

Characterizing Quantum Theory in terms of Information-Theoretic Constraints

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ABSTRACT

We show that three fundamental information-theoretic constraints—the impossibility of superluminal information transfer between two physical systems by performing measurements on one of them, the impossibility of broadcasting the information contained in an unknown physical state, and the impossibility of unconditionally secure bit commitment—suffice to entail that the observables and state space of a physical theory are quantum-mechanical. We demonstrate the converse derivation in part, and consider the implications of alternative answers to a remaining open question about nonlocality and bit commitment.

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Of John Wheeler’s ‘Really Big Questions’, the one on which most progress has been made is *It from Bit?*—does information play a significant role at the foundations of physics? It is perhaps less ambitious than some of the other Questions, such as *How Come Existence?*, because it does not necessarily require a metaphysical answer. And unlike, say, *Why the Quantum?*, it does not require the discovery of new laws of nature: there was room for hope that it might be answered through a better understanding of the laws as we currently know them, particularly those of quantum physics. And this is what has happened: the better understanding is the quantum theory of information and computation.

How might our conception of the quantum physical world have been different if *It From Bit* had been a motivation from the outset? No one knows how to derive *it* (the nature of the physical world) from *bit* (the idea that information plays a significant role at the foundations of physics), and I shall argue that this will never be possible. But we can do the next best thing: we can start from the qubit.

Introduction to David Deutsch’s (2003) ‘It From Qubit’

1 Introduction

Towards the end of the passage above, Deutsch is pessimistic about the prospects of deducing the nature of the quantum world from the idea that information plays a significant role at the foundations of physics. We propose to counter Deutsch’s pessimism by beginning with the assumption that we live in a world in which there are certain constraints on the acquisition, representation, and communication of information, and then deducing from these assumptions the basic outlines of the quantum-theoretic description of physical systems.¹

The three fundamental information-theoretic constraints we shall be interested in are:

- the impossibility of superluminal information transfer between two physical systems by performing measurements on one of them;
- the impossibility of perfectly broadcasting the information contained in an unknown physical state; and
- the impossibility of unconditionally secure bit commitment.

These three ‘no-go’s’ are all well-known consequences of standard nonrelativistic Hilbert space quantum theory. However, like Einstein’s radical re-derivation of Lorentz’s transformation based upon privileging a few simple principles, we here propose to raise the above constraints to the level of fundamental information-theoretic ‘laws of nature’ from which quantum theory can, we claim, be deduced. We shall do this by starting with a mathematically abstract characterization of a physical theory that includes, as special cases, all classical mechanical theories of both wave and particle varieties, and all variations on quantum theory, including quantum field theories (plus any hybrids of these theories). Within this framework, we are able to give general formulations of the three information-theoretic constraints above, and then show that they jointly entail:

¹Chris Fuchs and Gilles Brassard first suggested the project to one of us (JB) as a conjecture or speculation (Brassard’s preferred term) that quantum mechanics can be derived from two cryptographic principles: the possibility of secure key distribution and the impossibility of secure bit commitment [23, 24, 25, 7].

- that the algebras of observables pertaining to distinct physical systems must commute, usually called microcausality or (a term we prefer) *kinematic independence* (see Summers [46]);
- that any individual system’s algebra of observables must be nonabelian, i.e., *non-commutative*;
- that the physical world must be *nonlocal*, in that spacelike separated systems must at least sometimes occupy entangled states.

We shall argue that these latter three *physical* characteristics are definitive of what it means to be a quantum theory in the most general sense. Conversely, we would want to prove that these three physical characteristics entail the three information-theoretic principles from which we started, thereby providing a characterization theorem for quantum theory in terms of those principles. In this, we are only partly successful, because there remains an open question about bit commitment.

The fact that one can characterize quantum theory (modulo the open question) in terms of just a few simple information-theoretic principles not only goes some way towards answering Wheeler’s query ‘Why the Quantum?’ (without, *pace* Deutsch, the introduction of new laws), but lends credence to the idea that an information-theoretic point of view is the right perspective to adopt in relation to quantum theory. Notice, in particular, that our derivation links information-theoretic principles directly to the very features of quantum theory—noncommutativity and nonlocality—that are so conceptually problematic from a purely physical/mechanical point of view. We therefore suggest substituting for the conceptually problematic mechanical perspective on quantum theory an information-theoretic perspective. That is, we are suggesting that quantum theory be viewed, not as first and foremost a mechanical theory of waves and particles (cf. Bohr’s infamous dictum, reported in Petersen [39]: ‘There is no quantum world.’), but as a theory about the possibilities and impossibilities of information transfer.

We begin, in section 2, by laying out the mathematical framework within which our entire analysis will be conducted—the theory of C^* -algebras. After introducing the basics, we establish that the C^* -algebraic framework does indeed encompass both classical and quantum statistical theories, and go on to argue that the latter class of theories is most properly viewed as picked out solely in virtue of its satisfaction of kinematic independence, noncommutativity, and nonlocality—even though there is far more to the physical content of any given quantum theory than these three features. Section 3 contains our technical results. We first formulate the three information-theoretic constraints—no superluminal information transfer via measurement, no broadcasting, and no bit commitment—in C^* -algebraic terms, after briefly reviewing the concepts as they occur in standard nonrelativistic Hilbert space quantum theory. We show that these information-theoretic ‘no-go’ principles jointly entail kinematic independence, noncommutativity, and nonlocality. We demonstrate the converse derivation in part: that the physical properties of kinematic independence and noncommutativity jointly entail no superluminal information transfer via measurement and no broadcasting.

The remaining open question concerns the derivation of the impossibility of unconditionally secure bit commitment from kinematic independence, noncommutativity, and nonlocality in the theory-neutral C^* -algebraic framework. The proof of this

result in standard quantum mechanics (Mayers [37, 38], Lo and Chau [35]) depends on the biorthogonal decomposition theorem, which is not available in the more general framework. If the derivation goes through, then we have a characterization theorem for quantum theory in terms of the three information-theoretic principles, thereby considerably generalizing the known proofs of these principles within the standard nonrelativistic Hilbert space quantum theory framework. If not, then there must be quantum mechanical systems—perhaps systems associated with von Neumann algebras of some nonstandard type—that allow an unconditionally secure bit commitment protocol! Section 4 concludes with some further remarks about the significance of our information-theoretic characterization of quantum theory.

2 The C^* -Algebraic Approach to Physical Theory

2.1 Basic Concepts

We start with a brief review of abstract C^* -algebras and their relation to the standard formulation of quantum theory in terms of concrete algebras of operators acting on a Hilbert space.

A unital C^* -algebra is a Banach $*$ -algebra over \mathbb{C} containing the identity, where the involution and norm are related by $\|A^*A\| = \|A\|^2$. Thus, the algebra $\mathfrak{B}(\mathcal{H})$ of all bounded operators on a Hilbert space \mathcal{H} —which, of course, is used in the standard formulation of nonrelativistic quantum theory—is an example of a C^* -algebra, with $*$ taken to be the adjoint operation, and $\|\cdot\|$ the standard operator norm. Moreover, any $*$ -subalgebra of $\mathfrak{B}(\mathcal{H})$ containing the identity operator that is closed in the operator norm is a (unital) C^* -algebra. By a *representation* of a C^* -algebra \mathfrak{A} is meant any mapping $\pi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$ that preserves the linear, product, and $*$ structure of \mathfrak{A} . If, in addition, π is one-to-one (equivalently, $\pi(A) = 0$ implies $A = 0$), the representation is called *faithful*. In a faithful representation, $\pi(\mathfrak{A})$ provides an isomorphic copy of \mathfrak{A} . A representation is irreducible just in case the only closed subspaces of \mathcal{H} that are invariant under π are \mathcal{H} and the null space.

A von Neumann algebra \mathcal{R} is a concrete collection of operators on some fixed Hilbert space \mathcal{H} —specifically, a $*$ -subalgebra of $\mathfrak{B}(\mathcal{H})$ that contains the identity and satisfies $\mathcal{R} = \mathcal{R}''$. (Here \mathcal{R}'' is the double commutant of \mathcal{R} , where the commutant \mathcal{R}' is the set of all operators on \mathcal{H} that commute with every operator in \mathcal{R}). This is equivalent, via von Neumann’s double commutant theorem ([30, Theorem 5.3.1]), to the assertion that \mathcal{R} contains the identity and is closed in the strong operator topology (where $Z_n \rightarrow Z$ strongly just in case $\|(Z_n - Z)x\| \rightarrow 0$ for all $x \in \mathcal{H}$, and the norm here is the Hilbert space vector norm).

Every von Neumann algebra is also a C^* -algebra, but not every C^* -algebra of operators is a von Neumann algebra. A von Neumann algebra \mathcal{R} is termed a factor just in case its center $\mathcal{R} \cap \mathcal{R}'$ contains only multiples of the identity. This is equivalent to the condition $(\mathcal{R} \cup \mathcal{R}')'' = \mathfrak{B}(\mathcal{H})$, so \mathcal{R} induces a ‘factorization’ of the total Hilbert space algebra $\mathfrak{B}(\mathcal{H})$ into two subalgebras which together generate that algebra. Factors are classified into different types. The algebra $\mathfrak{B}(\mathcal{H})$ for any Hilbert space \mathcal{H} is a type I factor, and every type I factor arises as the algebra of all bounded operators on some

Hilbert space [30, Theorem 6.6.1]. Type II and type III factors, and subclassifications, have applications to the thermodynamic limit of quantum statistical mechanics and quantum field theory.

A *state* of a C^* -algebra \mathfrak{A} is taken to be any positive, normalized, linear functional $\rho : \mathfrak{A} \rightarrow \mathbb{C}$ on the algebra. For example, a state of $\mathfrak{B}(\mathcal{H})$ in standard quantum theory is obtained if we select a positive trace-one (density) operator D on \mathcal{H} , and define $\rho(A) = \text{Tr}(AD)$ for all $A \in \mathfrak{B}(\mathcal{H})$, which defines a linear functional that is positive (if $A = X^*X$, then $\text{Tr}(X^*XD) = \text{Tr}(XDX^*)$, and the latter is nonnegative because XDX^* is a positive operator), normalized (since $\text{Tr}(ID) = 1$), and linear (since operator composition and the trace operation are both linear). We can make the usual distinction between pure and mixed states, the former defined by the property that if $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$, with $\lambda \in (0, 1)$, then $\rho = \rho_1 = \rho_2$. In the concrete case of $\mathfrak{B}(\mathcal{H})$, a pure state of course corresponds to a density operator for which $D^2 = D$ —which is equivalent to the existence of a unit vector $|v\rangle \in \mathcal{H}$ representing the state of the system via $\rho(A) = \langle v|A|v\rangle$ ($A \in \mathfrak{B}(\mathcal{H})$).

One should note, however, that, because countable additivity is not presupposed by the C^* -algebraic notion of state (and, therefore, Gleason's theorem does not apply), there can be pure states of $\mathfrak{B}(\mathcal{H})$ not representable by vectors in \mathcal{H} . In fact, if A is any self-adjoint element of a C^* -algebra \mathfrak{A} , and $a \in \text{sp}(A)$, then there always exists a pure state ρ of \mathfrak{A} that assigns a *dispersion-free* value of a to A [30, Ex. 4.6.31]. Since this is true even when we consider a point in the continuous spectrum of a self-adjoint operator A acting on a Hilbert space, *without* any corresponding eigenvector, it follows that there *are* pure states of $\mathfrak{B}(\mathcal{H})$ in the C^* -algebraic sense that cannot be vector states (nor, in fact, representable by any density operator \mathcal{H}).

The primary reason countable additivity is not required of C^* -algebraic states is that it is a representation-dependent concept that presupposes the availability of the notion of an infinite convergent sum of orthogonal projections. That is: to say that a state ρ is countably additive is to say that $\sum_{i=1}^{\infty} P_i = I$ implies $\sum_{i=1}^{\infty} \rho(P_i) = 1$; yet the notion of convergence required for the first sum to make sense is strong operator convergence (where a sequence of operators A_i converges strongly to some operator A just in case, for all $x \in \mathcal{H}$, $A_i x \rightarrow Ax$ in *vector* norm). Since, obviously, the elements of a C^* -algebra \mathfrak{A} need not be thought of as operators acting on a Hilbert space of vectors, countable additivity is simply unavailable as a potential general constraint that could be imposed on the state space of \mathfrak{A} .

Nevertheless, it turns out that for any state ρ of a C^* -algebra \mathfrak{A} , there is always *some* representation of \mathfrak{A} in which ρ is representable by a vector (even if ρ is a mixed state). According to the Gelfand-Naimark-Segal theorem [30, Thm. 4.5.2], every state ρ determines a unique (up to unitary equivalence) representation $(\pi_\rho, \mathcal{H}_\rho)$ of \mathfrak{A} and vector $\Omega_\rho \in \mathcal{H}_\rho$ such that $\rho(A) = \langle \Omega_\rho, \pi_\rho(A)\Omega_\rho \rangle$ ($A \in \mathfrak{A}$), and such that the set $\{\pi_\rho(A)\Omega_\rho : A \in \mathfrak{A}\}$ is dense in \mathcal{H}_ρ . The triple $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ is called the GNS representation of \mathfrak{A} induced by the state ρ , and this representation is irreducible if and only if ρ is pure (equivalently, every bounded operator on \mathcal{H}_ρ is the strong limit of operators in $\pi_\rho(\mathfrak{A})$).

Now, by considering the collection of all pure states on \mathfrak{A} , and forming the direct sum of all the irreducible GNS representations these states determine, one obtains a highly reducible but faithful representation of \mathfrak{A} in which every pure state of \mathfrak{A} is

represented by a vector (as usual in standard nonrelativistic quantum theory). As a consequence of this construction, we obtain the Gelfand-Naimark theorem: every abstract C^* -algebra has a concrete faithful representation as a norm-closed $*$ -subalgebra of $\mathfrak{B}(\mathcal{H})$, for some appropriate Hilbert space \mathcal{H} [30, Remark 4.5.7]. So there is a sense in which C^* -algebras are no more general than algebras of operators on Hilbert spaces—apart from the fact that, when working with an abstract C^* -algebra, one does not privilege any particular concrete Hilbert space representation of the algebra (which turns out to be important not to do in quantum field theory, where one needs to allow for inequivalent representations of the canonical commutation relations—see, e.g., Clifton and Halvorson [12]).

2.2 Physical Generality of the C^* -Algebraic Language

If C^* -algebras supply little more than a way of talking abstractly about operator algebras, and the latter are characteristic of quantum theory, how can we possibly claim that the C^* -algebraic machinery supplies a universal language within which all main-line physical theories, including even *classical* mechanics, can be framed? The fallacy, here, is that the use of operator algebras is only relevant to quantum theories.

Take, as a simple example, the classical description of a system of n point particles. Focusing first on the kinematical content of the theory, the observables of the system are real-valued functions on its phase space \mathbb{R}^{6n} . These can be thought of as the self-adjoint elements of the C^* -algebra $\mathfrak{B}(\mathbb{R}^{6n})$ of all bounded, complex-valued measurable functions on \mathbb{R}^{6n} —where the multiplication law is just pointwise multiplication of functions, the adjoint is complex conjugation, and the norm of a function is the supremum of its absolute values. The statistical states of the system are given by probability measures μ on \mathbb{R}^{6n} , and pure states, corresponding to maximally complete information about the particles, are given by the individual points of \mathbb{R}^{6n} . Using a statistical state μ , we obtain the corresponding expectation functional, which is the system's state in the C^* -algebraic sense, by defining $\rho(f) = \int_{\mathbb{R}^{6n}} f d\mu$ ($f \in \mathfrak{B}(\mathbb{R}^{6n})$).

Turning now to dynamics, the Heisenberg picture of the time evolution of a state is determined by a group of bijective, Lebesgue measure-preserving, flow mappings $T_t : \mathbb{R}^{6n} \rightarrow \mathbb{R}^{6n}$ ($t \in \mathbb{R}$) that induce an automorphism group τ_t on $\mathfrak{B}(\mathbb{R}^{6n})$ via $\tau_t(f) = f \circ T_t$. State evolution when a measurement occurs is also fully analogous to the quantum case. The probability in state μ that the value of f will be found on measurement to lie in a Borel set Δ is given by $\rho(\chi_{f^{-1}(\Delta)})$ (note that $\chi_{f^{-1}(\Delta)}$ is a projection in $\mathfrak{B}(\mathbb{R}^{6n})$); and, should that be the case, the new post-measurement state is given by (see Bóna [2000] and Duvenhage [2002a,b]):

$$\rho'(g) = \frac{\rho(\chi_{f^{-1}(\Delta)} g \chi_{f^{-1}(\Delta)})}{\rho(\chi_{f^{-1}(\Delta)})} \quad (g \in \mathbb{R}^{6n})$$

Lastly, note that because classical n point particle mechanics employs a C^* -algebra (as we have seen, $\mathfrak{B}(\mathbb{R}^{6n})$), it follows from the Gelfand-Naimark theorem that classical mechanics can be done in Hilbert space! Yet this really is nothing new, having been pointed out long ago by Koopman [31] and von Neumann [47] (see Mauro [36] for an up-to-date discussion).

Of course, nothing we have said *proves* that all physical theories admit a C^* -algebraic formulation. Indeed, that would be absurd to claim: one can certainly conceive of theories whose algebra of observables falls short of being isomorphic to the self-adjoint part of a C^* -algebra and, instead, only instantiates some weaker mathematical structure, such as a Segal [45] algebra. To foreclose such possibilities, it could be of interest to pursue an axiomatic justification of the C^* -algebraic framework along lines similar to those provided by Emch [21, Ch. 1.2]. However, it suffices for present purposes simply to observe that all physical theories that have been found *empirically successful*—not just phase space and Hilbert space theories (Landsman [34]), but also theories based a manifold (Connes [13])—fall under this framework (whereas, for example, so-called ‘nondistributive’ Segal algebras permit violations of the Bell inequality far in excess of that permitted by standard quantum theory and observed in the laboratory—see Landau [33]).

2.3 Classical versus Quantum Theories

We must mention one final important representation theorem: every (unital) *abelian* C^* -algebra \mathfrak{A} is isomorphic to the set $C(X)$ of all continuous, complex-valued functions on a compact Hausdorff space X [30, Thm. 4.4.3]. This is called the *function representation* of \mathfrak{A} . The underlying ‘phase space’ X in this representation is none other than the pure state space $\mathcal{P}(\mathfrak{A})$ of \mathfrak{A} endowed with its weak- $*$ topology. (A sequence of states $\{\rho_n\}$ on \mathfrak{A} weak- $*$ converges to a state ρ just in case $\rho_n(A) \rightarrow \rho(A)$ for all $A \in \mathfrak{A}$.) The isomorphism maps an element $A \in \mathfrak{A}$ to the function \hat{A} (the Gelfand transformation of A) whose value at any $\rho \in \mathcal{P}(\mathfrak{A})$ is just the (dispersion-free) value that ρ assigns to A . Thus, not only does every classical phase space presentation of a physical theory define a C^* -algebra, but, conversely, behind every abstract abelian C^* -algebra lurks in its function representation a good old-fashioned classical phase space theory. All of this justifies treating a theory formulated in C^* -algebraic language as classical just in case its algebra is abelian. It follows that a necessary condition for thinking of a theory as a *quantum* theory is that its C^* -algebra be non-abelian. However, as we shall now explain, we do not believe this is sufficient unless something further is said about the presence of entangled states.

In 1935 and 1936, Schrödinger published an extended two-part commentary [43, 44] on the Einstein-Podolsky-Rosen argument [19], where he introduced the term ‘entanglement’ to describe the peculiar correlations of the EPR-state as [43, p. 555]: ‘*the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.*’ In the first part, he considers entangled states for which the biorthogonal decomposition is unique, as well as cases like the EPR-state, where the biorthogonal decomposition is non-unique. There he is concerned to show that suitable measurements on one system can fix the (pure) state of the entangled distant system, and that this state depends on what observable one chooses to measure, not merely on the outcome of that measurement. In the second part, he shows that a ‘sophisticated experimenter,’ by performing a suitable local measurement on one system, can ‘steer’ the distant system into any mixture of pure states representable by its reduced density operator. (So the distant system can be steered into any pure state in the support of the reduced density operator, with a nonzero probability that depends

only on the pure state.) For a mixture of linearly independent states, the steering can be done by performing a PV-measurement in a suitable basis. If the states are linearly dependent, the experimenter performs a POV-measurement, which amounts to enlarging the experimenter’s Hilbert space by adding an ancilla, so that the dimension of the enlarged Hilbert space is equal to the number of linearly independent states. Schrödinger’s result here anticipates the later result by Hughston, Jozsa, and Wootters [28] that underlies the ‘no go’ bit commitment theorem. (Similar results were proved by Jaynes [29] and Gisin [27].)

What Schrödinger found problematic—indeed, objectionable—about entanglement was this possibility of remote steering [43, p. 556]:

It is rather disconcerting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter’s mercy in spite of his having no access to it.

He conjectured that an entangled state of a composite system would almost instantaneously decay to a mixture as the component systems separated. (A similar possibility was raised and rejected by Furry [26].) There would still be correlations between the states of the component systems, but remote steering would no longer be possible [44, p. 451]:

It seems worth noticing that the [EPR] paradox could be avoided by a very simple assumption, namely if the situation after separating were described by the expansion (12), but with the additional statement that the knowledge of the *phase relations* between the complex constants a_k has been entirely lost in consequence of the process of separation. This would mean that not only the parts, but the whole system, would be in the situation of a mixture, not of a pure state. It would not preclude the possibility of determining the state of the first system by *suitable* measurements in the second one or *vice versa*. But it would utterly eliminate the experimenters influence on the state of that system which he does not touch.

Expansion (12) is the biorthogonal expansion:

$$\Psi(x, y) = \sum_k a_k g_k(x) f_k(y) \tag{1}$$

Schrödinger regarded the phenomenon of interference associated with noncommutativity in quantum mechanics as unproblematic, because he saw this as reflecting the fact that particles are wavelike. But he did not believe that we live in a world in which physical systems can exist nonlocally in entangled states, because such states would allow Alice to steer Bob’s system into any mixture of pure states compatible with Bob’s reduced density operator. Schrödinger did not expect that experiments would bear this out. On his view, entangled states, which the theory allows, are entirely local insofar as they characterize physical systems, and nonlocal entangled states are simply an artefact of the formalism.

Of course, it was an experimental question in 1935 whether Schrödinger’s conjecture was correct or not. We now know that the conjecture is false. A wealth of experimental evidence, including the experimentally confirmed violations of Bell’s inequality

(e.g., Aspect *et al* [1]), testify to this. The relevance of Schrödinger’s conjecture for our inquiry is this: it raises the possibility of a quantum-like world in which there is interference but no nonlocal entanglement. We will need to exclude this possibility on information-theoretic grounds.

As indicated, for a composite system, $A+B$, consisting of two component subsystems, A and B , we propose to show (i) that the ‘no superluminal information transfer via measurement’ condition entails that the C^* -algebras \mathfrak{A} and \mathfrak{B} , whose self-adjoint elements represent the observables of A and B , commute with each other, and (ii) that the ‘no broadcasting’ condition entails that \mathfrak{A} and \mathfrak{B} separately are noncommutative (nonabelian). Now, if \mathfrak{A} and \mathfrak{B} are nonabelian and mutually commuting (and C^* -independent²), it follows immediately that there are nonlocal entangled states on the C^* -algebra $\mathfrak{A} \vee \mathfrak{B}$ they generate (see Landau [32], who shows that there is a state ρ on $\mathfrak{A} \vee \mathfrak{B}$ that violates Bell’s inequality and hence is nonlocally entangled; also Summers and Werner [46] and Bacciagaluppi [3]). So, at least mathematically, the presence of nonlocal entangled states in the formalism is guaranteed, once we know that the algebras of observables are nonabelian. What does not follow is that these states actually occur in nature. For example, even though Hilbert space quantum mechanics allows for paraparticle states, such states are not observed in nature. In terms of our program, in order to show that entangled states are actually instantiated, and—contra Schrödinger—instantiated nonlocally, we need to derive this from some information-theoretic principle. This is the role of the ‘no bit commitment’ constraint.

Bit commitment is a cryptographic protocol in which one party, Alice, supplies an encoded bit to a second party, Bob. The information available in the encoding should be insufficient for Bob to ascertain the value of the bit, but sufficient, together with further information supplied by Alice at a subsequent stage when she is supposed to reveal the value of the bit, for Bob to be convinced that the protocol does not allow Alice to cheat by encoding the bit in a way that leaves her free to reveal either 0 or 1 at will.

In 1984, Bennett and Brassard [5] proposed a quantum bit commitment protocol now referred to as BB84. The basic idea was to associate the 0 and 1 commitments with two equivalent quantum mechanical mixtures represented by the same density operator. As they showed, Alice can cheat by adopting an Einstein-Podolsky-Rosen (EPR) attack or cheating strategy: she prepares entangled pairs of particles, keeps one of each pair (the ancilla) and sends the second particle (the channel particle) to Bob. In this way she can fake sending one of two equivalent mixtures to Bob, and reveal either bit at will at the opening stage, by effectively steering Bob’s particles into the desired mixture via appropriate measurements on her ancillas. Bob cannot detect this cheating strategy.

Mayers [37, 38], and Lo and Chau [35], showed that the insight of Bennett and Brassard can be extended to a proof that a generalized version of the EPR cheating strategy can always be applied, if the Hilbert space is enlarged in a suitable way by introducing additional ancilla particles. The proof of the ‘no go’ quantum bit commitment theorem exploits biorthogonal decomposition via the Hughston-Jozsa-Wootters result [28] (effectively anticipated by Schrödinger). Informally, this says that for a quantum

²See section 3.1.

mechanical system consisting of two (separated) subsystems represented by the tensor product of two type-I factors $\mathfrak{B}(\mathcal{H}_1) \otimes \mathfrak{B}(\mathcal{H}_2)$, any mixture of states on $\mathfrak{B}(\mathcal{H}_2)$ can be generated from a distance by performing an appropriate POV-measurement on the system represented by $\mathfrak{B}(\mathcal{H}_1)$, for an appropriate entangled state of the composite system $\mathfrak{B}(\mathcal{H}_1) \otimes \mathfrak{B}(\mathcal{H}_2)$. This is what makes it possible for Alice to cheat in her bit commitment protocol with Bob. It is easy enough to see this for the original BB84 protocol. Surprisingly, this is also the case for any conceivable quantum bit commitment protocol. (See Bub [9] for a discussion.)

Now, unconditionally secure bit commitment is impossible for classical systems, in which the algebras of observables are abelian. It might seem inappropriate, then, that we propose ‘no bit commitment’ as a constraint distinguishing quantum from classical theories. The relevant point to note here is that the insecurity of any bit commitment protocol in a nonabelian setting depends on considerations entirely different from those in a classical abelian setting. Classically, unconditionally secure bit commitment is impossible, essentially because Alice can send (encrypted) information to Bob that guarantees the truth of an exclusive classical disjunction (equivalent to her commitment to a 0 or a 1) only if the information is biased towards one of the alternative disjuncts (because a classical exclusive disjunction is true if and only if one of the disjuncts is true and the other false). No principle of classical mechanics precludes Bob from extracting this information. So the security of the protocol cannot be unconditional and can only depend on issues of computational complexity.

By contrast, in a situation of the sort envisaged by Schrödinger, in which the algebras of observables are nonabelian but composite physical systems cannot exist in nonlocal entangled states, if Alice sends Bob one of two mixtures associated with the same density operator to establish her commitment, then she is, in effect, sending Bob evidence for the truth of an exclusive disjunction that is not based on the selection of a particular disjunct. (Bob’s reduced density operator is associated ambiguously with both mixtures, and hence with the truth of the exclusive disjunction: ‘0 or 1’.) This is what noncommutativity allows: different mixtures can be associated with the same density operator. What thwarts the possibility of using the ambiguity of mixtures in this way to implement an unconditionally secure bit commitment protocol is the existence of nonlocal entangled states between Alice and Bob. This allows Alice to cheat by preparing a suitable entangled state instead of one of the mixtures, where the reduced density operator for Bob is the same as that of the mixture. Alice is then able to steer Bob’s systems into either of the two mixtures associated with the alternative commitments at will.

So what would allow unconditionally secure bit commitment in a nonabelian theory is the absence of physically occupied nonlocal entangled states. One can therefore take Schrödinger’s remarks as relevant to the question of whether or not secure bit commitment is possible in our world. In effect, Schrödinger believes that we live in a quantum-like world in which secure bit commitment is possible. Experiments such as those designed to test Bell’s inequality can be understood as demonstrating that this is not the case. The violation of Bell’s inequalities can then be seen as a criterion for the possibility of remote steering in Schrödinger’s sense.

3 Technical Results

3.1 Terminology and Assumptions

Our aim in this section is to show that the kinematic aspects of the quantum theory of a composite system $A+B$ (consisting of two component subsystems A and B) can be characterized in terms of information-theoretic constraints. The physical observables of A and B are represented, respectively, by self-adjoint elements of unital subalgebras \mathfrak{A} and \mathfrak{B} of a C^* -algebra \mathfrak{C} . We let $\mathfrak{A} \vee \mathfrak{B}$ denote the C^* -algebra generated by \mathfrak{A} and \mathfrak{B} . A state of A , B , or $A+B$ (i.e., a catalog of the expectation values of all observables) can be represented by means of a positive, normalized, linear functional on the respective algebra of observables. Recall that a state ρ is said to be pure just in case $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$, for $\lambda \in (0, 1)$, entails that $\rho = \rho_1 = \rho_2$; otherwise, ρ is said to be mixed. For simplicity, we will assume that the systems A and B are identically constituted — i.e., they have precisely the same degrees of freedom — and therefore that there is an isomorphism between the algebras \mathfrak{A} and \mathfrak{B} . (However, it will be clear that most of our results do not depend on this assumption.) We will hold this isomorphism fixed throughout our discussion so that we can use the same notation to denote ambiguously an operator in \mathfrak{A} and its counterpart in \mathfrak{B} . We will also use the same notation for a state of \mathfrak{A} and its counterpart in the state space of \mathfrak{B} .

The most general dynamical evolution of a system is represented by a completely positive, linear ‘operation’ mapping T of the corresponding algebra of observables. (We also require that $T(I) \leq I$. The operation T is said to be selective if $T(I) < I$, and nonselective if $T(I) = I$.) Recall that a linear mapping T of \mathfrak{A} is positive just in case $A \geq 0$ entails $T(A) \geq 0$, and is completely positive just in case, for each positive integer n , the mapping $T \otimes \iota$ of $\mathfrak{A} \otimes M_n(\mathbb{C})$ into itself, defined by $(T \otimes \iota)(A \otimes B) = T(A) \otimes B$, is positive. (Here $M_n(\mathbb{C})$ is the C^* -algebra of $n \times n$ matrices over the complex numbers.) If T is an operation of \mathfrak{A} , and ρ is a state of \mathfrak{A} such that $\rho(T(I)) \neq 0$, then the mapping $T^*\rho$ defined by

$$(T^*\rho)(A) = \frac{\rho(T(A))}{\rho(T(I))} \quad (A \in \mathfrak{A}) \quad (2)$$

is a state of \mathfrak{A} . The standard example of a selective operation is a collapsing von Neumann measurement of some observable O with spectral projection P . In that case, $T(A) = PAP$ ($A \in \mathfrak{A}$), and $T^*\rho$ is the final state obtained after measuring O in state ρ and ignoring all elements of the ensemble that do not yield as measurement result the eigenvalue of O corresponding to P . The standard example of a nonselective operation is a time evolution induced by a unitary operator $U \in \mathfrak{A}$, where $T(A) = U^*AU$ ($A \in \mathfrak{A}$) simply represents the Heisenberg picture of such evolution.

Finally, we must add one nontrivial independence assumption in order to capture the idea that A and B are physically *distinct* systems. (Our current assumptions would allow that $\mathfrak{A} = \mathfrak{B}$, which obviously fails to capture the situation we are intending to describe.) Various notions of independence for a pair $\mathfrak{A}, \mathfrak{B}$ of C^* -algebras have been developed in the literature [22, 46]. We are particularly interested in the notion of C^* -independence developed in [22], because it does not presuppose that \mathfrak{A} and \mathfrak{B} are kinematically independent (i.e., that $[A, B] = 0$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$). Thus, we

will assume that — whether or not \mathfrak{A} and \mathfrak{B} are kinematically independent — any state of \mathfrak{A} is compatible with any state of \mathfrak{B} . More precisely, for any state ρ_1 of \mathfrak{A} , and for any state ρ_2 of \mathfrak{B} , there is a state ρ of $\mathfrak{A} \vee \mathfrak{B}$ such that $\rho|_{\mathfrak{A}} = \rho_1$ and $\rho|_{\mathfrak{B}} = \rho_2$. This condition holds (i.e., \mathfrak{A} and \mathfrak{B} are C^* -independent) if and only if $\|AB\| = \|A\|\|B\|$, for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ [22, Prop. 3].

3.2 No Superluminal Information Transfer via Measurement and Kinematic Independence

We first show that \mathfrak{A} and \mathfrak{B} are kinematically independent if and only if the ‘no superluminal information transfer via measurement’ constraint holds. The sense of this constraint is that when Alice and Bob perform local measurements, Alice’s measurements can have no influence on the statistics for the outcomes of Bob’s measurements (and vice versa); for, otherwise, measurement would allow the instantaneous transfer of information between Alice and Bob. That is, the mere performance of a local measurement (in the nonselective sense) cannot, in and of itself, transfer information to a physically distinct system.

The most general nonselective measurement operation performable by Alice is given by

$$T(A) = \sum_{i=1}^n E_i^{1/2} A E_i^{1/2} \quad (A \in \mathfrak{A} \vee \mathfrak{B}) \quad (3)$$

where the E_i are positive operators in \mathfrak{A} such that $\sum_{i=1}^n E_i = I$. The restriction to nonselective measurements is justified here because selective operations can trivially change the statistics of observables measured at a distance, simply in virtue of the fact that the ensemble relative to which one computes statistics has changed.

We will say that an operation T conveys no information to Bob just in case T^* leaves the state of Bob’s system invariant (so that everything ‘looks the same’ to Bob after the operation as before, in terms of his expectation values for the outcomes of measurements on observables).

Definition. An operation T on $\mathfrak{A} \vee \mathfrak{B}$ conveys no information to Bob just in case $(T^*\rho)|_{\mathfrak{B}} = \rho|_{\mathfrak{B}}$ for all states ρ of \mathfrak{B} .

Note that each C^* -algebra has sufficient states to discriminate between any two observables (i.e., if $\rho(A) = \rho(B)$ for all states ρ , then $A = B$). Now $(T^*\rho)|_{\mathfrak{B}} = \rho|_{\mathfrak{B}}$ iff $\rho(T(B)) = \rho(B)$ for all $B \in \mathfrak{B}$ and for all states ρ of $\mathfrak{A} \vee \mathfrak{B}$. Since all states of \mathfrak{B} are restrictions of states on $\mathfrak{A} \vee \mathfrak{B}$, it follows that $(T^*\rho)|_{\mathfrak{B}} = \rho|_{\mathfrak{B}}$ if and only if $\omega(T(B)) = \omega(B)$ for all states ω of \mathfrak{B} , i.e., if and only if $T(B) = B$ for all $B \in \mathfrak{B}$.

It is clear that the kinematic independence of \mathfrak{A} and \mathfrak{B} entails that Alice’s local measurement operations cannot convey any information to Bob (i.e., $T(B) = \sum_{i=1}^n E_i^{1/2} B E_i^{1/2} = B$ for $B \in \mathfrak{B}$ if T is implemented by a positive operator valued resolution of the identity in \mathfrak{A}). Thus, we need only show that if Alice cannot convey any information to Bob by performing local measurement operations, then \mathfrak{A} and \mathfrak{B} are kinematically independent. In the standard Hilbert space case, our argument would

proceed as follows: Consider any ideal (Lüders), non-selective measurement of the form

$$T(A) = PAP + (I - P)A(I - P) \quad (A \in \mathfrak{A} \vee \mathfrak{B}) \quad (4)$$

where P is a projection in \mathfrak{A} . Then no superluminal information transfer via measurement entails that for any $B \in \mathfrak{B}$,

$$B = T(B) = PBP + (I - P)B(I - P) \quad (5)$$

and therefore

$$2PBP - PB - BP = 0 \quad (6)$$

Multiplying on the left (respectively, right) with P , and using the fact that $P^2 = P$, we then obtain $PBP - PB = 0$ (respectively, $PBP - BP = 0$). Subtracting these two equations gives $[P, B] = 0$. Thus, since \mathfrak{A} is spanned by its projections and \mathfrak{B} is spanned by its self-adjoint operators, it follows that \mathfrak{A} and \mathfrak{B} are kinematically independent.

In the more general C^* -algebraic framework, this argument is not available: Since the algebra \mathfrak{A} does not necessarily contain projection operators, we cannot assume that there are any measurement operations of the form given in Eqn. 4 where P is a projection. Instead, since C^* -algebras are spanned by their effects (positive operators), consider the simplest case of a POV measurement defined by

$$T_E(A) = E^{1/2}AE^{1/2} + (I - E)^{1/2}A(I - E)^{1/2} \quad (A \in \mathfrak{A} \vee \mathfrak{B}) \quad (7)$$

where E is some effect in \mathfrak{A} . We will now show that when B is self-adjoint, $T_E(B) = B$ entails that $[E, B] = 0$.

Theorem 1. *$T_E(B) = B$ for all effects $E \in \mathfrak{A}$ and self-adjoint operators $B \in \mathfrak{B}$ only if \mathfrak{A} and \mathfrak{B} are kinematically independent.*

For the proof of this theorem, recall that a derivation is a linear map such that $d(AB) = A(dB) + (dA)B$.

Proof. Suppose that $T_E(B) = B$ where E is an effect in \mathfrak{A} and B is a self-adjoint operator in \mathfrak{B} . Then a tedious but elementary calculation shows that

$$[E^{1/2}, [E^{1/2}, B]] = 0 \quad (8)$$

Clearly, the map $X \mapsto i[E^{1/2}, X]$ defines a derivation d of $\mathfrak{A} \vee \mathfrak{B}$. Moreover, since $d(dB) = 0$ and B is self-adjoint, it follows that $i[E^{1/2}, B] = dB = 0$ [10, Appendix A]. Thus, $[E, B] = 0$. Finally, since a C^* -algebra is spanned by its effects, if $T_E(B) = B$ for all effects $E \in \mathfrak{A}$ and self-adjoint operators $B \in \mathfrak{B}$, then \mathfrak{A} and \mathfrak{B} are kinematically independent. \square

Thus, the kinematic independence of \mathfrak{A} and \mathfrak{B} is equivalent to the ‘no superluminal information transfer by measurement’ constraint. In deriving our subsequent results, we assume kinematic independence.

3.3 No Broadcasting and Noncommutativity

In a cloning process, a ready state σ of system B, and the state to be cloned ρ of system A, are transformed into two copies of ρ . Thus, such a process creates no correlations between the states of A and B. By contrast, in a more general broadcasting process, a ready state σ , and the state to be broadcast ρ are transformed to a new state ω of A+B, where the marginal states of ω with respect to both A and B are ρ .³ In the context of elementary quantum mechanics, neither cloning nor broadcasting is generally possible: A pair of pure states can be cloned if and only if they are orthogonal (Wootters and Zurek [48], Dieks [15]), and (more generally) an arbitrary pair of states can be broadcast if and only if they are represented by mutually commuting density matrices (Barnum *et al* [4]). Thus, one might suspect that in a classical theory (in which all operators commute), all states can be broadcast. In this section, we show that this is indeed the case; and, in fact, the ability to broadcast states distinguishes classical systems from quantum systems.

We now introduce a general notion of broadcasting for a pair $\mathfrak{A}, \mathfrak{B}$ of kinematically independent C^* -algebras. But we must first establish the existence and uniqueness of product states of $\mathfrak{A} \vee \mathfrak{B}$.

A state ρ of $\mathfrak{A} \vee \mathfrak{B}$ is said to be a *product state* just in case $\rho(AB) = \rho(A)\rho(B) = \rho(BA)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Since \mathfrak{A} and \mathfrak{B} are both C^* -independent and kinematically independent, there is an isomorphism π from the $*$ -algebra generated by \mathfrak{A} and \mathfrak{B} onto the algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ such that $\pi(AB) = A \otimes B$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. We will occasionally omit reference to π and use $A \otimes B$ to denote the product of $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. It also follows that π can be extended to a continuous surjection $\bar{\pi}$ from $\mathfrak{A} \vee \mathfrak{B}$ onto the spatial tensor product $\mathfrak{A} \otimes \mathfrak{B}$ [22, Thm. 1]. Since the state space of $\mathfrak{A} \otimes \mathfrak{B}$ has a natural tensor product structure, we can use the mapping $\bar{\pi}$ to define product states of $\mathfrak{A} \vee \mathfrak{B}$. In particular, for each state ρ of $\mathfrak{A} \otimes \mathfrak{B}$ define the state $\pi^*\rho$ of $\mathfrak{A} \vee \mathfrak{B}$ by setting

$$(\pi^*\rho)(A) = \rho(\pi(A)) \quad (A \in \mathfrak{A} \vee \mathfrak{B}) \quad (9)$$

If ω is a state of \mathfrak{A} and ρ is a state of \mathfrak{B} , then $\pi^*(\omega \otimes \rho)$ is a product state of $\mathfrak{A} \vee \mathfrak{B}$ with marginal states ω and ρ . In fact, the following result shows that $\pi^*(\omega \otimes \rho)$ is the unique product state of $\mathfrak{A} \vee \mathfrak{B}$ with these marginal states.

Lemma 1. *Suppose that \mathfrak{A} and \mathfrak{B} are kinematically independent C^* -algebras. Then for any state ω of \mathfrak{A} and for any state ρ of \mathfrak{B} there is at most one state σ of $\mathfrak{A} \vee \mathfrak{B}$ such that $\sigma(AB) = \omega(A)\rho(B)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.*

Proof. The set \mathfrak{S} of finite sums of the form $\sum_{i=1}^n A_i B_i$ with $A_i \in \mathfrak{A}$ and $B_i \in \mathfrak{B}$ is a $*$ -algebra containing both \mathfrak{A} and \mathfrak{B} . Moreover, \mathfrak{S} is clearly contained in any $*$ -algebra that contains both \mathfrak{A} and \mathfrak{B} . Thus, \mathfrak{S} is the $*$ -algebra generated by \mathfrak{A} and \mathfrak{B} . Suppose then that σ_0 and σ_1 are states of $\mathfrak{A} \vee \mathfrak{B}$ such that $\sigma_0(AB) = \omega(A)\rho(B) = \sigma_1(AB)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Then

$$\sigma_0 \left(\sum_{i=1}^n A_i B_i \right) = \sum_{i=1}^n \omega(A_i)\rho(B_i) = \sigma_1 \left(\sum_{i=1}^n A_i B_i \right) \quad (10)$$

³We are indebted to Rob Spekkens for clarifying this distinction for us, and for supplying relevant references.

for all $A_i \in \mathfrak{A}$ and $B_i \in \mathfrak{B}$. Since $\mathfrak{A} \vee \mathfrak{B}$ is the closure of \mathfrak{G} in the norm topology, and since states are continuous in the norm topology, $\sigma_0 = \sigma_1$. \square

When it will not cause confusion, we will henceforth suppress reference to the state mapping π^* , so that $\omega \otimes \rho$ denotes the unique product state on $\mathfrak{A} \vee \mathfrak{B}$ with marginals ω and ρ .

Definition. Given two isomorphic, kinematically independent C^* -algebras \mathfrak{A} and \mathfrak{B} , we say that a pair $\{\rho_0, \rho_1\}$ of states of \mathfrak{A} can be *broadcast* just in case there is a standard state σ of \mathfrak{B} and a dynamical evolution represented by an operation T of $\mathfrak{A} \vee \mathfrak{B}$ such that $T^*(\rho_i \otimes \sigma)|_{\mathfrak{A}} = T^*(\rho_i \otimes \sigma)|_{\mathfrak{B}} = \rho_i$ ($i = 0, 1$). We say that a pair $\{\rho_0, \rho_1\}$ of states of \mathfrak{A} can be *cloned* just in case $T^*(\rho_i \otimes \sigma) = \rho_i \otimes \rho_i$ ($i = 0, 1$).

We show next that pairwise broadcasting is always possible in classical systems. Indeed, when the algebras of observables are abelian, there is a ‘universal’ broadcasting map that clones any pair of input pure states and broadcasts any pair of input mixed states.

Theorem 2. *If \mathfrak{A} and \mathfrak{B} are abelian then there is an operation T on $\mathfrak{A} \vee \mathfrak{B}$ that broadcasts all states of \mathfrak{A} .*

Proof. Since \mathfrak{A} is abelian, $\mathfrak{A} \vee \mathfrak{B}$ is naturally isomorphic to $\mathfrak{A} \otimes \mathfrak{B}$ [42, Theorem 2]. Since \mathfrak{A} and \mathfrak{B} are isomorphic abelian algebras, both are isomorphic to the space $C(X)$, where X is some compact Hausdorff space, and therefore $\mathfrak{A} \otimes \mathfrak{B} \cong C(X) \otimes C(X) \cong C(X \times X)$ [30, p. 849]. Define a mapping η from $X \times X$ into $X \times X$ by setting

$$\eta((x, y)) = (x, x) \quad (x, y \in X) \quad (11)$$

Since η is continuous, we can define a linear mapping T on $C(X \times X)$ by setting $Tf = f \circ \eta$. Since the range of Tf is a subset of the range of f , the mapping T is positive, and $T(I) = I$. Furthermore, every positive mapping whose domain or range is an abelian algebra is completely positive [30, Exercise 11.5.22]. Therefore, T is a nonselective operation on $C(X) \otimes C(X)$. To see that T broadcasts all states, note first that $T(f \otimes g) = fg \otimes I$ for any product function $f \otimes g$. In particular, $T(I \otimes f) = f \otimes I$, and thus

$$T^*(\rho \otimes \sigma)(I \otimes f) = (\rho \otimes \sigma)(f \otimes I) = \rho(f) \quad (12)$$

for any states ρ, σ of $C(X)$. That is, $T^*(\rho \otimes \sigma)|_{\mathfrak{B}} = \rho$. On the other hand, since $T(f \otimes I) = f \otimes I$, it follows that

$$T^*(\rho \otimes \sigma)(f \otimes I) = (\rho \otimes \sigma)(f \otimes I) = \rho(f) \quad (13)$$

for any states ρ, σ of $C(X)$. That is, $T^*(\rho \otimes \sigma)|_{\mathfrak{A}} = \rho$. \square

Barnum *et al* [4] note that if density operators D_0, D_1 on a Hilbert space \mathcal{H} can be simultaneously diagonalized, then there is a unitary operator U that broadcasts the corresponding pair of states. Thus, such states can be broadcast by a *reversible* operation, and the added strength of *irreversible* (general completely positive) operations is

not necessary. However, the broadcasting operation T defined in the previous theorem is patently irreversible, since it corresponds to a many-to-one mapping $(x, y) \mapsto (x, x)$ of the pure state space. Indeed, although there are many physically significant classical systems where broadcasting can be performed via reversible operations, this is not generally true. Consider the following two contrasting cases.

First, in the case of classical particle mechanics, systems A and B each have the phase space \mathbb{R}^{6n} , for some finite n . Thus, $\mathfrak{A} \vee \mathfrak{B} \cong C(\mathbb{R}^{6n}) \otimes C(\mathbb{R}^{6n}) \cong C(\mathbb{R}^{6n} \times \mathbb{R}^{6n})$, where $C(\mathbb{R}^{6n})$ is the set of bounded continuous functions from \mathbb{R}^{6n} into \mathbb{C} . Let the ready state of system B be the zero vector in \mathbb{R}^{6n} , and let η be the invertible linear transformation of $\mathbb{R}^{6n} \times \mathbb{R}^{6n}$ given by the matrix

$$\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

where I is the identity matrix of \mathbb{R}^{6n} . Since η is an autohomeomorphism of $\mathbb{R}^{6n} \times \mathbb{R}^{6n}$, the mapping $f \mapsto f \circ \eta$ defines an automorphism of $\mathfrak{A} \vee \mathfrak{B}$ [30, Thm. 3.4.3]. Moreover, $\eta((\mathbf{x}, \mathbf{0})) = (\mathbf{x}, \mathbf{x})$ for any pure state \mathbf{x} , and an argument similar to that used above shows that η broadcasts arbitrary states.

Second, suppose that systems A and B each have the phase space $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, where the open sets of \mathbb{N}^* consist of all finite subsets of \mathbb{N} , plus all cofinite sets containing ∞ . (That is, \mathbb{N}^* is the one-point compactification of \mathbb{N} .) Then $\mathfrak{A} \vee \mathfrak{B} \cong C(\mathbb{N}^*) \otimes C(\mathbb{N}^*) \cong C(\mathbb{N}^* \times \mathbb{N}^*)$, and every automorphism α of $\mathfrak{A} \vee \mathfrak{B}$ is induced by an autohomeomorphism η of $\mathbb{N}^* \times \mathbb{N}^*$ via the equation $\alpha(f) = f \circ \eta$ [30, Thm. 3.4.3]. However, it is not difficult to see that there is no autohomeomorphism of $\mathbb{N}^* \times \mathbb{N}^*$ that clones arbitrary pairs of pure states. In particular, for any $n \in \mathbb{N}^*$, there is an $m \in \mathbb{N}^*$ such that (n, m) is not mapped onto (m, m) by any autohomeomorphism of $\mathbb{N}^* \times \mathbb{N}^*$. Therefore, general classical systems do not permit broadcasting via a reversible operation.

We now show that general quantum systems do not permit broadcasting. In particular, we prove that if any two states of a system can be broadcast, then that system has an abelian algebra of observables. Our proof proceeds by showing that if any two states can be broadcast, then any two *pure* states can be cloned; and that if two pure states of a C^* -algebra can be cloned, then they must be orthogonal.

Two pure states ρ, ω of a C^* -algebra are said to be orthogonal just in case $\|\rho - \omega\| = 2$. More generally, the transition probability $p(\rho, \omega)$ is defined to be (see [41]):

$$p(\rho, \omega) = 1 - \frac{1}{4} \|\rho - \omega\|^2 \quad (14)$$

If ρ is a state of \mathfrak{A} , and U is a unitary operator in \mathfrak{A} , then we let ρ_U denote the state defined by

$$\rho_U(A) = \rho(U^* A U) \quad (A \in \mathfrak{A}) \quad (15)$$

In this case, ρ and ρ_U are said to be unitarily equivalent. Furthermore, if U is a unitary operator in \mathfrak{A} and V is a unitary operator in \mathfrak{B} , then

$$(\omega \otimes \rho)_{U \otimes V}(A \otimes B) = (\omega \otimes \rho)(U^* A U \otimes V^* B V) \quad (16)$$

$$= (\omega_U \otimes \rho_V)(A \otimes B) \quad (17)$$

for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Thus, the uniqueness of product states (Lemma 1) entails that $(\omega \otimes \rho)_{U \otimes V} = \omega_U \otimes \rho_V$.

For the following lemma, we will need to make use of the fact that $p(\rho, \rho_U) = |\rho(U)|^2$ for any pure state ρ [40, Lemma 2.4].

Lemma 2. *If ρ_0, ρ_1 are unitarily equivalent pure states of \mathfrak{A} , and σ is an arbitrary state of \mathfrak{B} then:*

$$p(\rho_0 \otimes \sigma, \rho_1 \otimes \sigma) = p(\rho_0, \rho_1) \quad (18)$$

$$p(\rho_0 \otimes \rho_0, \rho_1 \otimes \rho_1) = p(\rho_0, \rho_1)^2 \quad (19)$$

Proof. Since ρ_0 and ρ_1 are unitarily equivalent, there is a unitary operator $U \in \mathfrak{A}$ such that $\rho_1 = (\rho_0)_U$ and $p(\rho_0, \rho_1) = |\rho_0(U)|^2$. Thus $\rho_1 \otimes \sigma = (\rho_0 \otimes \sigma)_{(U \otimes I)}$, and therefore

$$p(\rho_0 \otimes \sigma, \rho_1 \otimes \sigma) = |(\rho_0 \otimes \sigma)(U \otimes I)|^2 = |\rho_0(U)|^2 = p(\rho_0, \rho_1) \quad (20)$$

Similarly, $\rho_0 \otimes \rho_0 = (\rho_1 \otimes \rho_1)_{(U \otimes U)}$, and therefore

$$p(\rho_0 \otimes \rho_0, \rho_1 \otimes \rho_1) = |(\rho_0 \otimes \rho_0)(U \otimes U)|^2 = |\rho_0(U)|^4 = p(\rho_0, \rho_1)^2 \quad (21)$$

□

Lemma 3. *Suppose that \mathfrak{A} and \mathfrak{B} are kinematically independent. If ρ is a state of $\mathfrak{A} \vee \mathfrak{B}$ such that $\rho|_{\mathfrak{A}}$ is pure or $\rho|_{\mathfrak{B}}$ is pure, then ρ is a product state.*

Proof. Let $\omega = \rho|_{\mathfrak{A}}$, and let B be an effect in \mathfrak{B} . Define positive linear functionals λ_1 and λ_2 on \mathfrak{A} by setting

$$\lambda_1(A) = \rho(B^{1/2}AB^{1/2}) = \rho(AB) \quad (22)$$

$$\lambda_2(A) = \rho((I-B)^{1/2}A(I-B)^{1/2}) = \rho(A(I-B)) \quad (23)$$

for all $A \in \mathfrak{A}$. It then follows that

$$\omega(A) = \rho(A) = \lambda_1(A) + \lambda_2(A) \geq \lambda_1(A) \quad (24)$$

Since ω is a pure state of \mathfrak{A} , λ_1 is a nonnegative multiple $k\omega$ of ω , and

$$k = k\omega(I) = \lambda_1(I) = \rho(B) \quad (25)$$

Accordingly,

$$\rho(AB) = \lambda_1(A) = k\omega(A) = \omega(A)\rho(B) \quad (26)$$

By linearity, the same equation holds when we replace B by an arbitrary element of \mathfrak{B} . Therefore, $\rho(AB) = \omega(A)\rho(B) = \rho(A)\rho(B)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. □

The proof of the next theorem turns on the fact that nonselective operations cannot *increase* the norm distance between states, and therefore cannot *decrease* the transition probabilities between states. That is, for any nonselective operation T ,

$$p(T^*\omega, T^*\rho) \geq p(\omega, \rho) \quad (27)$$

for all states ω, ρ . To see this, note that

$$\|T^*\omega - T^*\rho\| = \sup\{ |(\omega - \rho)(T(A))| : \|A\| \leq 1 \} \quad (28)$$

and the Russo-Dye theorem entails that if $\|A\| \leq 1$ then $\|T(A)\| \leq 1$.

Theorem 3. *If for each pair $\{\rho_0, \rho_1\}$ of states of \mathfrak{A} , there is an operation T on $\mathfrak{A} \vee \mathfrak{B}$ that broadcasts $\{\rho_0, \rho_1\}$, then \mathfrak{A} is abelian.*

Proof. We assume that for each pair $\{\rho_0, \rho_1\}$ of states of \mathfrak{A} , there is an operation T on $\mathfrak{A} \vee \mathfrak{B}$ that broadcasts $\{\rho_0, \rho_1\}$. Suppose for reductio ad absurdum that \mathfrak{A} is not abelian. Then there are pure states ρ_0, ρ_1 of \mathfrak{A} such that $0 < \|\rho_0 - \rho_1\| < 2$ [30, Exercise 4.6.26]. In this case, ρ_0 and ρ_1 are unitarily equivalent [30, Corollary 10.3.8]. By hypothesis, there is a standard state σ of \mathfrak{B} and an operation T on $\mathfrak{A} \vee \mathfrak{B}$ such that

$$T^*(\rho_0 \otimes \sigma)|_{\mathfrak{A}} = T^*(\rho_0 \otimes \sigma)|_{\mathfrak{B}} = \rho_0 \quad (29)$$

$$T^*(\rho_1 \otimes \sigma)|_{\mathfrak{A}} = T^*(\rho_1 \otimes \sigma)|_{\mathfrak{B}} = \rho_1 \quad (30)$$

Since ρ_0 and ρ_1 are pure, it follows from Lemma 3 that $T^*(\rho_0 \otimes \sigma) = \rho_0 \otimes \rho_0$ and $T^*(\rho_1 \otimes \sigma) = \rho_1 \otimes \rho_1$. Thus,

$$p(\rho_0, \rho_1) = p(\rho_0 \otimes \sigma, \rho_1 \otimes \sigma) \quad (31)$$

$$\leq p(T^*(\rho_0 \otimes \sigma), T^*(\rho_1 \otimes \sigma)) \quad (32)$$

$$= p(\rho_0 \otimes \rho_0, \rho_1 \otimes \rho_1) \quad (33)$$

$$= p(\rho_0, \rho_1)^2 \quad (34)$$

(The equalities in Equations 31 and 34 follow from Lemma 2, while the inequality in Equation 32 follows from the fact that transition probabilities cannot decrease under T^* .) However, the inequality $p(\rho_0, \rho_1) \leq p(\rho_0, \rho_1)^2$ contradicts the fact that $0 < p(\rho_0, \rho_1) < 1$. Therefore, \mathfrak{A} is abelian. \square

3.4 No Bit Commitment and Nonlocality

We show that the impossibility of unconditionally secure bit commitment between systems A and B, in the presence of the kinematic independence and noncommutativity of their algebras of observables, entails nonlocality: spacelike separated systems must at least sometimes occupy entangled states. Specifically, we show that if Alice and Bob have spacelike separated quantum systems, but cannot prepare any entangled state, then Alice and Bob can devise an unconditionally secure bit commitment protocol.

We first show that quantum systems are characterized by the existence of non-uniquely decomposable mixed states.

Lemma 4. *Let \mathfrak{A} be a C^* -algebra. Then \mathfrak{A} is nonabelian if and only if there are distinct pure states $\omega_{1,2}$ and ω_{\pm} of \mathfrak{A} such that $(1/2)(\omega_1 + \omega_2) = (1/2)(\omega_+ + \omega_-)$.*

Proof. If \mathfrak{A} is abelian then its states are in one-to-one correspondence with measures on its pure state space. In particular, if $\rho = (1/2)(\omega_1 + \omega_2)$, where ω_1, ω_2 are distinct pure states, then this decomposition is unique.

Conversely, suppose that there are $A, B \in \mathfrak{A}$ such that $[A, B] \neq 0$ (cf. [8, Example 4.2.6]). Then there is a pure state ρ of \mathfrak{A} such that $\rho([A, B]) \neq 0$ [30, Thm. 4.3.8]. Let $(\pi, \mathcal{H}, \Omega)$ be the GNS representation of \mathfrak{A} induced by ρ . The dimension of the Hilbert space \mathcal{H} must exceed one; for otherwise

$$\pi([A, B]) = [\pi(A), \pi(B)] = 0 \quad (35)$$

in contradiction with the fact that $\langle \Omega, \pi([A, B])\Omega \rangle = \rho([A, B]) \neq 0$. Thus, there is a pair x_1, x_2 of orthogonal unit vectors in \mathcal{H} . Define two states ω_i of \mathfrak{A} by setting

$$\omega_i(A) = \langle x_i, \pi(A)x_i \rangle \quad (A \in \mathfrak{A}) \quad (36)$$

Similarly, define, in the same way, another pair of states ω_{\pm} of \mathfrak{A} using the orthogonal unit vectors $2^{-1/2}(x_1 \pm x_2)$. Since ρ is pure, $(\pi, \mathcal{H}, \Omega)$ is irreducible, and all four of the states $\omega_{1,2}$ and ω_{\pm} are pure and distinct. Moreover, by construction $(1/2)(\omega_1 + \omega_2) = (1/2)(\omega_+ + \omega_-)$. \square

If each of A and B has a non-uniquely decomposable mixed state, then A+B has a pair $\{\rho_0, \rho_1\}$ of distinct (classically) correlated states whose marginals relative to B are identical. In particular, let

$$\rho_0 = (1/2)(\omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2) \quad (37)$$

$$\rho_1 = (1/2)(\omega_+ \otimes \omega_+ + \omega_- \otimes \omega_-) \quad (38)$$

The protocol then proceeds as follows: Alice and Bob arrange things so that at the commitment stage, Alice can make a choice that will determine that either ρ_0 or ρ_1 is prepared, the former corresponding to the commitment 0, and the latter corresponding to the commitment 1. Alice and Bob also agree that at the revelation stage, if Alice committed to 0 then she will instruct Bob to perform a measurement that will distinguish between states ω_1 and ω_2 , and if she committed to 1 then she will instruct Bob to perform a measurement that will distinguish between states ω_+ and ω_- . We must be cautious on this last point: neither Alice nor Bob will typically be able to perform a measurement that can discriminate with certainty between these states. However, for any $\epsilon > 0$, there is an effect $A \in \mathfrak{A}$ such that $\omega_1(A) > 1 - \epsilon$ and $\omega_2(A) < \epsilon$. Similarly, there is an effect $B \in \mathfrak{A}$ such that $\omega_+(B) > 1 - \epsilon$ and $\omega_-(B) < \epsilon$. That is, Alice and Bob can perform measurements that will discriminate with arbitrary accuracy between ω_1 and ω_2 , or between ω_+ and ω_- . Finally, Alice will verify her commitment by performing the corresponding measurement on her system and reporting the outcomes to Bob.

Our final theorem shows that if Alice and Bob have access only to classically correlated states (i.e., convex combinations of product states), then this bit commitment protocol is secure. In particular, we show that Alice cannot cheat by preparing some

state σ which she could later transform at will into either ρ_0 or ρ_1 . To be precise, the no superluminal information transfer by measurement constraint entails that Alice can perform an operation T on $\mathfrak{A} \vee \mathfrak{B}$ only if $T(B) = B$ for all $B \in \mathfrak{B}$, and $T(A) \in \mathfrak{A}$ for all $A \in \mathfrak{A}$. It follows then that Alice can transform product states only to other product states.

Theorem 4. *If \mathfrak{A} and \mathfrak{B} are nonabelian then there is a pair $\{\rho_0, \rho_1\}$ of states of $\mathfrak{A} \vee \mathfrak{B}$ such that:*

1. $\rho_0|_{\mathfrak{B}} = \rho_1|_{\mathfrak{B}}$.
2. *There is no classically correlated state σ of $\mathfrak{A} \vee \mathfrak{B}$ and operations T_0 and T_1 performable by Alice such that $T_0^* \sigma = \rho_0$ and $T_1^* \sigma = \rho_1$.*

For the proof of this theorem, we recall that two representations (π, \mathcal{H}) and (ϕ, \mathcal{K}) of a C^* -algebra are said to be quasi-equivalent just in case there is a $*$ isomorphism α from $\pi(\mathfrak{A})''$ onto $\phi(\mathfrak{A})''$ such that $\alpha(\pi(A)) = \phi(A)$ for each A in \mathfrak{A} . Similarly, states ω and ρ of \mathfrak{A} are said to be quasi-equivalent just in case their corresponding GNS representations are quasi-equivalent. Finally, quasi-equivalence is an equivalence relation, and is closed under finite convex combinations.

Proof. Let ρ_0 and ρ_1 be the states defined in Eqns. 37 and 38. Suppose that $\sigma = \sum_{i=1}^n \lambda_i (\alpha_i \otimes \beta_i)$, where the α_i are states of \mathfrak{A} , and the β_i are states of \mathfrak{B} . Let T_0 and T_1 be operations of $\mathfrak{A} \vee \mathfrak{B}$ that can be performed by Alice. Then for each $i \in [1, n]$, there are states α'_i and α''_i of \mathfrak{A} such that $T_0^*(\alpha_i \otimes \beta_i) = \alpha'_i \otimes \beta_i$ and $T_1^*(\alpha_i \otimes \beta_i) = \alpha''_i \otimes \beta_i$. Moreover, since T_0^* and T_1^* are affine,

$$\rho_0 = T_0^* \sigma = \sum_{i=1}^n \lambda_i (\alpha'_i \otimes \beta_i) \quad (39)$$

$$\rho_1 = T_1^* \sigma = \sum_{i=1}^n \lambda_i (\alpha''_i \otimes \beta_i) \quad (40)$$

Let μ denote the mixed state $(1/2)(\omega_1 + \omega_2) = (1/2)(\omega_+ + \omega_-)$ of \mathfrak{B} . Then $\sum_{i=1}^n \lambda_i \beta_i = \mu$, so that each β_i is quasi-equivalent to μ . Let (π, \mathcal{H}) be the representation of \mathfrak{B} defined in Lemma 4, let P_1 denote the projection onto x_1 , and let P_+ denote the projection onto $2^{-1/2}(x_1 + x_2)$. (Note that since $\omega_{1,2}, \omega_{\pm}$ are represented by vectors in \mathcal{H} , it follows that (π, \mathcal{H}) is unitarily equivalent to the GNS representations induced by these states. Moreover, (π, \mathcal{H}) is quasi-equivalent to the GNS representation induced by β_i .) Since $\pi(\mathfrak{B})$ is weakly dense in $\mathbf{B}(\mathcal{H})$, there are nets $\{A_i\} \subseteq \mathfrak{B}$ and $\{B_i\} \subseteq \mathfrak{B}$ such that $\pi(A_i)$ converges ultraweakly to P_1 and $\pi(B_i)$ converges ultraweakly to P_+ . (Moreover, we can choose these nets so that $0 \leq \pi(A_i), \pi(B_i) \leq I$ for all i .) Since each of the states $\omega_{1,2}, \omega_{\pm}$ is represented by a vector in \mathcal{H} , ultraweak continuity of normal states entails that:

$$\lim_i \rho_1(A_i \otimes (I - A_i)) = 0 \quad (41)$$

$$\lim_i \rho_1((I - A_i) \otimes A_i) = 0 \quad (42)$$

$$\lim_i \rho_0(B_i \otimes (I - B_i)) = 0 \quad (43)$$

$$\lim_i \rho_0((I - B_i) \otimes B_i) = 0 \quad (44)$$

Furthermore, since $\lim_i \mu(A_i) = 1/2$, there exists some $j \in [1, n]$ such that $\lim_i \beta_j(A_i) > 0$. Let $\beta = \beta_j$ and let $\alpha' = \alpha'_j$. Then, combining the previous equalities with Eqns. 39 and 40 gives:

$$\lim_i \alpha'(A_i)\beta(I - A_i) = 0 \quad (45)$$

$$\lim_i \alpha'(I - A_i)\beta(A_i) = 0 \quad (46)$$

$$\lim_i \alpha''(B_i)\beta(I - B_i) = 0 \quad (47)$$

$$\lim_i \alpha''(I - B_i)\beta(B_i) = 0 \quad (48)$$

Since $0 \leq \beta(A_i), \alpha'(A_i) \leq 1$ for all i , it follows that $\{\beta(A_i)\}$ and $\{\alpha'(A_i)\}$ have accumulation points. Thus, we may pass to a subnet in which $\lim_i \beta(A_i)$ and $\lim_i \alpha'(A_i)$ exist (and the preceding equations still hold). We now claim that $\lim_i \beta(A_i) = 1$. Indeed, since $\lim_i \alpha'(I - A_i)\beta(A_i) = 0$, if $\lim_i \beta(A_i) > 0$ then $1 - \lim_i \alpha'(A_i) = \lim_i \alpha'(I - A_i) = 0$. Moreover, since $\lim_i \alpha'(A_i)\beta(I - A_i) = 0$, it follows that $1 - \lim_i \beta(A_i) = \lim_i \beta(I - A_i) = 0$. Thus $\lim_i \beta(A_i) = 1$. An analogous argument shows that either $\lim_i \beta(B_i) = 0$ or $\lim_i \beta(B_i) = 1$.

Now, since the GNS representation induced by β is quasi-equivalent to the irreducible representation (π, \mathcal{H}) , there is a density operator D on \mathcal{H} such that $\beta(X) = \text{Tr}(DX)$ for all $X \in \mathfrak{B}$. Since density operator states are ultraweakly continuous,

$$\text{Tr}(DP_1) = \lim_i \text{Tr}(DA_i) = 1, \quad (49)$$

and therefore $D = P_1$. Thus, if $\lim_i \beta(B_i) = 0$ then we have a contradiction:

$$1/2 = \text{Tr}(P_1 P_+) = \text{Tr}(DP_+) = \lim_i \text{Tr}(DB_i) = 0. \quad (50)$$

But $\lim_i \beta(B_i) = 1$ would also result in the contradiction $1/2 = 1$. Therefore, there is no classically correlated state σ such that $T_0^* \sigma = \rho_0$ and $T_1^* \sigma = \rho_1$. \square

The converse result remains open: it is not known whether nonlocality—the fact that spacelike separated systems occupy entangled states—entails the impossibility of unconditionally secure bit commitment. As we indicated in the introduction, the proof of the corresponding result in elementary quantum mechanics (in which all algebras are type I von Neumann factors) depends on the biorthogonal decomposition theorem, via the theorem of Hughston, Jozsa, and Wootters [28]. Thus, proving the converse would amount to generalizing the Hughston-Jozsa-Wootters result to arbitrary nonabelian C^* -algebras. If, as we believe, the more general result holds, then quantum theory can be characterized in terms of our three information-theoretic constraints. So, either quantum theory can be characterized in terms of our information-theoretic constraints, or there are physical systems which permit an unconditionally secure bit commitment protocol.

4 Concluding Remarks

Within the framework of a class of theories broad enough to include both classical and quantum particle and field theories, and hybrids of these theories, we have shown that three information-theoretic constraints suffice to exclude the classical theories. Specifically, the information-theoretic constraints entail that the algebras of observables of distinct physical systems commute, that the algebra of observables of each individual system is noncommutative, and that spacelike separated systems occupy entangled states.

Conversely, from the three physical characteristics of a quantum theory in the most general sense—kinematic independence, noncommutativity, and nonlocality—we have derived two of the three information-theoretic constraints: the impossibility of superluminal information transfer between two physical systems by performing measurements on one of them, and the impossibility of perfectly broadcasting the information contained in an unknown physical state.

It remains an open question whether the third information-theoretic constraint—the impossibility of unconditionally secure bit commitment—can be derived as well. As we indicated above, this would involve something equivalent to an algebraic generalization of the Hughston-Jozsa-Wootters theorem [28] to cover cases of systems with an infinite number of degrees of freedom that arise in quantum field theory and the thermodynamic limit of quantum statistical mechanics (in which the number of microsystems and the volume they occupy goes to infinity, while the density defined by their ratio remains constant). The Stone-von Neumann theorem, which guarantees the existence of a unique representation (up to unitary equivalence) of the canonical commutation relations for systems with a finite number of degrees of freedom, breaks down for such cases, and there will be many unitarily inequivalent representations of the canonical commutation relations.

Since we intend our characterization of quantum theory to apply quite generally to these cases as well (including the quantum theoretical description of exotic phenomena such as Hawking radiation, black hole evaporation, Hawking information loss, etc.), we do not restrict the notion of a quantum theory to the standard quantum mechanics of a system represented on a single Hilbert space with a unitary dynamics. So it would not be an appropriate goal of our characterization project to derive the Schrödinger equation as a description of the dynamics of a quantum system from information-theoretic assumptions. A unitary dynamics will not be implementable in a quantum field theory on a curved space-time, for example, which might be a preliminary semi-classical step towards a quantum theory of gravity (see Arageorgis *et al* [2]).

The foundational significance of our derivation, as we see it, is that quantum mechanics should be interpreted as a *principle theory*, where the principles at issue are information-theoretic. The distinction between *principle* and *constructive* theories is introduced by Einstein in his discussion of the significance of the transition from Newtonian to relativistic physics [18]. As Einstein puts it, most theories in physics are constructive, with the aim of representing complex phenomena as constructed out of the elements of a simple formal scheme. So, for example, the kinetic theory of gases is a constructive theory of thermal and diffusion processes in terms of the movement of molecules. By contrast [18, p. 228], principle theories begin with empirically discov-

ered ‘general characteristics of natural processes, principles that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy.’ Einstein cites thermodynamics as the paradigm example of a principle theory. The methodology here is analytic, not synthetic, with the aim of deducing [18, p. 228] ‘necessary conditions, which separate events have to satisfy, from the universally experienced fact that perpetual motion is impossible.’

Einstein’s point is that the theory of relativity is to be understood as a principle theory. In the case of the special theory, there are two relevant principles: the equivalence of inertial frames for all physical laws (the laws of electromagnetic phenomena as well as the laws of mechanics), and the constancy of the velocity of light in vacuo for all inertial frames. These principles are irreconcilable in the Euclidean geometry of Newtonian space-time, where inertial frames are related by Galilean transformations. The required revision yields the special theory of relativity and Minkowski geometry, in which inertial frames are related by Lorentz transformations. In his ‘Autobiographical Notes’ [20, p. 57], Einstein characterizes the special principle of relativity, that the laws of physics are invariant with respect to Lorentz transformations from one inertial system to another, as ‘a restricting principle for natural laws, comparable to the restricting principle of the non-existence of the *perpetuum mobile* which underlies thermodynamics.’ In the case of the general theory of relativity, the group of allowable transformations includes all differentiable transformations of the space-time manifold onto itself.

A relativistic theory is a theory with certain symmetry or invariance properties, defined in terms of a group of space-time transformations. Following Einstein we understand this invariance to be a consequence of the fact that we live in a world in which natural processes are subject to certain constraints. A quantum theory is a theory in which the observables and states have a certain characteristic algebraic structure. Unlike relativity theory, quantum mechanics was born as a recipe or algorithm for calculating the expectation values of observables measured by macroscopic measuring instruments. These expectation values (or probabilities of ranges of values of observables) cannot be reduced to probability distributions over the values of dynamical variables (or probability distributions over properties of the system). Analogously, one might imagine that the special theory of relativity was first formulated geometrically by Minkowski rather than Einstein, as an algorithm for relativistic kinematics and the Lorentz transformation, which is incompatible with the kinematics of Newtonian space-time. What differentiates the two cases is that Einstein’s derivation provides an interpretation for relativity theory: a description of the conditions under which the theory would be true, in terms of certain principles that constrain the law-like behavior of physical systems. It is in this sense that our derivation of quantum theory from information-theoretic principles can be understood as an interpretation of quantum theory: the theory can now be seen as reflecting the constraints imposed on the theoretical representations of physical processes by these principles.

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