Incomplete Understanding of Complex Numbers

Girolamo Cardano – A Case Study in the Acquisition of Mathematical Concepts

1 Introduction

In this paper, I present the case of the discovery of complex numbers by Girolamo Cardano. Cardano acquires the concepts of (specific) complex numbers, complex addition, and complex multiplication. Describing his case as one of the acquisition of these concepts is supported by our natural interpretation of the historical facts and by the expertise of historians of mathematics. I argue that several strategies that deny Cardano possession of these concepts fail. Then I show that his acquisition of these concepts cannot be explained on the basis of Christopher Peacocke's original version of the *Conceptual Role Theory* of concept possession. Cardano's case shows how incomplete an individual's understanding of his own concepts can be. I suggest that Strong Conceptual Role Theories that are committed to specifying a set of transitions that is both necessary and sufficient for possession of mathematical concepts will always face counterexamples of the kind illustrated by Cardano. I close by explaining why relying more heavily on resources of *Anti-Individualism* yields a more promising framework for understanding the acquisition and possession of concepts of abstract subject matters.

2 Cardano's Discovery of Complex Numbers

In Renaissance Italy, when Girolamo Cardano (1501-1576) was writing, algebra was considered the art of finding the *regula della cosa* – the rule for solving some specific mathematical problem. The main achievement of mathematicians of this period was a deeper understanding of cubic and bi-quadratic equations. Understanding such equations was partly motivated by interests in banking, commerce, and gambling.

Cardano's main mathematical work - the *Ars Magna* from 1545 – is a long treatise on different kinds of equation and their solutions. (Cardano 1968) The book is both a compendium of other mathematician's findings and of original research by its author. The *Ars Magna* famously features the solution of the specific cubic equation

$$x^3 + p \times x = q \ .$$

Its solution is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \ .$$

Finding it was a major achievement, often credited to Cardano. But the author of the *Ars Magna* appropriated the solution from his fellow mathematician Niccolò Tartaglia (1500 – 1557). (Burton 1995, 320ff.; Oystein 1953)

Work on this and other cubic and biquadratic equations led to Cardano's discovery of the complex numbers. The importance of *this* discovery – arguably Cardano's greatest discovery – ironically remained hidden from Cardano during his life.

The solution to the above cubic yields complex numbers whenever

$$\left(\frac{q}{2}\right)^2 < \left(\frac{p}{3}\right)^3 \ .$$

Much like negative numbers, Cardano rejected complex numbers as results of his computations.

Nevertheless, in the context of a different problem, Cardano performed addition and multiplication on complex numbers. In *Ars Magna*, *Chapter 37*, *Rule I*, Cardano considers the task of finding numbers that multiplied with each other yield 40. Added to each other, those same numbers should yield 10. Solving this problem, Cardano makes use of the following rule. Let N be the result of the addition, Z be the result of the multiplication. Let $M=\frac{N}{2}$. Consider the expressions $M+\sqrt{M^2-Z}$ and $M-\sqrt{M^2-Z}$. The sum of these two terms is N and their formal product is Z, as required. If $M^2<Z$, then the terms denote complex numbers. In the section of *Ars Magna* entitled *Rule II* (Cardano 1968, 219), Cardano solves the problem with N = 10 and Z = 40, using the rule mentioned above. His rule yields $5+\sqrt{-15}$ and $5-\sqrt{-15}$ as the answer to the problem. Solving the problem in this way is to accept the truth of

(1) $(5+\sqrt{-15})+(5-\sqrt{-15})=10$

and equally that

(2) $(5+\sqrt{-15})\times(5-\sqrt{-15})=40$.

Cardano writes that it is by multiplying $5+\sqrt{-15}$ and $5-\sqrt{-15}$ that we get 40 and thus find two numbers that when added yield 10. Accepting the truth of (1) and (2) is to accept that the brackets in these two equations can be resolved as in the case of integers. Consequently, accepting (1) and (2) is to accept that

(3) $\sqrt{-15}+(-\sqrt{-15})=0$

and

(4) $\sqrt{-15}\times\sqrt{-15}=-15$.[1]

So, in solving this kind of problem, Cardano *prima facie* adds and multiplies complex numbers.

Cardano's acceptance of complex numbers, as we will see, was limited. Not only did he reject complex results like those for the cubic equation above. He also rejected subtraction and division for these numbers. Indeed, much of the time, he doubted that those were numbers at all. In spite of his use of complex numbers in computations, he remained convinced that they were an absurdity and, in the end, not

[1]Here and throughout the paper I use contemporary symbolic notation to present the computations performed by Cardano and others. I do so for the sake of convenience. I do not commit to the claim that Cardano's mathematics should be identified with contemporary mathematics without qualification.

numbers at all.

Raffaele Bombelli (1526-1572), another major figure in Renaissance mathematics, was deeply impressed with Cardano's work. In 1572 he published a successor to the *Ars Magna*, his *L'Algebra*. (Bombelli 1572; Wagner 2010; Burton 1995, Chap. 7; Bagni 2009; Hofmann 1972; Jayawardene 1973, 513; Crossley 1987) While treatments in earlier treatises had been mostly applied to specific problems, Bombelli confronted them in the abstract.

Bombelli, too, investigated cubic equations. Where Cardano rejected his complex results for the equation

$x^3 = 15 \times x + 4$,

Bombelli explicitly assumed that complex numbers would behave like other numbers. He conjectured that $\sqrt[3]{2+\sqrt{-121}}$ and $\sqrt[3]{2-\sqrt{-121}}$ might be related like the radicals themselves and differ only in sign. Thus they would equal $a+b\times\sqrt{-1}$ and $a-b\times\sqrt{-1}$. This step allowed him to show that

$$x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

$$x=\sqrt[3]{\left(2+\sqrt{-1}\right)^3}+\sqrt[3]{\left(2-\sqrt{-1}\right)^3}$$

$$x=\left(2+\sqrt{-1}\right)+\left(2-\sqrt{-1}\right) = 4.$$

With this calculation, Bombelli had explicitly shown that complex numbers could lead to real results. This realization made Bombelli systematically work out addition, multiplication, subtraction, and division-rules for complex numbers. He further provided a chart with all the different manipulations of the imaginary unit, *i*.[2] Bombelli thus provided a first explication – what I will later[3] introduce at greater lenght as the *arithmetical explication* – of the concept of complex numbers. He

[2]Bombelli comments: „It was a wild thought in the judgment of many; and I too for a long time was of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.“ (Burton 1995, 328)
[3]Cf. p. 19f.

provided an explication of the concepts Cardano discovered. Where Cardano had shunned away from his discovery, Bombelli now systematically pursued it.

The discussion in this paper will concern the *concepts* Cardano possesses. Concepts are a kind of representational content. Representational contents type representational psychological states.[4] Representational contents thus help identify what kind of psychological state a thinker is in. They type ways in which subject matters are represented by individuals. They type competencies for representing subject matters that are associated with being in the relevant mental states. Contents type psychological states by setting veridicality conditions for them – conditions under which a psychological state is accurate or inaccurate of a subject matter.

Contents of propositional states specify under which conditions an individual's psychological states are true or false. Concepts are components of propositional psychological states' contents – states that can enter propositional inference. Concepts' meanings approximately correspond to the lexical meaning expressed by nouns. The concept <u>dog</u>, for example, is the component of a representational psychological state that is (approximately)[5] expressed by the word "dog." Concepts's meanings are grounded primarily (but not exclusively) in the concepts' reference.

Concepts must be distinguished from individuals' *conceptions*.[6] The conception an individual associates with a concept roughly corresponds to how an individual would use, explicate or define the concept.

[4]Cf. (Burge 1993, 291ff.; Burge 2010). In what follows, I will indicate representational contents of thoughts by underlining them. I am here basically assuming the terminology and notation from (Burge 2010, Chapters 1&2).

[5]The relation between linguistic meaning and mental content is complicated. Often, several concepts are associated with one linguistic form. Often, it depends on context, which concept is expressed by a word. I here merely provide a sketch of the relationship between concepts and linguistic expressions.

[6]Philosophical tradition acknowledges the need to distinguish concepts from conceptions. Discussions of externalism and anti-individualism in the philosophy of mind strongly support making the distinction. I can here only sketch some of the motivations for this distinction. But see Kripke (1980), Putnam (1970; 1975; 1973), Burge (1993; 2012).

Differences in cognitive significance motivate the distinction of concepts from conceptions. Suppose the explication force is mass times acceleration captures an individual's conception of force. This explication is informative in a way the explication force is force is not. There is hence a difference in cognitive significance between the concept force, and mass times acceleration, the concept explicating force. The distinction between concepts and conceptions is motivated by considering the dynamic potential of concepts. Individuals misunderstand concepts in ways that allow for correction. Entire communities can achieve better understanding of their concepts. If the meaning of a concept were given by an individual's or a community's conception of it, such misunderstanding, correction, and improvement of understanding would be impossible.

Thought attributions are the primary means for identifying the contents of an individual's thoughts. The noun phrases embedded in true ascriptions of thoughts provide the thoughts' representational contents. As pointed out above, concepts are components of such contents. They roughly correspond to elements of the noun phrase. So, if a thought is correctly ascribed, then the individual thinks and hence possesses all the concepts that are components of the thought. Thus, if the attributions of the thought contents expressed by (1) – (4) are correct, then Cardano is thinking thoughts involving the concepts of individual complex numbers, complex addition, and complex multiplication. He thinks thoughts with these *concepts*, even though his *conception* of these concepts, as will become apparent, is incomplete and flawed.

Cardano's case has several features that make it especially interesting. Cardano's discovery of complex numbers was *accidental*. Cardano did not have the goal to discover complex numbers. He was interested in the solution to his specific problem. The discovery was nevertheless systematic. He used complex numbers and operations on them in computations leading to successful solutions to his problem.

Cardano *seriously considered the possibility* of the new kind of entity he discovers. He pondered whether the transitions he discovered were genuine. He *named or baptized* the new entities. Thus, Cardano stated that it seemed to him that " $\sqrt{-9}$ is neither +3 nor -3 but is some recondite third sort of thing." (Cardano 1968, 207) He called these things the "impossible quantities." (Kenney 1989, 196)

Cardano *doubted* and eventually *rejected* his discovery as genuine. His success in solving specific problems was evidence for the existence of the new entities. His doubts and rejection were inspired by theoretical preconceptions of his time about the nature of number, possible algebraic solutions, and proof. Mathematicians of his time assumed that a solution to an algebraic problem must be interpretable as a positive quantity. Genuine proof of a solution to a problem consisted in a geometrical demonstration of the solution. Cardano repeatedly insisted that "it is clear that this case is impossible." (Cardano 1969, 219) When attempting a geometrical proof of his solution, he famously commented: "putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making $25-(-15)$ which is +15. [Cardano means that $-(-15)$ equals +15, which should be added to 25, yielding 40. D.B.] Hence this product is 40. ... So progresses arithmetic subtlety, the end of which, as is said, is as refined as it is useless." (Cardano 1969, 219) Cardano referred to solutions obtained on the basis of calculations on complex numbers as "false solutions." (Kenney 1989, 201)

Throughout his investigations, Cardano made *mistakes*. His understanding of the entities at hand was *incomplete* and very rudimentary. Not only did he doubt and eventually reject the correct judgments he made. He rejected division and subtraction as operations on the new entities. With complex numbers, he wrote, "one cannot carry out the other operations.” (Cardano 1968, 220) He took his additions or multiplications to be incorrect, if his final result was a complex number. For instance, he rejected claims of the form $\sqrt{-15}+\sqrt{-15}=2\times\sqrt{-15}$ and $\sqrt{-9}=3\times\sqrt{-1}$.

Historians of mathematics universally attribute to Cardano the discovery of complex numbers, in spite of his ambivalent attitude towards them. The above account of Cardano's history, with its theoretical commitments, is the *standard description* of the case. (Boyer 1991; Kline 1972; Kenney 1989; Crossley 1987; Burton 1995; Stillwell 2001; Maracchia 2003) Historians describe Cardano as calculating with complex numbers, as adding and multiplying them. They support the *prima facie* plausible ascription of the thoughts expressed by (1) – (4) to Cardano. The historians' verdict is a refined version of the way we would naturally describe Cardano's case.[7]

---

[7]My claims about the standard description do not amount to a strong position in the ongoing debate between presentism and historicism in the history of mathematics. Historicism is, roughly, the view that one cannot interpret early mathematics in terms of contemporary mathematical problems, aims, and methods. Presentism maintains that one can, and indeed should, do so. (Hodgkin 2005, 5ff.)

The actual practice of historians of mathematics betrays a nuanced application of both historicist and presentist methods. In this paper I rely on verdicts by historians of mathematics that specifically investigate Cardano's case.

My argument relies on the claim that Cardano possessed concepts of complex numbers as explicated a few decades later by Bombelli. Thus I rely on the claim that the correct explication of Cardano's concepts was available only after Cardano discovered these concepts. I do *not* assume that Cardano was thinking thoughts with (all) the concept(s) or explications of complex numbers available in contemporary mathematics – e.g. the explication of complex numbers as points on a Riemann surface.

My argument is similarly unaffected by debates about revolutions in mathematics (Gillies 1995). Historians have both claimed (Dauben 1984) and denied (Crowe 1975) that mathematics undergoes proper revolutions or paradigm-changes. This debate is ongoing. Much of the debate concerns the proper notion of a revolution. According to one view, a revolution in mathematics would require abandoning (all of the relevant) earlier mathematics. According to another view, revolutions occur when earlier theories are not abandoned or overthrown, but their significance within mathematics is strongly altered. Neither of the views denies that there is a strong continuity between mathematical theories.

My argument would be affected by this debate if it turned out that there is *no* continuity in mathematics between Cardano and Bombelli. The argument would be affected if it could be argued that a discontinuity between the two mathematicians altered their concepts. Historians of mathematics specializing on this episode in the history of mathematics, however, instead support the claim that there is a continuity in Cardano and Bombelli's concepts.

My claims are similarly unaffected by recent theorizing about conceptual change in the history of science more generally. Thus (Friedman 2002, 185ff.) argues that some developments in the history of science lead to changes in theoretical frameworks or paradigms. Theories of conceptual change typically do not claim that a shift in framwork leads to a change in *all* concepts. Rather, they lead to a change in an important subset of concepts.

I believe that work on externalist theories of meaning by e.g. Kripke (1980), Putnam (1970;

In a striking passage, Emelie Kenney summarizes the special role historians of mathematics take Cardano to play and mentions two factors: "[Cardano] is generally credited with the discovery of complex numbers, probably because he was the first to distinguish complex numbers from other mathematical objects by naming them ... he called such objects "impossible quantities" – and he *used* these objects in certain calculations." (Kenney 1989, 196)

Plausibly, the following aspects of the case play a role in our and the historians' support for the ascriptions.

The discovery of the entities and their naming or baptism are naturally explained on the basis of Cardano's successful *reference* to them.

Computations over the complex numbers and the hypothetical consideration of their existence require that Cardano refer to them. Computations and hypothetical consideration are naturally explained by his thinking thoughts involving complex numbers and operations on them. Cardano seems to make systematically correct computations involving complex numbers. Their correctness is naturally explained on the basis of inferential relations between his concepts. These inferential relations are established by complex algebra.

Cardano is thinking *true thoughts* and false thoughts from complex algebra. His success and failure in thinking those thoughts is evaluated against the mathematical reality of complex numbers. The most natural way of explaining *why* his thoughts are to be thus evaluated is that they refer to the complex numbers. They do so in virtue of containing concepts of complex numbers, complex addition, and multiplication. Attributing those concepts to Cardano is part of the natural explanation

---

1975; 1973), Burge (1993; 2012) has shown that even during deep changes in scientific theorizing, many concepts typically do *not* change.

However, all the paper claims is a continuity of concepts between Cardano and Bombelli. There is no reason to assume that the truth of Friedman's general account would affect this claim as against historians of mathematics' verdict.

Thanks to an anonymous reviewer for prompting these clarifications.

of how we evaluate Cardano's thoughts.

Inferential relations between the thoughts Cardano entertains partly explain the *rationality* of Cardano's attitudes. His hypothetical considerations, doubts, and rejections are partly explained by assuming that he is entertaining propositional attitudes to contents involving the relevant concepts.

Cardano's work is the *actual historical origin* of complex algebra. He discovers and initiates research into *a new subject matter*. As pointed out before, Bombelli explicitly sets out to deepen Cardano's investigations into complex numbers. (Jayawardene 1973) Bombelli provides an *explication* of Cardano's concept of complex number. He does so by stating the addition, multiplication, subtraction, and division rules for complex numbers. He does so by providing a chart with all possible manipulations of the imaginary quantity *i.* Cardano and Bombelli share and ground, a "piece of the common intellectual heritage of mankind." (Williamson 2006, 31) They and later mathematicians could communicate their findings relying on the same concepts. They could disagree about the nature of the relevant entities and concepts, and help each other improve their understanding. We take Bombelli to be *sharing* thoughts about complex numbers with Cardano. We similarly take Cardano to be sharing such thoughts with Leibniz, Euler, and ourselves. Shared concepts enable a natural explanation of the origin, growth, and transmission of knowledge about a subject matter.

Cardano's discovery of complex numbers involves his acquisition of the concepts of complex numbers, complex addition, and complex multiplication. This verdict is supported by our intuitive, natural interpretation of the historical facts. It is further supported by the expertise undergirding the historians of mathematic's equivalent verdict. Cardano was in possession of the concepts of complex numbers, complex addition, and complex multiplication.

3 Reinterpretation

One may still be inclined to think that the ascriptions of the contents expressed by (1) – (4) should not be taken literally. I will discuss five reinterpretation strategies, focusing on Cardano's thoughts involving complex addition. I will argue that reinterpretation is implausible[8].

The dialectical situation must be noted. The historians' *standard description* of Cardano's case above provides strong *prima facie* support for making the ascriptions. Whoever wants to argue that Cardano does not in fact think thoughts with the contents expressed by (1) – (4) above, will not only have to argue for attributing whatever alternative thoughts he takes Cardano to have. He will also have to argue that the ordinary attributions on their ordinary interpretations are incorrect and should be abandoned. None of the reinterpretation strategies I am aware of can adequately meet this challenge.

The first reinterpretation strategy claims that Cardano is thinking the *same concepts* he has been thinking when he was doing algebra on the integers and reals. No new concepts have been acquired. Parts of his thoughts are about integers and real numbers. Parts that seem to involve concepts of complex numbers – such as $\sqrt{-1}$ – could be of two kinds. The concept of a square root on integers may here be 'applied' to negative numbers. Alternatively, no concepts may be expressed by the relevant symbols at all. In either case, parts of the contents of Cardano's thoughts lack reference. They yield false thoughts or thoughts without truth value.

[8] Much of the discussion in the present section is indebted to the methodology employed in (Burge 1978) and (Burge 1979). Cf. also (Williamson 2006) and (Williamson 2003).

Attributing falsehoods or thoughts without truth value is against the standard description of the case. It amounts to holding that Cardano thinks obvious falsehoods or thoughts without truth value that exactly resemble true, innovative mathematical thoughts. It requires an explanation as to why Cardano's solutions to his problems are successful. This explanation would have to be better than the historians' explanation. Providing such an explanation would have to be motivated. I do not see what the motivation might be.

Attributing obviously false thoughts to Cardano would require an alternative explanation as to why Cardano is not irrational in holding these thoughts.

Finally, claiming that Cardano's concepts are the same as before contradicts Cardano's own reflections about what he is doing. Cardano explicitly considers the possibility that expressions like $\sqrt{-1}$ do refer to novel entities. His historical achievement partly consists in his considered naming of those entities. The present reinterpretation strategy has to deny the importance of Cardano's reflections. It more generally has to deny Cardano's achievement.

The first reinterpretation strategy seems to bring no explanatory reward. The burden of proof is on it to motivate abandoning the standard description. This first proposal must be rejected.

The second reinterpretation strategy acknowledges that Cardano's concepts have changed. It suggests attributing notions that capture Cardano's *conception*. The contents of Cardano's thoughts, according to this proposal, are constituted by being elements in Cardano's conception or theory of his thoughts' referents. Cardano's conception was a *mis*conception. It contained the contents expressed by (1) – (4). Further, it contained thoughts to the effect that the entities referred to cannot be subtracted, divided, and are not numbers.

I provide three considerations as against this reinterpretation. First, we do not normally attribute thoughts involving misconceptions just because individuals incompletely understand their own thoughts. Suppose an individual claims that $5+\sqrt{-15}$ and $5-\sqrt{-15}$ cannot be subtracted from each other. Suppose he is thinking that $(5+\sqrt{-15})-(5-\sqrt{-15})=2\times\sqrt{-15}$ is incorrect. We will not normally assume that he is thinking thoughts different from ours. Rather, we will assume that the individual just gets things wrong. Normally, we will think that he has incompletely understood some of the concepts constituting the thought.

Second, finding a reinterpretation is usually non-trivial. There are no agreed-upon standards for deciding how to reformulate the attributed thoughts. I suggested the reformulation might be based on the conjunction of the contents expressed by (1) – (4), together with Cardano's misconceptions about the new entities. But why all of these, instead of only a subclass of the assumptions? Will changes in Cardano's views about the relevant entities lead to a change in the concepts we attribute? The reinterpretation proposal must avoid relying on *ad hoc* methodology. It is simpler and more informative to just attribute thoughts we would *prima facie* attribute.

Third, the present reinterpretation violates the assumption that Cardano refers to complex numbers. If Cardano's concepts and their referents are determined on the basis of Cardano's misconception, then the referents cannot be the complex numbers. The reinterpretation must find referents that allow a principled assessment of Cardano's thoughts as true or false. No subject matter exists that does make true Cardano's misconception. No subject matter other than complex algebra seems close enough to warrant attributing reference to it. The subject matter of the misconception might be stipulated. Such stipulation would be *ad hoc* and for the sake of saving the proposal.

Pending independent argument, this reinterpretation does not yield any explanatory reward. The second proposal for reinterpretation must be rejected.

The third reinterpretation strategy proposes that Cardano did not think thoughts involving concepts of complex numbers and their addition. Nor should we attribute thoughts based on Cardano's misconception of complex numbers. Rather, Cardano was thinking thoughts involving some other, *more primitive concept*.[9]

The standard description holds that Cardano's concept of complex numbers and addition is roughly that explicated by Bombelli. There is thus a sense in which Cardano's concept is more primitive than concepts explicated by modern conceptions of complex numbers.

The claim that Cardano's concept was explicated by Bombelli is plausible for several reasons. Bombelli's explication is true. It is still considered to be accurate today. His explication is a concept that was actually used in subsequent mathematics. It was the only true explication of complex numbers available to mathematicians for centuries to come. The claim that Bombelli explicated Cardano's concept is supported by their historical proximity, their working within the same mathematical tradition, their being driven by the same kinds of mathematical problems. Cardano and Bombelli apply similar methods. And Bombelli understands himself as developing Cardano's discovery. It is plausible to think that Cardano would have accepted Bombelli's explication of his discovery, had he had a chance to do so.

What further, yet more primitive, alternative concept might we attribute to Cardano? It cannot merely be grounded in Cardano's conception of complex numbers, his beliefs or abilities to manipulate these entities. For, Cardano's actual conception

[9]This strategy has been proposed to me by an anonymous reviewer. Guy Longworth independently suggested a similar reinterpretation strategy. I am indebted to both.

was a *mis*conception. And, as discussed above, this misconception plausibly should not determine our attribution of concepts to Cardano.

An alternative proposal might rely on a subset of Cardano's beliefs to ground its attribution of concepts to Cardano. But which subset should we pick? And on what grounds? The re-interpreter would have to provide a principled explanation as to why we should take Cardano to be thinking thoughts involving these particular concepts, as opposed to concepts grounded in another subset of Cardano's conception, or else her proposal seems *ad hoc*. I am not aware of any more primitive, more plausible, alternative attribution of thoughts that is grounded in this way.

The present reinterpretation strategy thus lacks a principled basis. Unless such a basis is provided, and pending independent motivation for the proposal, it should thus be rejected.

The fourth reinterpretation proposal is what has been called the *metalinguistic* strategy. On this proposal, the individual does not think object-level thoughts. Rather, he has false metalinguistic beliefs about the words or symbols expressing those concepts. Usually, the proposal is so construed as to make reference to the usage of a word in the individual's linguistic community. Cardano does not defer to his community's usage of a linguistic symbol. He is the first person to somewhat systematically use the symbols in the relevant way. The metalinguistic strategy could therefore be to provide the content of (1) as

Whatever "$\sqrt{-15}$" refers to, if anything at all, "+" refers to some type of addition such that (1) is equivalent to $5+5=10 \land \sqrt{-15}+(-\sqrt{-15})=0$.

As before, the present proposal has no *prima facie support*. We do not normally attribute only metalinguistic thoughts to individuals who have incompletely understood a concept.

The proposal would not have available the simple, natural explanation of the correctness and rationality of Cardano's attitudes and inferences. It would have to find, and motivate an alternative explanation.

Cardano seems to share thoughts with thinkers who have superior grasp of the concepts involved. The present proposal will have to deny that Cardano shares object-level thoughts about complex numbers with later mathematicians.

Cardano and Bombelli use different expressions – "R m:15" and "piu di meno" respectively – to denote complex numbers. (Cardano 1968) They use "p" (Kline 1972, 260) and "più" (Fauvel & Gray 1987, 265) to denote complex addition. Leibniz, who makes extensive excerpts from Bombelli and further develops the theory of complex numbers, refers to *i* as " $\sqrt{-1}$ " and to complex addition as "+ ." (Hofmann 1973, 232ff.) All these thinkers would have to be taken to be thinking different thoughts when they are thinking the thought expressed by (1). This consequence seems highly counterintuitive, contrary to how we attribute thoughts to individuals.

It is not implausible to think that at some point Cardano's object-level reasoning was accompanied by meta-reflection on the relevant symbols. It *is* implausible that Cardano never engages in object-level reasoning. Cardano's rule for solving the problem that leads to computing complex numbers is formulated on the object-level. There is no reason to think that in applying it Cardano did not reason in accordance with its formulation. It is the application of this rule that leads to results like $5+\sqrt{-15}$. There is no ground for thinking that the last step of Cardano's reasoning involves a sudden shift to the meta-level, and only for cases involving

$5+\sqrt{-15}$.

Pending further argument, reinterpretation does not yield any explanatory reward. The fourth reinterpretation proposal thus should be rejected.

The fifth and last reinterpretation strategy contends that Cardano does not think contentful thoughts when thinking (1) – (4). Instead, he is said to be manipulating symbols according to previously acquired *syntactic routines*.[10]

Again, it is not normally the case that we attribute contentless manipulations of symbols to individuals that incompletely understand thoughts. The proposal cannot evaluate Cardano's thoughts as true or false. Their rationality could not be explained in terms of their reference. Attributing contentless thoughts thus comes at a high cost. Unless some general argument is available to the effect that all thought is mere symbol-manipulation, lack of attribution of content in specific cases must be motivated.

The proposal must provide an explanation as to why Cardano follows only a select few of his syntactical routines. Cardano adds and multiplies complex numbers. He does not subtract or divide them. If Cardano is following acquired syntactic routines in the first two case, why does he not follow these syntactic routines in the other two? The proposal must provide an explanation of this discrepancy.

Cardano explains his abstinence from subtraction, and division in the natural way, by appeal to content. All his doubts and rejections are explained at the level of reference. He rejects subtraction, because he finds that this operation does not make sense for these entities. And he doubts the value of his calculations on the basis of worries about the ontological status of the referents of his numerals. The present

[10] I owe this objection to Shel Smith. Cf. also (Wilson 2007). I reject a view of mathematics as the mere manipulation of symbols. According to a version of this view, no mathematical thoughts have contents. It is beyond the scope of this paper to address the view at length. There might be independent reason for thinking that *Cardano* merely manipulates symbols. This is the challenge I discuss in the main text.

proposal would have to motivate heavy reinterpretation of a natural explanation of Cardano's behavior.

In absence of further argument this last reinterpretation strategy yields an explanatory disadvantage. It cannot motivate giving up the standard description of Cardano's case. It should thus be rejected.

None of the reinterpretation proposals considered seems to motivate abandoning the standard description of Cardano's case. They get the facts about Cardano wrong. They have no explanatory advantage over the standard description. Reinterpretation of Cardano’s thoughts that avoids ascribing concepts from complex algebra to Cardano is implausible. Accordingly, it should be accepted that Cardano entertained thoughts involving the concepts $\sqrt{-15}$, $5+\sqrt{-15}$, complex addition, and multiplication. The truth of this claim will be assumed from here on.

4 Peacocke's Conceptual Role Theory of Concept Possession

Christopher Peacocke proposes a *Conceptual Role Theory* of concept possession. The basic idea is that it is constitutive of an individual's having thoughts involving a concept, that she have certain dispositions to make judgments or inferences.

For Peacocke, mastery of a concept, deference to a community, and having an implicit conception of a concept jointly exhaust individuals' ways of *having* a concept.[11] (Peacocke 1992)

It is clear that Cardano cannot *defer* to any other thinker in his use of the relevant concepts. Nobody else is using the concepts. So there is no one he could defer to.

[11] I do here not discuss in what ways Peacocke's theory of concept possession has changed in (Peacocke 2008). He explicitly mentions implicit conceptions as partly constitutive of concept possession at (Peacocke 1998b, 131 & 140) and in (Peacocke 2003).

(Peacocke 1992, 19ff.)

In *A Study of Concepts*, Peacocke proposes an account of what it is to *master* a concept. For an individual to master a concept C she has to be *primitively compelled* to make the transitions mentioned in the mastery conditions. She must do so because they are of the form mentioned in the mastery conditions. (Peacocke 1992, 6) “To say that the thinker finds such transitions primitively compelling is to say this: (i) he finds them compelling; (ii) he does not find them compelling because he has inferred them from other premises and/or principles; (iii) for possession of the concept C in question ... he does not need to take the correctness of the transitions as answerable to anything else.” (Peacocke 1992, 6)

Peacocke demands that the transitions constituting a concept be strong enough to determine the right semantic value. They must do so by way of distinguishing it from all other semantic values. For a mathematical concept to be truly a concept of, say, some number, its possession conditions must be rich enough to distinguish it from all other numbers. For Peacocke, mastery conditions for mathematical concepts are thus based on a definition of those concepts. Basing possession conditions on a correct definition will ensure determination of a unique semantic value.[12] (Peacocke 1992, 17; Peacocke 1987, 153-200; Peacocke 1998a)

The complex numbers, like many mathematical concepts, have several equivalent definitions. Here I present elements of what I call the *arithmetical explication*. It is based on one standard definition of the complex numbers. (Ebbinghaus 1991, 65; Feferman 1989, 255 & 303) This explication of the concept can be understood as an expression of Bombelli's explication in modern notation.

[12] Peacocke does to my knowledge not explicitly state this requirement on mathematical concepts. But both his examples and constraints on possession conditions seem to imply the requirement.

(Bombelli 1572; Burton 1995, 327ff.; Hofmann 1972; Jayawardene 1973) The explication goes as follows:

A complex number is a number of the form $a+b\times i$, where $a, b\in R$, and $i=\sqrt{-1}$.

An individual whose mastery of the concept of a complex number reflects the arithmetical explication must understand the elements in the explication. He should thus meet three requirements. The individual should understand that the number is an instance of the general form given in the arithmetical explication. He should understand that the number is composed of a real and imaginary part. And he should understand that the complex number is part of a certain mathematical structure.[13]

Understanding that a complex number is an instance of the general form given in the explication requires acknowledging that complex numbers have a standard format. It requires acknowledging that both real and imaginary parts might equal zero. So, the individual should allow that both real numbers and the imaginary unit alone are complex numbers. The individual should allow that the imaginary unit alone is a legitimate item for his calculations.

Understanding that the complex number is composed of a real and an imaginary part requires acknowledging its imaginary part. Such acknowledgement requires understanding *i*. Understanding *i* requires not only understanding that $i=\sqrt{-1}$ is a legitimate item for calculations. It requires understanding *how* to manipulate the square root of a negative number. In particular, it requires grasping $i^2=-1$. This equation seems inconsistent with the identity $\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$,

[13] I think the individual should also understand that the individual complex numbers are *numbers*, hence are of the same ontological type as, say, the integers. Since Bombelli, too, had difficulties accepting complex numbers as numbers, I here only discuss the slightly weaker requirement that the explication requires understanding complex numbers as elements of certain mathematical structures.

$a, b \in R \geq 0$. It was highly non-trivial to see that there is no inconsistency. Understanding *i* finally requires understanding that the imaginary part of a complex number is a multiple of *i*. It requires understanding, for example, that $\sqrt{-9} = 3 \times \sqrt{-1}$.

Understanding that the complex number is part of a certain mathematical structure requires, in this case, acknowledging that basic arithmetical operations can be performed on complex numbers.

Certainly, thinking the thoughts expressed in (1) – (4) requires some understanding of this structure. Thoughts (1) – (4) involve addition and multiplication of complex numbers. The individual in possession of the concepts of complex numbers, complex addition, subtraction, multiplication, or division, should show understanding of the definitions of *those* concepts.

Addition on complex numbers is defined as the operation

$$(a + b \times i) + (c + d \times i) = (a + c) + (b + d) \times i , \; a, b, c, d \in R .$$

Multiplication on complex numbers is defined as the operation

$$(a + b \times i) \times (c + d \times i) = (a \times c - b \times d) + (a \times d + b \times c) \times i , \; a, b, c, d \in R .$$

Subtraction of complex numbers is defined as the operation

$$(a + b \times i) - (c + d \times i) = (a - c) + (b - d) \times i , \; a, b, c, d \in R .$$

Division of complex numbers is defined as the operation

$$\frac{a + b \times i}{c + d \times i} = \left(\frac{a + b \times i}{c + d \times i} \right) \times \left(\frac{c - d \times i}{c - d \times i} \right) , \; a, b, c, d \in R .$$

Understanding these operations on the complex numbers requires the ability to perform them on the complex numbers. Understanding these operations also requires accepting closure for the complex numbers – that complex numbers that are not elements of the set of real numbers can be results of computations.

Mastery of mathematical concepts requires understanding that reflects a definition or sufficiently rich explication of those concepts. Such understanding, for

Peacocke, consists in an individual's being *compelled* to judge and infer in accordance with the explications given in the last several paragraphs.

Cardano clearly is not compelled to make any of the required judgments or inferences. Cardano does not acknowledge a standard format for the complex numbers. He does not think of the complex numbers as composites of real and imaginary units. Cardano nowhere considers *i* by itself and is not in general compelled to manipulate *i* in the standard ways. Cardano is not compelled or able to perform the basic arithmetical operations on complex numbers. He rejects closure.

One might think that Cardano may have had some initial disposition to add or subtract when his calculations yielded formulae like those in (1) – (4). His understanding and skill with addition and multiplication may have disposed him to try these operations on anything that *prima facie* resembled a number. Is there then a sense in which Cardano was disposed to make the relevant judgments and inferences, after all?

Two responses to this worry are available. First, it conflicts with the history of the case to claim that Cardano was compelled to perform the relevant operations. For mathematicians of his time, negative results and roots of negative numbers were signs of error. Their occurrence in calculations would normally lead Cardano and his contemporaries to terminate their calculations and check for the source of error. So Cardano was not compelled to proceed with an addition or subtraction on complex numbers. Second, assume that some general disposition to *add* was present, even when faced with complex numbers. It would not still not have been a disposition to add *complex numbers*. The latter disposition would have to reflect the rich explication of addition of complex numbers from above. It would have to reflect the rich explication of complex numbers above. But we have seen that Cardano did not have such a rich disposition to calculate with complex numbers.

Where Cardano considers the possibility of operations on the complex numbers, his considerations are highly theoretical in nature. They are based on the observation that accepting the operations will yield desired results for the mathematical problem at issue. Backtracking from the results leads Cardano to entertain the possibility of (1) – (4). He reports his own disinclination to entertain that possibility.

Cardano, to conclude, does not have the relevant individual-level dispositions. He is not compelled to judge and infer in the required ways. Rather, he is compelled to reject the application of arithmetical operations to complex numbers. He is compelled to reject roots of negative numbers. His consideration of (1) – (4) is based entirely on the observation that their acceptance would yield a real result. Cardano hence has not mastered the concepts in question.

Implicit conceptions are (or entail the presence of) sub-individual dispositions.[14] The contents of implicit conceptions may, on the sub-individual level, be 'explicitly' represented, for example in some language of thought. Or they may be 'implicitly' represented, by being “grounded in the operation of a processor.” (Peacocke 2008, 142; cf. also Peacocke 1998b & Peacocke 2003)

For Peacocke, 'explicit' representation seems to require that the definition of the concept in question is actually represented in the individual's psychology. This representational state is then supposed to underly the individual's disposition. 'Implicit' representation seems to mean that the definition is implemented in the rules governing the functioning of a sub-individual processor or module. They are not themselves represented, but govern computations of that processor or module. The functioning of the latter constitutes the sub-individual disposition required for

[14]For criticism of the notion, cf. (Burge 2012, 578).

possession of the concept. The disposition is the sub-individual analogue of the disposition underlying mastery of the concepts.

Cardano does not seem to be disposed to infer and judged with complex number-concepts in the right ways. Peacocke might claim that Cardano has such dispositions, but that the dispositions are masked. The burden is then upon him to explain, on the basis of the facts of the case, why postulating such dispositions constitutes a better description of Cardano's case than the standard description. He has to explain, in particular, why Cardano has these dispositions and the masking dispositions. Such an explanation will either require an account of the acquisition of the sub-individual dispositions or a theory as to how they could be innate.

It is implausible that Cardano has a sub-individual disposition based on the arithmetical explication of complex numbers above. Only long after Cardano are complex numbers fully accepted. Only then is it plausible to think that individuals have *acquired* a full disposition derived from the explication above. Maybe Bombelli, upon having worked out the computations of *i* and the arithmetical operations on the complex numbers, has acquired such an implicit conception. Cardano *clearly* has not had the opportunity to acquire or learn the relevant disposition. He acquires the concepts in the context described. There is no evidence that he engages in the learning required for the acquisition of the full sub-individual disposition.

The relevant disposition is plausibly not *innate*. Empirical research strongly suggests that there is no relevant module or processor. Mathematical core cognition – the ensemble of innate mathematical processors – does not seem to represent complex arithmetic. (Carey 2009; Stanovich & West 2000)

Cardano does not meet any of Peacocke's conditions for having the concept of complex numbers. Peacocke's theory thus should deny that Cardano possesses the relevant concepts. It conflicts with the natural description of Cardano's case. Cardano's case provides a counterexample to Peacocke's theory.

5 Strong Conceptual Role Theories

The discovery of complex numbers by Cardano seems to be an instance of a general type of case known from the history of mathematics.[15] In the relevant kind of case, an individual accidentally discovers a new, abstract subject matter. The individual acquires relevant concepts of that subject matter. The individual is not, however, disposed to engage in some centrally important correct reasoning associated with the respective contents and subject matter. She understands the concepts incompletely, poorly, and is inclined to dismiss her discovery.

The extent to which Cardano's understanding of his own concepts is incomplete must be appreciated. Compare his case to Leibniz and Newton's discovery of the notion of the derivative. Leibniz and Newton were impressively competent with the concept <u>derivative</u>. They could calculate derivatives for a wide range of polynomial, exponential, and trigonometric functions. They were capable of applying the chain-rule and other complex rules for finding derivatives. They had available a geometric notion as a guide towards a fuller explication of <u>derivative</u>: the local rate of change of a function, given by the slope of the tangent. They arrived at an explication of the core of the notion <u>derivative</u> that at least approximates correct explications. Both mathematicians were driven by the goal of providing a full theory of this and adjacent notions.[16]

Cardano's competence is exhausted with a few instances of addition and

[15] There are plausibly other instances of this type of case. For example, Saccheri's development of non-Euclidean geometries – intended as *reductio ad absurdum* proofs of Euclid's fifth postulate. (Burton 1995, Chapter 11).

[16] Cf. Sheldon Smith, "Incomplete Understanding of Concepts: The Case of the Derivative" [*Ms*] for a fascinatingly rich discussion of the history of this case. Cf. also (Burton 1995, 393 & 413ff.; Burge 2012; Burge 1990; 258ff.).

multiplication of complex numbers. Even this competence is flawed. Nevertheless he thinks thoughts with the concepts in question. Cardano's case highlights how minimal an individual's competence with a concept can be without the individual's losing the ability to think thoughts with the concept.

Cardano's case thus invites a general observation. Many *Conceptual Role Theories of Mental Content* claim that mental states have their contents in virtue of the psychological transitions individuals are disposed to engage in. (E.g. Block 1986; Field 1977; Harman 1982; Boghossian 1996; Boghossian 2003; Horwich 1997) The theories at issue are committed to specifying a set of transitions that is both necessary and sufficient for possession of any given concept. This set of transitions must be jointly sufficient to uniquely determine the right referent(s) for the concept. So for specifying the concept's possession conditions, conceptual role theories must rely on a full explication or a definition. I call such conceptual role theories *Strong Conceptual Role Theories*. Such theories of mathematical concepts are open to counterexamples like Cardano's case.

It is a widespread misconception of mathematical discovery that it proceeds by stipulation and definition. The discoverers of many mathematical concepts had incomplete understanding of those concepts. They were not capable of providing a correct or even sophisticated explication of those concepts. They were incapable of making many accurate inferences and judgments involving those concepts. Those individuals were not disposed to engage in the complex transitions specified by a *specific, supposedly constitutive* explication or definition of those concepts. They nevertheless possessed the concepts. It seems likely that it will always be possible to find at least a *hypothetical* individual matching this description.

An individual must have some minimal competence with a concept, in order to think the concept. But there plausibly is no specific set of transitions that an individual has to be competent with. Suppose that Cardano had made inferences involving subtraction and division, instead of addition and multiplication, of complex numbers. These transitions might have been grounded in a competence *equally* sufficient for possession of the concepts of complex numbers.

Further reflection on the nature of mathematical concepts strengthens this general point. Many mathematical concepts have *different* explications, definitions, or elaborations. (Burge 2012, 582ff.) One explication of the concept complex number is the *arithmetical explication* stated above. The history of mathematics witnessed many more explications of this and adjacent concepts.[17] One particularly momentous one was what I will call the *geometrical explication* of complex number.[18] In the centuries following Cardano, complex numbers became more and more accepted as elements in calculations. Mathematicians were still reluctant, however, to accept their status as numbers. Even Euler, in his *Algebra* (1770), talks of them as "numbers, which by their nature are impossible [and] are ordinarily called imaginary, or fanciful numbers because they exist only in the imagination." (Burton 1995, 629)

Fuller understanding of these new numbers was sought in two directions. One was the attempt to provide a new explication of the concept number. Mathematicians tried to conceptualize a wider notion of arithmetic that would encompass the complex numbers. The understanding of (complex) numbers in terms of fields is one outcome of these attempts. Another line of thought tried to validate complex numbers by giving them a geometric interpretation. The Norwegian Caspar David Wessel is credited with first achieving such an interpretation in his *Essai sur la répresentation analytique de la direction* from 1797. Shortly later, in 1806, Jean Robert Argand accomplishes the

[17]E.g. as a field, as pairs of real numbers, as vectors, matrices, quaternions. (Burton 1995). Another important definition of complex numbers is Euler's: $z = r \times e^{i \times \theta}$.

[18]This explication is often called the polar representation of complex numbers. I propose the above term because it highlights the great difference between the approaches leading to the geometrical and the arithmetical explication.

same feat. Both accomplishments were widely ignored. Only Carl Friedrich Gauss's use of the geometrical interpretation in his *Theoria Residuorum Biquadraticum* in 1831 made the idea more widely accepted. There, Gauss replaces the number $a+b\times i$ by the point (a,b). He had already made use of this idea in his dissertation on the fundamental theorem of algebra in 1799.

According to the *geometrical explication*, the number $z=a+b\times i$ is identified with the point (a,b) in the complex plane. Switching to polar coordinates, we can let $a=r\times\cos(\theta)$, and $b=r\times\sin(\theta)$, where θ is the angle of the radius $((0,0)(a,b))$ with the real axis of the complex plane.

Assuming $a+b\times i\neq 0$, we can write $a+b\times i=r\times(\cos(\theta)+i\times\sin(\theta))$. The right-hand side of the equation provides an explication of complex numbers in polar terms:

$$z=r\times(\cos(\theta)+i\times\sin(\theta)) .$$

Alternative explications of a mathematical notion are often equally respectable.

It seems plausible that a different mathematical community, with a history differing from ours, might have first discovered the geometrical explication of complex numbers. That community would have nevertheless acquired concepts of complex numbers. It would have been performing operations on the complex numbers. It might have had a concept <u>number</u> referring to integers, reals, *and* complex numbers.

An individual, call him Wessel, in that different mathematical community should intuitively count as thinking thoughts involving concepts of complex numbers. Suppose Wessel has minimal competence with his geometrical explication of complex numbers. Assume that Wessel is referring to the complex numbers partly in virtue of exercising this competence. In this respect, Wessel's case is similar to Cardano's.

In Tyler Burge's terminology, Wessel, and Cardano would seem to share the *ur-concept(s)* of (certain) complex numbers, while (incompletely) understanding different explications of that ur-concept. (Burge 2012, 582ff. & 588/9) Wessel is competent with the geometrical explication of that concept. Cardano incompletely understands the arithmetical explication of that concept.

A Strong Conceptual Role Theory, insisting on some specific set of necessary and sufficient conditions for possession of these concepts will have to give principled grounds for choosing one, rather than any other, set of transitions. According to the Strong Conceptual Role Theory, Cardano and Wessel do *not* share concepts of complex numbers. It cannot accommodate our intuition that both individuals possess these concepts.

A more plausible theory of concept possession should allow that individuals can think mathematical concepts, even though they have incompletely understood them. And it should allow that individuals can share concepts even though their full, successful explications of these same concepts differ.

6 Anti-Individualism for Mathematical Concepts

The considerations from the foregoing sections together support relying more heavily on *Anti-Individualism* as a framework for investigating the constitution of mental states about abstract subject matters. In its canonical formulation, Anti-Individualism is the claim that “[t]he natures of many mental states constitutively depend on relations between a subject matter beyond the individual and the individual that has the mental states, where relevant relations help determine specific natures of those states.”[19] (Burge 2010, 61)

[19] For a full explanation of Anti-Individualism, cf. (Burge 2010, Chapter 3).

Strong Conceptual Role Theories introduced in the last section are in principle compatible with this anti-individualist claim. Strong Conceptual Role Theories can allow that mental states and events necessary for having the judgmental and inferential dispositions constitutive of concept possession are individuated on the basis of conditions beyond individuals' dispositions. For example, a judgment occurring in an inference constitutive of possessing some concept may require that the individual be related in the right way to her environment.[20]

A fuller application of Anti-Individualism emphasizes the central role relations to a subject matter play in individuating an individual's psychological states and the possession conditions for concepts. It allows that no specific set of dispositions are necessary or sufficient for an individual to possess some concept, if the individual is related to the subject matter in the right way. Such emphasis on relations to a subject matter over the individual's dispositions is *not* available to the Strong Conceptual Role Theories at issue.

For many empirical contents, anti-individualistic arguments have shown that individuals' dispositions do not constitute the contents of their mental states. Instead, a complex set of causal relations between the individual and her social or physical environment does. The content of those states is hence not fully constituted by conceptual role. (e.g. Burge 1979; Burge 2010, 61ff.) [21]

There is no causal connection to abstract subject matters that might constitute contents as in the case of empirical concepts. For this reason, many proponents and opponents of Strong Conceptual Role Theories applied to empirical concepts have accepted conceptual role as constitutive at least for contents of abstract subject matters. (Fodor 1990, 111) At least here, it is often assumed, constitutive inferential

[20]Christopher Peacocke takes his Conceptual Role Theory to be anti-individualist in this way. (Peacocke 1992; Peacocke 2008)

[21]For some examples of the causal relations that figure into the individuation of states with empirical contents, cf.(Burge 2010, 70ff. & 73-82). For perceptual contents, (Burge 2010, 82-108).

roles plausibly uniquely fix thoughts' contents and semantic values.

So the question arises, as to *what* the relevant relations to the subject matter might consist in. Burge proposes the following principle: "For an individual to have any representational state (such as a belief or perception) as of a subject matter, that state must be associated with some veridical representational states that bear referential, indicational, and attributional representational relations to a suitably related subject matter."[22] (Burge 2010, 68)

The content of an individual's states, then, would be (partly) constituted by semantic – not causal – relations to a subject matter. But many philosophers have deemed the existence of just such relations to the subject matter of mathematics mysterious. (Benacerraf 1973) It is mysterious to them because they require that such a relation be reduced to or explained in terms of causal relations. The unavailability of such a reduction or explanation makes Anti-Individualism's constitutive explanation seem problematic to them. Conceptual Role Theories appeal to such philosophers because they promise to explain thought about mathematics in terms of dispositions that *are* reducible to causal relations.

With Frege we may regard such worries as driven by ideology, rather than argument. Frege writes that our knowledge of logical structures depends on a psychological faculty: reason. (Frege 1967; Burge 1992, 301ff., 309ff., 314; Burge 1998) This remark suggests that having a mind, for Frege, is partly constituted by a disposition to acknowledge certain basic truths of logic. A similar rationalist position may be held with respect to mathematical truths. The representational relation between individuals and mathematical entities would thus not be *problematic*, but rather "individuative or *constitutive*" of minds.[23]

[22]For a full explanation of the principle, cf. (Burge 2010, 68-73). For an explanation as to how the two principles cited here apply to mathematical and logical contents, cf. *(*Burge 2010, 71ff., esp. 74).
[23]"Questions of 'access' to [the realm of mathematical entities] are on reflection seen to be misconceived." (Burge 1992, 316)

We can still ask under what circumstances constitutive representational relations to *some specific* abstract subject matter are established. Cardano's 'getting things right' about complex numbers is *explained* in terms of the truth of his thoughts. His thoughts are true because they contain singular constructions referring to the complex numbers. (Quine 1960, §§ 25&50) Why is Cardano described as 'getting things right' about the complex numbers? Answering this general question requires investigating a broader range of cases. But reflection on Cardano suggests some factors that seem relevant.

First, Cardano is engaged in the right kind of activity. Cardano is investigating cubic and biquadratic equations. There is a deep, non-accidental connection between the subject matter of his investigations and the complex numbers.

Second, Cardano is working at a point in history where the discovery of complex numbers was 'in the air.' There is a recognizable pattern of mathematical investigation during Cardano's period that comes close to discovering complex algebra. That mathematicians were approaching complex algebra can be seen in how naturally Bombelli is taking the next step in its development, only several decades after Cardano.

Most importantly, Cardano *does* have some *minimal competence* with the concepts at issue. He does get some things right when using them. Appeal to competence is a central, important explanatory device in psychology. In order to possess *any* concept, plausibly, an individual must be competent in making *some* transitions with that concept. Many different sets of transitions, however, may be disjointly sufficient for possession of a concept. No particular set may be necessary for possession of that concept. (Burge 1998, 318ff.; Burge 2012, 576ff.)

Anti-Individualism about the possession of mathematical concepts does not commit to necessary and sufficient conditions for concept possession. It provides a framework for thinking about the possession and acquisition of such concepts. *Anti-Individualism* offers a constitutive condition: the representational relation between an individual's thoughts and a subject matter. It invites further investigation into the conditions that are sufficient for having certain concepts in specific cases.

Literature

Bagni, G. (2009). Bombelli's Algebra (1572) and a New Mathematical Object. *For the Learning of Mathematics*, 29, 29-31.

Benacerraf, P. (1973). Mathematical Truth. *The Journal of Philosophy,* 70, 661-679.

Block, N. (1986). Advertisement for a Semantics for Psychology. *Midwest Studies in Philosophy* 10, 615-678.

Boghossian, P.A. (1996). Analyticity Reconsidered. *Nous,* 30(3), 360-391.

Boghossian, P.A. (2003). Blind Reasoning. *Proceedings of the Aristotelian Society Suppl. Vol. 77,* 225-248.

Bombelli, R. (1572). *L'Algebra.* Bologna.

Boyer, C.B. (1991). *A History of Mathematics.* New York: Wiley.

Burge, T. (1978). Belief and Synonymy. *The Journal of Philosophy* 75(3), 119-138.

Burge, T. (1979). Individualism and the Mental. In T. Burge (2007), *Foundations of Mind* (pp. 100-151), Oxford: Oxford University Press.

Burge, T. (1990). Frege on Sense and Linguistic Meaning. In T. Burge (2005), *Truth, Thought, Reason* (pp. 242-270), Oxford: Oxford University Press.

Burge, T. (1992). Frege on Knowing the Third Realm. In T. Burge (2005), *Truth, Thought, Reason* (pp. 299-316), Oxford: Oxford University Press.

Burge, T. (1993). Concepts, Definitions, and Meaning. In T. Burge (2007), *Foundations of Mind* (pp. 291-306). Oxford: Oxford University Press

Burge, T. (1998). Frege on Knowing the Foundation. In T. Burge (2005), *Truth, Thought, Reason* (pp. 317-355), Oxford: Oxford University Press.

Burge, T. (2010). *Origins of Objectivity.* Oxford: Oxford University Press.

Burge, T. (2012). Living Wages of Sinn. In T. Burge (2013), *Cognition Through Understanding,* Oxford: Oxford University Press.

Burton, D.M. (1995). *The History of Mathematics: An Introduction, 6th Edition 2005.* New York: McGraw-Hill

Cardano, C. (1968). *The Great Art or The Rules of Algebra, translated by T.R. Witmer*. Cambridge MA: MIT Press

Carey, S. (2009). *Origins of Concepts.* Oxford: Oxford University Press.

Crossley, J.N. (1987). *The Emergence of Number.* Singapore: World Scientific.

Crowe, M. (1975). Ten 'Laws' Concerning Patterns of Change in the History of Mathematics. In Gillies, D. (ed.) (1995). *Revolutions in Mathematics*. Oxford: Oxford University Press

Dauben, J. (1984). Conceptual Revolutions and the History of Mathematics: Two Studies in the Growth of Knowledge. In Gillies, D. (ed.) (1995). *Revolutions in Mathematics*. Oxford: Oxford University Press

Ebbinghaus, H.-D. (Ed.) (1991). *Numbers*, New York: Springer.

Fauvel J. & Gray J. (Ed.) (1987). *The History of Mathematics, A Reader.* London: Macmillan Press and Open University

Feferman, S. (1989). *The Number Systems. Foundations of Algebra and Analysis.* New York: Chelsea

Publishing
Field, H. (1977). Probabilistic Semantics. *Journal of Philosophy*, 74, 379-409.
Fodor, J. (1990). A Theory of Content I & II. In J. Fodor, *A Theory of Content and other Essays* (pp. 51-136, Cambridge MA: MIT.
Frege, G. (1967). *The Basic Laws of Arithmetic, trans. and ed. M. Furth*. Berkeley: University of California Press
Friedman, M. (2002). Kant, Kuhn, and the Rationality of Science. *Philosophy of Science* 69; 171-190
Gillies, D. (ed.) (1995). *Revolutions in Mathematics*. Oxford: Oxford University Press
Harman, G. (1982). Conceptual Role Semantics. *Notre Dame J. Formal Logic* 23, 242-256.
Hodgkin, L. (2005). *A History of Mathematics. From Mesopotamia to Modernity*. Oxford: Oxford University Press
Hofmann, J.E. (1972). Bombelli's Algebra – eine genialische Einzelleistung und ihre Einwirkung auf Leibniz. *Studia Leibnitiana, IV 3/4*, 198-252
Horwich, P. (1997). Implicit Definition, Analytic Truth, and Apriori Knowledge. *Nous* 31(4), 423-440
Jayawardene, S.A. (1973). The Influence of Practical Arithmetics on the Algebra of Rafael Bombelli. *Iris* (64)4, 510-523
Kenney, E. (1989). Cardano: Arithmetic Subtlety and Impossible Solutions. *Philosophia Mathematica*, (II)4, 195-216.
Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times Volume I.* New York: Oxford University Press.
Kripke, S. (1980). *Naming and Necessity*. Cambridge, MA: Harvard University Press
Maracchia, S. (2003). Algebra e Geometria in Cardano. In M. Baldi & G. Canziani (Eds.), *Cardano e la tradizione dei Saperi*, Milan: Angeli.
Oystein, O. (1953). *Cardano, the Gambling Scholar.* Princeton NJ: Princeton University Press.
Peacocke, C. (1987). Understanding Logical Constants: A Realist's Account. *Proceedings of the British Academy*, 73, 153-200.
Peacocke, C. (1992). *A Study of Concepts.* Cambridge, MA: MIT Press.
Peacocke, C. (1998a). The Concept of a Natural Number. *Australasian Journal of Philosophy*, 76, 105-109.
Peacocke, C. (1998b). Implicit Conceptions, the A priori, and the Identity of Concepts. In E. Villanueva (Ed.), *Concepts, Philosophical Issues 9,* Atascadero, CA: Ridgeview.
Peacocke, C. (2003). Implicit Conceptions, Understanding, and Rationality. In M. Hahn & B. Ramberg (Eds.), *Reflections and Replies: Essays on the Philosophy of Tyler Burge*, Cambridge MA: MIT Press.
Peacocke, C. (2008). *Truly Understood*, Oxford: Oxford University Press.
Putnam, H. (1970). Is Semantics Possible? In H. Putnam, *Philosophical Papers Vol. ii*, Cambridge: Cambridge University Press (1975)
Putnam, H. (1973). Explanation and Reference. In H. Putnam, *Philosophical Papers Vol. ii*, Cambridge: Cambridge University Press (1975)
Putnam, H. (1975). The Meaning of 'Meaning'.In H. Putnam, *Philosophical Papers Vol. ii*, Cambridge: Cambridge University Press (1975)
Quine, W.V.O. (1960). *Word and Object*, Cambridge MA, MIT Press.
Stanovich, K.E. & West, R.F. (2000). Individual Differences in Reasoning: Implications for the Rationality-Debate. *Behavioral and Brain Sciences* 23, 645-726.
Stillwell, J. (2001). *Mathematics and its History*, New York: Springer.
Wagner, R. (2010). The Natures of Numbers in and around Bombelli's L'Algebra. *Arch.Hist.Exact Sci* 64, 485-523.
Williamson, T. (2003). Understanding and Inference. *The Aristotelian Society* 77, 249-293.
Williamson, T. (2006). Conceptual Truth. *The Aristotelian Society Supplement* 80(1), 1-41.
Wilson, M. (2007). *Wandering Significance.* Oxford: Oxford University Press.