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Introduction of Implication and Generalization in Axiomatic Calculi

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Abstract

Introduction of implication and generalization rules have a close relationship, for which there is a key idea for clarifying how they are connected: *varying objects*. Varying objects trace how generalization rules are used along a demonstration in an axiomatic calculus. Some ways for introducing implication and for generalization are presented here, taking into account some basic properties that calculi can have.

1 Introduction

The rules for introducing material implication share the following form:
• If $\Gamma \cup \{\alpha\} \vdash \beta$, then, under certain conditions, $\Gamma \vdash \alpha \to \beta$.

In the most simplest case there are no other additional conditions for concluding that $\Gamma \vdash \alpha \to \beta$. It occurs for *closed* and *partial strong calculi*, in which no of their rules involve any generalization, and all of them preserve in a way the consequents of implications. Closed calculi can also simulate generalization rules in some way if they also have non primitive but *admissible rules* dealing with some generalization forms.

More complex cases, related to *open calculi*, require a kind of tracing of how generalization rules (such as introduction of universal quantifier, introduction of necessity, and so on) must be used for deducing that $\Gamma \cup \{\alpha\} \vdash \beta$,

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in order to conclude that $\Gamma \vdash \alpha \to \beta$.

It can be observed, by a careful examination of the classical logic books, that two distinct choices for the introduction of implication and generalization rules have been made standard:

1st) The rule for introducing implication has no restrictions, but there are constraints for introducing the universal quantifier and other operators of this kind. A calculus adopting this strategy is called *closed* in our context. It is more often used in calculi presented in natural deduction and sequent calculus style. Examples of closed calculi may be found in [1], [13], [6], [7] and [11]. However, this closed option may be very cumbersome when used to calculi presented in axiomatic style having varying objects other than variables, such as in modal logics.

2nd) The introduction of implication is done with restrictions, but the introduction of the universal quantifier and other analogous operators is unconditional. This strategy is more often adopted for axiomatic formulations. These calculi are called *open*. Examples may be found in [8], [9] and [12].

Some well known formulations of introduction of implication rule, in the context of open calculi, presented in axiomatic style, that can be found in the literature, present some undesirable features, such as:

- explicit use of the concept of demonstration, instead of an idea of a higher level dealing with syntactic consequence;
- lack of an adequate tracing to accompany the use of varying objects in rules of generalization, making difficult in many situations to know when introduction of implication is allowed.

Below we will give two examples of formulations for introduction of implication, commonly found in the literature, that suffer from the abovementioned ills:

- "For the predicate calculus (or the full number-theoretic formal system), if $\Gamma, A \vdash B$ with the free variables held constant for the last assumption formula A, then $\Gamma \vdash A \to B$." According to [8], pg. 97.
- "Assume that $\Gamma, A \vdash B$, where, in the deduction, no application of Gen to a wf which depends upon A has as its quantified variable a free variable of A. Then $\Gamma \vdash A \to B$." According to [9], pg. 63.

In [8], chapter 5, pages 94–106, there are some ideas which were the main basis from which the present paper evolved, specially the idea of free variables being held constant with respect to the premises along a demonstration in an axiomatic calculus. This idea was extended and systematized by us to a more general concept, named *varying objects*. We have developed a study about them in an abstract way, for dealing with a broad spectrum of calculi. This paper is a new version and expansion of two former ones, [4] and [3].

Varying objects are a kind of extension of the concept of variables. They can trace what kind of applications of generalization rules are used along a demonstration. There are two ways of tracing, name here *dependence* and *supporting*. Both only consider applications of generalization rules having a hypothesis depending in the considered demonstration on some premise in it. The first tracing method, dependence, also considers if the varying objects used in applications of generalization rules are free in some hypothesis, whereas the second tracing method, supporting, does not take it into account.

Besides the usual binary syntactic consequence relation between a collection of formulas Γ and a formula α, defined by an axiomatic calculus $\mathbf{C}$, noted here by $\Gamma \vdash_{\mathbf{C}} \alpha$, the two tracing methods define two new syntactic consequence relations, named here *dependence consequence* and *supporting consequence*, which are ternary relations, in the sense that they relate a collection of formulas Γ, a formula α, and a collection of varying objects $\mathcal{V}$. They are noted respectively by $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

These two new consequence relations share many common properties, but there are some properties exclusive to each one of them.

Under certain conditions, dependence consequence and supporting consequence can be partially (in *partial stable calculi*) or completely (in *stable calculi*) equivalent. Partial stable calculi have a weak formulation for generalization rules applied to supporting consequence, whereas stable calculi have a strong formulation for them.

Partial strong calculi have a weak formulation for introduction of implication with respect to supporting consequence, whereas *strong calculi* have a strong formulation for it.

The open calculi having the strongest formulations for introduction of implication and generalization rules are the *strong stable* ones.

The concepts here presented were already applied by the first author in the formulations and generalized proofs of metatheorems for some non classical calculi in [5], [10] and [2].

2 Basic Concepts

In this section we define some basic ideas related to axiomatic calculi, such as schemas, inference rules, axioms, applications and demonstrations, from which it is specified the basic consequence relation of an axiomatic calculi.

2.1 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:
- $\mathbf{L}$ is a formal language;
- $\alpha, \beta, \gamma, \delta$ are formulas of $\mathbf{L}$;
- Γ, ϑ, ζ are collections of formulas of $\mathbf{L}$.

2.2 Definition. A collection of formulas of $\mathbf{L}$ is also said a *schema* (*in* $\mathbf{L}$). An *inference rule* (*in* $\mathbf{L}$) is a collection of n-tuples of formulas (of $\mathbf{L}$), for some $n \geq 2$. A *postulate* (*in* $\mathbf{L}$) is a schema (in $\mathbf{L}$) or an inference rule (in $\mathbf{L}$). An (*axiomatic*) *calculus* (*in* $\mathbf{L}$) is a pair $\langle \mathbf{L}, \mathbb{P} \rangle$, whereon $\mathbb{P}$ is a collection of postulates in $\mathbf{L}$. If $\mathbf{C} = \langle \mathbf{L}, \mathbb{P} \rangle$ is a calculus, then $\mathbf{L}$ is said to be *its language*, and $\mathbb{P}$ is said to be *its basis.* A schema belonging to the basis of a calculus is said to be *a schema of it.* A rule belonging to the basis of a calculus is said to be *a rule of it.* A schema or a rule of a calculus are also said to be *a postulate of it.*

2.3 Notation. From now on, unless stated otherwise, $\mathbf{C} = \langle \mathbf{L}, \mathbb{P} \rangle$ is an axiomatic calculus.

2.4 Definition. An *application* (*of an inference rule*) (*in* $\mathbf{L}$) is an element of an inference rule (in $\mathbf{L}$).

2.5 Notation. If $\langle \alpha_1, \ldots, \alpha_n, \beta \rangle$ is an application, we also note it by $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$.

2.6 Definition. If $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$ is an application, $\alpha_1, \ldots, \alpha_n$ are said to be *their hypotheses*, and β is said to be *its conclusion* or *consequence.* We also say that β is a *conclusion* or *consequence over* $\alpha_1, \ldots, \alpha_n$ *by* this application.

2.7 Notation. When there is no possibility of confusion, we note a given schema simply by writing down a generic element of it. The same is done for rules – we note a given rule simply by writing down a generic element of it.

2.8 Examples. The schema $\{(\alpha \wedge \beta) \to \beta \mid \alpha, \beta$ are formulas in $\mathbf{L}\}$ is noted simply by $(\alpha \wedge \beta) \to \beta$. At the same way, we note the rule $\left\{ \dfrac{\alpha, \alpha \to \beta}{\beta} \mid \alpha, \beta \text{ are formulas in } \mathbf{L} \right\}$ by $\dfrac{\alpha, \alpha \to \beta}{\beta}$.

2.9 Definition. The *domain of an inference rule* is the collection of all tuples $\langle \alpha_1, \ldots, \alpha_n \rangle$ such that there exists β for which $\dfrac{\alpha_1, \ldots, \alpha_n}{\beta}$ is an application of this rule.

2.10 Definition. An inference rule is said to be *unary* if each application of it has only one hypothesis.

2.11 Definition. A formula belonging to a schema of $\mathbf{C}$ is said to be an *axiom of* $\mathbf{C}$. An application belonging to a rule of $\mathbf{C}$ is said to be an *application of* (*a rule of*) $\mathbf{C}$.

2.12 Definition. A *demonstration in* $\mathbf{C}$ *of* α *from* Γ is a finite non empty sequence $\mathcal{D}$ of formulas of $\mathbf{L}$ such that α is the last formula of $\mathcal{D}$ and, for each formula β of $\mathcal{D}$, at least one of the following conditions is satisfied:

- β is an axiom of $\mathbf{C}$;
- $\beta \in \Gamma$ (in this case we also say that β *is justified in* $\mathcal{D}$ *as being a premise*);
- there is an application $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ of $\mathbf{C}$ such that each β_i, for any $i \in \{1, \ldots, n\}$, precedes β in $\mathcal{D}$.

If there is a demonstration in $\mathbf{C}$ of α from Γ, we also note it by $\Gamma \vdash_{\mathbf{C}} \alpha$, and we say that α is a *consequence from* Γ *in* $\mathbf{C}$, or that α is a *theorem of* Γ *in* $\mathbf{C}$. We also note "$\emptyset \vdash_{\mathbf{C}} \alpha$" by "$\vdash_{\mathbf{C}} \alpha$". If $\vdash_{\mathbf{C}} \alpha$, α is also said to be a *thesis of* $\mathbf{C}$.

2.13 Theorem. A formula α is a consequence from Γ in $\mathbf{C}$ if, and only if, at least one of the following conditions is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there is an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule of $\mathbf{C}$ such that $\Gamma \overset{\mathcal{V}}{\vdash}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \overset{\mathcal{V}}{\vdash}_{\mathbf{C}} \alpha_n$.

2.14 Theorem. The following properties are valid for the relation "$\vdash_{\mathbf{C}}$":

(i) if α is an axiom of $\mathbf{C}$, then $\vdash_{\mathbf{C}} \alpha$;

(ii) if $\alpha \in \Gamma$, then $\Gamma \vdash_{\mathbf{C}} \alpha$;

(iii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$, then $\{\alpha_1, \ldots, \alpha_n\} \vdash_{\mathbf{C}} \alpha$;

(iv) if $\Gamma \vdash_{\mathbf{C}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathbf{C}} \alpha$;
(v) if $\Gamma \vdash_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash_{\mathbf{C}} \alpha_n$ and $\{\alpha_1, \ldots, \alpha_n\} \vdash_{\mathbf{C}} \beta$, then $\Gamma \vdash_{\mathbf{C}} \beta$;
(vi) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is $\Gamma' \subseteqq \Gamma$ such that Γ' is finite and $\Gamma' \vdash_{\mathbf{C}} \alpha$.

Next we provide two simple syntactic ideas which are used in some examples given in this paper.

2.15 Definition. In a language in which "$\forall$" is a quantifier, a *universal generalization of* α is a formula of the form $\forall x_1 \ldots \forall x_n \, \alpha$ $(n \geq 0)$.

2.16 Definition. In a language in which "$\Box$" is an unary connective, we say the $\Box$ *is free in* a given formula α if there is a subformula of α out of the scope of $\Box$ in it.

3 Variation, Dependence and Supporting

In this section the idea of inference rule and its applications is expanded by attaching to each application a set of varying objects. From this new departure two new consequence relations are defined. Their basic properties and the interrelationship between them and with the basic consequence relation are presented.

3.1 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:
- $\alpha, \beta, \gamma, \delta$ are formulas in $\mathbf{C}$;
- Γ, ϑ, ζ are collections of formulas in $\mathbf{C}$.

Below we extend the concept of application of a rule of inference by attaching to it a collection of things named *varying objects*.

3.2 Definition. For each application of a rule of inference, we attach to it a collection whose elements are named its *varying objects*. A rule whose applications do not have varying objects is said to be a *constant rule*; otherwise we say that it is a *varying rule*. We say that $\mathbf{o}$ is a *varying object in* $\mathbf{C}$ if there is an application of a rule in $\mathbf{C}$ such that $\mathbf{o}$ is a varying object of this application. For each calculus $\mathbf{C}$, it is specified when a varying object $\mathbf{o}$ is *free in* a given formula α. The following additional conditions are to be fulfilled:
- the number of varying objects of each application of a rule in $\mathbf{C}$ is finite;

- each varying object of an application of a rule is not free in the consequence of this application.

3.3 Examples. In practice, we find the following varying objects:
- variables used in universal quantification: "x" is the varying object of the application $\dfrac{\alpha}{\forall x\, \alpha}$ of the rule of universal generalization, which occurs in many quantificational logics;
- the hidden variable used for introducing connectives associated with modalities such as necessity; such variable can be indicated by the sign itself introduced by the rule: $\Box$ is the varying object of the rule $\dfrac{\alpha}{\Box\alpha}$, which is present in many modal logics.

3.4 Notation. From now on, unless stated otherwise, we adopt the following conventions for the following letters, with or without primes and/or subscripts:
- $\mathbf{o}$ is a varying object in $\mathbf{C}$;
- $\mathcal{V}, \mathcal{W}$ are collections of varying objects in $\mathbf{C}$.

3.5 Definition. A calculus is said to be *closed* if all their rules are constant, otherwise it is said to be *open*.

3.6 Definition. A given rule $\mathbf{r}$ is said to be *admissible in* a closed calculus $\mathbf{C}$ if it satisfies the following conditions:
- $\mathbf{r}$ is unary;
- the domain of $\mathbf{r}$ is the collection of all formulas of $\mathbf{C}$;
- $\mathbf{r}$ is a varying rule;
- $\mathbf{r}$ is not an inference rule of $\mathbf{C}$;
- if α is an axiom of $\mathbf{C}$ and $\dfrac{\alpha}{\alpha'}$ is an application of $\mathbf{r}$, then $\vdash_{\mathbf{C}} \alpha'$;
- if $\dfrac{\alpha}{\alpha'}$ is an application of $\mathbf{r}$ such that no varying object of it belongs to α, then $\alpha \vdash_{\mathbf{C}} \alpha'$;
- if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of a rule of $\mathbf{C}$ and $\alpha'_1, \ldots, \alpha'_n, \alpha'$ are respectively consequences of applications of $\mathbf{r}$ over $\alpha_1, \ldots, \alpha_n, \alpha$, using the same collection of varying objects, then $\alpha'_1, \ldots, \alpha'_n \vdash_{\mathbf{C}} \alpha'$.

3.7 Example. Let $\mathbf{C}$ be a calculus whose axioms have the following forms, including all their universal generalizations:
- $\alpha \to \forall x\, \alpha$, whereon x is not free in α;
- $\forall x\, (\alpha \to \beta) \to (\forall x\, \alpha \to \forall x\, \beta)$.

The only rule of $\mathbf{C}$ is $\dfrac{\alpha, \alpha \rightarrow \beta}{\beta}$, which is a constant rule.
We have that $\mathbf{C}$ is a closed calculus such that the varying rule $\dfrac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable, is admissible in $\mathbf{C}$.

3.8 Theorem.

- If $\begin{cases} * \ \mathbf{C} \text{ is a closed calculus,} \\ * \ \Gamma \vdash_{\mathbf{C}} \alpha, \\ * \ \dfrac{\alpha}{\alpha'} \text{ is an application of an admissible rule of } \mathbf{C}\text{, such that each of its} \\ \quad \text{varying objects are not free in } \Gamma, \end{cases}$

  then $\Gamma \vdash_{\mathbf{C}} \alpha'$.

Proof:
If α is an axiom, then, by definition 3.6, $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$.
If $\alpha \in \Gamma$, then no varying object of the application $\dfrac{\alpha}{\alpha'}$ is free in α, so, by definition 3.6, $\alpha \vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$.
If there is an application $\dfrac{\beta_1, \ldots, \beta_n}{\alpha}$ of $\mathbf{C}$ such that $\Gamma \vdash_{\mathbf{C}} \beta_1, \ldots, \Gamma \vdash_{\mathbf{C}} \beta_n$, and $\beta'_1, \ldots, \beta'_n$ are respectively consequences of $\beta_1, \ldots, \beta_n$ by the same rule in which α' is a consequence of α, using the same collection of varying objects, we have, by induction hypothesis, that $\Gamma \vdash_{\mathbf{C}} \beta'_1, \ldots, \Gamma \vdash_{\mathbf{C}} \beta'_n$. By definition 3.6, $\beta'_1, \ldots, \beta'_n \vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \vdash_{\mathbf{C}} \alpha'$. □

3.9 Definition. Let $\mathcal{D} = \alpha_1, \ldots, \alpha_n$ be a demonstration in $\mathbf{C}$. We say that α_i *is relevant to* α_j *in* $\mathcal{D}$ $(i, j \in \{1, \ldots, n\})$ if one of the following conditions is fulfilled:

- $i = j$ and α_j is justified in $\mathcal{D}$ as a premise;
- α_j is justified in $\mathcal{D}$ as a consequence of an application $\dfrac{\beta_1, \ldots, \beta_p}{\alpha_j}$ of a rule in $\mathbf{C}$ and there exists a hypothesis β_k $(k \in \{1, \ldots, p\})$ of this application such that α_i is relevant to β_k in $\mathcal{D}$.

3.10 Definition. We say that a demonstration $\mathcal{D}$ in $\mathbf{C}$ *depends on* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects $\mathbf{o}$ of applications of rules in $\mathcal{D}$ having a hypothesis in which $\mathbf{o}$ is free such that there is a formula, justified as a premise in $\mathcal{D}$, whereon $\mathbf{o}$ is free too, relevant to this hypothesis in $\mathcal{D}$. If there is a demonstration in $\mathbf{C}$ of α from Γ such that it depends on $\mathcal{V}$, we say that α depends on $\mathcal{V}$ from Γ in $\mathbf{C}$, and we note this by $\Gamma \overset{\mathcal{V}}{\vdash_{\mathbf{C}}} \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \geq 1$, we also note this by $\Gamma \overset{\mathbf{o}_1, \ldots, \mathbf{o}_n}{\vdash_{\mathbf{C}}} \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is an *unvarying* *demonstration in* **C**. If α depends on $\emptyset$ from Γ in **C**, we say that it is an *unvarying consequence of* Γ *in* **C**.

3.11 Theorem. A formula α depends on $\mathcal{V}$ from Γ in **C** if, and only if, at least one of the following conditions is fulfilled:

- α is an axiom of **C**;
- $\alpha \in \Gamma$;
- there is an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in **C** such that $\Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_n$ and, for every varying object **o** of this application such that $\mathbf{o} \notin \mathcal{V}$ and for every α_i $(1 \leq i \leq n)$, if **o** is free in α_i, then there is $\Gamma' \subseteq \Gamma$, such that **o** is not free in Γ' and $\Gamma' \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_i$.

If $\mathcal{V} = \emptyset$, we can replace the third clause above by the following condition:

- there exists an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in **C**, such that $\Gamma \mathrel{|\!\frac{\emptyset}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{|\!\frac{\emptyset}{\mathbf{C}}} \alpha_n$ and, for every varying object **o** of this application and for every α_i $(1 \leqslant i \leqslant n)$, if **o** is free in α_i, then there exists $\Gamma' \subseteq \Gamma$ such that **o** is not free in Γ' and $\Gamma' \mathrel{|\!\frac{\emptyset}{\mathbf{C}}} \alpha_i$.

3.12 Examples. Let **C** be a calculus without schemas whose inference rules are the following:

- $\dfrac{\alpha}{\forall x \alpha}$, whereon the varying object of each application is the quantified variable;
- $\dfrac{\alpha}{\Box \alpha}$, whereon $\Box$ is the varying object of all applications.

The following propositions provide examples of dependence consequence:

- $p(x, y) \mathrel{|\!\frac{x,y}{\mathbf{C}}} \forall x \forall y \forall z\, p(x, y)$;
- $p(x, y, z) \mathrel{|\!\frac{x,y,z}{\mathbf{C}}} \forall x \forall y \forall z\, p(x, y, z)$;
- $p(x) \mathrel{|\!\frac{x,\Box}{\mathbf{C}}} \Box \forall x\, p(x)$;
- $\Box p(x) \mathrel{|\!\frac{x}{\mathbf{C}}} \Box \forall x\, \Box p(x)$;
- $\Box p(x, y) \mathrel{|\!\frac{x,y}{\mathbf{C}}} \Box \forall x \forall y\, \Box p(x, y)$.

3.13 Theorem. The following properties are valid for the relation "$\mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}}$":

(i) if there is a demonstration $\mathcal{D}$ in **C** of α from Γ whose collection of varying objects of applications of rules of **C** in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$;

(ii) if $\Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$, then $\Gamma \mathrel{|\!\frac{}{\mathbf{C}}} \alpha$;

(iii) if $\Gamma \mathrel{|\!\frac{}{\mathbf{C}}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$;

(iv) $\mathrel{|\!\frac{}{\mathbf{C}}} \alpha$ iff $\mathrel{|\!\frac{\emptyset}{\mathbf{C}}} \alpha$;

(v) if $\mathbf{C}$ is closed, then $\Gamma \mid_{\overline{\mathbf{C}}} \alpha$ iff $\Gamma \mid\frac{\emptyset}{\mathbf{C}} \alpha$;

(vi) if α is an axiom of $\mathbf{C}$, then $\mid\frac{\emptyset}{\mathbf{C}} \alpha$;

(vii) if $\alpha \in \Gamma$, then $\Gamma \mid\frac{\emptyset}{\mathbf{C}} \alpha$;

(viii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, then $\{\alpha_1, \ldots, \alpha_n\} \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$;

(ix) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$ such that $\mathcal{W}$ is the collection of all varying objects $\mathbf{o}$ of this application in which $\mathbf{o}$ is free in some of their hypotheses, then $\{\alpha_1, \ldots, \alpha_n\} \mid\frac{\mathcal{W}}{\mathbf{C}} \alpha$;

(x) if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$ and $\mathcal{V} \subseteqq \mathcal{V}'$, then $\Gamma \mid\frac{\mathcal{V}'}{\mathbf{C}} \alpha$;

(xi) if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$ and $\Gamma \subseteqq \Gamma'$, then $\Gamma' \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$;

(xii) if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$, then there is $\mathcal{V}' \subseteqq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \mid\frac{\mathcal{V}'}{\mathbf{C}} \alpha$;

(xiii) if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$, then there is $\Gamma' \subseteqq \Gamma$ such that Γ' is finite and $\Gamma' \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$;

(xiv) if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha$ and, for each $\mathbf{o} \in \mathcal{W}$, $\mathbf{o}$ is not free in Γ, then $\Gamma \mid\frac{\mathcal{V}-\mathcal{W}}{\mathbf{C}} \alpha$;

(xv) if $\begin{cases} * \ \Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha, \\ * \ \text{for each } \mathbf{o} \in \mathcal{W}, \text{ there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o} \text{ is not free in } \Gamma' \text{ and} \\ \quad \Gamma' \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha, \end{cases}$

then $\Gamma \mid\frac{\mathcal{V}-\mathcal{W}}{\mathbf{C}} \alpha$.

3.14 Example. The following assertions are not valid for the relation "$\mid\frac{\mathcal{V}}{}$":

- if $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha_1, \ldots, \Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \mid\frac{\mathcal{V}}{\mathbf{C}} \beta$, then $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \beta$;
- if $\begin{cases} * \ \Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha_1, \ldots, \Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \mid\frac{\mathbf{o}_1, \ldots, \mathbf{o}_n}{\mathbf{C}} \beta, \\ * \ \text{for all } i \in \{1, \ldots, n\} \text{ and for all } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o}_i \notin \mathcal{V} \text{ and } \mathbf{o}_i \text{ is free in} \\ \quad \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o}_i \text{ is not free in } \Gamma' \text{ and } \Gamma' \mid\frac{\mathcal{V}}{\mathbf{C}} \alpha_j, \end{cases}$

  then $\Gamma \mid\frac{\mathcal{V}}{\mathbf{C}} \beta$.

Proof:
Let $\mathbf{C}$ be a calculus whose schemas are "$\alpha \to \alpha \vee \beta$" and "$\forall x\, \alpha \to \alpha(x|t)$", and whose rules of inference are

$$\frac{\alpha,\ \alpha \to \beta}{\beta} \text{ and } \frac{\alpha}{\forall x\, \alpha},$$

such that the first is a constant rule and the second is a varying rule in which the varying object of each application is the corresponding quantified variable.

We have that $\begin{cases} \{\forall y\, q(y,z),\, q(y,z) \to r(y)\} \mid\!\frac{\emptyset}{\mathbf{C}}\, r(y) \\ r(y) \mid\!\frac{\emptyset}{\mathbf{C}}\, \forall z\,(r(y) \vee s(z)) \end{cases}$, however it is not true that $\{\forall y\, q(y,z),\, q(y,z) \to r(y)\} \mid\!\frac{\emptyset}{\mathbf{C}}\, \forall z\,(r(y) \vee s(z))$, from which we have a counterexample for the first proposition.
Likewise, we have that $\begin{cases} \{\forall y\, q(y,z),\, \forall y\, q(y,z) \to r(y)\} \mid\!\frac{\emptyset}{\mathbf{C}}\, r(y) \\ r(y) \mid\!\frac{y}{\mathbf{C}}\, \forall y \forall z\,(r(y) \vee s(z)) \end{cases}$, nevertheless it is not true that $\{\forall y\, q(y,z),\, q(y,z) \to r(y)\} \mid\!\frac{\emptyset}{\mathbf{C}}\, \forall y \forall z\,(r(y) \vee s(z))$, from which we have a counterexample for the second proposition. $\square$

3.15 Definition. We say that a demonstration $\mathcal{D}$ in $\mathbf{C}$ *is supported by* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects of applications of rules in $\mathcal{D}$ such that, for each conclusion of such applications, there exists a premise relevant to it in $\mathcal{D}$. If there exists a demonstration in $\mathbf{C}$ of α from Γ such that $\mathcal{D}$ is supported by $\mathcal{V}$, we say that α *is supported by* $\mathcal{V}$ *from* Γ *in* $\mathbf{C}$, and we note this by $\Gamma \parallel\!\frac{\mathcal{V}}{\mathbf{C}}\, \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \geq 1$, we also note $\Gamma \parallel\!\frac{\mathcal{V}}{\mathbf{C}}\, \alpha$ by $\Gamma \parallel\!\frac{\mathbf{o}_1,\ldots,\mathbf{o}_n}{\mathbf{C}}\, \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is a *stable demonstration in* $\mathbf{C}$. If α is supported by $\emptyset$ from Γ in $\mathbf{C}$, we say that α is a *stable consequence of* Γ *in* $\mathbf{C}$.

3.16 Theorem. A formula α is supported by $\mathcal{V}$ from Γ in $\mathbf{C}$ if, and only if, at least one of the following clauses is fulfilled:
- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there exists an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \parallel\!\frac{\mathcal{V}}{\mathbf{C}}\, \alpha_1, \ldots, \Gamma \parallel\!\frac{\mathcal{V}}{\mathbf{C}}\, \alpha_n$ and, if there is a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, then $\mid\!\frac{}{\mathbf{C}}\, \alpha_1, \ldots, \mid\!\frac{}{\mathbf{C}}\, \alpha_n$.

3.17 Examples. Let $\mathbf{C}$ be the calculus defined in 3.12. The following propositions provide examples of supporting consequence:
- $p(x,y) \parallel\!\frac{x,y,z}{\mathbf{C}}\, \forall x \forall y \forall z\, p(x,y)$;
- $p(x,y,z) \parallel\!\frac{x,y,z}{\mathbf{C}}\, \forall x \forall y \forall z\, p(x,y,z)$;
- $p(x) \parallel\!\frac{x,\square}{\mathbf{C}}\, \square \forall x\, p(x)$;
- $\square p(x) \parallel\!\frac{x,\square}{\mathbf{C}}\, \square \forall x\, \square p(x)$;
- $\square p(x,y) \parallel\!\frac{x,y,\square}{\mathbf{C}}\, \square \forall x \forall y\, \square p(x,y)$.

3.18 Theorem. The following properties are valid for the relation "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$":

(i) if there exists a demonstration $\mathcal{D}$ in $\mathbf{C}$ of α from Γ whose collection of varying objects of applications of rules of $\mathbf{C}$ in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma \vdash_{\mathbf{C}} \alpha$;

(iii) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(iv) $\vdash_{\mathbf{C}} \alpha$ iff $\Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(v) if $\mathbf{C}$ is closed, then $\Gamma \vdash_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vi) if α is an axiom of $\mathbf{C}$, then $\Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(vii) if $\alpha \in \Gamma$, then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(viii) if $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ is an application of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, then $\{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ix) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\mathcal{V} \subseteq \mathcal{V}'$, then $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(x) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xi) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(xii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\Gamma' \subseteq \Gamma$ such that Γ' is finite and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(xiii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

3.19 Example. The following assertions are not valid for the relation "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$":

- if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and, for each $\mathbf{o} \in \mathcal{W}$, $\mathbf{o}$ is not free in Γ, then $\Gamma \Vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$;
- if
  - $*$ $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,
  - $*$ for each $\mathbf{o} \in \mathcal{W}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$,

  then $\Gamma \Vdash^{\mathcal{V}-\mathcal{W}}_{\mathbf{C}} \alpha$.
- if
  - $*$ $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \beta$,
  - $*$ for all $i \in \{1, \ldots, n\}$ and for all $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin \mathcal{V}$ and $\mathbf{o}_i$ is free in α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not free in Γ' and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof:
Let $\mathbf{C}$ be a calculus with no schemas, and whose rules of inference are

$$\frac{\alpha \wedge \beta}{\alpha}, \quad \frac{\forall x\, \alpha}{\alpha(x|t)} \text{ and } \frac{\alpha}{\forall x\, \alpha},$$

such that the first two ones are constant rules and the third one is a varying rule in which the varying object of each application is the quantified variable.
We have that $p(x) \Vdash^{y}_{\mathbf{C}} \forall y\, p(y)$, but it doesn't imply that $p(x) \Vdash^{\emptyset}_{\mathbf{C}} \forall y\, p(y)$, so we have a counterexample for the first two propositions.
Likewise, we have that $\begin{cases} \forall x\, p(x) \wedge p(y) \Vdash^{\emptyset}_{\mathbf{C}} \forall x\, p(x) \\ \forall x\, p(x) \Vdash^{z}_{\mathbf{C}} \forall z\, p(z) \end{cases}$, however it is not true that $\forall x\, p(x) \wedge p(y) \Vdash^{\emptyset}_{\mathbf{C}} \forall z\, p(z)$, therefore we have a counterexample for the third proposition. □

3.20 Theorem. The following proposition describes a way of expansion for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$" in a generic calculus.

- If $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

3.21 Theorem. If $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.
Proof:
If α is an axiom of $\mathbf{C}$ or $\alpha \in \Gamma$, there is nothing to prove.
Let us suppose then that there is an application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a rule of $\mathbf{C}$ fulfilling the conditions of theorem 3.16. By induction hypothesis, we have that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$. Given a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, we have $\vdash_{\mathbf{C}} \alpha_1, \ldots, \vdash_{\mathbf{C}} \alpha_n$, and hence $\vdash_{\mathbf{C}} \alpha$, which is, according to propositions 4, 10 and 11 of theorem 3.13, a sufficient condition for concluding that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. □

3.22 Example. Consider again $\mathbf{C}$ the calculus defined in 3.12. We have that $p(x, y) \vdash^{x,y}_{\mathbf{C}} \forall x \forall y \forall z\, p(x, y)$, but it does not imply that $p(x, y) \Vdash^{x,y}_{\mathbf{C}} \forall x \forall y \forall z\, p(x, y)$; we have only that $p(x, y) \Vdash^{x,y,z}_{\mathbf{C}} \forall x \forall y \forall z\, p(x, y)$, so $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ does not always imply that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

4 Special Axiomatic Calculi

In this section some conditions are presented by which dependence and supporting consequences can be partially or completely equivalent, and by which generalization rules and introduction of implication can work in a weaker or in a stronger way.

4.1 Definition. A calculus $\mathbf{C}$ is said to be *partial stable* if the following conditions are valid:

- each varying rule of $\mathbf{C}$ is unary, its domain is the collection of all formulas in $\mathbf{C}$, and each of its applications has exactly one varying object;
- for each application $\dfrac{\alpha'}{\alpha}$ of a varying rule in $\mathbf{C}$, if its varying object is not free in α', then $\alpha' \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;
- for each application $\dfrac{\alpha_1, \ldots, \alpha_n}{\alpha}$ of a constant rule in $\mathbf{C}$, if $\alpha'_1, \ldots, \alpha'_n, \alpha'$ are respectively conclusions of applications of a varying rule over $\alpha_1, \ldots, \alpha_n, \alpha$, using the same varying object, then $\alpha'_1, \ldots, \alpha'_n \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

4.2 Example. Let $\mathbf{C}$ be a calculus whose schemas are the following:

- $\alpha \rightarrow \forall x\, \alpha$, whereon x is not free in α;
- $\alpha \rightarrow \Box\alpha$, whereon $\Box$ is not free in α;
- $\forall x\, (\alpha \rightarrow \beta) \rightarrow (\forall x\, \alpha \rightarrow \forall x\, \beta)$;
- $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

The rules of $\mathbf{C}$ are the following:

- $\dfrac{\alpha, \alpha \rightarrow \beta}{\beta}$, which is a constant rule;
- $\dfrac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable;
- $\dfrac{\alpha}{\Box\alpha}$, whereon $\Box$ is the varying object of each application.

We have that $\mathbf{C}$ is partial stable.

4.3 Theorem.

- If
  - $*$ $\mathbf{C}$ is partial stable,
  - $*$ $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$,
  - $*$ $\dfrac{\alpha}{\alpha'}$ is an application of a varying rule in $\mathbf{C}$ such that its varying object is not free in Γ,

  then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

Proof: It is similar to the proof of theorem 4.16. $\Box$

4.4 Theorem. If $\mathbf{C}$ is partial stable, then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$.

Proof: It is similar to the proof of theorem 4.17. $\Box$

4.5 Theorem. If **C** is partial stable, then "$\Vert\!\frac{\emptyset}{\mathbf{C}}$" has the following additional property:

- if $\begin{cases} * \ \Gamma \Vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_1, \ldots, \Gamma \Vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \Vert\!\frac{\mathbf{o_1}, \ldots, \mathbf{o_n}}{\mathbf{C}}\, \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \text{ is free} \\ \quad \text{in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not free in } \Gamma' \\ \quad \text{and } \Gamma' \Vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_j, \end{cases}$

  then $\Gamma \Vert\!\frac{\emptyset}{\mathbf{C}}\, \beta$.

Proof: It is similar to the proof of theorem 4.18. □

4.6 Corollary. If **C** is partial stable, then the following additional properties are valid for the relation "$\vert\!\frac{\emptyset}{\mathbf{C}}$":

- $\Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_1, \ldots, \Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \vert\!\frac{\emptyset}{\mathbf{C}}\, \beta$, then $\Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \beta$;
- if $\begin{cases} * \ \Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_1, \ldots, \Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \Vert\!\frac{\mathbf{o_1}, \ldots, \mathbf{o_n}}{\mathbf{C}}\, \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \text{ is free} \\ \quad \text{in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not free in } \Gamma' \\ \quad \text{and } \Gamma' \vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha_j, \end{cases}$

  then $\Gamma \vert\!\frac{\emptyset}{\mathbf{C}}\, \beta$.

Proof: It suffices to use theorems 4.5 and 4.4, together with the proposition 13 of theorem 3.18. □

4.7 Definition. A calculus **C** is said to be *partial strong* if the following clauses are satisfied:

- $\vert\!\frac{}{\mathbf{C}}\, \alpha \to \alpha$;
- $\beta \Vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha \to \beta$;
- $\alpha, \alpha \to \beta \Vert\!\frac{\emptyset}{\mathbf{C}}\, \beta$;
- for each application $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ of a constant rule in **C**,

$$\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \Vert\!\frac{\emptyset}{\mathbf{C}}\, \alpha \to \beta.$$

4.8 Example. Let **C** be a calculus whose schemas are the following:

- $\alpha \to (\beta \to \alpha)$;
- $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$.

The only inference rule of **C** is $\dfrac{\alpha, \alpha \to \beta}{\beta}$, which is a constant rule.
We have that **C** is partial strong.

4.9 Example. Let $\mathbf{C}'$ be a calculus obtained from the calculus $\mathbf{C}$ of the preceding example by adding to it two new rules:
- $\frac{\alpha}{\forall x\, \alpha}$, whereon the varying object of each application is the quantified variable;
- $\frac{\alpha}{\Box\alpha}$, whereon $\Box$ is the varying object of each application.

We have that $\mathbf{C}'$ is also partial strong.

4.10 Theorem. The following propositions are equivalent:
- $\mathbf{C}$ is partial strong;
- for any Γ, α and β, $\Gamma \cup \{\alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$.

Proof: It is similar to the proof of theorem 4.23. $\Box$

4.11 Corollary. If $\mathbf{C}$ is closed, then the following propositions are equivalent:
- $\mathbf{C}$ is partial strong;
- for any Γ, α and β, $\Gamma \cup \{\alpha\} \vdash_{\mathbf{C}} \beta$ iff $\Gamma \vdash_{\mathbf{C}} \alpha \to \beta$.

Proof: It suffices to use theorem 4.10 and proposition 5 of theorem 3.18. $\Box$

4.12 Scholium. If the first, second and fourth clauses of definition 4.7 are valid for $\mathbf{C}$, then $\Gamma \cup \{\alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$ implies that $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$.

4.13 Corollary. If $\mathbf{C}$ is partial stable, then the following propositions are equivalent:
- $\mathbf{C}$ is partial strong;
- $\Gamma \cup \{\alpha\} \vdash^{\emptyset}_{\mathbf{C}} \beta$ iff $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$.

Proof: It suffices to use theorems 4.4 and 4.10. $\Box$

4.14 Definition. A partial stable calculus $\mathbf{C}$ is said to be *stable* if it has the following additional property:
- for each application $\frac{\beta}{\alpha}$ of a varying rule in $\mathbf{C}$, whereon $\mathbf{o}$ is its varying object, if β' and α' are respectively conclusions of applications of a varying rule in $\mathbf{C}$ over β and over α using a same varying object distinct from $\mathbf{o}$, then $\beta' \Vdash^{\mathbf{o}}_{\mathbf{C}} \alpha'$.

4.15 Example. The calculus defined in example 4.2 is partial stable, but it is not stable. If we add to it the schemas "$\forall x\, \alpha \to \alpha$" and "$\Box\alpha \to \alpha$", then we obtain a stable calculus.

4.16 Theorem.

- If $\begin{cases} * \ \mathbf{C} \text{ is stable,} \\ * \ \Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha, \\ * \ \dfrac{\alpha}{\alpha'} \text{ is an application of a varying rule in } \mathbf{C} \text{ such that its varying object} \\ \ \text{is not free in } \Gamma, \end{cases}$

  then $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$.

Proof:
Let $\frac{\alpha}{\alpha'}$ be an application of a varying rule in $\mathbf{C}$, whose varying object, denoted by $\mathbf{o}'$ from now on, is not free in Γ.
If α is an axiom of $\mathbf{C}$, then $\mathrel{|\!\!\frac{}{\mathbf{C}}} \alpha$, so $\mathrel{|\!\!\frac{}{\mathbf{C}}} \alpha'$, therefore $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$.
If $\alpha \in \Gamma$, then $\mathbf{o}'$ is not free in α, so, as $\mathbf{C}$ is stable, $\alpha \mathrel{\|\!\!\frac{\emptyset}{\mathbf{C}}} \alpha'$, therefore $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$.
If there is an application of a constant rule $\frac{\alpha_1, \ldots, \alpha_n}{\alpha}$ in $\mathbf{C}$ such that $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_n$, we have, by induction hypothesis, that $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'_1, \ldots, \Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'_n$, whereon $\alpha'_1, \ldots, \alpha'_n$ are respectively consequences over $\alpha_1, \ldots, \alpha_n$ by applications of the same rule from which $\frac{\alpha}{\alpha'}$ is an application, using the same varying object $\mathbf{o}'$. As $\mathbf{C}$ is stable, it follows that $\alpha'_1, \ldots, \alpha'_n \mathrel{\|\!\!\frac{\emptyset}{\mathbf{C}}} \alpha'$, hence $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$.
Let us suppose now that there exists an application $\frac{\beta}{\alpha}$ of a varying rule in $\mathbf{C}$, whose varying object is $\mathbf{o}$, such that $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$. Consider β' a consequence of β by an application of the same rule in which α' is consequence of α, using the same varying object $\mathbf{o}'$. By induction hypothesis, $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \beta'$. If $\mathbf{o} \in \mathcal{V}$ and $\mathbf{o} = \mathbf{o}'$, then $\mathbf{o}' \in \mathcal{V}$, hence, from the hypothesis $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$, we have that $\Gamma \mathrel{|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$. If $\mathbf{o} \in \mathcal{V}$ and $\mathbf{o} \neq \mathbf{o}'$, then, as $\mathbf{C}$ is stable, $\beta' \mathrel{\|\!\!\frac{\mathbf{o}}{\mathbf{C}}} \alpha'$, therefore $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$. If $\mathbf{o} \notin \mathcal{V}$, then $\mathrel{|\!\!\frac{}{\mathbf{C}}} \alpha$, thence $\mathrel{|\!\!\frac{}{\mathbf{C}}} \alpha'$, therefore $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha'$. $\square$

4.17 Theorem. If $\mathbf{C}$ is stable, then $\Gamma \mathrel{|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$ iff $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$.

Proof:
By theorem 3.21, we have that $\Gamma \mathrel{\|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$ implies $\Gamma \mathrel{|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$, so it remains to prove the converse.
Let us suppose that $\Gamma \mathrel{|\!\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$.
Let $\mathcal{D}$ be a demonstration of α from Γ depending on $\mathcal{V}$, β the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application of a varying

rule $\dfrac{\beta'}{\beta}$ such that its varying object does not belong to $\mathcal{V}$ and some premise is relevant to β' in $\mathcal{D}$. Let $\mathbf{o}$ be the varying object of this application.
If $\mathbf{o}$ is not free in β', then, as $\mathbf{C}$ is stable, we have that $\beta' \Vdash^{\emptyset}_{\mathbf{C}} \beta$, hence, as the considered occurrence of β' precedes β in $\mathcal{D}$, we have that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta'$, and therefore, by transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
If $\mathbf{o}$ is free in β', then, as $\mathbf{o} \notin \mathcal{V}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta'$, hence, as $\mathbf{C}$ is stable and in accordance with theorem 4.16, $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
In any case, there is a demonstration $\mathcal{D}_\beta$ in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of β in $\mathcal{D}$ by $\mathcal{D}_\beta$, we obtain, given $\mathcal{D}$, a demonstration in $\mathbf{C}$ of α from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and whose hypotheses have premises relevant to them in the new demonstration, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of α from Γ supported by $\mathcal{V}$, or rather, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. □

4.18 Theorem. If $\mathbf{C}$ is stable, then "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$" has the following additional property:

- if $\begin{cases} * \ \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o_1}, \ldots, \mathbf{o_n}}_{\mathbf{C}} \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \notin \mathcal{V} \\ \quad \text{and } \mathbf{o_i} \text{ is free in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not free} \\ \quad \text{in } \Gamma' \text{ and } \Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j, \end{cases}$

  then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof:
Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ be respectively demonstrations in $\mathbf{C}$ of $\alpha_1, \ldots, \alpha_p$ from Γ supported by $\mathcal{V}$, and let $\mathcal{E}$ be a demonstration in $\mathbf{C}$ of β from $\{\alpha_1, \ldots, \alpha_p\}$ supported by $\{\mathbf{o_1}, \ldots, \mathbf{o_n}\}$. Concatenating $\mathcal{D}_1, \ldots, \mathcal{D}_p, \mathcal{E}$, we obtain a demonstration $\mathcal{D}$ of β in $\mathbf{C}$ from Γ.
Let γ be the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application $\dfrac{\gamma'}{\gamma}$ of a varying rule, such that its varying object does not belong to $\mathcal{V}$ and some element of Γ is relevant to γ' in $\mathcal{D}$. As $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are demonstrations supported by $\mathcal{V}$, we have that the considered occurrence of

γ' appears in $\mathcal{E}$, hence, considering $\mathbf{o}$ the varying object of the application, we get that $\mathbf{o} \in \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$.
Let ϑ and ζ be defined by

$$\vartheta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is free in } \alpha_j\},$$
$$\zeta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is not free in } \alpha_j\}.$$

It is easy to verify that there exists a finite Γ', such that $\Gamma' \subseteq \Gamma$, $\mathbf{o}$ is not free in Γ' and, for every $\delta \in \vartheta$, $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \delta$. Therefore, by the construction of ζ, $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$, and $\mathbf{o}$ is not free in $\Gamma' \cup \zeta$.
As the considered occurrence of γ' precedes γ in $\mathcal{D}$, we have that $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma'$, and hence, by transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", we get $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma'$, and therefore, by theorem 4.16, $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma$.
For every $\delta \in \Gamma' \cup \zeta$, we have that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \delta$, and hence, once again due to transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma$. Or rather, there exists a demonstration $\mathcal{D}_\gamma$ in $\mathbf{C}$ of γ from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of γ in $\mathcal{D}$ by $\mathcal{D}_\gamma$, we have a new demonstration $\mathcal{D}'$ in $\mathbf{C}$ of β from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and each hypothesis has some premise relevant to it in $\mathcal{D}'$, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$, or rather, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$. □

4.19 Corollary. If $\mathbf{C}$ is stable, then the following additional properties are valid for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$":

- if $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$;
- if
  - $*$ $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \beta$,
  - $*$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin \mathcal{V}$ and $\mathbf{o}_i$ is free in α_j, then exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not free in Γ' and $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof: It suffices to use theorems 4.18 and 4.17, together with the proposition 13 of theorem 3.18. □

4.20 Definition. A partial strong calculus $\mathbf{C}$ is said to be *strong* if it has the following additional property:

- for each application $\dfrac{\beta_1,\ldots,\beta_n}{\beta}$ of a varying rule of **C** whose collection of varying objects is $\mathcal{V}$, if no element of $\mathcal{V}$ is free in α, then $\{\alpha\to\beta_1,\ldots,\alpha\to\beta_n\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$.

4.21 Example. The calculus defined in example 4.8 is also strong.

4.22 Example. The calculus defined in example 4.9 is partial strong, but it is not strong. Consider **C** a new calculus obtained from it by adding two new schemas:
- $\forall x(\alpha\to\beta)\to(\alpha\to\forall x\,\beta)$, whereon x is not free in α;
- $\Box(\alpha\to\beta)\to(\alpha\to\Box\beta)$, whereon $\Box$ is not free in α.

We have that **C** is strong.

4.23 Theorem. The following propositions are equivalent:
(i) **C** is strong;
(ii) for any Γ, α, β and $\mathcal{V}$, such that each $\mathbf{o}\in\mathcal{V}$ is not free in α, $\Gamma\cup\{\alpha\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta$ iff $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$.

(i) implies (ii):
Let us suppose that **C** is a strong calculus and that each $\mathbf{o}\in\mathcal{V}$ is not free in α.
If $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$, then, due to clause (iii) of definition 4.7, $\Gamma\cup\{\alpha\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta$.
Consider now that $\Gamma\cup\{\alpha\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta$.
If β is an axiom of **C**, then $\mathrel{|\!\!\frac{}{\mathbf{C}}}\beta$, hence, according to clause (ii) of definition 4.7, $\Gamma\mathrel{|\!\!\frac{}{\mathbf{C}}}\alpha\to\beta$, therefore $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$.
If $\beta\in\Gamma$, then $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta$, hence, according to clause (ii) of definition 4.7, $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$.
If $\beta=\alpha$, then, according to clause (i) of definition 4.7, $\Gamma\mathrel{|\!\!\frac{}{\mathbf{C}}}\alpha\to\alpha$, therefore $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$.
If there is an application $\dfrac{\beta_1,\ldots,\beta_n}{\beta}$ of a rule of **C** such that $\Gamma\cup\{\alpha\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta_1,\ldots,\Gamma\cup\{\alpha\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\beta_n$, we have, by induction hypothesis, $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta_1,\ldots,\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta_n$. If there is a varying object of this application that does not belong to $\mathcal{V}$, then, according to theorem 3.16, $\mathrel{|\!\!\frac{}{\mathbf{C}}}\beta$, hence, once again by clause (ii) of definition 4.7, $\mathrel{|\!\!\frac{}{\mathbf{C}}}\alpha\to\beta$, therefore $\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$. If every varying object of this application belongs to $\mathcal{V}$, then, as **C** is strong, $\{\alpha\to\beta_1,\ldots,\alpha\to\beta_n\}\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$, therefore $\Gamma\mathrel{\Vert\!\!\frac{\mathcal{V}}{\mathbf{C}}}\alpha\to\beta$. $\Box$

(ii) implies (i):
Let us suppose that for any Γ, α, β and $\mathcal{V}$ such that each $\mathbf{o} \in \mathcal{V}$ is not free in α, $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ iff $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.
As $\alpha \vdash_{\mathbf{C}} \alpha$, we have that $\vdash_{\mathbf{C}} \alpha \to \alpha$.
As $\{\beta, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$, we get $\beta \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$.
As $\alpha \to \beta \Vdash^{\emptyset}_{\mathbf{C}} \alpha \to \beta$, we have that $\{\alpha, \alpha \to \beta\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$.
Finally, let $\dfrac{\beta_1, \ldots, \beta_n}{\beta}$ be an application of a rule of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, and α a formula in $\mathbf{C}$ where no element of $\mathcal{V}$ is free.
We have that

$$\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta_1, \ldots, \{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta_n,$$

hence, as $\{\beta_1, \ldots, \beta_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, we have that $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n, \alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, therefore $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$. □

4.24 Scholium. If the first, second and fourth clauses of definition 4.7, together with the only clause of definition 4.20, are valid for $\mathbf{C}$, then the following proposition is true:

- if $\begin{cases} \text{each } \mathbf{o} \in \mathcal{V} \text{ is not free in } \alpha, \\ \Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta, \end{cases}$ then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

4.25 Theorem. If $\mathbf{C}$ is stable, then the following propositions are equivalent:

- $\mathbf{C}$ is a strong calculus;
- for any Γ, α, β and $\mathcal{V}$, such that each $\mathbf{o} \in \mathcal{V}$ is not free in α, $\Gamma \cup \{\alpha\} \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ iff $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

Proof: It suffices to use theorems 4.23 and 4.17. □

5 Conclusion

We have presented general formulations for generalization rules and for introduction of implication, valid for a large family of axiomatic calculi, from closed to open ones.

Closed calculi have the simplest formulation for introduction of implication, and generalization rules can be simulated through admissible rules. For these calculi, both these procedures can be performed by using only the basic consequence relation.

The same does not happen with respect to open calculi. In them, for managing the interrelationship between introduction of implication and generalization rules, the basic consequence relation is not sufficient for tracing how varying objects are used along a demonstration, so it is necessary to annotate them for all applications of inference rules, taking into account two adequate consequence relations, which can be partially or completely equivalent, depending on the particular calculus. In practice, at the worst case, it is necessary to work in a simultaneous way with two consequence relations for tracing varying objects, the dependence and supporting ones.

These results are very important for modelling new calculi that should have properties related to weaker or stronger forms of generalization and introduction of implication. Some of them were essential for obtaining an abstract completeness proof for a broad group of calculi with respect to their semantics, in [2], pgs. 72-88. A future paper will present a concise exposition of this proof.

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