to the Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\mu)$, then the following ρ relations are satisfied, one for every parameter on which the group depends:

$$\sum_i E_i^{\mathcal{L}} \xi_{ki} = d_\mu j_k^\mu \tag{17}$$

where

$$j_k^\mu = -\sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \xi_{ki} \right) - \mathcal{L} \eta_k^\mu + \zeta_k^\mu. \tag{18}$$

In other words, certain linear combinations of the Euler expressions become divergences under the Noether condition for global variational symmetries.[15]

4 Conservation laws

When Hamilton's stationarity principle can be applied to every one of the fields ϕ_i, so the Euler-Lagrange equation of motion (12) holds for all i, the left-hand side of (17) vanishes, and it follows from Noether's first theorem that there exist ρ conserved currents:

$$d_\mu j_k^\mu = 0. \tag{19}$$

It now follows from Gauss' theorem, and the assumption that the fields vanish at spatial infinity, that the integral over all space of the zero component of j_k^μ is time independent:

$$\frac{d}{dt} \int j_k^0 d\mathbf{x} \equiv \frac{dQ_k}{dt} = 0 \tag{20}$$

where Q_k is the conserved Noether charge associated with the variational symmetry and the parameter ω_k.

Discussions of familiar cases of conserved quantities and their associated symmetries (such as conservation of linear and angular momentum associated with spatial homogeneity and isotropy respectively in particle physics) abound, and we will not add to them. The point we want to stress at this point is that it is not necessary in order to obtain the continuity equation (19) that all the Euler expressions on the LHS of (17) vanish (although this is ordinarily the case). Obviously, if the coefficient ξ_i^k happens to vanish in each case where the Euler expression $E_i^{\mathcal{L}}$ fails to, the continuity equation will nonetheless hold.

[15] For fuller treatments of Noether's first theorem, see, besides the references found in footnote 2 above, [13], [15], pp. 219-222, and [16], pp. 565–567.

A nice example is found in Trautman's 1962 discussion[16] of the case of a set of tensor fields interacting with a single particle in a possibly curved, non-dynamical, or absolute background space-time. (It is not, in other words, being assumed that the Euler expression $E^{\mathcal{L}}_{\mu\nu}$ associated with the metric field $g_{\mu\nu}$ vanishes.) Trautman shows that globally conserved quantities do not generally follow from invariance of the action under the general group of coordinate transformations (including rigid, or global transformations). However, non-trivial integral conservation laws do arise when the space-time has 'motions' (Killing vectors): when there exist coordinate transformations such that the form variation of the metric field $\delta_0 g_{\mu\nu}$ vanishes. This ensures that the LHS of (13) and hence (17) vanishes in this case, since it is assumed that the Euler expressions vanish for all the fields other than $g_{\mu\nu}$. In the particular case of space-time with constant curvature ten conservation laws hold.[17] Note that even in the case of a space-time with no Killing vectors, Trautman's treatment serves to remind us that a Noether-type analysis is perfectly meaningful when not all the dependent variables (fields) are subject to Hamilton's principle—a point which will be relevant below.

5 The significance of the first theorem

5.1 Preliminaries

As stated earlier, connection between (global) symmetries and conservation principles was hardly news in 1918. Within the field of particle mechanics, it had been appreciated in the previous century by Lagrange, Hamilton, Jacobi and Poincaré. An anticipation of Noether's first theorem in the special cases of the 10-parameter Lorentz and Galilean groups had been given by Herglotz in 1911 and Engel in 1916, respectively.[18] What was significant about Noether's work was its unprecedented degree of generality, and the possibility it raised for the systematic derivation of conservation laws in cases where the variational symmetries are relatively easy to classify. In recent years progress has been made particularly in the study of systems with 'generalized' Noether transformations. In particular, in this case a complete one-to-one correspondence between one-parameter groups of Noether transformations and conservation laws is provided by Noether's first theorem, stimulating the hope that complete classification of conservation laws can be obtained by constructive symmetry group methods.[19]

[16] [3], section 5-3. What follows is a reconstruction, rather than a summary, of Trautman's argument.

[17] For a related discussion of the case of electromagnetism in an arbitrary background space-time see [12], section II.

[18] A very useful study of the historical background to Noether's 1918 paper is found in [17].

[19] See [5], p. 287.

Having highlighted the strengths of Noether's first theorem, we feel it important not to loose sight of its subtleties and limitations. The easily misunderstood connection with dynamical symmetries will be treated shortly. In the meantime, we note the following.

1. Long ago, Wigner [18] warned of a 'facile identification' of symmetries with conservations principles. Wigner's point was that not all dynamical systems are amenable to a Lagrangian formulation, in which case Noether's first theorem does not apply. Indeed, Wigner gave a simple example of a system with time translation symmetry but no corresponding conserved quantity. (The attempt to treat the symmetry-conservation connection without reliance on the Lagrangian formalism has led to significant work.[20])

2. We have seen that continuity equations do not follow automatically from Noether's first theorem. But even when such equations do hold, the boundary conditions specified above leading to the existence of conserved charges may not obtain.[21] Moreover, the continuity equations in some cases actually coincide with the (Euler-Lagrange) equations of motion.[22]

3. A significant and perhaps little-known feature of the Noether program is the fact that the variational symmetry one associates with a given conservation principle can depend on the choice of the Lagrangian in cases where the dynamical system has 'inequivalent' Lagrangians (i.e. Lagrangians not equivalent up to a divergence term). A simple example is the two-dimensional harmonic oscillator,[23] whose alternative Lagrangians are

$$L_1 = \frac{1}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2\left(q_1^2 + q_2^2\right)\right] \tag{21}$$

   and the less familiar

$$L_2 = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2. \tag{22}$$

   The shared Euler-Lagrange equations ensure conservation of angular momentum, and in the case of Lagrangian (22) the corresponding Noether transformations are the familiar O(2) rotations, but in

[20] See in this connection e.g. [19], p. 235 and the references in footnote 14 therein, and [20].

[21] An example is given in [14], section 5.

[22] An example in elastostatics is discussed in [5], p. 276. Examples in quantum mechanics and electromagnetism are given in [14]. These latter examples also illustrate that neither the Lagrangian nor the conserved charges need be real-valued, and that the Noether symmetry transformation may not have an''active' interpretation: it may not carry states of the dynamical system into states.

[23] See e.g. [21], pp. 203–204.

the case of Lagrangian (23) they are the 'squeeze' transformations $q_1' = e^{\eta}q_1$, $q_2' = e^{-\eta}q_2$ for arbitrary constant η. (Note that while both the O(2) rotations and squeezes preserve the form of the equations of motion for q_1 and q_2, the rotations are not a variational symmetry relative to (22) and nor are the squeeze transformations for (21).)

An even more striking case was found by Sudbery. The symmetries normally associated with the conservation of energy and momentum are the homogeneity of time and space, but consider the unusual electromagnetic Lagrangian density in the form of a 4-vector field:

$$\mathcal{L}_\alpha = {}^*F^{\mu\nu}\partial_\nu F_{\mu\alpha} - F^{\mu\nu}\partial_\nu {}^*F_{\mu\alpha}, \tag{23}$$

where $F_{\mu\nu}$ is the usual antisymmetric electromagnetic field tensor and ${}^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is its dual. Assuming that the four associated actions are stationary under arbitrary variations of $F_{\mu\nu}$ vanishing on the boundary, then the ensuing Euler-Lagrange equations are equivalent to the full set of Maxwell's equations in the source-free case:

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu {}^*F^{\mu\nu} = 0. \tag{24}$$

Now consider the so-called 'duality rotations':

$$F_{\mu\nu} \to F_{\mu\nu}\cos\theta + {}^*F_{\mu\nu}\sin\theta \tag{25}$$

and hence

$${}^*F_{\mu\nu} \to {}^*F_{\mu\nu}\cos\theta - F_{\mu\nu}\sin\theta. \tag{26}$$

Each $\mathcal{L}_\alpha$ is strictly invariant under these transformations, which depend on the single parameter θ. For each of the four Lagrangians there is a continuity equation $d_\mu j^\mu_\alpha = 0$, where the 4-current takes the form

$$j^\mu_\alpha = 2F_{\alpha\lambda}F^{\mu\lambda} - \frac{1}{2}F_{\kappa\lambda}F^{\kappa\lambda}\delta^\mu_\alpha. \tag{27}$$

This is proportional to the usual energy-momentum tensor for the free Maxwell field! In this case the corresponding Noether transformations are not space-time symmetries, but 'internal' symmetries (affecting only the dependent variables). Conversely, the conservation law associated with space-time translations—which are Noether transformations with respect to the Lagrangians $\mathcal{L}_\alpha$—has to do with a third-rank tensor which is closely related to the so-called Lipkin tensor.[24]

[24]For details see [22].

5.2 The connection with dynamical symmetries

Let us suppose that the group of transformations (1) takes solutions of the Euler-Lagrange equations into solutions, and hence constitutes *dynamical* symmetry transformations. Then the first point is that it is not every dynamical symmetry group is a Noether group in respect of the relevant Lagrangian. (Recall that being a Noether group involves quasi-invariance of the action, not the more restrictive condition of invariance.) The existence of "non-Noetherian" dynamical symmetries is widely known. Typical cases involve a rescaling of variables which leaves the Euler-Lagrange equations unaltered but results in an overall multiplicative constant appearing in the action.[25] Other cases involve transformations for which the variation δS only satisfies the quasi-invariance condition (7) 'on-shell', i.e. when Hamilton's principle is applied to the appropriate fields. (Yet another kind of case is found in item (3) in the previous subsection.) The important conclusion for our purposes is that specifically using a Noether-type analysis, it is simply not possible to infer that the existence of a (global) dynamical symmetry is always associated with the existence of a conservation principle, even when the dynamics is Lagrangian.

It is interesting that not all commentators agree as to precisely how the notion of a dynamical symmetry needs to be supplemented in order to justify the quasi-invariance condition. We shall not enter here into the details of this issue, interesting though they may be.[26] What interests us is the possibility that behind, and motivating, this discussion is a lingering view that a conservation principle is *explained* by the existence of a variational symmetry, which—at least in some cases—can be related to the existence of a dynamical symmetry.

Although we cannot be sure how popular this view is, it seems to us that it is wrong, and not borne out by the nature of Noether's theorem. The very notion of explanation involved is misguided. Noether was not attempting to explain conservation principles in terms of variational symmetries; indeed she stressed that her first 1918 theorem can be proved in reverse.[27] How should we understand its significance?

It is curious to us that more emphasis has not been given in the Noether-related literature to the result that, subject to an important caveat (see below), all transformations which constitute a variational symmetry (Noether

[25]For a recent example in quantum mechanics, see [14], section 4.

[26]Quasi-nvariance involves two distinct conditions for the action, *viz.* quasi-scalarity and quasi-form-invariance, and the issue is how these are related to the existence of dynamical symmetries and where the divergence term appears. Readers may like to compare the treatment in the influential 1952 paper by Hill [13] (see also [23], [24] and [15], pp. 190-91) with that found in Trautman [3] and Anderson [25], p. 91. We side with the latter approach; the reason is found in [7], section 3.

[27]For discussion of the inverse of Noether's first theorem which allows for quasi-invariance of the action, see [26].

group) also constitute a dynamical symmetry relative to the same Lagrangian, even if the converse is not the case. Let us call this result the symmetry theorem. Putting the nature of its limitations aside briefly, the symmetry theorem allows us to see the Noether theorem in its true light. Noether starts with the existence of a variational symmetry. Her first theorem allows us to infer, under ordinary circumstances for global symmetries, the existence of certain conserved charges, or at least a set of continuity equations. The symmetry theorem separately allows us to infer the existence of a dynamical symmetry group. We have now established a *correlation* between certain dynamical symmetries and certain conservation principles.[28] Neither of these two kinds of thing is conceptually more fundamental than, or used to explain the existence of, the other (though as noted earlier if it is easier to establish the variational symmetry group, then a method for calculating conserved charges is provided). After all, the real physics is in the Euler-Lagrange equations of motion for the fields, from which the existence of dynamical symmetries and conservation principles, if any, jointly spring.

We finish with a remark on the status of the symmetry theorem. The most systematic treatment of it to our knowledge is given by Olver.[29] We regard it important to emphasize that the proof requires all the fields to be subject to Hamilton's principle, so that no fields are non-dynamical. A counterexample to the theorem is easy to construct if this dynamical condition is not met. Consider again the case of the two-dimensional harmonic oscillator discussed in the previous subsection. It was mentioned that the action based on the familiar Lagrangian (21) is invariant under O(2)-rotations, which are characterized by

$$q'_1 = q_1 \sin\theta + q_2 \cos\theta \tag{28}$$
$$q'_2 = q_1 \cos\theta - q_2 \sin\theta \tag{29}$$

for a given angle θ. It is easy to show that this transformation is a dynamical symmetry of the Euler-Lagrange equation of motion obtained by applying Hamilton's principle to the variable q_1

$$\ddot{q}_1 - \omega^2 q_1 = 0 \tag{30}$$

[28] As Rosen [24] and others have pointed out, transformations that are more general than what we have called Noether groups can be associated with conservation principles within the Lagrangian formalism, and for such transformations the connection with dynamical symmetries is far less clear. But we resist Rosen's ([24] p. 349) conclusion that 'symmetry transformations appear in general to be unconnected with conservation laws'. Recall that conservation laws can in principle be derived directly from the equations of motion of the system, without appealing to the existence of dynamical symmetries at all. If this point does not detract from the importance of Noether's original result—and it doesn't—then neither does Rosen's 'generalization' of Noether's first theorem render her original result (along with the symmetry theorem above) incapable of establishing a correlation between certain conservation principles and dynamical symmetries.

[29] See [5], theorem 4.14 p. 255, theorem 4.34 p. 278, and particularly theorem 5.53 p. 332. A much more informal treatment can be found in [27].

only if the corresponding Euler-Lagrange equation holds for q_2.

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