Separable Social Welfare Evaluation
for Multi-Species Populations[*]

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Abstract

If non-human animals experience wellbeing and suffering, then such welfare consequences arguably should be included in a social welfare evaluation. Yet economic evaluations almost universally ignore non-human animals, in part because axiomatic social choice theory has not characterized multi-species social welfare functions. Here we propose axioms and characterize a range of functional forms to fill this gap. Among these, we identify a new characterization of additively-separable generalized (multi-species) total utilitarianism. We provide examples to illustrate that non-separability across species is implausible in a multi-species setting, in part because good lives for different species are at very different welfare levels.

Keywords: animals, animal welfare, social welfare functions, population ethics, utilitarianism
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1 Introduction

If non-human animals experience wellbeing and suffering, then such welfare consequences arguably should be included in a social welfare evaluation. Welfarists from Bentham (1780) to Singer (1975) have recognized that welfarist social evaluation must incorporate animal welfare. Yet exactly how to do so is a deeply open question—even compared with the many open question in human welfare economics (Blackorby, Bossert and Donaldson, 2005; Budolfson, Fischer, and Scovronick, 2023). Economic evaluations almost universally ignore non-human animals, in part because axiomatic social choice theory has failed to propose and characterize multi-species social welfare functions. Which multi-species social welfare functions are consistent with attractive normative principles? Finding an answer is important for many economic policy questions including climate change, food policy, and medical research.

We contribute the first axiomatic characterization of a multi-species social welfare function, using explicitly multi-species axioms. Some prior studies articulate a multi-species social welfare function without characterizing it and then use it to address economic or ethical questions.[1] Others build upon the observation that animal welfare has implications for welfarist social evaluation if some humans care about animals.[2] We characterize social orderings of variable-population, multi-species vectors of individual lifetime utilities. These vectors include the lifetime utilities of animals, valued for their own wellbeing and not merely for their consequences on humans.

A core concept is *separability*, which in public economics and welfare evaluations holds that the implications of a change can be evaluated by considering only the individuals for whom it has consequences. Unaffected individuals are *separable* from such an evaluation. Separability generates an additive functional form for social welfare evaluation. The economics literature identifies additive social welfare with utilitarianism. Additively-separable utilitarianism is a workhorse of practical public economics, but remains controversial (Fleurbaey, 2010; Eden, 2023), so novel justifications for separability could address long-running, policy-relevant debates.

Therefore we focus particularly on the arguments for, possibilities for, and alternatives to cross-species separability (even while, in some characterizations, permitting within-species non-separability). It is well-known in welfare economics that assuming individual-

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[1]These include Blackorby and Donaldson (1992), the pioneer welfare economics research to incorporate non-human animals, as well as Clarke and Ng (2006), Budolfson and Spears (2019), and Espinosa and Treich (2021).

[2]Including Cowen (2006) and Fleurbaey and Van der Linden (2021).

level independence yields a fully additively-separable, generalized utilitarian social welfare function. We show that weaker, species-level separability achieves the same utilitarian outcome, in the context of individual-level anonymity. This result adds to the debate about separability in the welfare economics literature and increases the theoretical cost of choosing a non-separable form.

The possibility of welfare-relevant non-human animals, this result suggests, offers a new reason for social evaluation to be additively-separable across individuals, even when comparing a policy choice that would only have consequences for humans. In this way, we show that attending to animal welfare—long a hallmark of the informal, philosophical literature on utilitarianism—provides a novel justification of utilitarianism in economics' formal, additive sense.

2   Motivating examples

In this section we offer simple numerical examples. We intend these to motivate the intuition for cross-species separability. In particular, we propose that species-level independence is attractive in a multi-species, variable-population setting, in part because different species have very different utilities for individuals living a good life. As a result, there is no clear "denominator" that a non-separable evaluation could use to combine individuals of very different species.

First consider generalized, multi-species average utilitarianism of the form: $\frac{\sum_s \Xi_n^s \times n(s)}{\sum_s n(s)}$, where s indexes species, Ξ_n^s is average[3] utility of species s of size n, and $n(s)$ is the size of species s. Average utilitarianism is separable across species for fixed-population cases but non-separable for variable population cases because of its denominator. Assume the world consists of:

- 1 million mammals, each living great lives for a mammal at utility 10;

- 400,000 birds, each living great lives for a bird at utility 5; and

- 300,000 lizards, each living great lives for a lizard at utility 3.

Would it be an improvement or a worsening to add 10 birds each at utility 6? Assume that this utility level, although lower than that of the mammals (perhaps because it represents a shorter life), is an excellent life for a bird. According to average utilitarianism, adding

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[3]This argument goes through unchanged if Ξ_n^s is replaced with any other equally-distributed-equivalent representative utility (Fleurbaey, 2010), instead of average utility.

these birds, which lower average welfare, would be a worsening — so it would be better to slightly worsen the lives of each mammal and lizard in order to prevent these birds from being born. We find this implausible.

Averagism is not unique in these implications. Consider the following example of species-blind, rank-dependent (and therefore non-separable) utilitarianism: $\sum_r \beta^r u_r$, where r is an individual's wellbeing rank from the worst-off, u_r is the individual's lifetime utility, and β is a constant between 0 and 1. For simplicity, imagine a world containing:

- 1 mammal, living a great mammal life at 10;

- 1 bird, living a great bird life at 5; and

- 1 lizard, living a great lizard life at 3.

Would it be an improvement or a worsening to improve the bird's life to 5.1, add a second bird also at 5.1, and leave the mammal and the lizard unchanged? If $\beta = \frac{1}{3}$, then such an improvement-and-addition would make the population worse — again recommending harming the mammal and the lizard, if necessary, to prevent it. In this case, the same recommendation — rejecting the improvement-and-addition — could be made by species-blind, number-sensitive egalitarianism of the form $n^\alpha \times g^{-1}\left(\frac{1}{n}\sum g(u_i)\right)$ for a concave g such as $g(u) = \ln(u)$, even with some positive values for the constant α.

Should there be no happy dogs in a world of happy humans, merely because the dogs' lives are shorter? Should there be no humans in a galaxy of blissful aliens? Perhaps not, according to average utilitarianism, rank-dependent utilitarianism, and egalitarianism.

What these examples reveal is a consequence of the fact that different species can have very different welfare profiles, even in good lives. A life can be good *for a species* without being good *relative to the full, multi-species population*. But it strikes us as normatively implausible — and arguably speciesist — to consider an extra individual a worsening when its own life is worth living, is excellent by the distribution of its species, and has a highly favorable balance of pleasure and suffering. This speciesist outcome is not escaped merely by a social welfare function being species-blind.

The fact that these and other implausible implications can be avoided by assuming independence across species motivates our novel axioms. A multi-species setting offers a new version of classic arguments for independence, such as by Blackorby, Bossert and Donaldson (2005). Their paper can be read as arguing that birth cohort is a plausible dimension of independent separability (especially for past people who are dead); by analogy, our paper can be read as arguing that species is a plausible dimension of independent separability. In both cases, additive separabiltiy follows.

3 Framework

The set of positive integers is denoted by $\mathbb{N}$. The set of all real numbers is denoted by $\mathbb{R}$. The set of non-negative (resp. positive) real numbers is denoted by $\mathbb{R}_+$ ($\mathbb{R}_{++}$).

There is a finite set of species S. For simplicity, we assume that $S = \{1, \cdots, T\}$ for some $T \in \mathbb{N}$ with $T \geq 3$. For each species, a variable number of individuals may exist. If population size is $n \in \mathbb{N}$, the wellbeing vector for individuals is $u = (u_1, \cdots, u_n)$, where u_i is individual i's the lifetime utility. The set of all possible wellbeing vector is $U = \bigcup_{k \in \mathbb{N}} \mathbb{R}^k$. For $u \in U$, we write $n(u)$ the population size, that is $n(u) = k$ if $u \in \mathbb{R}^k$, and $N(u) = \{1, \cdots, n(u)\}$. For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we denote $x \cdot \mathbb{1}_k = (x, \cdots, x) \in \mathbb{R}^k$.

An alternative will give a wellbeing vector for each species. This can be modeled as a mapping $a : S \to U$, where for each $s \in S$ $a(s) \in U$ is the wellbeing vector in species s. We denote A the set of all possible mappings $a : S \to U$, so that A is the set of alternatives. When $a(s) = u \in U$, we denote $a_i(s) = u_i$ for each $i \in N(a(s))$ the wellbeing of individual i of species s in alternative a.

For a subset of species $\bar{S} \subset S$ and for two alternatives $a, a' \in A$, we denote $a_{\bar{S}} a'$ the alternative $\hat{a} \in A$ such that $\hat{a}(s) = a(s)$ for all $s \in \bar{S}$ and $\hat{a}(s) = a'(s)$ for all $s \in (S \setminus \bar{S})$. When $\bar{S} = \{s\}$ we write $a_s a'$ instead of $a_{\{s\}} a'$.

For any $a \in A$, let us denote $\mathbf{n}(a) = \big(n(a(1)), \cdots, n(a(T))\big)$ the vector of species population sizes. For $\mathbf{n} = (n_1, \cdots, n_T)$ an element of $\mathbb{N}^T$, we denote $A_{\mathbf{n}}$ the set of alternatives in A such that $\mathbf{n}(a) = \mathbf{n}$. These are alternatives with a given population size n_s for each species s.

3.1 Basic principles

We study a social ordering $\succsim$ on A. We assume two basic principles throughout. The first is a Pareto principle applied to situations where population size is fixed for each species. It says that if all individuals of each species is at least as well off and all individual of some species are better off, then the situation is socially better. Hence we do not necessarily judge that an improvement for one individual is enough to increase social welfare, but an improvement for all individuals in a species is enough.

Pareto. For all $a, a' \in A$, if $n(a(s)) = n(a'(s))$, $a(s) \geq a'(s)$ for all $s \in S$ and $a(t) \gg a'(t)$ for some $t \in S$, then $a \succ a'$.

For any $\mathbf{n} \in \mathbb{N}^T$, we can associate to any $a \in A_{\mathbf{n}}$ an element in $\prod_{s=1}^{T} \mathbb{R}^{n_s}$, that is a can be viewed as a collection $(a(1), \cdots, a(T)) \in \prod_{s=1}^{T} \mathbb{R}^{n_s}$. We can therefore define the

topology of $A_{\mathbf{n}}$ based on the product topology on $\prod_{s=1}^{T} \mathbb{R}^{n_s}$ to define the following notion of continuity.

Extended continuity. For any $\mathbf{n}, \hat{\mathbf{n}} \in \mathbb{N}^T$ and $a \in A_{\mathbf{n}}$, the sets $\{a' \in A_{\hat{\mathbf{n}}} | a \succsim a'\}$ and $\{a' \in A_{\hat{\mathbf{n}}} | a \precsim a'\}$ are closed.

All the orderings we will consider satisfy those two properties. We describe any ordering that does as "regular":

Definition 1 (Regularity). *A social welfare ordering $\succsim$ is regular if it satisfies Pareto and Extended continuity.*

4 Separable ethics

Social welfare economics has long debated the normative attractiveness of individual-level independence. The core of our paper is a weakening of this classic axiom that only assumes independence across species, while permitting non-separability among individuals within a species.

Species separability. For all $a, a', \hat{a}, \hat{a}' \in A$ and for all $\bar{S} \subset S$, $a_{\bar{S}}\hat{a} \succsim a'_{\bar{S}}\hat{a}$ if and only if $a_{\bar{S}}\hat{a}' \succsim a'_{\bar{S}}\hat{a}'$.

The examples in Section 2 are violations of Species separability.

Within-species egalitarian dominance. For each $s \in S$, there exists a wellbeing level $c^s \in \mathbb{R}$ such that the following properties are verified,

    (1) For any real numbers $x > y \geq c^s$ and natural numbers $k > l$, there exist a, a' and $\hat{a} \in A$ such that $a(s) = x \cdot \mathbb{1}_k$, $a'(s) = y \cdot \mathbb{1}_l$ and $a_s\hat{a} \succ a'_s\hat{a}$.

    (2) For any real numbers $x < y \leq c^s$ and natural numbers $k > l$, there exist a, a' and $\hat{a} \in A$ such that $a(s) = x \cdot \mathbb{1}_k$ and $a'(s) = y \cdot \mathbb{1}_l$ then $a_s\hat{a} \prec a'_s\hat{a}$.

Within-species egalitarian dominance represents that for each species there exists a level of wellbeing c^s such that, when wellbeing is higher than c^s, a larger egalitarian population of with higher wellbeing is better than a smaller egalitarian population with lower wellbeing (at least in some cases). On the other hand, when wellbeing is lower than c^s, a larger egalitarian population with lower wellbeing is worse than a smaller egalitarian population with higher wellbeing (at least in some cases). The level c^s can be interpreted as the level for a good enough life, in the sense that we may want to add

people if they have more than this level, but not if they have less. Like in our motivating examples, we allow for the fact that c^s may be different between species, so that an excellent life for a bird may be at a lower level than a good life for a mammal.

Species separability and Within-species egalitarian dominance are enough to obtain an additive representation, when $\succsim$ is regular. To state our first result, let us introduce the following definition.

Definition 2 (Equally-distributed equivalent function). *A function $F : U \to \mathbb{R}$ is an equally-distributed equivalent function if it satisfies the following properties:*

1. *For all $k \in \mathbb{N}$ and $u, v \in \mathbb{R}^k$, $F(u) \geq F(v)$ whenever $u, v \in \mathbb{R}^k$ are such that $u \geq v$, and $F(u) > F(v)$ whenever $u, v \in \mathbb{R}^k$ are such that $u \gg v$;*

2. *The restriction of function F to each $\mathbb{R}^k$ is continuous;*

3. *$F(x \cdot \mathbb{1}_k) = x$ for all $x \in \mathbb{R}$ and all $k \in \mathbb{N}$.*

The equally-distributed equivalent function represents the social ethics of the distribution of wellbeing within a species. If $F(u) = \frac{1}{n(u)} \sum_{i=1}^{n(u)} u_i$, we obtain the classical utilitarian view. A generalization is the prioritarian view with $F(u) = g^{-1} \left(\frac{1}{n(u)} \sum_{i=1}^{n(u)} g(u_i) \right)$ where g is some increasing and concave function. Two other prominent views are the egalitarian maximin view ($F(u) = \min u_i$) and the rank-dependent utilitarian view ($F(u) = \frac{1-\beta}{1-\beta^{n(u)}} \sum_{i=1}^{n(u)} \beta^{r(i)-1} u_{r(i)}$).

Proposition 1 characterize a large class of additively separable social welfare orderings.

Proposition 1. *The following statements are equivalent:*

(1) *$\succsim$ is a regular social welfare ordering that satisfies Species separability and Within-species egalitarian dominance.*

(2) *For each $s \in S$, there exist an equally-distributed equivalent function $\Xi^s : U \to \mathbb{R}$, a function $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ that is continuous and increasing in its second argument, and a real number c^s such that $V^s(1, c^s) = 0, V^s(k+1, x) \geq V^s(k, x)$ (resp. $V^s(k+1, x) \leq V^s(k, x)$) for all $x \geq c^s$ (resp. $x \leq c^s$) and for all $k \in \mathbb{N}$, such that for all $a, a' \in A$*

$$a \succsim a' \iff \sum_{s \in S} V^s \Big(n\big(a(s)\big), \Xi^s\big(a(s)\big) \Big) \geq \sum_{s \in S} V^s \Big(n\big(a'(s)\big), \Xi^s\big(a'(s)\big) \Big).$$

Proof. See the Appendix. □

A key feature of the social orderings in Proposition 1 is that we have a (possibly) different social evaluation function for each species, as described by functions V^s and Ξ^s. We can be utilitarian for cows, rank-dependent for pigs, prioritarian for cats. Also, the value of population size may vary from one species to another. If $V^s(n, x) = n \times x$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we obtain something like a totalist approach. On the contrary, if $V^s(n, x) = x$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, population size as such does not matter and only the (generalized) average is important. Proposition 1 allows to be totalist for humans but averageist for fishes.

5   Replication invariance

For any $l, k \in \mathbb{N}$ and any vector $u \in \mathbb{R}^l$ we say that $v \in \mathbb{R}^{kl}$ is a k-replica of u if for all $i \in \{1, \cdots, l\}$ and $m \in \{1, \cdots, k\}$ we have $v_{(i-1)k+m} = u_i$. We denote $k \star u$ a k-replica of u. Similar, for any $a \in A$ we say that a' is a k-replica of a, denoted $k \star a$, if for each $s \in S$ we have $a'(s) = k \star a(s)$.

We introduce the following property, familiar in the population ethics literature, that guarantees that judgements are invariant to replications.

Strong replication invariance. For all $a, a' \in A$, for all $k \in \mathbb{N}$, $a \succsim a'$ if and only if $k \star a \succsim k \star a'$.

One motivation for replication invariance is as follows. Imagine that species populations are composed of successive generations (finitely many). Imagine also a kind of "stationarity" situation where the population in a species is the same for each generation: same population size and same distribution of wellbeing for individuals within the species. Strong replication invariance means that we can assess the intertemporal situation by just assessing what happens within a single generation.

With Strong replication invariance, we obtain the following characterization.

Proposition 2. *The following statements are equivalent:*

(1) $\succsim$ *is a regular social welfare ordering that satisfies Species separability, Within-species egalitarian dominance, and Strong replication invariance.*

(2) *There exist $\alpha \geq 0$, for each $s \in S$ a continuous and increasing function $F^s : \mathbb{R} \to \mathbb{R}$ such that $F^s(c^s) = 0$ for some $c^s \in \mathbb{R}$, and an equally-distributed equivalent function $\Xi^s : U \to \mathbb{R}$*

satisfying $\Xi^s(u) = \Xi^s(k \star u)$ *for all* $k \in \mathbb{N}$ *and* $u \in U$, *such that for all* $a, a' \in A$

$$a \succsim a' \Longleftrightarrow \sum_{s=1}^{S} \big[n(a(s))\big]^{\alpha} \times F^s\Big[\Xi^s(a(s))\Big] \geq \sum_{s=1}^{S} \big[n(a'(s))\big]^{\alpha} \times F^s\Big[\Xi^s(a'(s))\Big].$$

Proof. See the Appendix. $\qquad\square$

This form is reminiscent of number-dampened utilitarian social welfare criteria proposed by Ng (1989).

6 Anonymity and generalized utilitarianism

The last step in our characterization of generalized total utilitarianism is to assume individual-level anonymity. Implicitly, this axiom assumes that lifetime utilities are not only interpersonally comparable but also are comparable across species.

Anonymity. For all $\mathbf{n} \in \mathbb{N}^T$ and all $a, a', \hat{a} \in A_{\mathbf{n}}$, if there exists two species $s, t \in S$ and $i \in \{1, \cdots, n_s\}$, $j \in \{1, \cdots, n_t\}$, such that $a_i(s) = a'_j(t)$, $a'_i(s) = a_j(t)$, $a_k(s) = a'_k(s)$ for all $k \neq i$ and $a_\ell(t) = a'_\ell(t)$ for all $\ell \neq j$, then $a_{\{s,t\}}\hat{a} \sim a'_{\{s,t\}}\hat{a}$.

Proposition 3. *A regular social welfare ordering* $\succsim$ *satisfies Species separability, Within-species egalitarian dominance and Anonymity if and only if there exists a continuous and increasing function* $\phi : \mathbb{R} \to \mathbb{R}$ *and real numbers* $c^S \in \mathbb{R}$ *such that, for all* $a\ a' \in A$,

$$a \succsim a' \Longleftrightarrow \sum_{s=1}^{S} \sum_{i=1}^{n(a(s))} \big[\phi(a_i(s)) - \phi(c^s)\big] \geq \sum_{s=1}^{S} \sum_{i=1}^{n(a'(s))} \big[\phi(a'_i(s)) - \phi(c^s)\big].$$

Proof. See the Appendix. $\qquad\square$

Although none of our axioms directly assume individual-level independence, the intuition of the proof is that individual-level anonymity can be combined with species-level separability to construct individual-level independence, by hypothetically cycling people in and out of alternative species. This completes our construction of multi-species utilitarianism.

7 Discussion of possible extensions

7.1 Good lives and the critical-level parameter

Within-species egalitarian dominance allows us to have different definitions of a good life for different species. Indeed, for a certain level of wellbeing x it may be the case that we want to extend species s at level x (adding on people at level x to an already existing population $x \cdot \mathbb{1}_k$) but that we may not want to expend species t at the level: this happens when $c^s < x < c^t$. This makes it possible to have different critical levels in our characterization result of Prop. 3.

Requiring an additional principle would imply more uniformity in the critical-levels, at least in some situations. A first principle is Complementarity for species size and well-being. The axiom says that, in allocating two species population sizes between two species, each with a different perfectly-egalitarian per-person welfare level, it is better to assign the larger population size to the species with better-off individuals.

Complementarity for species size and well-being. For all a, a', $\hat{a} \in A$, if there exists real numbers $x > y$, two natural numbers $k > l$ and two species $s, t \in S$ such that $a(s) = x \cdot \mathbb{1}_l$, $a(t) = y \cdot \mathbb{1}_k$, $a'(s) = x \cdot \mathbb{1}_k$ and $a'(t) = y \cdot \mathbb{1}_l$ then $a_{\{s,t\}}\hat{a} \prec a'_{\{s,t\}}\hat{a}$.

Under species separability, Complementarity for species size and well-being implies that parameter c^s in the statement of Within-species egalitarian dominance should be the same for all species. Hence, in Proposition 1, we must also have that there exists a unique number $c \in \mathbb{R}$ such that $V^s(k, c) = 0$ for all $s \in S$ and $k \in \mathbb{N}$. And in Proposition 2, we must have there exists a unique number $c \in \mathbb{R}$ such that $F^s(c) = 0$ for all $s \in S$. Remark that it does not mean that the critical level will always be the same for two species: Propositions 1 and 2 allow that we use average utilitarianism for each species $(V^s(n(u), \Xi^s(u)) = \frac{1}{n(u)} \sum u_i)$ so that we want to add an individual to a population if their utility is higher than the average utility in their species (which may be different from one species to the other). However, if Complementarity for species size and well-being is added to the axioms in Proposition 3, then we must have one common critical level $c \in \mathbb{R}$ for all species and in all cases.

Within-species priority for lives worth living. For all a, a' and $\hat{a} \in A$, for all $s \in S$, for all natural numbers k, l, if there exists numbers $x > 0 > y$ such that $a(s) = x \cdot \mathbb{1}_k$, $a'(s) = y \cdot \mathbb{1}_l$, then $a_s\hat{a} \succ a'_s\hat{a}$.

Priority for lives worth living is discussed in the case of a single specie by (Blackorby, Bossert and Donaldson, 2005, p. 135-136). The priority requires all alternatives in which

each person is above neutrality to be ranked as better than all those in which each person is below it. Blackorby, Bossert and Donaldson (2005) argues that this capture the intuition behind the fact that we should avoid various "sadistic conclusions" (Arrhenius, 2000). Under Priority for lives worth living and Species separability, parameter c^s in the statement of Within-species egalitarian dominance should be 0 for all species.

7.2 Weakening separability or continuity

More social welfare functions become possible if we weaken Species separability to require separability only when one species is concerned.

Weak species separability. For all a, a', $\hat{a}$, $\hat{a}' \in A$ and for all $s \in S$, $a_s\hat{a} \succsim a'_s\hat{a}$ if and only if $a_s\hat{a}' \succsim a'_s\hat{a}'$.

In that case, we can find other social welfare criteria that satisfy Complementarity for species size and well-being, Within-species priority for lives worth living and Strong replication invariance. A first example uses a rank-dependent aggregator across species. Let Π be the set of permutations from S to S.

Example 1: Rank-dependence across species. Let $\beta_1 > \cdots > \beta_n$, preferences $\succsim$ are represented by the following social welfare function:

$$W(a) = \max_\pi \sum_{s \in S} \beta_{\pi(s)} \times \left[n\big(a(s)\big)\right]^\alpha \times F\Big(\Xi^s\big(a_i(s)_{i \in N(a(s))}\big)\Big)$$

with F any continuous and increasing functions, and $\Xi^s : \mathbb{R}^n \to \mathbb{R}$ continuous non-decreasing normalized functions such that $\Xi^s(u) = \Xi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$.

The rank-dependent aggregator assigns the highest weight β_1 to the species with highest welfare as measured by $\left[n\big(a(s)\big)\right]^\alpha \times F\Big(\Xi^s\big(a_i(s)_{i \in N(a(s))}\big)\Big)$, and generality assigns weights in decreasing order of welfare. This is exactly the opposite of the standard rank-dependent case where we give more weight to worse off people. This suggests an anti-egalitarian feature of the aggregation. We conjecture that this anti-egalitarian feature is embedded in Complementarity for species size and well-being (we prefer to give more to better off species).

It is also possible to weaken Extended continuity. using a leximax aggregator, we could satisfy Species separability without additivity. For a vector $v = (v_1, \cdots, v_{|S|}) \in \mathbb{R}^{|S|}$,

we denote $\tilde{v} \in \mathbb{R}^{|S|}$ the re-ordering of v in a non-increasing fashion, so that there exists $\pi \in \Pi$ such that $\tilde{v}_s = v_{\pi(s)}$ for all $s = 1, \cdots, |S|$ and $\tilde{v}_s \geq \tilde{v}_{s+1}$ for all $s = 1, \cdots, |S| - 1$. For two vector $v, v' \in \mathbb{R}^{|S|}$, we write $v >_{\text{lexmax}} v'$ if there is $s \in \{1, \cdots, |S|\}$ such that $\tilde{v}_t = \tilde{v}'_t$ for all $t = 1, \cdots, s-1$ and $\tilde{v}_s > \tilde{v}'_s$. We write $v =_{\text{lexmax}} v'$ if $\tilde{v} = \tilde{v}'$. And we write $v \geq_{\text{lexmax}} v'$ if either $v >_{\text{lexmax}} v'$ or $v =_{\text{lexmax}} v'$.

Example 2: Leximax across species. There exist continuous and increasing function F, and continuous non-decreasing normalized functions Ξ_s such that, for all $a, a' \in A$, $a \succsim a'$ if and only if

$$\left(\left[n\big(a(s)\big) \right]^{\alpha} \times F\left(\Xi^s \big(a_i(s)_{i \in N(a(s))} \big) \right) \right)_{s \in S} \geq_{\text{lexmax}} \left(\left[n\big(a'(s)\big) \right]^{\alpha} \times F\left(\Xi^s \big(a'_i(s)_{i \in N(a'(s))} \big) \right) \right)_{s \in S}.$$

A Appendix

A.1   Proof of Proposition 1

The fact that statement (2) implies statement (1) is easily checked. Let us prove that statement (1) implies statement (2).

Step 1: A representation for a within-species preorder. Let $a^\star \in A$ be such that $n\big(a^\star(s)\big) = 1$ and $a_1^\star(s) = 0$ for each $s \in S$. For each $s \in S$, let us define the ordering $\succsim^s$ on $U = \bigcup_{k \in \mathbb{N}} \mathbb{R}^k$ as follows: for any $u, v \in U$, $u \succsim^s v$ if and only if $a_s a^\star \succsim \hat{a}_s a^\star$ where $a(s) = u$ and $\hat{a}(s) = v$. Given that $\succsim$ is transitive, reflexive and complete, so is $\succsim^s$. By Pareto, the preorder $\succsim^s$ satisfies the monotonicity property: if $u, v \in \mathbb{R}^n$ (for some $n \in \mathbb{N}$) are such that $u \geq v$ then $u \succsim^s v$; if $u, v \in \mathbb{R}^k$ (for some $k \in \mathbb{N}$) are such that $u \gg v$ then $u \succ^s v$. By Extended continuity, $\succsim^s$ also satisfies an extended continuity property: for any $k, \ell \in \mathbb{N}$ and any $u \in \mathbb{R}^k$, the sets $\{v \in \mathbb{R}^\ell | u \succsim^s v\}$ and $\{v \in \mathbb{R}^\ell | v \succsim^s u\}$ are closed. Given that $\succsim^s$ satisfies the monotonicity and extended continuity properties, by Theorem 2 in Blackorby, Bossert and Donaldson (1984), there exists a function $W^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ continuous and increasing in its second argument, and an equally-distributed equivalent function $\Xi^s : U \to \mathbb{R}$ such that for any $u, v \in U$, $u \succsim^s v$ if and only $W^s\big(n(u), \Xi^s(u)\big) \geq W^s\big(n(v), \Xi^s(v)\big)$.

Remark that, by Species Separability and definition of $\succsim^s$, for any a, a' and $\hat{a} \in A$,

$$a_s \hat{a} \succsim a'_s \hat{a} \iff a(s) \succsim^s a'(s) \iff W^s\big(n(a(s)), \Xi^s(a(s))\big) \geq W^s\big(n(a'(s)), \Xi^s(a'(s))\big).$$

By Within-species egalitarian dominance, there exists a wellbeing level $c^s \in \mathbb{R}$ such that For any real numbers $x > y \geq c^s$ and natural numbers $k > l$, there exist a, a' and $\hat{a} \in A$ such that $a(s) = x \cdot \mathbb{1}_k$, $a'(s) = y \cdot \mathbb{1}_l$ and $a_s \hat{a} \succ a'_s \hat{a}$. By the result above, it implies that $x \cdot \mathbb{1}_k \succ^s y \cdot \mathbb{1}_l$ for any $x > y \geq c^s$ and $k > l$. Similarly, by the second part of Within-species egalitarian dominance, it is the case that $x \cdot \mathbb{1}_k \prec^s y \cdot \mathbb{1}_l$ for any $x < y \leq c^s$ and $k > l$.

Let us show that $x \cdot \mathbb{1}_{k+1} \succsim^s x \cdot \mathbb{1}_k$ for any $x \geq c^s$ and $k \in \mathbb{N}$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. We have shown that $(x + \varepsilon_n) \cdot \mathbb{1}_{k+1} \succ^s x \cdot \mathbb{1}_k$ for any ε_n and $k \in \mathbb{N}$. Hence, by Extended continuity, $x \cdot \mathbb{1}_{k+1} \succsim^s x \cdot \mathbb{1}_k$ Similarly, we can show that $x \cdot \mathbb{1}_{k+1} \precsim^s x \cdot \mathbb{1}_k$ for any $x \leq c^s$ and $k \in \mathbb{N}$. Remark that we thus have $c^s \cdot \mathbb{1}_{k+1} \succsim^s c^s \cdot \mathbb{1}_k$ and $c^s \cdot \mathbb{1}_{k+1} \precsim^s c^s \cdot \mathbb{1}_k$ for each $k \in \mathbb{N}$, so that, by transitivity, $c^s \cdot \mathbb{1}_k \precsim^s c^s \cdot \mathbb{1}_\ell$ for any $k, \ell \in \mathbb{N}$. We normalize W^s by setting $W^s(1, c^s) = 0$. For each $k \in \mathbb{N}$ we have $W^s(k, c^s) = 0$. Also, $V^s(k + 1, x) \geq V^s(k, x)$ (resp. $V^s(k + 1, x) \leq V^s(k, x)$) for all $x \geq c^s$ (resp. $x \leq c^s$) and for all $k \in \mathbb{N}$.

Step 2: An additively separable representation. For each $s \in S$ and $k \in \mathbb{N}$ let us define $O_k^s = \big\{z \in \mathbb{R} | \exists u \in \mathbb{R}^k, W^s\big(k, \Xi^s(u)\big) = z\big\}$. Given that Ξ^s is continuous and W^s is continuous in

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its second argument, O_k^s is an open interval in $\mathbb{R}$. Also, given that $W^s(k, c^s) = 0, 0 \in O_k^s$. Define $O^s = \bigcup_{k=1}^{\infty} O_k^s$. Given that each O_k^s is an open interval in $\mathbb{R}$ and that $O_k^s \cup O_{k+1}^s \neq \emptyset$ (because $0 \in O_k^s \cup O_{k+1}^s$), it is also the case that O^s is an open interval in $\mathbb{R}$.

Let us define a relation $\widehat{\succsim}$ on $\prod_{s \in S} O^s$ as follows. Let $\mathbf{x} = (x^1, \cdots, x^T)$ and $\mathbf{y} = (y^1, \cdots, y^T)$ be elements of $\prod_{s \in S} O^s$. We have $\mathbf{x} \widehat{\succsim} \mathbf{y}$ if and only if there exists $a, a' \in A$ such that $W^s\Big(n(a(s)), \Xi^s(a(s))\Big) = x^s$ and $W^s\Big(n(a'(s)), \Xi^s(a'(s))\Big) = y^s$ for each $s \in S$, and $a \succsim a'$.

Let us show that $\widehat{\succsim}$ is a well-defined reflexive, complete and transitive relation. Let a and $a' \in A$ be such that for each $s \in S$ we have

$$W^s\Big(n(a(s)), \Xi^s(a(s))\Big) = W^s\Big(n(a'(s)), \Xi^s(a'(s))\Big) = x^s \in O^s.$$

We need to prove that $a \sim a'$ so that, by definition, $(x^1, \cdots, x^T) \widehat{\sim} (x^1, \cdots, x^T)$ (i.e. $\widehat{\succsim}$ is well-defined and reflexive). $W^s\Big(n(a(s)), \Xi^s(a(s))\Big) = W^s\Big(n(a'(s)), \Xi^s(a'(s))\Big)$ implies $a(s) \sim a'(s)$ so that $a_s \hat{a} \sim a'_s \hat{a}$ by Species separability. Let $a^1, \cdots, a^{T-1}$ be a sequence of alternatives such that, for each $t \in \{1, \cdots, T-1\}$, $a^t(s) = a'(s)$ for all $s \leq t$ and $a^t(s) = a(s)$ for all $s > t$. Using the reasoning involving Species separability, we have $a \sim a^1, a^t \sim a^{t+1}$ for all $t \in \{1, \cdots, T-2\}$ and $a^{T-1} \sim a'$. So, by transitivity, $a \sim a'$.

The reflexivity of $\widehat{\succsim}$ and its definition result in the following property: for any a, $a' \in A$:

$$a \succsim a' \iff \Big(W^s\Big(n(a(1)), \Xi^1(a(1))\Big), \cdots, W^s\Big(n(a(S)), \Xi^S(a(S))\Big)\Big) \tag{A.1}$$
$$\widehat{\succsim} \Big(W^s\Big(n(a'(1)), \Xi^1(a'(1))\Big), \cdots, W^s\Big(n(a'(S)), \Xi^S(a'(S))\Big)\Big).$$

Given that result and the definition of each O^s, completeness and transitivity of $\widehat{\succsim}$ follow easily. Then, the fact that $\widehat{\succsim}$ is totally separable results from the fact that $\succsim$ satisfies Species separability. The fact that each set of factors $T \subset S$ is essential results from the fact that $\succsim$ satisfies Pareto. Hence, by Lemma 1 in the Online Supplementary Appendix (which is Theorem 3 in Debreu (1960)), there exist there exists continuous functions $\Phi^s : O^s \to \mathbb{R}$ such that

$$\mathbf{x} \widehat{\succsim} \mathbf{y} \iff \sum_{s \in S} \Phi^s(x^s) \geq \sum_{s \in S} \Phi^s(y^s). \tag{A.2}$$

We can normalize without loss of generality the Φ^s functions so that $\Phi^s(0) = 0$ for all $s \in S$.

Step 3: Conclusion. By Equations (A.1) and (A.2) in Step 2, we know that, for any $a, a' \in A$:

$$a \succsim a' \iff \sum_{s \in S} \Phi^s \circ W^s \Big(n(a(s)), \Xi^s(a(s)) \Big) \geq \sum_{s \in S} \Phi^s \circ W^s \Big(n(a'(s)), \Xi^s(a'(s)) \Big), \quad \text{(A.3)}$$

where the Φ^s functions are continuous and increasing (as defined in Step 2), the W^s functions are continuous and increasing in their second argument, and the Ξ^s functions are equally-distributed equivalent functions (as defined in Step 1). Define $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by $V^s := \Phi^s \circ W^s$. Functions V^s are continuous and increasing in their second argument. Furthermore, given that $\Phi^s(0) = 0$ for all $s \in S$ and $W^s(k, c^s) = 0$ for all $k \in \mathbb{N}$ and $s \in S$, we obtain that there exists $c^s \in \mathbb{R}$ such that $V^s(k, c^s) = 0$ for all $k \in \mathbb{N}$. Also, $V^s(k + 1, x) \geq V^s(k, x)$ (resp. $V^s(k + 1, x) \leq V^s(k, x)$) for all $x \geq c^s$ (resp. $x \leq c^s$) and for all $k \in \mathbb{N}$.

A.2   Proof of Proposition 2

The fact that statement (2) implies statement (1) is easily checked. Let us prove that statement (1) implies statement (2).

Given that $\succsim$ is a regular social welfare ordering that satisfies Species separability and Within-species egalitarian dominance, we know by Prop. 1 that for each $s \in S$, there exist an equally-distributed equivalent function $\Xi^s : U \to \mathbb{R}$, a function $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ that is continuous and increasing in its second argument, and a real number c^s such that $V^s(k, c^s) = 0$, $V^s(k + 1, x) \geq V^s(k, x)$ (resp. $V^s(k + 1, x) \leq V^s(k, x)$) for all $x \geq c^s$ (resp. $x \leq c^s$) and for all $k \in \mathbb{N}$, such that for all $a, a' \in A$

$$a \succsim a' \iff \sum_{s \in S} V^s \Big(n\big(a(s)\big), \Xi^s\big(a(s)\big) \Big) \geq \sum_{s \in S} V^s \Big(n\big(a'(s)\big), \Xi^s\big(a'(s)\big) \Big).$$

Consider any $\mathbf{n} \in \mathbb{N}^T$ and $k \in \mathbb{N}$. Let us define the ordering $\succsim_\mathbf{n}$ on $\mathbb{R}^S$ defined as follows: for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^S$, $\mathbf{x} \succsim_\mathbf{n} \mathbf{y}$ if and only if $a \succsim a'$, where $a(s) = x_s \cdot \mathbb{1}_{n_s}$ and $a'(s) = y_s \cdot \mathbb{1}_{n_s}$ for all $s \in S$. By definition,

$$\mathbf{x} \succsim_\mathbf{n} \mathbf{y} \iff \sum_{s \in S} V^s(n_s, x_s) \geq \sum_{s \in S} V^s(n_s, x_s)$$

But by Strong replication invariance, and by definition, it is also the case that $\mathbf{x} \succsim_\mathbf{n} \mathbf{y} \iff$

$\mathbf{x} \succsim_{k\mathbf{n}} \mathbf{y}$. Hence,

$$\mathbf{x} \succsim_{\mathbf{n}} \mathbf{y} \iff \sum_{s \in S} V^s(n_s, x_s) \geq \sum_{s \in S} V^s(n_s, x_s)$$

$$\iff \sum_{s \in S} V^s(kn_s, x_s) \geq \sum_{s \in S} V^s(kn_s, x_s)$$

By the unicity of additive representations up to a positive affine transformation (**?**), there must exist $\gamma_k^{\mathbf{n}} > 0$ and $\beta_k^{s,\mathbf{n}}$ such that: $V^s(kn_s, x_s) = \gamma_k^{\mathbf{n}} V^s(n_s, x_s) + \beta_k^{s,\mathbf{n}}$ for all $s \in S$ and for all $x_s \in \mathbb{R}$. Given that, for all $s \in S$, $V^s(k, c^s) = 0$ for all $k \in \mathbb{N}$, we must have $\beta_k^{s,\mathbf{n}} = 0$.

Next, if we pick any $\mathbf{n}' \in \mathbb{N}$ such that $n_s = n_s'$, we obtain by a similar reasoning $V^s(kn_s, x_s) = \gamma_k^{\mathbf{n}'} V^s(n_s, x_s))$ for some $\gamma_k^{\mathbf{n}'} > 1$. Hence we must have $\gamma_k^{\mathbf{n}'} = \gamma_k^{\mathbf{n}}$ whenever there exists $s \in S$ such that $n_s = n_s'$. This actually implies that $\gamma_k^{\mathbf{n}'} = \gamma_k^{\mathbf{n}}$ for any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}$.[4] Pick any arbitrary $\mathbf{n_0} \in \mathbb{N}$ and let $\gamma_k = \gamma_k^{\mathbf{n_0}}$, we obtain that for any $s \in S$, $\ell, k \in \mathbb{N}$ and $x \in \mathbb{R}$: $V^s(k\ell, x) = \gamma_k V^s(\ell, x)$.

Denote $\rho : \mathbb{N} \to \mathbb{R}_{++}$ the function such that $\rho(k) = \gamma_k$ for each $k \in \mathbb{N}$. The fact that $V^s(k\ell, x) = \rho(k)V^s(\ell, x)$ implies that $V^s(k\ell, x) = \rho(k)\rho(\ell)V^s(1, x)$ and also that $V^s(k\ell, x) = \rho(k\ell)V^s(1, x)$ for any $x \in \mathbb{R}$. Therefore $\rho(k\ell) = \rho(k)\rho(\ell)$ for all l and $k \in \mathbb{N}$. Given that $V^s(\ell, x) \geq V^s(m, x)$ for all $x > c^s$ and $l > m$, we must also have that ρ is non-decreasing.[5] By Theorem A in Howe (1986), we know that there must exist $\alpha \in \mathbb{R}_+$ such that $\rho(k) = k^\alpha$. And therefore $V^s(k, x) = k^\alpha V^s(1, x)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$.

For any $s \in S$, $k \in \mathbb{N}$, $u, v \in U$ such that $n(u) = n(v)$, considering a, a' and $\hat{a} \in A$ such that $a(s) = u$ and $\hat{a}(s) = v$, Strong replication invariance implies that $a_s\hat{a} \succsim a_s'\hat{a}$ if and only if $k \star a_s k \star \hat{a} \succsim k \star a_s' k \star \hat{a}$. By all the results up to now, this means that:

$$V^s\left(1, \Xi^s(u)\right) \geq V^s\left(1, \Xi^s(v)\right) \iff V^s\left(1, \Xi^s(k \star u)\right) \geq V^s\left(1, \Xi^s(k \star v)\right),$$

for all $u, v \in U$ such that $n(u) = n(v)$. Given that Ξ^s is normalized, we must have $\Xi^s(u) = \Xi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$. Denoting F^s the function $V^s(1, .)$, we obtain the result in the Proposition because F^s is continuous and increasing.

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[4]For any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}$, let $\bar{\mathbf{n}} \in \mathbb{N}$ be such that $\bar{n}_1 = n_1$ and $\bar{n}_2 = n_2'$. Our results imply that $\gamma_k^{\bar{\mathbf{n}}} = \gamma_k^{\mathbf{n}}$ and $\gamma_k^{\bar{\mathbf{n}}} = \gamma_k^{\mathbf{n}'}$.

[5]Remark that, given that $V(k, 0) = 0$ and V is increasing in its second argument, $V(k, x) > 0$ for any $x > 0$.

A.3 Proof of Proposition 3

If there exists a continuous and increasing function $\phi : \mathbb{R} \to \mathbb{R}$ and real numbers $c^s \in \mathbb{R}$ such that, for all $a\, a' \in A$,

$$a \succsim a' \iff \sum_{s=1}^{S} \sum_{i=1}^{n(a(s))} \left[\phi(a_i(s)) - \phi(c^s) \right] \geq \sum_{s=1}^{S} \sum_{i=1}^{n(a'(s))} \left[\phi(a'_i(s)) \phi(c^s) \right],$$

it is clear that $\succsim$ is a regular social welfare ordering that satisfies Within-species priority for lives worth living, Species separability and Anonymity. Let us show the converse.

Step 1: The within-species preorder $\succsim^s$ is generalized utilitarian.

Let us define the ordering $\succsim^s$ on $U = \bigcup_{k \in \mathbb{N}} \mathbb{R}^k$ like in the first step of proof of Proposition 1. We have shown in that step that, if $\succsim$ is a regular social ordering satisfying Species separability and Within-species egalitarian dominance, then $\succsim^s$ is transitive, reflexive and complete, it satisfies an extended continuity property and monotonicity property. Furthermore, there exists c^s such that for any $k, l \in \mathbb{N}$, $c^s \cdot \mathbb{1}_k \sim^s c^s \cdot \mathbb{1}_l$.

We can also prove that $\succsim^s$ satisfies the following Anonymity property: for any $k \in \mathbb{N}$, any $u \in \mathbb{R}^k$ and any permutation $\pi : \{1, \cdots, k\} \to \{1, \cdots, k\}$, $u \sim^s (u_{\pi(1)}, \cdots, u_{\pi(1)})$. This results for repeated applications of Anonymity.

For any $u, v \in U$, let us denote $uv \in \mathbb{N}$ the vector $w \in U$ such that $n(w) = n(u) + n(v)$, $w_i = u_i$ for all $i \in \{1, \cdots, n(u)\}$ and $w_{n(u)+j} = v_j$ for all $j \in \{1, \cdots, n(v)\}$. Let us prove that $\succsim^s$ satisfies Utility Independence: For all $u, v \in U$, for all $l \in \mathbb{N}$, and for all $w, \hat{w} \in \mathbb{R}^l$, $uw \succsim^s vw \iff u\hat{w} \succsim^s v\hat{w}$. To see that, let $a, a', \hat{a}$, and $\hat{a}' \in A$ be such that:

- $a(s) = uw$, $a'(s) = vw$, $\hat{a}(s) = u\hat{w}$, $\hat{a}'(s) = v\hat{w}$;

- $a(t) = a'(t) = \hat{w}$, $\hat{a}(t) = \hat{a}'(t) = w$ for some $t \neq s$;

- $a(s') = a'(s') = \hat{a}(s') = \hat{a}'(s')$ for all $s' \in S \setminus \{s, t\}$.

By (repeated applications of) Anonymity and transitivity, $a \sim \hat{a}$ and $a' \sim \hat{a}'$. By Species separability, $a \succsim a' \iff uw \succsim^s vw$ and $\hat{a} \succsim \hat{a}' \iff u\hat{w} \succsim^s v\hat{w}$. Therefore, by transitivity, $uw \succsim^s vw \iff u\hat{w} \succsim^s v\hat{w}$.

Let us prove that $\succsim^s$ satisfies Strong Pareto: for any $k \in \mathbb{N}$, any u and $v \in \mathbb{R}^k$, if $u > v$ then $u \succ^s v$. We already now that if $u \gg v$ then $u \succ^s v$. This implies that for any real numbers $x > y$, $x \cdot \mathbb{1}_1 \succ^s y \cdot \mathbb{1}_1$. Let us show that for any $k \in \mathbb{N}$, if $u \in \mathbb{R}^k$ is such that $u_1 = z$ and $u_j = c^s$ for all $j > 1$, then $z \cdot \mathbb{1}_1 \sim^s u$. Indeed, let $a, a' \in A$ are such that:

- $a(s) = z \cdot \mathbb{1}_1$, $a'(s) = u$, $\hat{a}(s) = c^s \cdot \mathbb{1}_1$, $\hat{a}'(s) = c^s \cdot \mathbb{1}_k$;

- $a(t) = a'(t) = c^s \cdot \mathbb{1}_1$, $a'(t) = \hat{a}'(t) = z \cdot \mathbb{1}_1$ for some $t \neq s$;

- $a(s') = a'(s') = \hat{a}(s') = \hat{a}'(s')$ for all $s' \in S \setminus \{s, t\}$.

By Anonymity, $a \sim \hat{a}$ and $a' \sim \hat{a}'$. Given that $c^s \cdot \mathbb{1}_1 \sim^s c^s \cdot \mathbb{1}_k$, Species separability implies that $\hat{a} \sim \hat{a}'$. By transitivity we get $a \sim a'$ and by Species separability $z \cdot \mathbb{1}_1 \sim^s u$. Given this result and $x \cdot \mathbb{1}_1 \succ^s y \cdot \mathbb{1}_1$, it must be the case that $u \succ^s v$ whenever $u, v \in \mathbb{R}^k$, $u_1 = x$, $v_1 = y$ and $u_j = v_j = c^s$ for all $j \in \{2, \cdots, k\}$. By Utility Independence and Anonymity, this implies that for any $k \in \mathbb{N}$, any $u, v \in \mathbb{R}^k$, we have $u \succ^s v$ whenever there exists i such that $u_i > v_i$ and $u_j = v_j$ for all $j \neq i$. By repeated application of this finding and transitivity, we obtain that for any $k \in \mathbb{N}$, any u and $v \in \mathbb{R}^k$, if $u > v$ then $u \succ^s v$.

In sum, the pre-order $\succsim^s$ satisfies continuity, Anonymity, Utility independence, Strong Pareto, and Intermediate existence of critical levels.[6] By Theorem 6.5 in Blackorby, Bossert and Donaldson (2005), we thus know that there exists a continuous and increasing function $\phi^s : \mathbb{R} \to \mathbb{R}$ such that $\phi^s(c^s) = 0$, and real number $(A_k^s)_{k \in \mathbb{N}}$ such that for all u, $v \in U$:

$$u \succsim^s v \iff \sum_{i=1}^{n(u)} \phi^s(u_i) - A_{n(u)}^s \geq \sum_{i=1}^{n(v)} \phi^s(v_i) - A_{n(v)}$$

But $c^s \cdot \mathbb{1}_k \sim^s c^s \cdot \mathbb{1}_{k+1}$ for any $k \in \mathbb{N}$ implies that $A_k = A_{k+1}$ for any $k \in \mathbb{N}$ in the above formula. So, in the end, for all $u, v \in U$:

$$u \succsim^s v \iff \sum_{i=1}^{n(u)} \phi^s(u_i) \geq \sum_{i=1}^{n(v)} \phi^s(v_i), \tag{A.4}$$

with $\phi^s : \mathbb{R} \to \mathbb{R}$ a continuous and increasing function such that $\phi^s(c^s) = 0$.

Step 2: Conclusion.

Consider any $a, a' \in A$. Let $\bar{n}^s = \max\{n(a(s)), n(a'(s))\}$ for all $s \in S$, $\bar{N}^t = \sum_{s=1}^t \bar{n}^s$, and $\hat{a}$ and $\hat{a}' \in A$ be defined as follows:

- $n(\hat{a}(1)) = \bar{N}^T$, $\hat{a}_i(1) = a_i(1)$ for all $i \in \{1, \cdots, n(a(1))\}$ and $\hat{a}_j(1) = c^1$ for all $j > n(a(1))$;

- $n(\hat{a}'(1)) = \bar{N}^T$, $\hat{a}_i'(1) = a_i'(1)$ for all $i \in \{1, \cdots, n(a'(1))\}$ and $\hat{a}_j'(1) = c^1$ for all $j > n(a'(1))$;

- for all $s > 1$, $n(\hat{a}(s)) = \bar{n}^s$, $\hat{a}_i(s) = a_i(s)$ for all $i \in \{1, \cdots, n(a(s))\}$ and $\hat{a}_j(s) = c^s$ for all $j > n(a(s))$;

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[6]Which is implied by the fact that $0 \cdot \mathbb{1}_k \sim^s 0 \cdot \mathbb{1}_{k+1}$ for any $k \in \mathbb{N}$.

- for all $s > 1$, $n(\hat{a}'(1)) = \bar{n}^s$, $\hat{a}'_i(s) = a'_i(s)$ for all $i \in \{1, \cdots, n(a'(1))\}$ and $\hat{a}'_j(s) = c^s$ for all $j > n(a'(s))$;

Eq. A.4 implies that $a(s) \sim^s \hat{a}(s)$ and $a'(s) \sim^s \hat{a}'(s)$ for each $s \in S$ (we add people at level c^s whenever necessary). By repeated applications of Species separability, we obtain that $a \sim \hat{a}$ and $a' \sim \hat{a}'$. Next, let $\tilde{a}$ and $\tilde{a}' \in A$ be defined as follows:

- $n(\tilde{a}(1)) = \bar{N}^T$, $\tilde{a}_i(s) = \hat{a}_i(s)$ for all $i \in \{1, \cdots, \bar{n}^1\}$, and for any $t > 1$ and for any $j \in \{1, \cdots, \bar{n}^t\}$, $\tilde{a}_{\bar{N}^{t-1}+j}(1) = \hat{a}_j(t)$;

- $n(\tilde{a}'(1)) = \bar{N}^T$, $\tilde{a}'_i(1) = \hat{a}'_i(s)$ for all $i \in \{1, \cdots, \bar{n}^1\}$, and for any $t > 1$ and for any $j \in \{1, \cdots, \bar{n}^t)\}$, $\tilde{a}'_{\bar{N}^{t-1}+j}(s) = \hat{a}'_j(t)$;

- for all $s > 1$, $n(\tilde{a}(s)) = n(\tilde{a}'(s)) = \bar{n}^s$ and $\tilde{a}_i(s) = \tilde{a}'_i(s) = c^1$ for all $i \in \{1, \cdots, \bar{n}^s\}$.

The allocation $\tilde{a}$ is obtained from $\hat{a}$ by permuting the wellbeing of each individual of a species $t > 1$ with an individual at level c^1 in species 1 (one that has been added when creating allocation $\hat{a}$ from allocation a). Similarly, the allocation $\tilde{a}'$ is obtained from $\hat{a}'$ by permuting the wellbeing of each individual of a species $t > 1$ with an individual at level c^1 in species 1. By Anonymity, $\tilde{a} \sim \hat{a}$ and $\tilde{a}' \sim \hat{a}'$.

By construction, $\tilde{a}(s) = \tilde{a}'(s)$ for all $s > 1$ so that, by Species separararability, $\tilde{a} \succsim \tilde{a}'$ if and only $\tilde{a}(1) \succsim^1 \tilde{a}'(1)$. Summing up, by transitivity, we have:

$$a \succsim a' \iff \hat{a} \succsim \hat{a}' \iff \tilde{a} \succsim \tilde{a}' \iff \tilde{a}(1) \succsim^1 \tilde{a}'(1)$$

$$\iff \sum_{i=1}^{n(\tilde{a}(1))} \phi^1(\tilde{a}_i(1)) \geq \sum_{i=1}^{n(\tilde{a}'(1))} \phi^1(\tilde{a}'_i(1))$$

$$\iff \sum_{i=1}^{\bar{n}^1} \phi^1(\tilde{a}_i(1)) + \sum_{t=2}^{T} \sum_{j=1}^{\bar{n}^t} \phi^1(\tilde{a}_{\bar{N}^{t-1}+j}(1)) \geq \sum_{i=1}^{\bar{n}^1} \phi^1(\tilde{a}'_i(1)) + \sum_{t=2}^{T} \sum_{j=1}^{\bar{n}^t} \phi^1(\tilde{a}'_{\bar{N}^{t-1}+j}(1))$$

$$\iff \sum_{i=1}^{\bar{n}^1} \phi^1(\hat{a}_i(1)) + \sum_{t=2}^{T} \sum_{j=1}^{\bar{n}^t} \phi^1(\hat{a}_j(t)) \geq \sum_{i=1}^{\bar{n}^1} \phi^1(\hat{a}'_i(1)) + \sum_{t=2}^{T} \sum_{j=1}^{\bar{n}^t} \phi^1(\hat{a}'_j(t))$$

$$\iff \sum_{s=1}^{T} \left[\sum_{i=1}^{n(a(s))} \phi^1(a_i(s)) + \sum_{j=n(a(s))+1}^{\bar{n}^s} \phi^1(c^s) \right] \geq \sum_{s=1}^{T} \left[\sum_{i=1}^{n'(a(s))} \phi^1(a'_i(s)) + \sum_{j=n(a'(s))+1}^{\bar{n}^s} \phi^1(c^s) \right]$$

$$\iff \sum_{s=1}^{T} \sum_{i=1}^{n(a(s))} \left[\phi^1(a_i(s)) - \phi^1(c^s) \right] \geq \sum_{s=1}^{T} \sum_{i=1}^{n(a'(s))} \left[\phi^1(a'_i(s)) - \phi^1(c^s) \right],$$

where the last step is obtained by substracting $\sum_{s=1}^{T} \sum_{i=1}^{\bar{n}^s} \phi^1(c^s)$ from both sides. Denoting $\phi := \phi^1$, so that ϕ is a continuous and increasing function, we obtain the result.

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Online Supplementary Appendix: Debreu (1960)

Let E be a set of factors and for each $e \in E$ X_e be an open connected and separable space. Let $\succsim$ be a complete, reflexive and transitive relation on $\prod_{e \in E} X_e$. An element of $\mathbf{x} \in \prod_{e \in E} X_e$ is a collection $\mathbf{x} = (x_e)_{e \in E}$. Let $J \subset N$. We say that J is $\succsim$-separable if for any $\mathbf{x}, \hat{\mathbf{x}}, \mathbf{y}$ and $\hat{\mathbf{y}} \in \prod_{e \in E} X_e$ such that (i) $x_e = \hat{x}_e$ and $y_e = \hat{y}_e$ for all $e \in J$; and (ii) $x_{e'} = y_{e'}$ and $\hat{x}_{e'} = \hat{y}_{e'}$ for all $e' \in (E \setminus J)$; we have $\mathbf{x} \succsim \mathbf{y} \iff \hat{\mathbf{x}} \succsim \hat{\mathbf{y}}$. We say that J is strictly $\succsim$-essential if, for any $\mathbf{y} \in \prod_{e \in E} X_e$, there exist $\mathbf{x}, \hat{\mathbf{x}} \in \prod_{e \in E} X_e$ such that $y_e = x_e = \hat{x}_e$ for all $e \in (E \setminus J)$ but $\mathbf{x} \succ \hat{\mathbf{x}}$.

We say that $\succsim$ is totally separable if every subset $J \subset N$ is $\succsim$-separable. We have the following well-known result.

Lemma 1 (Debreu, 1960). *If $\succsim$ is a continuous, totally separable preference order on $\prod_{e \in E} X_e$, and every coordinate e is strictly $\succsim$-essential, then $\succsim$ has a fully additive utility representation: there exists continuous functions $\phi_e : X_e \to \mathbb{R}$ such that*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{e \in E} \phi_e(x_e) \geq \sum_{e \in E} \phi_e(y_e).$$

Proof. See Theorem 3 in Debreu (1960). $\qquad\square$