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A NEW SYSTEM OF PROOF-THEORETIC ORDINAL FUNCTIONS

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In this paper we present a family of ordinal functions ψ_v $(v \leq \omega)$, which seems to provide the so far simplest method for denoting large constructive ordinals. These functions are a simplified version of the θ-functions, but nevertheless have the same strength as those. This will be shown at the end of the paper (Theorem 3.7) by using proof-theoretic results from [1], [2], [5].—In Section 1 we define the functions ψ_v and prove their main properties. In Section 2 we define a primitive recursive notation system $(OT, <)$ based on the functions ψ_v. This system has the great advantage that its ordering relation $<$ is very simple and can be defined without reference to sets of coefficients or any similar concept. OT is introduced as a subset of a larger set T of terms, which plays an important role in Section 3. There we show that the statement $\mathrm{PRWO}(\psi_0\Omega_\omega)$, which says that there exist no primitive recursive infinite descending sequences in $(\{x \in OT: x < \psi_0\Omega_\omega\}, <)$, is not provable in Π^1_1-CA_0. This result is essentially used in Simpson [6] to establish the unprovability of a certain theorem of finite combinatorics. The proof of $\Pi^1_1\text{-CA}_0 \nvdash \mathrm{PRWO}(\psi_0\Omega_\omega)$ is based on the following results from [1]:

$$\mathrm{ID}_\nu \nvdash \forall n\, \exists k\, c^n_\nu(k) = 0 \quad (\nu \leq \omega)$$

where $c^n_\nu(k) \in T$, for all $n, k \in \mathbb{N}$; and every sequence $(c^n_\nu(k))_{k \in \mathbb{N}}$ is primitive recursive.

In Section 3 we will prove $c^n_\nu(k) \in OT$ and $(c^n_\nu(k) \neq 0 \Rightarrow c^n_\nu(k+1) < c^n_\nu(k))$. Since for all $\nu < \omega$ we have $c^n_\nu(k) < \psi_0\Omega_\omega$, it follows that $\mathrm{PRWO}(\psi_0\Omega_\omega)$ implies $\forall \nu < \omega\, \forall n\, \exists k\, c^n_\nu(k) = 0$. Since this can be proved in Peano Arithmetic and since Π^1_1-CA_0 is conservative over $\bigcup_{\nu<\omega} \mathrm{ID}_\nu$ with respect to arithmetic sentences, we obtain now $\Pi^1_1\text{-CA}_0 \nvdash \mathrm{PRWO}(\psi_0\Omega_\omega)$.

For readers unfamiliar with ordinal notations we give a short description of the basic ideas in the construction of Feferman's θ-functions and then indicate how our ψ-functions are related to this construction. The functions $\theta_\alpha : \mathrm{On} \to \mathrm{On}$ $(\alpha \in \mathrm{On})$ constitute a hierarchy of normal functions extending the usual Veblen hierarchy $(\varphi_\alpha)_{\alpha<\Gamma_0}$. Usually one writes $\theta\alpha\beta$ instead of $\theta_\alpha(\beta)$ and considers θ as a binary function. The ordinals $\theta\alpha\beta$ are defined by transfinite recursion on α in such a way that—intuitively spoken—as many ordinals as possible become denotable in terms of the constants $0, \aleph_1, \ldots, \aleph_\omega$ and the function symbols $+$ and θ. Suppose that $\theta\xi\eta$ has been defined for all $\xi<\alpha$, $\eta \in \mathrm{On}$. Then for each $\beta \in \mathrm{On}$ we consider the set $C(\alpha, \beta)$ of all ordinals γ which can be generated from ordinals $<\beta$ and the constants $0, \aleph_1, \ldots, \aleph_\omega$ by successive application of the functions $+$ and $\theta \restriction \{\xi : \xi < \alpha\} \times \mathrm{On}$. An ordinal β is called α-critical iff $\beta \notin C(\alpha, \beta)$, and $\theta_\alpha : \mathrm{On} \to \mathrm{On}$ is introduced as the ordering function of the class of all α-critical ordinals. After $\theta\alpha\beta$ has been defined for all $\alpha, \beta \in \mathrm{On}$ let $\theta(\omega+1)$ denote the set of all ordinals representable in terms of $0, \aleph_1, \ldots, \aleph_\omega$, $+$, θ. Surprisingly it turned out that the following subset $\theta^*(\omega+1)$ of $\theta(\omega+1)$ has essentially the same ordertype as $\theta(\omega+1)$:

Inductive definition of $\theta^*(\omega+1)$

(i) $0 \in \theta^*(\omega+1)$.

(ii) $\xi, \eta \in \theta^*(\omega+1) \Rightarrow \xi+\eta \in \theta^*(\omega+1)$.

(iii) $\alpha \in \theta^*(\omega+1)\ \&\ v \leq \omega \Rightarrow \theta\alpha\aleph_v \in \theta^*(\omega+1)$.

So by using only the functions $\alpha \mapsto \theta\alpha\aleph_v$ $(v=0, 1, \ldots, \omega)$ instead of $(\alpha, \beta) \mapsto \theta\alpha\beta$ one obtains a system of ordinal notations which has almost the same strength as the full system $\theta(\omega+1)$. This suggests to define directly a family of ordinal functions ψ_v $(v \leq \omega)$ corresponding to $\alpha \mapsto \theta\alpha\aleph_v$ $(v \leq \omega)$ such that the system of all ordinals representable in terms of $0, +, \psi_0, \ldots, \psi_\omega$ will be isomorphic to $\theta^*(\omega+1)$. So we are led to the following definition of $\psi_v\alpha$:

$$\psi_v\alpha := \min\{\gamma : \gamma \notin C_v(\alpha)\},$$

where $C_v(\alpha)$ denotes the set of all ordinals which can be generated from ordinals $<\aleph_v$ by the functions $+$ (addition) and $\psi_u \restriction \{\xi : \xi < \alpha\}$ $(u \leq \omega)$.

1. The functions ψ_v $(v \leq \omega)$

Preliminaries. We are working in ZFC. The letters $\alpha, \beta, \gamma, \delta, \xi, \eta, \zeta$ always denote ordinals. 'On' denotes the class of all ordinals and 'Lim' the class of all limit ordinals. Each ordinal α is identified with the set of its predecessors so that $\alpha = \{x \in \mathrm{On} : x < \alpha\}$ and $\alpha < \beta \Leftrightarrow \alpha \in \beta$. As usual $\alpha \mapsto \aleph_\alpha$ enumerates the class of all infinite cardinals. We define

$$\Omega_\xi := \begin{cases} 1 & \text{if } \xi = 0, \\ \aleph_\xi, & \text{if } \xi > 0. \end{cases}$$

We denote by P the class of all additive principal numbers, i.e.,

$$P = \{\alpha \in \mathrm{On} : 0 < \alpha \wedge \forall \xi, \eta < \alpha\, (\xi + \eta \in \alpha)\} = \{\omega^\xi : \xi \in \mathrm{On}\}.$$

Definition of $P(\alpha)$. (1) $P(0):=\emptyset$.

(2) For $\alpha>0$ there are uniquely determined $\alpha_0, \ldots, \alpha_n \in P$ with $\alpha = \alpha_0 + \cdots + \alpha_n$ and $\alpha_n \leqslant \cdots \leqslant \alpha_0$; we set $P(\alpha):=\{\alpha_0, \ldots, \alpha_n\}$.

Definition. For $\alpha_0, \ldots, \alpha_n \in P$ we set $\alpha_0 \# \cdots \# \alpha_n := \alpha_{\pi(0)} + \cdots + \alpha_{\pi(n)}$, where π is a permutation of $(0, \ldots, n)$ with $\alpha_{\pi(0)} \geqslant \cdots \geqslant \alpha_{\pi(n)}$.

1.1. Proposition. (a) $\alpha \notin P \Leftrightarrow P(\alpha) \subseteq \alpha$.

(b) $\gamma \in P \Rightarrow (P\alpha \subseteq \gamma \Leftrightarrow \alpha < \gamma)$.

(c) $P(\beta) \subseteq P(\alpha+\beta) \subseteq P(\alpha) \cup P(\beta)$.

(d) $\Omega_\xi \in P$, *for all* $\xi \in \mathrm{On}$.

Definition of sets of ordinals $C_v(\alpha)$ *and ordinals* $\psi_v\alpha$ $(v \leqslant \omega)$

The definition proceeds by transfinite recursion on α simultaneously for all $v \leqslant \omega$. Suppose that $C_v(\xi)$ and $\psi_v\xi$ are defined for all $\xi < \alpha$, $v \leqslant \omega$.

Then we set

$$C_v(\alpha) := \bigcup_{n<\omega} C_v^n(\alpha), \qquad \psi_v\alpha := \min\{\gamma : \gamma \notin C_v(\alpha)\},$$

where $C_v^n(\alpha)$ is defined by induction on n as follows

$$\begin{aligned} C_v^0(\alpha) &:= \Omega_v, \\ C_v^{n+1}(\alpha) &:= C_v^n(\alpha) \cup \{\gamma : P(\gamma) \subseteq C_v^n(\alpha)\} \\ &\qquad \cup \{\psi_u\xi : \xi \in \alpha \cap C_v^n(\alpha) \wedge \xi \in C_u(\xi) \wedge u \leqslant \omega\}. \end{aligned}$$

Remark. The condition "$\xi \in C_u(\xi)$" in the definition of $C_v^{n+1}(\alpha)$ is included since it makes the important properties of the functions ψ_v easier to prove. But it can be shown that by omitting this condition one does not change the sets $C_v(\alpha)$. Hence $C_v(\alpha)$ can be characterized as the least set X with:

(C1) $\Omega_v \subseteq X$,

(C2) $\forall \xi, \eta \in X\, (\xi + \eta \in X)$,

(C3) $\forall \xi \in X \cap \alpha\, \forall u \leqslant \omega\, (\psi_u\xi \in X)$.

In the following the letters u, v, w shall always denote ordinals $\leqslant\omega$.

1.2. Lemma. (a) $\psi_v 0 = \Omega_v$.

(b) $\psi_v\alpha \in P$.

(c) $\Omega_v \leqslant \psi_v\alpha < \Omega_{v+1}$.

(d) $\alpha \leqslant \beta \Rightarrow C_v(\alpha) \subseteq C_v(\beta)$ *and* $\psi_v\alpha \leqslant \psi_v\beta$.

(e) $\gamma \in C_v(\alpha) \Leftrightarrow P(\gamma) \subseteq C_v(\alpha)$.

(f) $\xi, \eta \in C_v(\alpha) \Rightarrow \xi + \eta \in C_v(\alpha)$.

(g) $\xi + \eta \in C_v(\alpha) \Rightarrow \eta \in C_v(\alpha)$.

(h) $\alpha_0 < \alpha$ *and* $\forall \xi\, (\alpha_0 \leqslant \xi < \alpha \Rightarrow \xi \notin C_v(\alpha_0)) \Rightarrow C_v(\alpha_0) = C_v(\alpha)$.

Proof. (a) By induction on n we get $C^n_v(0) = \Omega_v$.

(b) Assume $\psi_v\alpha \notin P$. Then $P(\psi_v\alpha) \subseteq \psi_v\alpha \subseteq C_v(\alpha)$ and thus $\psi_v\alpha \in C_v(\alpha)$. Contradiction.

(c) From $\Omega_v \subseteq C_v(\alpha)$ it follows that $\Omega_v \leqslant \psi_v\alpha$. Obviously the cardinality of $C_v(\alpha)$ is less than Ω_{v+1}. Hence there exists $\gamma < \Omega_{v+1}$ with $\gamma \notin C_v(\alpha)$ and therefore $\psi_v\alpha < \Omega_{v+1}$.

(d) Trivial.

(e) Using the fact that $\psi_u\xi \in P$ one proves $\forall\gamma \in C^n_v(\alpha)\,(P(\gamma) \subseteq C^n_v(\alpha))$ by induction on n. On the other side, if $P(\gamma) \subseteq C_v(\alpha)$, then $P(\gamma) \subseteq C^n_v(\alpha)$ for some $n \in \mathbb{N}$ (since $P(\gamma)$ is finite and $C^i_v(\alpha) \subseteq C^{i+1}_v(\alpha)$) and thus $\gamma \in C^{n+1}_v(\alpha) \subseteq C_v(\alpha)$.

(f) From $\xi, \eta \in C_v(\alpha)$ we obtain $P(\xi + \eta) \subseteq P(\xi) \cup P(\eta) \subseteq C_v(\alpha)$ and then $\xi + \eta \in C_v(\alpha)$.

(g) From $\xi + \eta \in C_v(\alpha)$ we obtain $P(\eta) \subseteq P(\xi + \eta) \subseteq C_v(\alpha)$ and then $\eta \in C_v(\alpha)$.

(h) Suppose $\alpha_0 < \alpha$ and $\forall\xi\,(\alpha_0 \leqslant \xi < \alpha \rightarrow \xi \notin C_v(\alpha_0))$. Then we get $C_v(\alpha_0) \subseteq C_v(\alpha)$ by 1.2(d), and $\forall\gamma\,(\gamma \in C^n_v(\alpha) \rightarrow \gamma \in C_v(\alpha_0))$ by induction on n.

1.3. Lemma. *$\alpha < \beta$ and $\alpha \in C_v(\alpha) \Rightarrow \psi_v\alpha < \psi_v\beta$.*

Proof. From the premise we conclude $\psi_v\alpha \leqslant \psi_v\beta$ and $\psi_v\alpha \in C_v(\beta)$. Hence $\psi_v\alpha < \psi_v\beta$, since $\psi_v\beta \notin C_v(\beta)$.

1.4. Lemma. (a) *$\gamma = \psi_{u_i}\xi_i$ and $\xi_i \in C_{u_i}(\xi_i)$ for $i = 0, 1 \Rightarrow u_0 = u_1$, $\xi_0 = \xi_1$.*

(b) *$\gamma \in C_v(\alpha)$ and $\Omega_v \leqslant \gamma \in P \Rightarrow \exists u, \xi\,(\gamma = \psi_u\xi$ and $\xi \in \alpha \cap C_v(\alpha) \cap C_u(\xi))$.*

(c) *$\Omega_v \leqslant \psi_u\xi \in C_v(\alpha)$ and $\xi \in C_u(\xi) \Rightarrow \xi \in \alpha \cap C_v(\alpha)$.*

Proof. (a) follows immediately from 1.2(c) and 1.3.

(b) We have $P(\gamma) = \{\gamma\}$ and $\gamma \in C^{n+1}_v(\alpha) \setminus C^n_v(\alpha)$ for some $n \in \mathbb{N}$. Hence $\gamma = \psi_u\xi$ with $\xi \in \alpha \cap C^n_v(\alpha)$ and $\xi \in C_u(\xi)$.

(c) Let $\gamma := \psi_u\xi$. By (b) we obtain $\gamma = \psi_w\zeta$ with $\zeta \in \alpha \cap C_v(\alpha) \cap C_w(\zeta)$. Now by (a) it follows that $w = u$ and $\xi = \zeta \in \alpha \cap C_v(\alpha)$

1.5. Lemma. *$C_v(\alpha) \cap \Omega_{v+1} = \psi_v\alpha$.*

Proof. $\psi_v\alpha \subseteq C_v(\alpha) \cap \Omega_{v+1}$ holds by definition and 1.2(c).

Now let $\gamma \in C_v(\alpha) \cap \Omega_{v+1}$. We have to show that $\gamma < \psi_v\alpha$.

1. $\gamma < \Omega_v$: Then $\gamma < \psi_v\alpha$ holds by 1.2(c).

2. $\gamma \in P$: Then $\gamma = \psi_u\xi$ with $\xi < \alpha$ and $\xi \in C_u(\xi)$ (1.4(b)).

By 1.2(c) we have $u \leqslant v$. If $u < v$, then $\gamma < \Omega_{u+1} \leqslant \Omega_v \leqslant \psi_v\alpha$. If $u = v$, then $\gamma = \psi_v\xi < \psi_v\alpha$ by 1.3.

3. $\Omega_v \leqslant \gamma \notin P$: Then $\gamma_0 := \max P(\gamma) \in C_v(\alpha) \cap \Omega_{v+1}$, and by 2. we obtain $\gamma_0 < \psi_v\alpha$. Hence $\gamma < \psi_v\alpha$, since $\psi_v\alpha \in P$.

1.6. Lemma

(a) $\psi_v(\alpha+1) = \begin{cases} \min\{\gamma \in P : \psi_v\alpha < \gamma\}, & \textit{if } \alpha \in C_v(\alpha), \\ \psi_v\alpha, & \textit{otherwise}. \end{cases}$

(b) $\alpha \in \mathrm{Lim} \Rightarrow \psi_v\alpha = \sup\{\psi_v\xi : \xi < \alpha \textit{ and } \xi \in C_v(\xi)\}$.

Proof. (a) 1. $\alpha \in C_v(\alpha)$: by 1.2(b) and 1.3 we have $\psi_v\alpha < \psi_v(\alpha+1) \in P$. Suppose $\psi_v\alpha \leqslant \gamma < \psi_v(\alpha+1)$ and $\gamma \in P$. Then by 1.4(b) we have $\gamma = \psi_u\xi$ with $\xi \leqslant \alpha$ and $\xi \in C_u(\xi)$. From $\psi_v\alpha \leqslant \psi_u\xi < \psi_v(\alpha+1)$ we get $u = v$. From $\psi_v\alpha \leqslant \psi_v\xi$ and $\xi \in C_v(\xi)$ it follows by 1.3 that $\alpha \leqslant \xi$. Hence $\alpha = \xi$ and $\gamma = \psi_v\alpha$.

2. If $\alpha \notin C_v(\alpha)$, then $C_v(\alpha) = C_v(\alpha+1)$ by 1.2(h).

(b) By 1.3 we have $\psi_v\xi < \psi_v\alpha$ for all $\xi < \alpha$ with $\xi \in C_v(\xi)$. Suppose now that $\psi_v 0 \leqslant \gamma < \psi_v\alpha$, and let $\gamma_0 := \max P(\gamma)$. Then $\Omega_v \leqslant \gamma_0 \in C_v(\alpha)$ and therefore $\gamma_0 = \psi_v\xi$ with $\xi < \alpha$ and $\xi \in C_v(\xi)$. Since $1 = \psi_0 0$ and $0 \in C_0(0) \subseteq C_v(\xi+1)$, we obtain $\xi+1 \in C_v(\xi+1)$. By 1.3 we also have $\gamma_0 = \psi_v\xi < \psi_v(\xi+1)$ and therefore $\gamma < \psi_v(\xi+1)$.

1.7. Lemma. (a) $\alpha < \varepsilon_0 \Rightarrow \alpha \in C_0(\alpha)$ *and* $\psi_0\alpha = \omega^\alpha$.

(b) $\alpha < \varepsilon_{\Omega_v+1},\ v \neq 0 \Rightarrow \alpha \in C_v(\alpha)$ *and* $\psi_v\alpha = \omega^{\Omega_v+\alpha}$.

Proof. By transfinite induction on α: We set

$$\varepsilon(v) := \begin{cases} \varepsilon_0, & \text{for } v = 0, \\ \varepsilon_{\Omega_v+1}, & \text{for } v > 0, \end{cases} \qquad \alpha * v := \begin{cases} \alpha, & \text{for } v = 0, \\ \Omega_v + \alpha, & \text{for } v > 0. \end{cases}$$

1. We have $0 \in C_v(0)$ and $\psi_v 0 = \Omega_v = \omega^{0*v}$.

2. Suppose $\alpha \in C_v(\alpha)$ and $\psi_v\alpha = \omega^{\alpha*v}$. Then also $\alpha+1 \in C_v(\alpha+1)$ and $\psi_v(\alpha+1) = \omega^{\alpha*v+1} = \omega^{(\alpha+1)*v}$ by 1.6(a).

3. Suppose $\alpha \in \varepsilon(v) \cap \mathrm{Lim}$ and $\forall \xi < \alpha\, (\xi \in C_v(\xi) \wedge \psi_v\xi = \omega^{\xi*v})$. Then by 1.6(b) we obtain $\psi_v\alpha = \sup\{\omega^{\xi*v} : \xi < \alpha\} = \omega^{\alpha*v}$. It remains to prove that $\alpha \in C_v(\alpha)$. For $\alpha < \Omega_v$ this is trivial. For $\alpha = \Omega_v$ we have $\alpha = \psi_v 0 > 0$ and thus $\alpha \in C_v(\alpha)$, since $0 \in C_v(0) \subseteq C_v(\alpha)$. For $\Omega_v < \alpha < \varepsilon(v)$ we have $P(\alpha) \subseteq \alpha$ and therefore by I.H. (induction hypothesis) $\xi \in C_v(\xi) \subseteq C_v(\alpha)$ for all $\xi \in P(\alpha)$. This yields $\alpha \in C_v(\alpha)$.

1.8. Lemma. (a) $C_v(\alpha) \subseteq \varepsilon_{\Omega_\omega+1}$. (b) $\varepsilon_{\Omega_\omega+1} \leqslant \alpha \Rightarrow C_v(\varepsilon_{\Omega_\omega+1}) = C_v(\alpha)$.

Proof. (a) Using 1.7(b) and 1.2(c) one proves $C_v^n(\alpha) \subseteq \varepsilon_{\Omega_\omega+1}$ by induction on n.

(b) follows from (a) and 1.2(h).

Definition of $G_u\gamma$. For every $\gamma \in C_0(\varepsilon_{\Omega_\omega+1})$ we define a finite set $G_u\gamma \subseteq \mathrm{On}$ in such a way that, for each α, $\gamma \in C_u(\alpha) \Leftrightarrow G_u\gamma \subseteq \alpha$. These sets will be used in Section 2 to define the set OT of ordinal notations. The definition of $G_u\gamma$ proceeds by induction on $\min\{n \in \mathbb{N} : \gamma \in C_0^n(\varepsilon_{\Omega_\omega+1})\}$:

(1) $\gamma \notin P$: $\quad G_u\gamma := \bigcup\{G_u\xi : \xi \in P(\gamma)\}$.

(2) $\gamma = \psi_v\xi$ with $\xi \in C_v(\xi)$: $\quad G_u\xi := \begin{cases} \{\xi\} \cup G_u\xi, & \text{if } u \leq v, \\ \emptyset, & \text{if } v < u, \end{cases}$.

1.9. Lemma. *If $\gamma \in C_0(\varepsilon_{\Omega_\omega+1})$, then $\gamma \in C_u(\alpha)$ holds if, and only if, $G_u\gamma \subseteq \alpha$.*

Proof. By induction on $\min\{n \in \mathbb{N} : \gamma \in C_0^n(\varepsilon_{\Omega_\omega+1})\}$:

1. $\gamma \notin P$: By I.H. we have $\xi \in C_u(\alpha) \Leftrightarrow G_u\xi \subseteq \alpha$, for every $\xi \in P(\gamma)$. Hence $P(\gamma) \subseteq C_u(\alpha) \Leftrightarrow G_u\gamma \subseteq \alpha$. By 1.2(e) we have $\gamma \in C_u(\alpha) \Leftrightarrow P(\gamma) \subseteq C_u(\alpha)$.

2. $\gamma = \psi_v\xi$ with $\xi \in C_v(\xi)$:

2.1. $u \leq v$: Then by I.H. we have $\xi \in C_u(\alpha) \Leftrightarrow G_u\xi \subseteq \alpha$, and by 1.4(c), $\gamma \in C_u(\alpha) \Leftrightarrow \xi \in \alpha \cap C_u(\alpha)$. From this we obtain $\gamma \in C_u(\alpha) \Leftrightarrow \{\xi\} \cup G_u\xi \subseteq \alpha$. But $G_u\gamma = \{\xi\} \cup G_u\xi$.

2.2. $v < u$: In this case we have $\gamma \in \Omega_u \subseteq C_u(\alpha)$ and $G_u\gamma = \emptyset$.

2. The notation system (OT, <)

In this section we introduce a primitive recursive set OT of formal terms together with a primitive recursive ordering on OT such that $(OT, <)$ is isomorphic to $(C_0(\varepsilon_{\Omega_\omega+1}), <)$.

Let $D_0, D_1, \ldots, D_\omega$ be a sequence of formal symbols.

Inductive definition of a set T of terms

(T1) $0 \in T$.

(T2) If $a \in T$ and $v \leq \omega$, then $D_v a \in T$; we call $D_v a$ a *principal term.*

(T3) If $a_0, \ldots, a_k \in T$ are principal terms and $k \geq 1$, then $(a_0, \ldots, a_k) \in T$.

In the following the letters a, b, c, d will always denote elements of T.

For principal terms a we set: $(a) := a$.

Inductive definition of $a < b$ for $a, b \in T$

(<1) $b \neq 0 \Rightarrow 0 < b$.

(<2) $u < v$ or $(u = v$ and $a < b) \Rightarrow D_u a < D_v b$.

(<3) Let $a = (a_0, \ldots, a_n)$, $b = (b_0, \ldots, b_m)$, $1 \leq m + n$. Then $a < b$ iff one of the following two cases holds:

(i) $n < m$ and $a_i = b_i$ for $i \leq n$.

(ii) $\exists k \leq \min\{n, m\}(a_k < b_k$ and $a_i = b_i$ for $i < k)$.

2.1. Lemma. *< is a linear ordering on T.*

Proof. Straightforward.

Abbreviations. Let $a \in T$ and $M, M' \subseteq T$:

$$M \preccurlyeq M' \quad :\Leftrightarrow \quad \forall x \in M\, \exists y \in M'\, (x \preccurlyeq y),$$
$$M \prec a \quad :\Leftrightarrow \quad \forall x \in M\, (x \prec a),$$
$$a \preccurlyeq M \quad :\Leftrightarrow \quad \exists x \in M\, (a \preccurlyeq x).$$

Inductive definition of $G_u a \subseteq T$ for $a \in T$

(G1) $G_u 0 := \emptyset$.

(G2) $G_u(a_0, \ldots, a_k) := G_u a_0 \cup \cdots \cup G_u a_k$.

(G3) $G_u D_v b := \begin{cases} \{b\} \cup G_u b, & \text{if } u \leqslant v, \\ \emptyset, & \text{if } v < u. \end{cases}$

Inductive definition of the subset OT of T

(OT1) $0 \in OT$.

(OT2) If $a_0, \ldots, a_k \in OT$ $(k \geqslant 1)$ are principal terms with $a_k \preccurlyeq \cdots \preccurlyeq a_0$, then $(a_0, \ldots, a_k) \in OT$.

(OT3) If $b \in OT$ with $G_v b \prec b$, then $D_v b \in OT$.

The elements of OT are called *ordinal terms.*

Proposition. $a \in OT \Rightarrow G_u a \subseteq OT$.

Inductive definition of an ordinal $o(a)$ for $a \in T$

(o.1) $o(0) := 0$.

(o.2) $o((a_0, \ldots, a_k)) := o(a_0) \# \cdots \# o(a_k)$ $(k \geqslant 1)$.

(o.3) $o(D_v b) := \psi_v o(b)$.

2.2. Lemma. *For $a, c \in OT$ we have*:

(a) $o(a) \in C_0(\varepsilon_{\Omega_\omega + 1})$,

(b) $G_u o(a) = \{o(x) : x \in G_u a\}$,

(c) $a \prec c \Rightarrow o(a) < o(c)$.

Proof. By induction on the length of a, simultaneously for (a), (b), (c): Let $\varepsilon := \varepsilon_{\Omega_\omega + 1}$.

1. $a = 0$: trivial.

2. $a = D_v b$: Then $G_v b \prec b$ and $b \in OT$.

(a) By I.H. we have $o(b) \in C_0(\varepsilon)$ and $G_v o(b) = \{o(x) : x \in G_v b\} \subseteq o(b)$. From this we obtain $o(b) \in \varepsilon \cap C_v(o(b))$ by 1.8, 1.9 and then $o(a) = \psi_v o(b) \in C_0(\varepsilon)$.

(b) Since $o(b) \in C_v(o(b))$, we have

$$G_u o(a) = \begin{cases} \{o(b)\} \cup G_u o(b), & \text{if } u \leqslant v, \\ \emptyset, & \text{if } v < u. \end{cases}$$

by I.H. we have $G_u o(b) = \{o(x) : x \in G_u b\}$. Hence $G_u o(a) = \{o(x) : x \in G_u a\}$.

(c) We make a subsidiary induction on the length of c:

(i) $c = D_u d$ with $v < u$: $o(a) < \Omega_{v+1} \leq \Omega_u \leq \psi_u o(d) = o(c)$.

(ii) $c = D_v d$ with $b < d$: By the I.H. we get $o(b) < o(d)$ and, as shown above, $o(b) \in C_v(o(b))$. This yields $\psi_v o(b) < \psi_v o(d)$.

(iii) $c = (c_0, \ldots, c_m)$ with $m \geq 1$ and $a \leq c_0$: By the subsidiary I.H. we get $o(a) \leq o(c_0)$ and thus $o(a) < o(c_0) \# o(c_1) \leq o(c)$.

3. $a = (a_0, \ldots, a_n)$ with $n \geq 1$ and $a_n \leq \cdots \leq a_0$:

(a) By I.H. we have $P(o(a)) = \{o(a_0), \cdots, o(a_n)\} \subseteq C_0(\varepsilon)$ and therefore $o(a) \in C_0(\varepsilon)$.

(b) By I.H. we have $G_u o(a_i) = \{o(x) : x \in G_u(a_i)\}$ for $i = 0, \ldots, n$. Hence

$$G_u o(a) = \bigcup_{i=0}^{n} G_u o(a_i) = \left\{ o(x) : x \in \bigcup_{i=0}^{n} G_u a_i \right\} = \{o(x) : x \in G_u a\}.$$

(c) Let $c = (c_0, \ldots, c_m)$ with $m \geq 0$.

(i) $n < m$ and $a_i = c_i$ for $i \leq n$: $o(a) = o(c_0) \# \cdots \# o(c_n) < o(c)$.

(ii) $k \leq \min\{n, m\}$ with $a_k < c_k$ and $a_i = c_i$ for $i < k$: By I.H. we have $o(a_n) \leq \cdots \leq o(a_k) < o(c_k)$ and thus $o(a_k) \# \cdots \# o(a_n) < o(c_k) \leq o(c_k) \# \cdots \# o(c_m)$. Hence

$$o(a) = o(c_0) \# \cdots \# o(c_{k-1}) \# o(a_k) \# \cdots \# o(a_n) < o(c).$$

2.3. Lemma. (a) $C_0(\varepsilon_{\Omega_\omega+1}) = \{o(x) : x \in OT\}$

(b) *For every* $a \in OT$ *with* $a < D_1 0$ *holds*: $o(a) =$ *the ordertype of* $(\{x \in OT : x < a\}, <)$.

(c) $\psi_0 \varepsilon_{\Omega_\omega+1} =$ *the ordertype of* $(\{x \in OT : x < D_1 0\}, <)$.

Proof. Let $\varepsilon := \varepsilon_{\Omega_\omega+1}$.

(a) By induction on n we prove: $\alpha \in C_0^n(\varepsilon) \Rightarrow \exists a \in OT\ (\alpha = o(a))$. (Together with 2.2(a) this yields $C_0(\varepsilon) = \{o(x) : x \in OT\}$.) for $n = 0$ the assertion is trivial. Let $\alpha \in C_0^{n+1}(\varepsilon) \setminus C_0^n(\varepsilon)$.

1. $\alpha = \alpha_0 + \cdots + \alpha_k$ with $\alpha_0, \ldots, \alpha_k \in C_0^n(\varepsilon)$ and $\alpha_k \leq \cdots \leq \alpha_0$: By I.H. there are $a_0, \ldots, a_k \in OT$ with $o(a_i) = \alpha_i$ $(i = 0, \ldots, k)$. By 2.1 and 2.2(c) we obtain $a_k \leq \cdots \leq a_0$ and thus $a := (a_0, \ldots, a_k) \in OT$. Now $o(a) = o(a_0) \# \cdots \# o(a_k) = \alpha$.

2. $\alpha = \psi_v \xi$ with $\xi \in C_0^n(\varepsilon) \cap C_v(\xi)$: By I.H. there exists $b \in OT$ with $o(b) = \xi$. By 2.2(b) and 1.9 we obtain $\{o(x) : x \in G_v b\} = G_v \xi \subseteq \xi = o(b)$. Hence $G_v b < b$ by 2.1 and 2.2(c). It follows that $D_v b \in OT$ and $o(D_v b) = \alpha$.

(b), (c) By (a) and 2.2(c) the system $(\{x \in OT : x < a\}, <)$ is isomorphic to $(C_0(\varepsilon) \cap o(a), <)$, for each $a \in OT$. By 1.5 we have $C_0(\varepsilon) \cap o(D_1 0) = C_0(\varepsilon) \cap \Omega_1 = \psi_0 \varepsilon$. This yields part (c). For $a < D_1 0$ we have $o(a) \in C_0(\varepsilon) \cap \Omega_1 = \psi_0 \varepsilon$ and thus $C_0(\varepsilon) \cap o(a) = o(a)$.

3. Unprovability of PRWO($\psi_0\Omega_\omega$) in Π^1_1-CA$_0$

Let $\alpha \leqslant \psi_0\varepsilon_{\Omega_\omega+1}$. By PRWO($\alpha$) we denote the statement that there are no primitive recursive infinite descending sequences in $(\{x \in OT : o(x) < \alpha\}, <)$. Using a result from [1] we will prove the following theorem.

3.1. Theorem. $\mathrm{ID}_\nu \nvdash \mathrm{PRWO}(\psi_0\varepsilon_{\Omega_\nu+1})$ $(0 < \nu \leqslant \omega)$.

Since $\psi_0\varepsilon_{\Omega_\nu+1} < \psi_0\Omega_\omega$, for all $\nu < \omega$, and since Π^1_1-CA$_0$ proves the same arithmetic sentences as $\bigcup_{\nu<\omega} \mathrm{ID}_\nu$, we get from 3.1:

Corollary. $\Pi^1_1\text{-CA}_0 \nvdash \mathrm{PRWO}(\psi_0\Omega_\omega)$.

Remark. In Pohlers [5] it was shown that TI($\boldsymbol{\nu}$), i.e. the principle of transfinite induction up to $\theta\varepsilon_{\Omega_\nu+1}0$, is not provable in ID_ν, Theorem 3.1 improves this result (for $\nu \leqslant \omega$) in so far as PRWO($\psi_0\varepsilon_{\Omega_\nu+1}$) is a Π^0_2-sentence while the complexity of TI($\boldsymbol{\nu}$) is Π^1_1. Moreover PRWO($\psi_0\varepsilon_{\Omega_\nu+1}$) is a consequence of TI($\boldsymbol{\nu}$).

We repeat now some definitions from [1]. As before the letters a, b, c, d shall always denote elements of T.

Definition of $a + b$ and $a \cdot n$

$$a + 0 := 0 + a := a,$$

$$(a_0, \ldots, a_n) + (b_0, \ldots, b_m) := (a_0, \ldots, a_n, b_0, \ldots, b_m),$$

$$a \cdot 0 := 0, \qquad a \cdot (n+1) := a \cdot n + a.$$

Proposition. $(a + b) + c = a + (b + c)$.

Definition of T_v for $v \leqslant \omega$

$$T_v := \{0\} \cup \{(D_{u_0}a_0, \ldots, D_{u_n}a_n) : n \geqslant 0, a_0, \ldots, a_n \in T, u_0, \ldots, u_n \leqslant v\}.$$

Remark. $T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_\omega = T$, and $T_u = \{x \in T : x < D_{u+1}0\}$ for $u < \omega$.

Abbreviation. $1 := D_0 0$.
We idenitfy $\mathbb{N}$ with the subset $\{0, 1, 1+1, 1+1+1, \ldots\}$ of $OT \cap T_0$.

Definition of dom(a) *and* $a[z]$ *for* $a \in T$, $z \in \mathrm{dom}(a)$

([ ].0) $\mathrm{dom}(0) := \emptyset$.
([ ].1) $\mathrm{dom}(1) := \{0\}$; $1[0] := 0$.
([ ].2) $\mathrm{dom}(D_{u+1}0) := T_u$; $(D_{u+1}0)[z] := z$.

([ ].3) $\mathrm{dom}(D_\omega 0) := \mathbb{N}$; $(D_\omega 0)[n] := D_{n+1}0$.

([ ].4) Let $a = D_v b$ with $b \neq 0$:

(i) $\mathrm{dom}(b) = \{0\}$: $\mathrm{dom}(a) := \mathbb{N}$: $a[n] := (D_v b[0] \cdot (n+1)$.

(ii) $\mathrm{dom}(b) = T_u$ with $v \leq u < \omega$: $\mathrm{dom}(a) := \mathbb{N}$; $a[n] := D_v b[D_u b[1]]$.

(iii) $\mathrm{dom}(b) \in \{\mathbb{N}\} \cup \{T_u : u < v\}$: $\mathrm{dom}(a) := \mathrm{dom}(b)$; $a[z] := D_v b[z]$.

([ ].5) $a = (a_0, \ldots, a_k)(k \geq 1)$: $\mathrm{dom}(a) := \mathrm{dom}(a_k)$;

$$a[z] := (a_0, \ldots, a_{k-1}) + a_k[z].$$

Definition. $0[n] := 0$,
$a[n] := a[0]$, if $\mathrm{dom}(a) = \{0\}$.

3.2. Lemma. (a) $z \in \mathrm{dom}(a) \Rightarrow a[z] < a$.

(b) $z, z' \in \mathrm{dom}(a) = T_u$ *and* $z < z' \Rightarrow a[z] < a[z']$.

(c) $0 \neq a \in T_v \Rightarrow \mathrm{dom}(a) \in \{\{0\}, \mathbb{N}\} \cup \{T_u : u < v\}$, *and* $a[z] \in T_v$ *for all* $z \in \mathrm{dom}(a)$.

Proof. Straightforward by induction on the length of a.

3.3. Lemma. $a, z \in OT$ *and* $z \in \mathrm{dom}(a) \Rightarrow a[z] \in OT$.

Before we are going to prove this lemma we want to give the

Proof of Theorem 3.1. Let $0 < v \leq \omega$,

$$c_v^n := D_0 \overbrace{D_v \cdots D_v}^{n} 0, \qquad c_v^n(k) := c_v^n[1][2] \cdots [k].$$

In [1, Corollary 4.0] we have shown:

(1) $\mathrm{ID}_v \nvdash \forall n \, \exists k \, c_v^n(k) = 0$.

One easily proves that $c_v^n \in OT \cap T_0$; this can be done in PA (Peano Arithmetic). Since the proofs of 3.2 and 3.3 can also be formalized in PA, we obtain:

(2) $\mathrm{PA} \vdash \forall n \, \forall k \, (c_v^n(k) \in OT \wedge (c_v^n(k) \neq 0 \rightarrow c_v^n(k+1) < c_v^n(k)))$.

Obviously the sequences $(c_v^n(k))_{k \in \mathbb{N}}$ are primitive recursive, and by 1.3, 1.7 we have $o(c_v^n) < \psi_0 \varepsilon_{\Omega_v + 1}$. Together with (2) this yields:

(3) $\mathrm{PA} \vdash \mathrm{PRWO}(\psi_0 \varepsilon_{\Omega_v+1}) \rightarrow \forall n \, \exists k \, c_v^n(k) = 0$.

From (1) and (3) we obtain Theorem 3.1.

For the proof of 3.3 we need the following definitions and lemmata.

Definition.

$$G_u^0 a := G_u a \cup \{0\}$$

$$b \lhd_z a :\Leftrightarrow b \prec a \quad \text{and} \quad \forall u\, \forall c\, (b \preccurlyeq c \preccurlyeq a \Rightarrow G_u b \preccurlyeq G_u c \cup G_u^0 z).$$

3.4. Lemma. *$b \lhd_z a$, $G_u a \prec a$, $G_u z \prec b \Rightarrow G_u b \prec b$.*

Proof. We have $G_u b \preccurlyeq G_u a \cup G_u^0 z \prec a$.

Assumption: $b \preccurlyeq G_u b$. Then there exists a subterm d of b with minimal length such that $b \preccurlyeq G_u d \prec a$. By the minimality of d we have $d = D_v c$ with $G_u c \prec b \preccurlyeq c \prec a$. Using $b \lhd_z a$ and $G_u z \prec b$ we obtain $G_u b \preccurlyeq G_u c \cup G_u^0 z \prec b$. Contradiction.

3.5. Lemma. *$b_0 \lhd_z b \Rightarrow a + b_0 \lhd_z a + b$ and $D_v b_0 \lhd_z D_v b$.*

Proof. 1. Suppose $a + b_0 \preccurlyeq c \preccurlyeq a + b$. Then $c = a + c_0$ with $b_0 \preccurlyeq c_0 \preccurlyeq b$. Hence

$$G_u(a + b_0) = G_u a \cup G_u b_0 \preccurlyeq G_u a \cup G_u c_0 \cup G_u^0 z = G_u c \cup G_u^0 z.$$

2. Suppose $D_v b_o \preccurlyeq c \preccurlyeq D_v b$. Then $c = (D_v c_0) + c_1$ with $b_0 \preccurlyeq c_0 \preccurlyeq b$. Using the premise $b_0 \lhd_z b$ we obtain $G_u b_0 \preccurlyeq G_u c_0 \cup G_u^0 z$. Now, for $v \geqslant u$, we have

$$G_u(D_v b_0) = \{b_0\} \cup G_u b_0 \preccurlyeq \{c_0\} \cup G_u c_0 \cup G_u^0 z \subseteq G_u c \cup G_u^0 z.$$

If $v < u$, then $G_u(D_v b_0) = \emptyset$.

3.6. Lemma. *$a \in T$ and $z \in \mathrm{dom}(a) \Rightarrow a[z] \lhd_z a$*

Proof. By induction on the length of a:

By 3.2 we have $a[z] \prec a$. —Suppose $a[z] \preccurlyeq c \preccurlyeq a$. We have to prove $G_u a[z] \preccurlyeq G_u c \cup G_u^0 z$.

1. $a = 1$ or $a = D_{w+1}0$: trivial.

2. $a = D_\omega 0: G_u a[z] = G_u D_{z+1} 0 \subseteq \{0\}$.

3. $a = D_v b$ with $\mathrm{dom}(b) = \{0\}$: Then $a[z] = (D_v b[0]) \cdot (z + 1)$ and $G_u a[z] = G_u D_v b[0]$. By I.H. and 3.5 we get $D_v b[0] \lhd_0 D_v b = a$. We also have $D_v b[0] \prec c \preccurlyeq a$ and therefore $G_u D_v b[0] \preccurlyeq G_u c \cup \{0\}$.

4. $a = D_v b$ and $\mathrm{dom}(b) = T_w$ and $v \leqslant w < \omega$: Then $a[z] = D_v b[x]$ with $x := D_w b[1]$. Suppose $u \leqslant v$, since otherwise $G_u a[z] = \emptyset$. From $a[z] \preccurlyeq c \preccurlyeq a$ it follows that $c = (D_v c_0) + c_1$ with $b[x] \preccurlyeq c_0 \preccurlyeq b$. By I.H. we have $b[x] \lhd_x b$, $b[1] \lhd_1 b$. Since $b[1] \preccurlyeq b[x] \preccurlyeq c_0 \preccurlyeq b$, we obtain

$$\begin{aligned} G_u a[z] &= \{b[x]\} \cup G_u b[x] \preccurlyeq \{c_0\} \cup G_u c_0 \cup G_u^0 x \\ &= \{c_0\} \cup G_u c_0 \cup \{b[1]\} \cup G_u^0 b[1] \preccurlyeq \{c_0\} \cup G_u c_0 \cup G_u^0 1 \subseteq G_u c \cup G_u^0 z. \end{aligned}$$

5. $a = D_v b$ and $\mathrm{dom}(b) \in \{\mathbb{N}\} \cup \{T_w : w < v\}$: By I.H. we get $b[z] \lhd_z b$ and then $a[z] = D_v b[z] \lhd_z D_v b = a$ by 3.5.

6. $a = (a_0, \ldots, a_k)$ $(k \geqslant 1)$: By I.H. we get $a_k[z] \lhd_z a_k$ and then $a[z] = (a_0, \ldots, a_{k-1}) + a_k[z] \lhd_z (a_0, \ldots, a_{k-1}) + a_k = a$ by 3.5.

Proof of Lemma 3.3. By induction on the length of a:

1. $a = (a_0, \ldots, a_k) \in OT$: Then $a_0, \ldots, a_k \in OT$ and $a_k[z] < a_k \leqslant \cdots \leqslant a_0$. By I.H. we have $a_k[z] \in OT$. Hence $a[z] = (a_0, \ldots, a_{k-1}) + a_k[z] \in OT$.

2. $a = D_v b \in OT$: Then $b \in OT$ and $G_v b < b$.

2.1 $\mathrm{dom}(b) = \{0\}$: By I.H. and 3.6 we obtain $b[0] \in OT$ and $b[0] \lhd_0 b$. From $b[0] \lhd_0 b$ and $G_v b < b$ we get $G_v b[0] < b[0]$ by 3.4. Hence $a[z] = (D_v b[0]) \cdot (z + 1) \in OT$.

2.2. $\mathrm{dom}(b) = T_u$ with $v \leqslant u < \omega$: We have to show $D_v b[x] \in OT$, where $x := D_u b[1]$. — By I.H. we have $b[1] \in OT$ and $(x \in OT \Rightarrow b[x] \in OT)$. By 3.6 we have $b[1] \lhd_1 b$. From this together with $G_v b < b$ and $G_v 1 < b[1]$ we obtain $G_v b[1] < b[1]$ by 3.4. Since $v \leqslant u$, $G_u b[1] \subseteq G_v b[1]$. Hence $x = D_u b[1] \in OT$, and therefore also $b[x] \in OT$. It remains to show that $G_v b[x] < b[x]$. But this follows immediately from $b[x] \lhd_x b$ (3.6), $G_v b < b$, $G_v x = \{b[1]\} \cup G_v b[1] \leqslant b[1] < b[x]$ by 3.4.

2.3. $\mathrm{dom}(b) \in \{\mathbb{N}\} \cup \{T_u : u < v\}$: By I.H. and 3.6 we have $b[z] \in OT$ and $b[z] \lhd_z b$. Since $z \in \mathrm{dom}(b) \in \{\mathbb{N}\} \cup \{T_u : u < v\}$, we have $G_v z < b[z]$. By 3.4 from $b[z] \lhd_z b$, $G_v b < b$, $G_v z < b[z]$ we get $G_v b[z] < b[z]$. Hence $a[z] = D_v b[z] \in OT$.

Finally we want to show that the ψ-functions have essentially the same strength as the θ-functions.

3.7. Theorem. $\theta \varepsilon_{\Omega_\nu + 1} 0 = \psi_0 \varepsilon_{\Omega_\nu + 1}$ $(0 < \nu \leqslant \omega)$

Proof. By [2] and [5] we have $\theta \varepsilon_{\Omega_\nu+1} 0 = |\mathrm{ID}_\nu|$. The proof of $\theta \varepsilon_{\Omega_\nu+1} 0 \leqslant |\mathrm{ID}_\nu|$ given in [2] can easily be adapted to the ψ-functions; so we get $\psi_0 \varepsilon_{\Omega_\nu+1} \leqslant |\mathrm{ID}_\nu|$, and its remains to prove $|\mathrm{ID}_\nu| \leqslant \psi_0 \varepsilon_{\Omega_\nu+1}$.

In the appendix of [1] we have proved:

$$|\mathrm{ID}_\nu| = \sup\{\mathrm{rk}(c_\nu^k) : k \in \mathbb{N}\}, \quad \text{where } c_\nu^k = D_0 \overbrace{D_\nu \cdots D_\nu}^{k} 0, \tag{1}$$

and $\mathrm{rk}(a) = \sup\{\mathrm{rk}(a[n]) + 1 : n \in \mathrm{dom}(a)\}$, for all $a \in T_0$.

By 3.2 and 3.3 we have:

$$0 \neq a \in OT \cap T_0 \quad \Rightarrow \quad a[n] < a \text{ and } a[n] \in OT \cap T_0. \tag{2}$$

From (2) and 2.2(c) we obtain by transfinite induction on a:

$$\mathrm{rk}(a) \leqslant o(a), \quad \text{for all } a \in OT \cap T_0. \tag{3}$$

From 1.2(d), 1.6(b), 1.7(b) we obtain:

$$\psi_0 \varepsilon_{\Omega_\nu+1} = \sup\{o(c_\nu^k) : k \in \mathbb{N}\}. \tag{4}$$

As already mentioned in the proof of 3.1 we have:

$$c_\nu^k \in OT \cap T_0. \tag{5}$$

Now from (1), (3), (4), (5) it follows that $|\mathrm{ID}_\nu| \leqslant \psi_0 \varepsilon_{\Omega_\nu+1}$.

Remark. The functions ψ_v ($v \leqslant \omega$) were first defined in an unpublished manuscript (1981) by the author. Later on this approach was extended by Jäger [4] and Schütte [3].

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