The Strength of the Grätzer-Schmidt Theorem

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Abstract. The Grätzer-Schmidt theorem of lattice theory states that each algebraic lattice is isomorphic to the congruence lattice of an algebra. A lattice is algebraic if it is complete and generated by its compact elements. We show that the set of indices of computable lattices that are complete is Π^1_1-complete; the set of indices of computable lattices that are algebraic is Π^1_1-complete; and that there is a computable lattice L such that the set of compact elements of L is Π^1_1-complete. As a corollary, there is a computable algebraic lattice that is not computably isomorphic to any computable congruence lattice.
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Introduction

The Grätzer-Schmidt theorem [2], also known as the *congruence lattice representation theorem*, states that each algebraic lattice is isomorphic to the congruence lattice of an algebra. It established a strong link between lattice theory and universal algebra. In this article we show that this theorem as stated fails to hold effectively in a very strong way.

We use notation associated with partial computable functions, φ_e, $\varphi_{e,s}$, $\varphi^\sigma_{e,s}$, φ^f_e as in Odifreddi [3]. Following Sacks [5] page 5, a Π^1_1 subset of ω may be written in the form

$$C_e = \{n \in \omega \mid \forall f \in \omega^\omega \ \varphi^f_e(n) \downarrow\}.$$

A subset $A \subseteq \omega$ is Π^1_1-*hard* if each Π^1_1 set is m-reducible to A; that is, for each e, there is a computable function f such that for all n, $n \in C_e$ iff $f(n) \in A$. A is Π^1_1-*complete* if it is both Π^1_1 and Π^1_1-hard. It is well known that such sets exist. Fix for the rest of the paper a number e_0 so that C_{e_0} is Π^1_1-complete. With each n, the set C_{e_0} associates a tree T'_n defined by

$$T'_n = \{\sigma \in \omega^{<\omega} \mid \varphi^\sigma_{e_0,|\sigma|}(n) \uparrow\}.$$

Note that T'_n has no infinite path iff $n \in C_e$.

A *computable lattice* $(L, \preceq)$ has underlying set $L = \omega$ and an lattice ordering $\preceq$ that is formally a subset of ω^2.

We will use the symbol $\preceq$ for lattice orderings, and reserve the symbol $\leq$ for the natural ordering of the ordinals and in particular of ω. Meets and joins corresponding to the order $\preceq$ are denoted by $\wedge$ and $\vee$. Below we will seek to build computable lattices from the trees T'_n; since for many n, T'_n will be finite, and a computable lattice must be infinite according to our definition, we will work with the following modification of T'_n:

$$T_n = T'_n \cup \{\langle i \rangle : i \in \omega\} \cup \{\varnothing\}$$

where $\varnothing$ denotes the empty string and $\langle i \rangle$ is the string of length 1 whose only entry is i. This ensures that T_n has the same infinite paths as T'_n, and each T_n is infinite. Moreover the sequence $\{T_n\}_{n\in\omega}$ is still uniformly computable.

1 Computational strength of lattice-theoretic concepts

1.1 Completeness

Definition 1. *A lattice* $(L, \preceq)$ *is* complete *if for each subset* $S \subseteq L$*, both* $\sup S$ *and* $\inf S$ *exist.*

Lemma 1. *The set of indices of computable lattices that are complete is* Π^1_1.

Proof. The statement that $\sup S$ exists is equivalent to a first order statement in the language of arithmetic with set variable S:

$$\exists a[\forall b(b \in S \to b \preceq a) \ \ \& \ \ \forall c((\forall b(b \in S \to b \preceq c) \to a \preceq c)].$$

The statement that $\inf S$ exists is similar, in fact dual. Thus the statement that L is complete consists of a universal set quantifier over S, followed by an arithmetical matrix.

Example 1. In set-theoretic notation, $(\omega+1, \leq)$ is complete. Its sublattice $(\omega, \leq)$ is not, since $\omega = \sup \omega \notin \omega$.

Proposition 1. *The set of indices of computable lattices that are complete is* Π^1_1*-hard.*

Proof. Let L_n consist of two disjoint copies of T_n, called T_n and T^*_n. For each $\sigma \in T_n$, its copy in T^*_n is called σ^*. Order L_n so that T_n has the prefix ordering

$$\sigma \preceq \sigma^\frown\tau,$$

T^*_n has the reverse prefix ordering, and $\sigma \prec \sigma^*$ for each $\sigma \in T_n$. We take the transitive closure of these axioms to obtain the order of L_n; see Figure 1.

Next, we verify that L_n is a lattice. For any σ, $\tau \in T_n$ we must show the existence of (1) $\sigma \vee \tau$, (2) $\sigma \wedge \tau$, (3) $\sigma \vee \tau^*$, and (4) $\sigma \wedge \tau^*$; the existence of $\sigma^* \vee \tau^*$ and $\sigma^* \wedge \tau^*$ then follows by duality.

We claim that for any strings α, $\sigma \in T_n$, we have $\alpha^* \succeq \sigma$ iff α is comparable with σ; see Figure 1. In one direction, if $\alpha \succeq \sigma$ then $\alpha^* \succeq \alpha \succeq \sigma$, and if $\sigma \succeq \alpha$

then $\alpha^* \succeq \sigma^* \succeq \sigma$. In the other direction, if $\alpha^* \succeq \sigma$ then by the definition of $\preceq$ as a transitive closure there must exist ρ with $\alpha^* \succeq \rho^* \succeq \rho \succeq \sigma$. Then $\alpha \preceq \rho$ and $\sigma \preceq \rho$, which implies that α and ρ are comparable.

Using the claim we get that (1) $\sigma \vee \tau$ is $(\sigma \wedge \tau)^*$, where (2) $\sigma \wedge \tau$ is simply the maximal common prefix of σ and τ; (3) $\sigma \vee \tau^*$ is $\sigma^* \vee \tau^*$ which is $(\sigma \wedge \tau)^*$; and (4) $\sigma \wedge \tau^*$ is $\sigma \wedge \tau$.

It remains to show that $(L_n, \preceq)$ is complete iff T_n has no infinite path. So suppose T_n has an infinite path S. Then $\sup S$ does not exist, because S has no greatest element, S^* has no least element, each element of S^* is an upper bound of S, and there is no element above all of S and below all of S^*.

Conversely, suppose T_n has no infinite path and let $S \subseteq L_n$. If S is finite then $\sup S$ exists. If S is infinite then since T_n has no infinite path, there is no infinite linearly ordered subset of L_n, and so S contains two incomparable elements σ and τ. Because T_n is a tree, $\sigma \vee \tau$ is in T_n^*. Now the set of all elements of L_n that are above $\sigma \vee \tau$ is finite and linearly ordered, and contains all upper bounds of S. Thus there is a least upper bound for S. Since L_n is self-dual, i.e. $(L_n, \preceq)$ is isomorphic to $(L_n, \succeq)$ via $\sigma \mapsto \sigma^*$, infs also always exist. So L_n is complete.

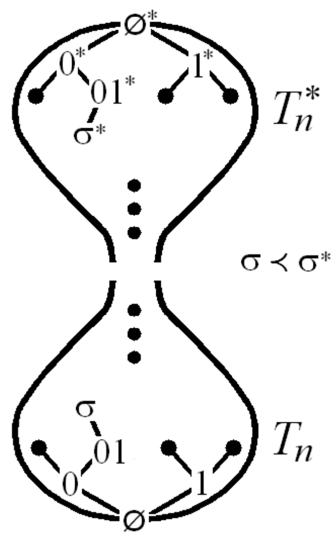

Fig. 1. The lattice L_n from Proposition 1.

1.2 Compactness

Definition 2. *An element $a \in L$ is* compact *if for each subset $S \subseteq L$, if $a \preceq \sup S$ then there is a finite subset $S' \subseteq S$ such that $a \preceq \sup S'$.*

Lemma 2. *In each computable lattice L, the set of compact elements of L is Π^1_1.*

Proof. Similarly to the situation in Lemma 1, the statement that a is compact consist of a universal set quantifier over S followed by an arithmetical matrix.

Example 2. Let $L[a] = \omega + 1 \cup \{a\}$ be ordered by $0 \prec a \prec \omega$, and let the element a be incomparable with the positive numbers. Then a is not compact, because $a \preceq \sup \omega$ but $a \not\preceq \sup S'$ for any finite $S' \subseteq \omega$.

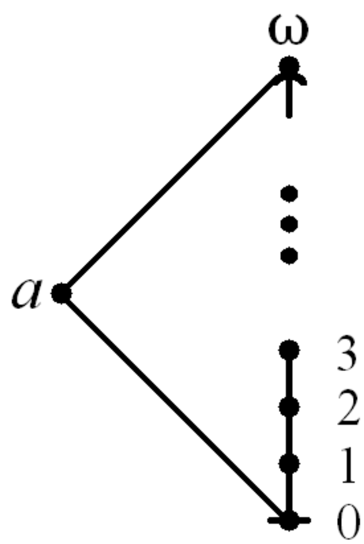

Fig. 2. The lattice $L[a]$ from Example 2.

The following result will be useful for our study of the Grätzer-Schmidt theorem.

Proposition 2. *There is a computable algebraic lattice L such that the set of compact elements of L is Π^1_1-hard.*

Proof. Let L consist of disjoint copies of the trees T_n, $n \in \omega$, each having the prefix ordering; least and greatest elements 0 and 1; and elements a_n, $n \in \omega$, such that $\sigma \prec a_n$ for each $\sigma \in T_n$, and a_n is incomparable with any element not in $T_n \cup \{0, 1\}$ (see Figure 3).

Suppose T_n has an infinite path S. Then $a_n = \sup S$ but $a_n \not\preceq \sup S'$ for any finite $S' \subseteq S$, since $\sup S'$ is rather an element of S. Thus a_n is not compact.

Conversely, suppose T_n has no infinite path, and $a_n \preceq \sup S$ for some set $S \subseteq L$. If S contains elements from $T_m \cup \{a_m\}$ for at least two distinct values of m, say $m_1 \neq m_2$, then $\sup S = 1 = \sigma_1 \vee \sigma_2$ for some $\sigma_i \in S \cap (T_{m_i} \cup \{a_{m_i}\})$, $i = 1, 2$. So $a_n \preceq \sup S'$ for some $S' \subseteq S$ of size two. If S contains 1, there is nothing to prove. The remaining case is where S is contained in $T_m \cup \{a_m, 0\}$ for some m. Since $a_n \preceq \sup S$, it must be that $m = n$. If S is finite or contains a_n, there is nothing to prove. So suppose S is infinite. Since T_n has no infinite path, there must be two incomparable elements of T_n in S. Their join is then a_n, since T_n is a tree, and so $a_n \preceq \sup S'$ for some $S' \subseteq S$ of size two.

Thus we have shown that a_n is compact if and only if T_n has no infinite path. There is a computable presentation of L where a_n is a computable function of n, for instance we could let $a_n = 2n$. Thus letting $f(n) = 2n$, we have that T_n has no infinite path iff $f(n)$ is compact, i.e. $\{a \in L : a \text{ is compact}\}$ is Π^1_1-hard.

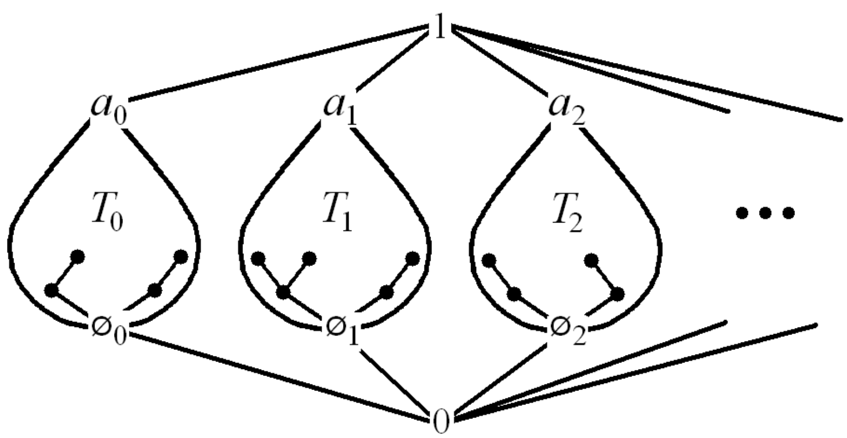

Fig. 3. The lattice L from Proposition 2.

1.3 Algebraicity

Definition 3. *A lattice* $(L, \preceq)$ *is* compactly generated *if* $C = \{a \in L : a \text{ is compact}\}$ *generates* L *under sup, i.e., each element is the supremum of its compact predecessors. A lattice is* algebraic *if it is complete and compactly generated.*

Lemma 3. *The set of indices of computable lattices that are algebraic is* Π^1_1.

Proof. L is algebraic if it is complete (this property is Π^1_1 by Lemma 1) and *each element is the least upper bound of its compact predecessors*, i.e., any element that is above all the compact elements below a is above a:

$$\forall b(\forall c(c \in C \ \& \ c \preceq a \to c \preceq b) \to a \preceq b)$$

Equivalently,

$$\forall b(\exists c(c \in C \;\;\&\;\; c \preceq a \;\;\&\;\; c \not\preceq b) \text{ or } a \preceq b)$$

This is equivalent to a Π^1_1 statement since, by the Axiom of Choice, any statement of the form $\exists c\, \forall S\; A(c, S)$ is equivalent to $\forall (S_c)_{c\in\omega}\; \exists c\; A(c, S_c)$

Example 3. The lattice $(\omega + 1, \leq)$ is compactly generated, since the only non-compact element ω satisfies $\omega = \sup \omega$. The lattice $L[a]$ from Example 2 and Figure 2 is not compactly generated, as the non-compact element a is not the supremum of $\{0\}$.

Proposition 3. *The set of indices of computable lattices that are algebraic is Π^1_1-hard.*

Proof. Let the lattice $T_n[a]$ consist of T_n with the prefix ordering, and additional elements $0 \prec a \prec 1$ such that a is incomparable with each $\sigma \in T_n$, and 0 and 1 are the least and greatest elements of the lattice. Note that $T_n[a]$ is always complete, since any infinite set has supremum equal to 1. We claim that $T_n[a]$ is algebraic iff T_n has no infinite path.

Suppose T_n has an infinite path S. Then $a \preceq \sup S$, but $a \not\preceq \sup S'$ for any finite $S' \subseteq S$. Thus a is not compact, and so a is not the sup of its compact predecessors (0 being its only compact predecessor), which means that $T_n[a]$ is not an algebraic lattice.

Conversely, suppose $T_n[a]$ is not algebraic. Then some element of $T_n[a]$ is not the join of its compact predecessors. In particular, some element of $T_n[a]$ is not compact. So there exists a set $S \subseteq T_n[a]$ such that for all finite subsets $S' \subseteq S$, $\sup S' < \sup S$. In particular S is infinite. Since each element except 1 has only finitely many predecessors, we have $\sup S = 1$. Notice that $T_n[a] \backslash \{1\}$ is actually a tree, so if S contains two incomparable elements then their join is already 1, contradicting the defining property of S. Thus S is linearly ordered, and infinite, which implies that T_n has an infinite path.

2 Lattices of equivalence relations

Let $\mathrm{Eq}(A)$ denote the set of all equivalence relations on A. Ordered by incusion, $\mathrm{Eq}(A)$ is a complete lattice. In a sublattice $L \subseteq \mathrm{Eq}(A)$, we write $\sup_L$ for the supremum in L when it exists, and sup for the supremum in $\mathrm{Eq}(A)$, and note that $\sup \leq \sup_L$.

A *complete sublattice of Eq*(A) is a sublattice L of $\mathrm{Eq}(A)$ such that $\sup_L = \sup$ and $\inf_L = \inf$. A sublattice of $\mathrm{Eq}(A)$ that is a complete lattice is not necessarily a complete sublattice in this sense. The following lemma is well known. A good reference for lattice theory is the monograph of Grätzer [1].

Lemma 4. *Suppose A is a set and $(L, \subseteq)$ is a complete sublattice of Eq(A). Then an equivalence relation E in L is a compact member of L if and only if E is finitely generated in L.*

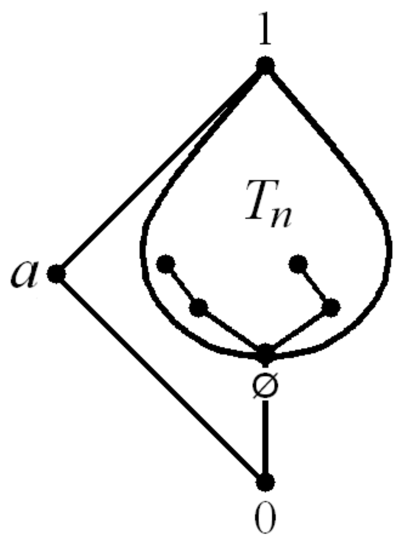

Fig. 4. The lattice $T_n[a]$ from Proposition 3.

Proof. One direction only uses that L is a sublattice of $\mathrm{Eq}(A)$ and L is complete as a lattice. Suppose E is not finitely generated in L. Let $C_{(a,b)}$ denote the infimum of all equivalence relations in L that contain (a, b). Then $E \subseteq \sup_L\{C_{(a,b)} : aEb\}$, but E is not below any finite join of the relations $C_{(a,b)}$. So E is not compact.

Suppose E is finitely generated in L. So there exists an n and pairs (a_1, b_1),. . ., (a_n, b_n) such that a_iEb_i for all $1 \leq i \leq n$, and for all equivalence relations F in L, if a_iFb_i for all $1 \leq i \leq n$ then $E \subseteq F$. Suppose $E \subseteq \sup_L\{E_i : 1 \leq i < \infty\}$ for some $E_1, E_2, \ldots \in L$. Since L is a complete sublattice of $\mathrm{Eq}(A)$, $\sup_L = \sup$, so $E \subseteq \sup\{E_i : 1 \leq i < \infty\}$. Note that $\sup\{E_i : 1 \leq i < \infty\}$ is the equivalence relation generated by the relations E_i under transitive closure. So there is some $j = j_n < \infty$ such that $\{(a_i, b_i) : 1 \leq i \leq n\} \subseteq \bigcup_{i=1}^{j} E_i$ and hence $E \subseteq \bigcup_{i=1}^{j} E_i$. Thus E is compact.

A *computable complete sublattice of Eq*(ω) is a uniformly computable collection $\mathcal{E} = \{E_i\}_{i\in\omega}$ of distinct equivalence relations on ω such that $(\mathcal{E}, \subseteq)$ is a complete sublattice of $\mathrm{Eq}(\omega)$. We say that the lattice $L = (\omega, \preceq)$ is *computably isomorphic* to $(\mathcal{E}, \subseteq)$ if there is a computable function $\varphi : \omega \to \omega$ such that for all i, j, we have $i \preceq j \leftrightarrow E_{\varphi(i)} \subseteq E_{\varphi(j)}$.

Lemma 5. *The indices of compact congruences in a computable complete sublattice of Eq*(ω) *form a* Σ_2^0 *set.*

Proof. Suppose the complete sublattice is $\mathcal{E} = \{E_i\}_{i\in\omega}$. By Lemma 4, E_k is compact if and only if it is finitely generated, i.e.,

$$\exists n\ \exists a_1,\ldots,a_n\ \exists b_1,\ldots,b_n\ \left[\bigwedge_{i=1}^{n} a_iE_kb_i\ \ \&\ \forall j\left(\bigwedge_{i=1}^{n} a_iE_jb_i \to E_k\subseteq E_j\right)\right].$$

Here $E_k \subseteq E_j$ is Π^0_1: $\forall x\forall y\ (xE_ky \to xE_jy)$, so the formula is Σ^0_2.

Theorem 1. *There is a computable algebraic lattice that is not computably isomorphic to any computable complete sublattice of Eq(ω).*

Proof. Let L be the lattice of Proposition 2, and let f be the m-reduction of Proposition 2. Suppose φ is a computable isomorphism between L and a computable complete sublattice of Eq(ω), $(\mathcal{E},\subseteq)$. Since being compact is a lattice-theoretic property, it is a property preserved under isomorphisms. Thus an element $a \in L$ is compact if and only if $E_{\varphi(a)}$ is a compact congruence relation. This implies that T_n has no infinite path if and only if $f(n)$ is a compact element of L, if and only if $E_{\varphi(f(n))}$ is a compact congruence relation. By Lemma 5, this implies that $C_{e_0} = \{n : T_n \text{ has no infinite path}\}$ is a Σ^0_2 set, contradicting the fact that this set is Π^1_1-complete.

3 Congruence lattices

An *algebra* $\mathfrak{A}$ consists of a set A and functions $f_i : A^{n_i} \to A$. Here i is taken from an index set I which may be finite or infinite, and n_i is the arity of f_i. Thus, an algebra is a purely functional model-theoretic structure. A *congruence relation* of $\mathfrak{A}$ is an equivalence relation on A such that for each unary f_i and all $x, y \in A$, if xEy then $f_i(x)Ef_i(y)$, and the natural similar property holds for f_i of arity greater than one.

The congruence relations of $\mathfrak{A}$ form a lattice under the inclusion (refinement) ordering. This lattice Con($\mathfrak{A}$) is called the *congruence lattice* of $\mathfrak{A}$.

The following lemma is well-known and straight-forward.

Lemma 6. *If $\mathfrak{A}$ is an algebra on A, then Con($\mathfrak{A}$) is a complete sublattice of Eq(A).*

Thus, may define a *computable congruence lattice* to be a computable complete sublattice of Eq(ω) which is also Con($\mathfrak{A}$) for some algebra $\mathfrak{A}$ on ω.

Theorem 2. *There is a computable algebraic lattice that is not computably isomorphic to any computable congruence lattice.*

Proof. By Theorem 1, there is a computable algebraic lattice that is not even computably isomorphic to any computable complete sublattice of Eq(ω).

Thus, we have a failure of a certain effective version of the following theorem.

Theorem 3 (Grätzer-Schmidt [2]**).** *Each algebraic lattice is isomorphic to the congruence lattice of an algebra.*

Remark 1. Let A be a set, and let L be a complete sublattice of $\mathrm{Eq}(A)$. Then L is algebraic [1], and so by Theorem 3 L is *isomorphic* to $\mathrm{Con}(\mathfrak{A})$ for some algebra $\mathfrak{A}$ on some set, but it is not in general possible to find $\mathfrak{A}$ such that L is *equal* to $\mathrm{Con}(\mathfrak{A})$. Thus, Theorem 1 is not a consequence of Theorem 2.

Remark 2. The proof of Theorem 2 shows that not only does the Grätzer-Schmidt theorem not hold effectively, it does not hold arithmetically. We conjecture that within the framework of reverse mathematics, a suitable form of Grätzer-Schmidt may be shown to be equivalent to the system Π^1_1-CA_0 (Π^1_1-comprehension) over the base theory ACA_0 (arithmetic comprehension). On the other hand, W. Lampe has pointed out that the Grätzer-Schmidt theorem is normally proved as a corollary of a result which may very well hold effectively: each upper semilattice with least element is isomorphic to the collection of compact congruences of an algebra.

Conjecture 1. An upper semilattice with least element has a computably enumerable presentation if and only if it is isomorphic to the collection of compact congruences of some computable algebra.

The idea for the *only if* direction of Conjecture 1 is to use analyze and slightly modify Jónsson and Pudlák's construction [4]. The *if* direction appears to be straightforward.

Remark 3. The lattices used in this paper are not modular, and we do not know if our results can be extended to modular, or even distributive, lattices.

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