ON RATIONAL LIMITS OF SHELAH-SPENCER GRAPHS

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Abstract. Given a sequence $\{\alpha_n\}$ in $(0, 1)$ converging to a rational, we examine the model theoretic properties of structures obtained as limits of Shelah-Spencer graphs $G(m, m^{-\alpha_n})$. We show that in most cases the model theory is either extremely well-behaved or extremely wild, and characterize when each occurs.

1. Introduction

The study of random graphs, initiated by Erdös and Renyi in 1959, was subsequently examined from a logical viewpoint – notably in papers of Shelah, Spencer, and Baldwin ([6], [1]). In particular, for α irrational in $(0, 1)$ the model theory connected with the graphs $G(n, n^{-\alpha})$ has been extensively studied. The latter objects are probability spaces whose events consist of all order n graphs – each of these occurs with a probability uniquely determined by demanding that every potential edge occurs independently with probability $n^{-\alpha}$. A key result of Shelah and Spencer [6] is the following 0-1 law: For σ any first order sentence in the language of graphs, $\lim_{n\to\infty} Pr[G(n, n^{-\alpha}) \models \sigma]$ is 0 or 1. Thus, for a fixed irrational α the *almost sure theory*, denoted T^α, is complete. More recently, the second author gave an $\forall\exists$-axiomatization for T^α (see [5]).

Baldwin and Shelah pointed out in [1] that models of the resulting theory could be obtained via Hrushovski's amalgamation construction. This proceeds by amalgamating a class of finite structures to obtain a *generic* of the class. The amalgamation is controlled by a notion of "strong substructure", which is in turn often determined by a *pre-dimension function*. In the current context such a function limits the proportion of new edges to new vertices in a strong extension.

Arguably, the crucial observation in the connection between the probabilistic and model-theoretic approaches is that the expected number of copies of a given extension is determined by precisely such a function. Specifically, if a given graph A almost surely occurs as a subgraph of $G(n, n^{-\alpha})$ in the limit, then the expected number of copies of an extension B is asymptotic to $n^{\delta(B/A)}$, where $\delta(B/A)$ is a predimension function given by $|B \setminus A| - \alpha e(B/A)$, for $e(B/A)$ the number of edges in AB that aren't in A. When α is irrational, $\delta(B/A)$ is never zero, hence $n^{\delta(B/A)}$ has asymptotic limit zero or one. However, when α is rational, this need not be the case, and indeed Spencer demonstrated in [7] that $G(n, n^{-\alpha})$ does not have a 0-1 law in this case.

This paper examines the rational case from a model-theoretic perspective. There are two distinct ways of handling pairs of graphs $A \subseteq B$ with $\delta(B/A) = 0$, which leads to two distinct classes, which we denote by $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}^+_\alpha, \preccurlyeq_\alpha)$. On the

Date: August 30, 2011.
Partially supported by NSF grants DMS-0600217 and DMS-0901336.

surface, these classes are similar. They both satisfy Baldwin and Shi's notion of full amalgamation and have generic structures, each unique up to isomorphism, which we denote by M_α and M_α^+, respectively.

However, when we look at the theories of these structures, we see a Jekyll and Hyde dichotomy. The theory of M_α is tame, being decidable, $\forall\exists$-axiomatizable, and $\aleph_0$-stable. On the other hand, Section 3 is devoted to showing that the theory of M_α^+ is wild. There is a subtheory Σ_α^+ that allows coding of finite sets, interprets a fragment of arithmetic, and is essentially undecidable.

In the final section, we give a partial explanation for the lack of a 0-1 law when α is rational. We see that any ultraproduct of generics coded by a sequence $\{\alpha_n\}$ converging to α **from below** has a theory that is elementarily equivalent to the 'nice' M_α. On the other hand, any ultraproduct of generics coded by a sequence $\{\alpha_n\}$ converging to α **from above** has a theory that is elementarily equivalent to the 'uncouth' M_α^+. This gives a model-theoretic parallel to Spencer's result in [7] that the blockage of a 0-1 law is really a one-sided phenomenon, with convergence to α from above being the problematic part.

2. Parameterized families of finite graphs

For the purposes of this paper, we restrict our attention to classes of graphs. In particular, we work in the language $L = \{E\}$ of a single, binary relation and all L-structures we consider have E being symmetric and irreflexive. However, by using coding techniques developed by Ikeda, Kikyo, and Tsuboi in Lemma 3.6 of [4], all of our results extend to any finite, relational language in which each relation is symmetric in its variables.

As notation, we denote $A \cup B$ simply by AB, and write $A \subseteq_\omega M$ to indicate that A is a *finite* substructure of M. For any finite graph A, we implicitly fix an enumeration of A and denote its quantifier free type by $\Delta_A(\bar{x})$.

We begin by defining two separate parametrized families of classes $(\mathbf{K}, \leq)$ of finite graphs. Fix a real number $\alpha \in (0,1)$ and define

$$\delta_\alpha(A) = v(A) - \alpha e(A)$$

for any finite (symmetric) graph A, where $v(A)$ denotes the number of vertices of A, and $e(A)$ denotes the number of edges. As notation, if $A \subseteq B$, let $\delta_\alpha(B/A) = \delta_\alpha(B) - \delta_\alpha(A)$. If A, B, C are finite graphs satisfying $B \cap C = A$, the *free join of B, C over A*, denoted $B \oplus_A C$, is the graph D with vertices $B \cup C$, and edges $E^D = E^B \cup E^C$. More generally, if $\{B_i : i < n\}$ satisfy $B_i \cap B_j = A$ for all $i \neq j$, then $\oplus_{i<n}(B_i/A)$ has universe $\bigcup_{i<n} B_i$ and edge set $\bigcup_{i<n} E^{B_i}$.

The following computations are routine:

Lemma 2.1. *For all $\alpha \in (0,1)$ and for all finite graphs A, B, B_i, C,*

(1) $\delta_\alpha(\oplus_{i<n}(B_i/A)) = \sum_{i<n} \delta_\alpha(B_i/A)$ *and*
(2) **[Monotonicity]** *If $A \subseteq B$ and $B \cap C = \varnothing$, then $\delta_\alpha(BC/AC) \leq \delta_\alpha(B/A)$.*

Definition 2.2. Our two parametrized classes are $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$, where

- $\mathbf{K}_\alpha = \{$finite $A : \delta_\alpha(B) \geq 0$ for every substructure $B \subseteq A\}$ and $A \leq_\alpha B$ if and only if $A \subseteq B$ and $\delta_\alpha(C/A) \geq 0$ for all $A \subseteq C \subseteq B$.
- $\mathbf{K}_\alpha^+ = \{$finite $A : \delta_\alpha(B) > 0$ for every nonempty $B \subseteq A\}$ and $A \preccurlyeq_\alpha B$ if and only if $A \subseteq B$ and $\delta_\alpha(C/A) > 0$ for all $A \subsetneq C \subseteq B$.

Obviously, $\mathbf{K}_\alpha^+ \subseteq \mathbf{K}_\alpha$, and $A \preccurlyeq_\alpha B$ implies $A \leq_\alpha B$. Furthermore, if α is irrational, then the classes $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ and $(\mathbf{K}_\alpha, \leq_\alpha)$ are identical. The results of the following Lemma are well known for the classes $(\mathbf{K}_\alpha, \leq_\alpha)$, but, using Lemma 2.1, the verifications for $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ are equally routine.

Lemma 2.3. *For every $\alpha \in (0,1)$, both of the classes $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ satisfy the following axioms of a class $(\mathbf{K}, \leq)$:*

1. $\mathbf{K}$ *is closed under isomorphisms and substructures;*
2. *The relation $A \leq B$ is invariant under isomorphisms of the pair (A, B);*
3. $\leq$ *is a partial order on $\mathbf{K}$, with $\varnothing \leq A$ for all $A \in \mathbf{K}$;*
4. $(\mathbf{K}, \leq)$ *is a* full amalgamation class *in the sense of Baldwin-Shi* [3]*, i.e., if $A, B, C \in \mathbf{K}$ and $A \leq B$, $A \subseteq C$, then $C \leq D$, where $D = B \oplus_A C$*
5. *If $A, B, C \in \mathbf{K}$ and $A \leq B$, then $A \cap C \leq B \cap C$.*

Definition 2.4. Given a class $(\mathbf{K}, \leq)$ of finite structures, an element $A \in \mathbf{K}$, and a (possibly infinite) structure M such that every finite substructure of M is an element of $\mathbf{K}$, a *strong embedding* $f : A \to M$ is an isomorphic embedding satisfying $A \leq B$ for all finite B satisfying $A \subseteq B \subseteq M$. A (countable) structure M is $(\mathbf{K}, \leq)$*-generic* if it satisfies the following three conditions:

1. There are $\langle A_n : n \in \omega \rangle$ from $\mathbf{K}$ such that $A_0 \leq A_1 \leq \ldots$ and $M = \bigcup_{n\in\omega} A_n$;
2. Every $A \in \mathbf{K}$ embeds strongly into M; and
3. For all pairs $A \leq B$ from $\mathbf{K}$, every strong embedding $f : A \to M$ extends to a strong embedding $g : B \to M$.

It is well known that if $(\mathbf{K}, \leq)$ is a class of finite structures closed under isomorphism, substructure, joint embedding, and amalgamation, then there is a $(\mathbf{K}, \leq)$-generic structure M. Moreover, M is unique up to isomorphism. In our context, all of our classes $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ satisfy amalgamation by Lemma 2.3(4) and joint embedding since $\varnothing \leq A$ for every A in all of our families.

Definition 2.5. For each $\alpha \in (0,1)$, let M_α denote the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic, and let M_α^+ denote the $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$-generic.

To understand these structures, we proffer three axiom schemata for a given amalgamation class $(\mathbf{K}, \leq)$; note that collectively these encode Baldwin and Shelah's notion of a semi-generic structure in [1]:

Definition 2.6.

- Universal sentences $\sigma(m)$, indexed by $m \in \omega$, asserting that every m-element substructure is an element of $\mathbf{K}$;
- $\forall\exists$ sentences $\psi(A, B)$, indexed by all pairs $A \leq B$ from $\mathbf{K}$ stating that every embedding of A into a model M of $\psi(A, B)$ extends to an embedding of B into M;
- $\forall\exists\forall$ sentences $\theta(A, B, m)$, indexed by $A \leq B$ from $\mathbf{K}$ and $m \in \omega$, asserting that every embedding of A into a model M of $\theta(A, B, m)$ extends to an embedding of B into M such that for every C satisfying $B \subseteq C \subseteq M$ with $|C \setminus B| \leq m$, either $B \leq C$ or the substructure of M with universe $BC \cong B \oplus_A (C \setminus B)A$.

If $(\mathbf{K}, \leq)$ is a full amalgamation class (i.e., satisfies the conclusions of Lemma 2.3) then the $(\mathbf{K}, \leq)$-generic structure M satisfies all of the axioms stated above. (To see that $M \models \theta(A, B, m)$, choose any $A' \subseteq M$ isomorphic to A. Choose n least such that $A' \subseteq A_n$, where $A_n \leq M$. Let $D = A_n \oplus_A B$. Then $A_n \leq D$, so choose a strong embedding $g : D \to M$ that is the identity on A_n. Then $g|B$ has the required property.)

As notation, we let S_α and S_α^+ denote the first two schemata with respect to $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$. Dually, let Σ_α and Σ_α^+ denote all three schemata with respect to $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$, respectively.

The next two Theorems are culminations of work by many authors.

Theorem 2.7. *If $\alpha \in (0, 1)$ is irrational, then $Th(M_\alpha)$ is axiomatized by S_α. It is decidable, stable, but not superstable, with nfcp, and has the Dimensional Order Property (DOP). Furthermore, it is precisely the almost sure theory of the sequence of random graphs $G(n, n^{-\alpha})$.*

Proof. Shelah and Spencer (see e.g., [6] and [7]) showed that the almost sure theory of the random graphs $G(n, n^{-\alpha})$ contains Σ_α, and that Σ_α is complete and therefore decidable. The connection with model theory was first noted by Baldwin. In [1] and [2] Baldwin and Shelah proved that the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic $M_\alpha \models \Sigma_\alpha$, and that the theory is strictly stable, with nfcp and has the DOP. In [5], the second author proved that S_α entails Σ_α. □

Theorem 2.8. *If $\alpha \in (0, 1]$ is rational, then again $Th(M_\alpha)$ is axiomatized by S_α. This theory is decidable and $\aleph_0$-stable.*

Proof. For rational α, Shelah and Spencer proved that there is no complete, almost sure theory of the random graphs $G(n, n^{-\alpha})$. Nevertheless, Baldwin and Shi [3] showed that a $(\mathbf{K}_\alpha, \leq_\alpha)$-generic exists, and proved that its theory is $\aleph_0$-stable. In [1], Baldwin and Shelah proved that Σ_α is equivalent to $Th(M_\alpha)$, thereby yielding decidability of the theory. Later, Ikeda, Kikyo, and Tsuboi [4] proved that S_α is equivalent to the theory of the generic. □

Note that for α irrational, the classes $(\mathbf{K}_\alpha, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ coincide, hence M_α is isomorphic to M_α^+. However, as we see in the next section, when α is rational, the theory of M_α^+ is substantially different from the theory of M_α.

3. 0-Extensions and the theory Σ_r^+

For the whole of this section, we fix a rational $r \in (0, 1)$ and investigate the theory Σ_r^+. We see that this theory is far from being decidable and has $2^{\aleph_0}$ distinct completions. The engine that is driving the distinction between the well-behaved theory Σ_r and the wild Σ_r^+ is how they handle minimal 0-extensions, which we now define.

Definition 3.1. Fix a rational $r \in (0, 1)$. A *minimal 0-extension* is a pair of structures $A \subsetneq B$ from $\mathbf{K}_r^+$ such that $\delta_r(B/A) = 0$, but $A \preccurlyeq_r C$ for any proper $A \subseteq C \subsetneq B$. A *biminimal 0-extension* (A, B) is a minimal 0-extension in which every element of A is connected to at least one element of $B \setminus A$.

The notion of a minimal 0-extension is a special type of *minimal pair*, which has been used by many authors. The notion of biminimality is used by Wagner in his exposition of the Hrushovski constructions [9].

Note that if (A, B) is a minimal 0-extension in $\mathbf{K}_r^+$, then the same pair satisfies $A \leq_r B$ from the point of view of $(\mathbf{K}_r, \leq_r)$. Thus, in the $(\mathbf{K}_r, \leq_r)$-generic M_r, for every embedding $f : A \to M_r$, there are infinitely many embeddings $g_i : B \to M_r$ extending f. This can be thought of as a type of homogeneity possessed by M_r. On the other hand, since $A \not\preccurlyeq_r B$ for a minimal 0-extension, given an embedding $f : A \to M_r^+$ into the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic, the number of embeddings $g : B \to M_r^+$ extending f can vary. As we will see with Theorem 3.5, this freedom allows us to interpret Robinson's R into any model of Σ_r^+. It is necessary to consider biminimal 0-extensions because we want to 'recover A' from an arbitrary embedding of B into a model of Σ_r^+.

We begin by showing that biminimal 0-extensions exist in abundance.

Lemma 3.2. *Fix any rational $r \in (0, 1)$. For every integer $n > 0$, there is an integer m such that for any $A \in \mathbf{K}_r^+$ of size n, there is a graph $C \supseteq A$ such that $|C \setminus A| = m$ and (A, C) is a biminimal 0-extension.*

Proof. First, note that if (A, C) is a biminimal 0-extension, and $f : A \to A'$ is any bijection (possibly not preserving the edge relation) then there is a biminimal 0-extension (A', C'), where the subgraph $C' \setminus A'$ is isomorphic to the subgraph $C \setminus A$, and where $E^{C'}(c', f(a))$ holds if and only if $E^C(c, a)$.

As a consequence, if we can construct a biminimal 0-extension (A, C^*) for any $A \in \mathbf{K}_r^+$ with $|A| = n$, then taking $m = |C^* \setminus A|$, the construction will yield a biminimal 0-extension (A', C') with $|C' \setminus A'| = m$ for every $A' \in \mathbf{K}_r$ with $|A'| = n$. Thus we fix a nonempty $A \in \mathbf{K}_r^+$ with $|A| = n$ and define an appropriate C^*. Before doing so, we introduce some other relevant graphs.

In [4], Ikeda, Kikyo, and Tsuboi demonstrate the existence of a 'minimal, proper, $(1 + r)$-component' $D(a, b)$. That is, D is a graph, $a, b \in D$, $a \neq b$, $\neg E(a, b)$, $\delta_r(X) \geq 1 + r$ for any X satisfying $\{a, b\} \subseteq X \subseteq D$ with equality holding only when $X = D$, and $\delta_r(X) \geq 1$ for every nonempty $X \subseteq D$, $\delta_r(X) \geq 1$.

For any $k \geq 1$, let $\{b_0, \ldots, b_k\}$ be a null graph, and for each $i < k$ let $D(b_i, b_{i+1})$ be isomorphic to $D(a, b)$, with $D(b_i, b_{i+1}) \cap D(b_j, b_{j+1}) = \{b_i, b_{i+1}\} \cap \{b_j, b_{j+1}\}$ for all $i \neq j$, and let

$$D_k(b_0, \ldots, b_k) = D(b_0, b_1) \oplus_{b_1} D(b_1, b_2) \oplus_{b_2} \cdots \oplus_{b_{k-1}} D(b_{k-1}, b_k)$$

We make two assertions:

(1) For nonempty $X \subseteq D_k(b_0, \ldots, b_k)$, $\delta_r(X) \geq 1 + (l - 1)r$, where $l = |X \cap \{b_0, \ldots, b_k\}|$;
(2) If $\{b_0, \ldots, b_k\} \subseteq X \subseteq D_k(b_0, \ldots, b_k)$, then $\delta_r(X) = 1 + kr$ if and only if $X = D_k(b_0, \ldots, b_k)$.

For the first assertion, note that since $r < 1$, it suffices to partition X into its connected components and handle each component separately. So assume X is nonempty and connected, and let $l = |X \cap \{b_0, \ldots, b_k\}|$. If $l = 0$, then $X \subseteq D(b_i, b_{i+1})$ for some i, hence $\delta_r(X) \geq 1$ by the definition of being a $(1 + r)$-component. If $l \neq 0$, then by connectedness, $X \cap \{b_0, \ldots, b_k\} = \{b_i, \ldots, b_{i+l-1}\}$ for some i. Let $X_{-1} = \{b_0\}$, $X_j = X \cap D(b_j, b_{j+1})$ for each $j < k$, and $X_{k+1} = \{b_k\}$. Then $X = X_{i-1} \oplus_{b_i} \cdots \oplus_{b_{i+l-1}} X_{i+l}$, where $\delta_r(X_{i-1}), \delta_r(X_{i+l}) \geq 1$ and $\delta_r(X_j) \geq 1+r$ for all $i \leq j < i + l$. It follows that $\delta_r(X) \geq 2 + (l - 1)(1 + r) - l = 1 + (l - 1)r$, as desired.

For the second assertion, suppose $\{b_0, \ldots, b_k\} \subseteq X$. Then we can write $X = X_0 \oplus_{b_1} X_1 \oplus \cdots \oplus_{b_{k-1}} X_{k-1}$, where $X_i = D(b_i, b_{i+1})$. Since $\{b_i, b_{i+1}\} \subseteq X_i$ for each i $\delta_r(X_i) \geq 1+r$, with equality holding if and only if $X_i = D(b_i, b_{i+1})$ and the second assertion follows.

Next, let $C_k = C_k(b^*, b_1, \dots, b_{k-1})$ be the graph formed from $D_k(b_0, \dots, b_k)$ by identifying the nodes b_0 and b_k into a new node b^*. That is, $E(b^*, y)$ holds in C_k if and only if either $E(b_0, y)$ or $E(b_k, y)$ holds in $D_k(b_0, \dots, b_k)$. One can think of C_k as a 'circle'. In particular, there is an automorphism of C_k fixing $\{b^*, \dots, b_{k-1}\}$ setwise formed by $b^* \mapsto b_1$, $b_i \mapsto b_{i+1}$, $b_{k-1} \mapsto b^*$, and extended by choosing isomorphisms between $D(b_i, b_{i+1})$ and $D(b_{i+1}, b_{i+2})$, etc.

We claim that for any nonempty $Y \subseteq C_k$, $\delta_r(Y) \geq mr$, where $m = Y \cap \{b^*, b_1, \dots, b_{k-1}\}$, with equality holding only when $Y = C_k$.

To see this, first suppose that $b^* \notin Y$. Then Y can be construed as a subgraph of $D_k(b_0, \dots, b_k)$, and $Y \cap \{b^*, b_1, \dots, b_{k-1}\} = Y \cap \{b_0, \dots, b_k\}$, so $\delta_r(Y) \geq 1 + (m-1)r > mr$ since $r < 1$. Second, by iterating the automorphism above, we again obtain $\delta_r(Y) > mr$ unless $\{b^*, b_1, \dots, b_{k-1}\} \subseteq Y$. Finally, assume that $m = k$, i.e., $\{b^*, b_1, \dots, b_{k-1}\} \subseteq Y$. In this case, let X^* be the 'unpacking' of Y in $D_k(b_0, \dots, b_k)$, i.e., the nodes of X^* are $Y \setminus \{b^*\} \cup \{b_0, b_k\}$. Then $v(X^*) = v(Y) + 1$, $e(X^*) = e(Y)$, and $|X \cap \{b_0, \dots, b_k\}| = k + 1$. From above, $\delta_r(X^*) \geq 1 + kr$, with equality holding if and only if $X^* = D_k(b_0, \dots, b_k)$. Thus, $\delta_r(Y) \geq kr$, with equality holding if and only if $Y = C_k$.

We can now produce a graph $C^* \supseteq A$ such that (A, C^*) is a biminimal 0-extension. Fix an enumeration $\{a_0, \dots, a_{n-1}\}$ of A. Define C^* to be the graph with universe the disjoint union $A \cup C_n(b^*, \dots, b_{n-1})$ and edges defined by $E^{C^*}(x, y)$ iff $(x, y) \in A$ and $E^A(x, y)$, or $(x, y) \in C_n$ and $E^{C_n}(x, y)$, or $(x, y) = (a_0, b^*)$, or $(x, y) = (a_i, b_i)$ for some $1 \leq i < n$. Then for any $Y \subseteq C_n$, if $l = |Y \cap \{b^*, \dots, b_{n-1}\}|$, then $\delta_r(AY/A) = \delta_r(Y) - lr$. From above, this number is always nonnegative, and is zero precisely when Y is empty or equal to C_n. □

We now use the existence of a biminimal 0-extensions (A, B) with $|A| = k + 1$ to be able to 'mark' any desired subset $X \subseteq [S]^k$ of any finite subset S of any model M of Σ_r^+. The idea of using the existence of an extension to code arbitrary subsets of a given finite set was used by Spencer in [7]. Recall that for any set Y, $[Y]^k$ denotes the subsets of Y with cardinality precisely k.

Proposition 3.3 (Definability of Finite Relations). *For every rational $r \in (0, 1)$ and every integer $k \geq 1$ there is a definable $R^k(x_1, \dots, x_k, v)$, symmetric in the first k variables, such that for every $M \models \Sigma_r^+$, for every finite $S \subseteq M$, and for every subset $X \subseteq [S]^k$, there is $v \in M$ such that for any $\bar{a} \in S^k$ with k distinct entries,*

$$M \models R^k(\bar{a}, v) \qquad \textit{if and only if} \qquad \text{range}(\bar{a}) \in X$$

Proof. Fix a rational $r \in (0, 1)$ and an integer $k \geq 1$. Let m be the integer from Lemma 3.2 corresponding to $k + 1$. Let $R^k(x_1, \dots, x_k, v)$ assert that there exist $\bar{y}$ with $|\bar{y}| = m$ such that all of the entries in $\bar{x}\bar{y}v$ are distinct, $\neg E(x_i, v)$ for all i, and the pair $(\bar{x}v, \bar{x}v\bar{y})$ is a biminimal 0-extension.

To see that this works, let $M \models \Sigma_r^+$, let $S \subseteq M$ be finite, and let $X \subseteq [S]^k$ be given. For the moment, view the subgraph S as a graph in its own right. Choose an element e (not necessarily in M) and let Se be the one-point extension of S with e unconnected to any vertex of S. For each $A \in X$, let $C_A \supseteq Ae$ be such that (Ae, C_A) is a biminimal 0-extension, $|C_A \setminus Ae| = m$, and $C_A \cap Se = Ae$.

Claim 1. $A \preccurlyeq_r C_A$ for each $A \in X$.

Proof. Choose $Z \neq A$ such that $A \subseteq Z \subseteq C_A$. If $Z = Ae$, then $\delta_r(Z/A) = 1$. If $Z = C_A$, then $\delta_r(Z/A) = \delta_r(Z/Ae) + \delta_r(Ae/A) = 0 + 1 = 1$. If $Z = C_A \setminus \{e\}$, then since

e is connected to some element of Z by biminimality, $\delta_r(Z/A) \geq \delta_r(Ze/Ae)+r \geq r$. In all other cases, $Ze \neq Ae, C_A$, so $\delta_r(Z/A) \geq \delta_r(Ze/Ae) > 0$. □

Letting $W = \oplus_A C_A$, it follows immediately from Claim 1 that $S \preccurlyeq_r W$. As well, note that for any V satisfying $Se \subseteq V \subseteq W$, we have $\delta_r(V/Se) \geq 0$, with equality holding if and only if $V = Se \cup \{C_A : A \in Y\}$ for some subset $Y \subseteq X$.

Since $M \models \theta(S, W, m)$, there is an embedding $g : W \to M$ such that $g|S = id$ and for any $B \subseteq M \setminus g(W)$ with $|B| \leq m$, either $B \preccurlyeq_r W$ or the subgraph of M with universe $Bg(W)$ is isomorphic to $BS \oplus_S g(W)$.

We now work entirely within M. By abuse of notation, we write W for $g(W)$, e for $g(e)$, and C_A for $g(C_A)$ for each $A \in X$. That is, e, W, and each C_A are from M.

Clearly, for every $A \in X$ and every enumeration $\bar{a}$ of A, $M \models R^k(\bar{a}, e)$, as witnessed by C_A. Conversely, choose any $B \in [S]^k$ and any enumeration $\bar{b}$ of B such that $M \models R^k(\bar{b}, e)$. Choose any graph $C^* \subseteq M$ witnessing this, i.e., $Be \subseteq C^*$, $|C^* \setminus Be| = m$, and (Be, C^*) is a biminimal 0-extension. We argue that $B \in X$.

Begin by partitioning C^* into three sets,

$$P = C^* \cap Se, \qquad Q = (C^* \cap SW) \setminus P, \qquad R = C^* \setminus SW$$

Claim 2. $RSW \cong RS \oplus_S W$.

Proof. This is trivial if R is empty, so assume otherwise. If $R \neq \varnothing$, then $Be \subseteq C^* \cap SW \subsetneq C^*$, so $\delta_r(C^*/C^* \cap SW) \leq 0$, hence $\delta_r(SWR/SW) \leq 0$ by monotonicity. Thus, $SW \not\preccurlyeq_r SWR$. But also, $|R| \leq m$, so the last statement follows from our choice of embedding g. □

We next argue that $Q \neq \varnothing$. Since (Be, C^*) is biminimal, there is some $c \in C^*$ such that $E(c, e)$ holds. E is irreflexive and e is unconnected to any $c \in S$, so $c \notin P$. However, $c \in R$ contradicts the conclusion of Claim 2. Thus, $c \in Q$, so Q is nonempty.

Claim 3. $RP = Be$, i.e., $C^* = BeQ$.

Proof. Since $RSW \cong RS \oplus_S W$, it follows that $C^* = RPQ \cong RP \oplus_P QP$. So

$$0 = \delta_r(C^*/Be) = \delta_r(RP/Be) + \delta_r(QPe/Be) - \delta_r(P/Be)$$

As well, $\delta_r(QSe/Se) \geq 0$, hence $\delta_r(QP/P) \geq 0$. Since $\delta_r(QPe/Be) - \delta_r(P/Be) = \delta_r(QP/P)$, we must have $\delta_r(RP/Be) \leq 0$. But $Be \subseteq RP \subseteq C^*$, so $\delta_r(RP/Be) \geq 0$, and is only equal to zero when $RP = Be$ or $RP = C^*$. However, the latter is impossible since Q is nonempty. □

Now $Q \subseteq W \setminus Se$, and $W \setminus Se$ is a free join of sets C_A over Se. Choose $A \in X$ such that $Q \cap C_A \neq \varnothing$.

Claim 4. $Q = (C_A \setminus Se)$.

Proof. We first argue that $Q \subseteq C_A$. If this were not the case, then let $Q_A = Q \cap C_A$ and $Q' = Q \setminus C_A$. Then $C^* = BeQ$ would be isomorphic to $Q_A Be \oplus_{Be} Q'Be$. But then,

$$0 = \delta_r(C^*/Be) = \delta_r(Q_A Be/Be) + \delta_r(Q'Be/Be)$$

which is impossible, since both summands are strictly positive by the minimality of (Be, C^*). Thus, $Q \subseteq C_A$. But Q is disjoint from Se and $|Q| = m = |C_A \setminus Se|$, hence $Q = C_A \setminus Se$. □

By biminimality and the construction of C_A, $\{s \in S : s$ is connected to some node in $C_A \setminus Se\} = A$. But also, by biminimality, every element of B is connected to some node of $Q = (C_A \setminus Se)$. Thus, $B = A$ and $A \in X$, as desired. $\square$

This coding technique is enough to show that any model of Σ_r^+ interprets Robinson's R, which we now define.

Definition 3.4. Let $L_R = \{+, \cdot, \leq, \eta_k\}_{k\in\omega}$. Robinson's R is the following L_R-theory: For every $k, \ell \in \omega$:

(1) $\eta_k + \eta_\ell = \eta_{k+\ell}$
(2) $\eta_k \cdot \eta_\ell = \eta_{k\ell}$
(3) $\eta_k \neq \eta_\ell$ for $k \neq \ell$
(4) $\forall x(x \leq \eta_k \to x = \eta_0 \vee \ldots \vee x = \eta_k)$
(5) $\forall x(x \leq \eta_k \vee \eta_k \leq x)$

Theorem 3.5. *For any rational $r \in (0, 1)$, the theory Σ_r^+ interprets Robinson's R.*

Proof. Fix a rational $r \in (0, 1)$. For ease of discourse, also fix a model $M \models \Sigma_r^+$. The interpretation $(\Omega, +, \cdot, \leq, \eta_k)_{k\in\omega}$ we give will be uniform in M. Also fix a minimal 0-extension (A, B). That is, $A, B \in \mathbf{K}_r^+$, $A \subsetneq B$, $\delta_r(B) = \delta_r(A)$, but $\delta_r(B') > \delta_r(A)$ for every $A \subsetneq B' \subsetneq B$. As notation, let C be the subgraph with universe $B \setminus A$. Suppose $|A| = n$ and $|C| = m$.

Let $\mathcal{A} = \{a \in M^n$: there is an isomorphism $f : A \to a$ such that for all embeddings $g_1, g_2 : B \to M$ extending f, either $g_1(B) = g_2(B)$ (setwise) or else $g_1(B) \cap g_2(B) = a\}$.

For $a \in \mathcal{A}$, let $\mathcal{C}_a = \{c \in M^m : (a, ac) \cong (A, B)\}$ and let $\mathcal{C}_a^* = \bigcup \mathcal{C}_a$. It follows from our definition of $\mathcal{A}$ that $c \cap c' = \varnothing$ for all distinct $c, c' \in \mathcal{C}_a$. A subset $\mathcal{B}_a \subseteq \mathcal{C}_a^*$ is a *basis* for $\mathcal{C}_a$ if it consists of exactly one element from each $c \in \mathcal{C}_a$.

We wish to define the notion of '$\mathcal{C}_a$ and $\mathcal{C}_{a'}$ having the same cardinality.' Proposition 3.3 allows us to succeed, at least when the sets $\mathcal{C}_a$ and $\mathcal{C}_{a'}$ are finite. For a fixed parameter u, let $R_u^k(\bar{x})$ denote the relation $R^k(\bar{x}, u)$. Then for $a, a' \in \mathcal{A}$, say $a \sim a'$ if and only if $\exists u, v, w[R_u^1(M) \cap \mathcal{C}_a^*$ is a basis for $\mathcal{C}_a$, $R_v^1(M) \cap \mathcal{C}_{a'}^*$ is a basis for $\mathcal{C}_{a'}^*$, and $R_w^2(M)$ codes a bijection between $R_u^1(M) \cap \mathcal{C}_a^*$ and $R_v^1(M) \cap \mathcal{C}_{a'}^*]$.

More formally, say that a set R of unordered pairs 'codes a bijection between the sets S and T' if the following three conditions hold: (1) $\{x, y\} \in R$ implies either $[x \in S \setminus T$ and $y \in T \setminus S]$ or $[x \in T \setminus S$ and $y \in S \setminus T]$, (2) $\bigcup R = S \triangle T$, and (3) if $\{x, y\}, \{x', y'\} \in R$ are distinct sets, then they are disjoint.

Claim: If $a \in \mathcal{A}$ and $\mathcal{C}_a$ is finite, then for every $a' \in \mathcal{A}$, $a \sim a'$ if and only if $|\mathcal{C}_a| = |C_{a'}|$.

Proof. It is evident that if $a, a' \in \mathcal{A}$ and and $\mathcal{C}_a$ is finite, then for any bases $\mathcal{B}_a$ and $\mathcal{B}_{a'}$ of $\mathcal{C}_a$ and $\mathcal{C}_{a'}$, respectively, the existence of a bijection between $\mathcal{B}_a$ and $\mathcal{B}_{a'}$ implies $|\mathcal{C}_a| = |\mathcal{C}_{a'}|$. Thus, $a \sim a'$ implies $|\mathcal{C}_a| = |\mathcal{C}_{a'}|$.

Conversely, if $|\mathcal{C}_a| = |\mathcal{C}_{a'}|$ and is finite, then let $\mathcal{B}_a$ and $\mathcal{B}_{a'}$ be any two bases for $\mathcal{C}_a$ and $\mathcal{C}_{a'}$, respectively. As these sets are finite and of the same cardinality, regardless of their intersection there is a set $W \subseteq [\mathcal{C}_a^* \cup \mathcal{C}_{a'}^*]^2$ of unordered pairs 'coding a bijection' as described above. But, by Proposition 3.3, since all three sets are finite there are elements $u, v, w \in M$ such that $R_u^1(M) \cap \mathcal{C}_a^* = \mathcal{B}_a$, $R_v^1(M) \cap \mathcal{C}_{a'}^* = \mathcal{B}_{a'}$, and $R_w^2(M) = W$, so $a \sim a'$. $\square$

Despite the fact that $\sim$ is well-behaved whenever $\mathcal{C}_a$ is finite, it need not be an equivalence relation on all of $\mathcal{A}$. It is visibly symmetric on $\mathcal{A}$, but since we have no control over the coding of infinite sets, it need not be either reflexive or transitive on all of $\mathcal{A}$. To remedy this, let

$$\mathcal{A}' = \{a \in \mathcal{A} : a \sim a \wedge \forall a', a'' \in \mathcal{A}[(a \sim a' \wedge a \sim a'') \to (a' \sim a'')]\}$$

Clearly, $\mathcal{A}'$ is a definable subset of M^n and $\sim$ is an equivalence relation on $\mathcal{A}'$. Let

$$\Omega = \mathcal{A}'/\sim$$

For each $k \in \omega$, let $D_k = \oplus_{i<k}(B/A)$ be the free join of k copies of B over A. Each $D_k \in \mathbf{K}_r^+$, so since $M \models \theta(\varnothing, D_k, m)$, there is $a_k \in \mathcal{A}$ such that $|\mathcal{C}_a| = k$. By the Claim, $a_k \in \mathcal{A}'$ as well, so define $\eta_k = a_k/\sim$. Again, by the Claim, this is well-defined. That is, for any k, $\eta_k = a/\sim$ for any $a \in \mathcal{A}'$ such that $|\mathcal{C}_a| = k$.

To define $\leq$ on Ω, by analogy with coding bijections above, say that a set $R \subseteq [S \cup T]^2$ *codes an injection from S into T* if (1) $\{x, y\} \in R$ implies either $[x \in S \setminus T$ and $y \in T \setminus S]$ or $[x \in T \setminus S$ and $y \in S \setminus T]$, (2) $(S \setminus T) \subseteq \bigcup R$, and (3) if $\{x, y\}, \{x', y'\} \in R$ are distinct sets, then they are disjoint.

Then, for $\mathbf{a}, \mathbf{a}' \in \Omega$, define $\mathbf{a} \leq \mathbf{a}'$ if and only if there exist $a \in \mathbf{a}$, $a' \in \mathbf{a}'$ and there exist u, v, w such that $R_u^1(M) \cap \mathcal{C}_a^*$ is a basis for $\mathcal{C}_a$, $R_v^1(M) \cap \mathcal{C}_{a'}^*$ is a basis for $\mathcal{C}_{a'}^*$, and $R_w^2(M)$ codes an injection between $R_u^1(M) \cap \mathcal{C}_a^*$ and $R_v^1(M) \cap \mathcal{C}_{a'}^*$.

It is not at all clear that $\leq$ defines a partial order on all of Ω, but this is not relevant. It is clear that for $k, \ell \in \omega$, $\eta_k \leq \eta_\ell$ if and only if $k \leq \ell$ and that $\eta_k \leq a/\sim$ for any $a \in \mathcal{A}'$ such that $|\mathcal{C}_a| \geq k$.

To define addition and multiplication on Ω requires one additional idea. Call a triple $\{\mathbf{a}, \mathbf{a}', \mathbf{a}''\} \subseteq \Omega$ *mutually separable* if there exist $a \in \mathbf{a}$, $a' \in \mathbf{a}'$, $a'' \in \mathbf{a}''$ with pairwise disjoint bases $\mathcal{B}_a, \mathcal{B}_{a'}, \mathcal{B}_{a''}$ for $\mathcal{C}_a, \mathcal{C}_{a'}, \mathcal{C}_{a''}$, respectively.

Note that for any $j, k, \ell \in \omega$, the graph $D_{j,k,\ell} := D_j \oplus_\varnothing D_k \oplus_\varnothing D_\ell$ is in $\mathbf{K}_r^+$, so since $M \models \theta(\varnothing, D_{j,k,\ell}, m)$, there exist $a_j, a_k, a_\ell \in M^n$ such that $\mathcal{C}_{a_j}^*, \mathcal{C}_{a_k}^*, \mathcal{C}_{a_\ell}^*$ are pairwise disjoint. Thus, any triple $\{\eta_j, \eta_k, \eta_\ell\}$ of 'standard' elements from Ω is mutually separable.

Now define addition on Ω by $\mathbf{a} + \mathbf{a}' = \mathbf{a}''$ if and only if EITHER they are mutually separable and there exist $a \in \mathbf{a}, a' \in \mathbf{a}', a'' \in \mathbf{a}''$ and disjoint bases $\mathcal{B}_a, \mathcal{B}_{a'}, \mathcal{B}_{a''}$ of $\mathcal{C}_a, \mathcal{C}_{a'}, \mathcal{C}_{a''}$ and there exist u_1, u_2, u_3, w such that $R_{u_1}^1(M) \cap \mathcal{C}_a^* = \mathcal{B}_a$, $R_{u_2}^1(M) \cap \mathcal{C}_{a'}^* = \mathcal{B}_{a'}$, $R_{u_3}^1(M) \cap \mathcal{C}_{a''}^* = \mathcal{B}_{a''}$ and R_w^2 codes a bijection between $\mathcal{B}_a \cup \mathcal{B}_{a'}$ and $\mathcal{B}_{a''}$ OR either they are not mutually separable or such u_1, u_2, u_3, w do not exist and $\mathbf{a}'' = \mathbf{a}$.

From the above, it is clear that for any $k, \ell \in \omega$, the first clause holds in evaluating $\eta_k + \eta_\ell$, and it is easily checked that $\eta_k + \eta_\ell = \eta_{k+\ell}$.

The definition of multiplication on Ω is almost identical. The only change is that R_w^3 codes a bijection between unordered pairs from $\mathcal{B}_a \cup \mathcal{B}_{a'}$ (one element from each set) and $\mathcal{B}_{a''}$. Again, this function is well behaved on the 'standard part' i.e., $\eta_k \cdot \eta_\ell = \eta_{k\ell}$ for all $k, \ell \in \omega$.

Thus, the structure $(\Omega, +, \cdot, \leq, \eta_k)_{k \in \omega} \models R$. □

Recall that a consistent theory T is essentially undecidable if every consistent extension of T is undecidable.

Corollary 3.6. *Σ_r^+ is essentially undecidable.*

Proof. It is shown in Part II, Theorem 9 of [8] that R is essentially undecidable. Tarski shows that essential undecidability is transferred by interpretations in Part I, Theorem 7. □

Remark 3.7. It is worth noting that while the interpreted model satisfies Robinson's Q when M is the generic, this is not generally true. In particular, for the ultraproducts $\prod_{\mathcal{U}} M_{\alpha_n}$ with $\{ \alpha_n \}$ a sequence of decreasing irrationals converging to r and M_{α_n} the Shelah-Spencer graph of weight α_n, there is a sentence σ which says that there is some copy of A which has the maximal possible number of copies of B embedded over it. This sentence is true in the ultraproduct as well, and the order type of the interpreted $(\omega, \leq)$ has a copy of ω^* (ω reversed) as a tail.

Finally, we conclude this section by showing that Σ_r^+ has the maximal number of completions.

Corollary 3.8. *Σ_r^+ has $2^{\aleph_0}$ completions.*

Proof. As any finite extension of Σ_r^+ is recursively axiomatizable, it cannot be complete. Using this, we define a tree $\{T_\eta : \eta \in {}^{<\omega}2\}$ of consistent, finite extensions of Σ_r^+, with $T_\eta \subseteq T_\nu$ whenever $\eta \lhd \nu$ and $T_{\eta^\frown 0} \cup T_{\eta^\frown 1}$ inconsistent for each η. Each of the branches extends to a complete extension. □

4. Going up, coming down, and general ultraproducts

Recall that for any α, $\mathbf{K}_\alpha^+ \subseteq \mathbf{K}_\alpha$ and $A \preccurlyeq_\alpha B$ implies $A \leq_\alpha B$, with equality holding whenever α is irrational. Furthermore, it follows immediately from the definition of the dimension functions that for all $\alpha < \beta$, $\mathbf{K}_\beta \subseteq \mathbf{K}_\alpha^+$ and $A \leq_\beta B$ implies $A \preccurlyeq_\alpha B$ for all finite graphs A, B.

4.1. Increasing sequences.

Lemma 4.1. *Let α^* be given. Then*

1. *If $A \notin \mathbf{K}_{\alpha^*}$ then there is $\epsilon > 0$ such that $A \notin \mathbf{K}_\alpha^+$ for all $\alpha \in (\alpha^* - \epsilon, \alpha^*)$*
2. *If $A, B \in \mathbf{K}_{\alpha^*}$ and $A \leq_{\alpha^*} B$, then there is $\epsilon > 0$ such that for any $\alpha \in (\alpha^* - \epsilon, \alpha^*)$, $A, B \in \mathbf{K}_\alpha$ and $A \leq_\alpha B$.*

Theorem 4.2. *Assume that $\{\alpha_n\}$ is a strictly increasing sequence from $(0, 1)$ that converges to some $\alpha^* \in (0, 1]$. For each $n \in \omega$, let N_n be elementarily equivalent to either M_{α_n} or $M_{\alpha_n}^+$ and let $\mathcal{U}$ be any non-principal ultrafilter on ω. Then the ultraproduct $\Pi_{\mathcal{U}} N_n$ is elementarily equivalent to M_{α^*}. In particular, its theory is decidable, Π_2-axiomatized by S_{α^*} and is stable. The theory is $\aleph_0$-stable if α^* is rational, but strictly stable if α^* is irrational.*

Proof. In light of Theorems 2.7 and 2.8 we need only show that $\Pi_{\mathcal{U}} N_n \models S_{\alpha^*}$. For each m, let $\sigma(m)$ be the universal sentence in S_{α^*} prescribing the substructures of size m. Since up to isomorphism there are only finitely many graphs of size m, and since each finite graph has only finitely many subgraphs, Lemma 4.1(1) implies that there is an $\epsilon > 0$ such that $N_n \models \sigma(m)$ whenever $\alpha_n \in (\alpha^* - \epsilon, \alpha^*)$. Thus, $N_n \models \sigma(m)$ for cofinitely many n, so $\Pi_{\mathcal{U}} N_n \models \sigma(m)$ by Łoś's theorem since $\mathcal{U}$ is non-principal.

The justification that $\Pi_{\mathcal{U}} N_n \models \psi(A, B)$ for each $A \leq_{\alpha^*} B$ is identical, using Lemma 4.1(2). □

4.2. **Decreasing sequences.** In this subsection we consider decreasing sequences $\{\alpha_n\}$ from $(0,1)$. We first note a special case. What distinguishes it from the other cases is the lack of 0-extensions, minimal or otherwise.

Remark 4.3. A special case occurs when $\alpha^* = 0$. Note that $\mathbf{K}_0^+ = \mathbf{K}_0$ is the class of all finite graphs, and for any graphs A, B, $A \leq_0 B$ if and only if $A \preccurlyeq_0 B$ if and only if $A \subseteq B$. Thus, the generic M_0 of $(\mathbf{K}_0, \leq_0) = \mathbf{K}_0^+, \preccurlyeq_0)$ is the Fraïsse limit of the class of all finite graphs and is usually referred to as the 'Random Graph.' Its theory is decidable, being axiomatized by the set of $\forall\exists$ axioms asserting that for any two finite, disjoint subsets F and G, there exists an element connected to every point in F, and to no point of G. As well, the theory of the Random Graph is ω-categorical but unstable, being the paradigm of a theory with the Independence Property.

Theorem 4.4. *Assume that $\{\alpha_n\}$ is a strictly decreasing sequence from $(0,1)$ that converges to some α^*. For each $n \in \omega$, let N_n be elementarily equivalent to either M_{α_n} or $M_{\alpha_n}^+$ and let $\mathcal{U}$ be any non-principal ultrafilter on ω. Then the ultraproduct $\Pi_{\mathcal{U}} N_n \models \Sigma_{\alpha^*}^*$ satisfies one of the following three conditions, depending on α^*:*

- *If α^* is irrational, then its theory is decidable, $\forall\exists$-axiomatizable, and strictly stable.*
- *If α^* is rational and positive, then $\Pi_{\mathcal{U}} N_n$ interprets Robinson's R, its theory is unstable, and some subtheory is essentially undecidable.*
- *If $\alpha^* = 0$, then the theory of the ultraproduct is equivalent to the theory of the Random Graph, which is decidable, $\forall\exists$-axiomatizable, ω-categorical, and unstable.*

Proof. We begin by showing that $\Pi_{\mathcal{U}} N_n \models \Sigma_{\alpha^*}^*$. Clearly, for each m and $\sigma(m) \in \Sigma_{\alpha^*}^+$, every $N_n \models \sigma(m)$, so $\Pi_{\mathcal{U}} N_n \models \sigma(m)$ as well. As $\theta(A, B, m)$ implies $\psi(A, B)$, it now suffices to prove that $\Pi_{\mathcal{U}} N_n \models \theta(A, B, m)$ for all $A, B \in \mathbf{K}_{\alpha^*}^+$ satisfying $A \preccurlyeq_{\alpha^*} B$. So fix $A, B \in \mathbf{K}_{\alpha^*}^+$ such that $A \preccurlyeq_{\alpha^*} B$. Arguing as in Lemma 4.1 with the inequalities reversed, there is an $\epsilon > 0$ such that $A, B \in K_\alpha$ and $A \preccurlyeq_{\alpha_n} B$ for all $\alpha \in (\alpha^*, \alpha^* + \epsilon)$. Furthermore, since there are only finitely many graphs $C \supseteq B$ satisfying $|C \setminus B| \leq m$, by shrinking ϵ we may additionally assume that $B \preccurlyeq_{\alpha^*} C \Leftrightarrow B \preccurlyeq_\alpha C$ for each of these C's and for every $\alpha \in (\alpha^*, \alpha^* + \epsilon)$. Since each $N_n \models \theta(A, B, m)$ whenever $A \preccurlyeq_{\alpha_n} B$, it follows that $N_n \models \theta(A, B, m)$ for cofinitely many $n \in \omega$. Thus, $\Pi_{\mathcal{U}} N_n \models \theta(A, B, m)$ since $\mathcal{U}$ is non-principal.

Now that $Th(\Pi_{\mathcal{U}} N_n)$ extends $\Sigma_{\alpha^*}^+$, the itemized cases follow from Theorem 2.7, Theorem 3.5, and Remark 4.3, respectively. □

4.3. **General ultraproducts.** We extend our results to arbitrary ω-sequences $\{\alpha_n\}$ and arbitrary untrafilters $\mathcal{U}$ on ω.

Definition 4.5. Suppose $\{\alpha_n\}$ is an ω-sequence of real numbers from $(0,1)$, $\mathcal{U}$ is an ultrafilter on ω, and $\alpha^* \in [0,1]$. We say that

- $\{\alpha_n\}$ *$\mathcal{U}$-converges to α^** if $\{n \in \omega : |\alpha_n - \alpha^*| < \epsilon\} \in \mathcal{U}$ for every $\epsilon > 0$;
- $\{\alpha_n\}$ *$\mathcal{U}$-converges to α^* from below* if $\{n \in \omega : \alpha_n \in (\alpha^* - \epsilon, \alpha^*)\} \in \mathcal{U}$ for every $\epsilon > 0$;
- $\{\alpha_n\}$ *$\mathcal{U}$-converges to α^* from above* if $\{n \in \omega : \alpha_n \in (\alpha^*, \alpha^* + \epsilon)\} \in \mathcal{U}$ for every $\epsilon > 0$; and
- $\{\alpha_n\}$ *$\mathcal{U}$-concentrates on α^** if $\{n \in \omega : \alpha_n = \alpha^*\} \in \mathcal{U}$.

Lemma 4.6. *Let $\{\alpha_n\}$ be any ω-sequence of real numbers from $(0,1)$ and let $\mathcal{U}$ be any ultrafilter on ω. Then $\{\alpha_n\}$ $\mathcal{U}$-converges to a unique real number $\alpha^* \in [0,1]$. Moreover, the $\mathcal{U}$-convergence is either from above, from below, or $\{\alpha_n\}$ $\mathcal{U}$-concentrates on α^*, and these possibilities are mutually exclusive.*

Proof. For each $k \geq 1$ and $i < 2^k$, let Q_i denote the half-open interval $[i/2^k, (i+1)/2^k)$ and let $\overline{Q_i}$ denote its closure. For each $k \geq 1$, as $\{Q_i : i < 2^k\}$ is a finite partition of $[0,1)$, there is a unique $i(k)$ such that $\{n \in \omega : \alpha_n \in Q_{i(k)}\} \in \mathcal{U}$. Moreover, for each k, $Q_{i(k+1)} \subseteq Q_{i(k)}$ and the diameters are decreasing to zero, so there is a unique real number $\alpha^* \in \bigcap_{k\geq 1} \overline{Q_{i(k)}}$. The trichotomy follows immediately, since the three sets $\{n \in \omega : \alpha_n < \alpha^*\}$, $\{n \in \omega : \alpha_n = \alpha^*\}$, and $\{n \in \omega : \alpha_n > \alpha^*\}$ obviously partition ω. □

Remark 4.7. Clearly, if the ultrafilter is non-principal, and either $\{\alpha_n\}$ $\mathcal{U}$-converges to 1 or the sequence $\{\alpha_n\}$ is strictly increasing, then the $\mathcal{U}$-convergence is from below. By inspecting the proof of Theorem 4.2, it is easily checked that if $\{\alpha_n\}$ $\mathcal{U}$-converges to α^* from below, then the ultraproduct $\Pi_{\mathcal{U}} N_n$ is elementarily equivalent to M_{α^*}.

Remark 4.8. Dually, if $\mathcal{U}$ is non-principal and either $\{\alpha_n\}$ $\mathcal{U}$-converges to 0 or the sequence $\{\alpha_n\}$ is strictly decreasing, then the $\mathcal{U}$-convergence is from above. As well, the proof of Theorem 4.4 holds whenever$\{\alpha_n\}$ $\mathcal{U}$-converges to α^* from above.

We summarize our results with the Theorem below.

Theorem 4.9. *Let $\{\alpha_n\}$ be any sequence of reals in $(0,1)$, for each $n \in \omega$ let N_n be elementarily equivalent to either M_{α_n} or $M^+_{\alpha_n}$, and let $\mathcal{U}$ be any ultrafilter on ω. Then exactly one of the four possibilities hold:*

1. *For some rational $\alpha^* \in (0,1]$, $\Pi_{\mathcal{U}} N_n \equiv M_{\alpha^*}$. Its theory is decidable, Π_2-axiomatized by S_{α^*}, and $\aleph_0$-stable.*
2. *For some irrational $\alpha^* \in (0,1)$, $\Pi_{\mathcal{U}} N_n \equiv M_{\alpha^*}$. Its theory is decidable, Π_2-axiomatized by S_{α^*}, and stable, unsuperstable.*
3. *For some rational $\alpha^* \in (0,1)$, $\Pi_{\mathcal{U}} N_n \models \Sigma^+_{\alpha^*}$ and interprets Robinson's R. Its theory is unstable and contains an essentially undecidable subtheory.*
4. *$\Pi_{\mathcal{U}} N_n$ is elementarily equivalent to the classical 'Random graph'.*

Proof. Let $\{\alpha_n\}$, $\{N_n\}$, and $\mathcal{U}$ be given. Let $\alpha^* \in [0,1]$ be the unique value to which $\{\alpha_n\}$ $\mathcal{U}$-converges. If $\{\alpha_n\}$ $\mathcal{U}$-converges to α^* from below, then by Remark 4.7 $\Pi_{\mathcal{U}} N_n$ is elementarily equivalent to M_{α^*}, and we are in either Case 1 or Case 2, depending on whether α^* is rational or irrational.

If $\{\alpha_n\}$ $\mathcal{U}$-converges to α^* from above and $\alpha^* \neq 0$, then Remark 4.8 places us into Case 2 or Case 3, again depending on the rationality of α^*.

Next, suppose that $\{\alpha_n\}$ $\mathcal{U}$-concentrates on α^*. If α^* is irrational, then we are visibly in Case 2. On the other hand, if α^* is rational, then we are in either Case 2 or Case 3, depending on which of the sets $\{n \in \omega : N_n \equiv M_{\alpha^*}\}$, $\{n \in \omega : N_n \equiv M^+_{\alpha^*}\}$ is in the ultrafilter.

Finally, if $\alpha^* = 0$, then the $\mathcal{U}$-convergence must be from above and Remark 4.3 applies, placing us in Case 4. □

We close by remarking that this Theorem demonstrates that almost no property, positive or negative, is preserved under ultraproducts. For example, it may be that each N_n has an ω-stable, decidable theory, yet the ultraproduct interprets

Robinson's R. Conversely, we may have models N_n, each interpreting Robinson's R, for which the theory of the ultraproduct is ω-stable and decidable.

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