Logica Universalis

Two Semantical Approaches to Paraconsistent Modalities

Juliana Bueno-Soler

Abstract. In this paper we extend the *anodic systems* introduced in Bueno-Soler (J Appl Non Class Logics 19(3):291–310, 2009) by adding certain paraconsistent axioms based on the so called *logics of formal inconsistency*, introduced in Carnielli et al. (Handbook of philosophical logic, Springer, Amsterdam, 2007), and define the classes of systems that we call *cathodic*. These classes consist of modal paraconsistent systems, an approach which permits us to treat with certain kinds of conflicting situations. Our interest in this paper is to show that such systems can be semantically characterized in two different ways: by *Kripke-style semantics* and by *modal possible-translations semantics*. Such results are inspired in some universal constructions in logic, in the sense that cathodic systems can be seen as a kind of fusion (a particular case of fibring) between modal logics and non-modal logics, as discussed in Carnielli et al. (Analysis and synthesis of logics, Springer, Amsterdam, 2007). The outcome is inherently within the spirit of universal logic, as our systems semantically intermingles modal logics, paraconsistent logics and many-valued logics, defining new blends of logics whose relevance we intend to show.

Mathematics Subject Classification (2000). Primary 03B45, 03B53; Secondary 03B50, 03B62.

Keywords. Modal logics, paraconsistent logics, completeness, Kripke semantics, possible-translations semantics.

1. Introduction

The alliance between paraconsistent logics and modal logics is not new. Indeed, to deal with contradictions that occur in certain discourses, S. Jaśkowski already in 1948 used a sort of modal environment to try to explain contradictions in dialogues by means of his discussive (or discursive) logics: the well-known system **D2** of Jaśkowski is, in fact, a legitimate Logic of Formal Inconsistency (LFI), as shown in [7] (example 93). However, it was only in 1986 that the first modal paraconsistent system was proposed in [13], with the

aim to deal with deontic paradoxes. That system was a modal extension of da Costa's C_1.

This approach has being extended by means of deontic modalities combined with LFIs, as developed in [10] and [12]. But the potentiality in combining paraconsistency and modalities goes far beyond deontic conundrums or norms: not only some problems described in [15] can be thought in paraconsistent terms, but also certain problems and paradoxes in epistemic and doxastic logics gain a new insight when regarded from the paraconsistent perspective. For instance, Fitch's paradoxes of knowability are treated in terms of paraconsistent epistemic logics in [8]; while other oddities of doxastic logics that can in principle be treated in paraconsistent terms are explained in [11], chapter 7.

The present paper introduces the classes $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ of modal paraconsistent systems, called here *cathodic logics*, by extending the respective propositional paraconsistent systems $\mathbf{PI}$, $\mathbf{mbC}$, $\mathbf{bC}$ and $\mathbf{Ci}$ through certain basic modal axioms plus the axiom schema $\mathbf{G}^{k,l,m,n}$ of Lemmon and Scott (see [16]). In this way one can obtain *modal paraconsistent fragments* of familiar modal systems as $\mathbf{T}$, $\mathbf{D}$, $\mathbf{S4}$, $\mathbf{S5}$ and so on, from the classes $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$. The careful construction of these systems permit us to maintain the desired features of the paraconsistent systems, as well as the character of the modal systems that are being combined; this kind of procedure is relevant to understand universal constructions used to combine logics.

Finally, we systematize the construction of classes of cathodic systems (which we call cathodic classes) and show that these classes can be semantically characterized in two different ways: by means of Kripke-style semantics, and also by means of modal possible-translations semantics.

The paper is divided into four sections (besides Introduction): Sect. 2 presents the basic notions on cathodic systems, while Sect. 3 defines the cathodic classes $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$. Section 4 discusses the semantic characterization of the classes of cathodic systems with respect to Kripke-style semantics and Sect. 5 introduces a new semantics to modal paraconsistent systems based on possible-translations semantics, also showing that the cathodic systems are also characterized with respect to modal possible-translations semantics.

2. Preliminaries

A *propositional language* for a system $\mathbf{S}$ is composed by a denumerable set Var of sentential variables $p, q, r, \ldots$ and operators in the set $\Sigma = \{\supset, \wedge, \square, \lozenge, \neg, \circ\}$. The special connective $\circ$ plays a crucial role in our systems, as it expresses the notion of consistency of a formula in the object-language level (more details in [7]). All of our systems contain $\circ$ in the language, with the exception of the ones in $\mathbf{PI}^{k,l,m,n}$.

The collection of *formulas* or *sentences* of $\mathbf{S}$ is obtained from propositional variables, as usual in modal logic, and is denoted by For. The elements

of For are represented by lowercase Greek letters α, β, γ, and subsets of For are represented by uppercase Greek letters Γ, Δ, Π. When necessary, the collection of sentences will be denoted by $For_{\mathbf{S}}$ instead of For only. Consider $\Gamma \cup \{\alpha\} \subseteq For$ and let $\vdash \subseteq \wp(For) \times For$ be a *consequence relation*, where $\wp(For)$ is the power set of the set For. A consequence relation is taken to be *tarskian*, *finitary* and *structural* as in [3], which is granted by the axioms, rules, and the standard definition of proofs and derivations. In this way, a logical system $\mathbf{S}$ is a pair $\langle For, \vdash \rangle$ governed by certain axioms and rules. When necessary, we shall write $\vdash_{\mathbf{S}}$ to denote a consequence relation in a specific system $\mathbf{S}$.

We use the notation $\alpha(p)$ to indicate that p occurs in α, and $\alpha[p/\beta]$ to express the substitution in α of each occurrence of p by β. Consider the following stock of propositional axioms and rules:

(**A1**)	$p \supset (q \supset p)$
(**A2**)	$(p \supset q) \supset [(p \supset (q \supset r)) \supset (p \supset r)]$
(**A3**)	$(p \supset r) \supset [((p \supset q) \supset r) \supset r]$
(**A4**)	$p \supset [q \supset (p \wedge q)]$
(**A5**)	$(p \wedge q) \supset p$
(**A6**)	$(p \wedge q) \supset q$
(**PI**)	$(p \vee \neg p)$
(**mbC**)	$\circ p \supset [p \supset (\neg p \supset q)]$
(**bC**)	$\neg\neg p \supset p$
(**Ci**)	$\neg \circ p \supset (p \wedge \neg p)$
(**MP**)	$\alpha, \; \alpha \supset \beta$ implies β
(**US**)	$\vdash \alpha(p)$ implies $\vdash \alpha[p/\beta]$

The system $\mathbf{PC}^{\supset}$ is a fragment of the propositional calculus $\mathbf{PC}$, introduced by L. Henkin in [14], and is composed by the axioms (**A1**)–(**A3**) and by the rule (**MP**). The disjunction connective is defined as:

$$(\alpha \vee \beta) \overset{\text{Def}}{=} (\alpha \supset \beta) \supset \beta$$

From such definition we can obtain the expected propositional properties of disjunction, such as: *expansion*, *commutativity*, *associativity*, *Dummett's law* and *proof by cases*. The completeness of $\mathbf{PC}^{\supset}$ was shown in [14] with respect to classical valuations for implication.

The system $\mathbf{PC}^{\supset,\wedge}$ is obtained from $\mathbf{PC}^{\supset}$ by adding (**A4**)–(**A7**) and (**US**). This new fragment can also be characterized by extending Henkin's results, adding to the proofs the cases involving conjunction. The main useful results valid in $\mathbf{PC}^{\supset,\wedge}$ are the Deduction Theorem and the *distributivity* of $\vee$ over $\wedge$. All details about $\mathbf{PC}^{\supset,\wedge}$ appear in [5].

From $\mathbf{PC}^{\supset,\wedge}$ it is possible to obtain several well-known paraconsistent systems: the system $\mathbf{PI}$,[1] introduced by Batens in [1], can be obtained from $\mathbf{PC}^{\supset,\wedge}$ by adding (**PI**); the system $\mathbf{mbC}$, introduced by Carnielli and Marcos in [9], is obtained from $\mathbf{PI}$ by adding (**mbC**); the system $\mathbf{bC}$ is obtained from

---

[1] The version of $\mathbf{PI}$ used here is an equivalent version of the original $\mathbf{PI}$ in [1]; the other paraconsistent systems in this paper are also equivalent versions of LFIs in [7].

mbC by adding (**bC**); and the system **Ci** is obtained from **mbC** by adding (**Ci**). All these paraconsistent systems were introduced in [9].

PI is the only system treated here which fails to be an LFI, since its language does not contain the consistency operator ∘, as discussed in [7]. It is known that from **mbC** a form of classical negation can be defined, commonly called *strong negation*, defined as:

$$\sim \alpha \overset{\text{Def}}{=} \alpha \supset [p \wedge (\neg p \wedge \circ p)]$$

From this definition all the relevant properties of classical negation are derivable, what is useful to show several natural results about cathodic systems involving classical negation.

Let $v : For \longrightarrow \{0,1\}$ be a collection of *bi-valuation functions*, where 1 denotes the "true" value and 0 denotes the "false" value, defined as follows:

$(Biv.1)$	$p \in Var$	implies	$v(p) = 1$ or $v(p) = 0$;
$(Biv.2)$	$v(\alpha \supset \beta) = 1$	iff	$v(\alpha) = 0$ or $v(\beta) = 1$;
$(Biv.3)$	$v(\alpha \wedge \beta) = 1$	iff	$v(\alpha) = 1$ and $v(\beta) = 1$;
$(Biv.4)$	$v(\alpha) = 0$	implies	$v(\neg\alpha) = 1$;
$(Biv.5)$	$v(\circ\alpha) = 1$	implies	$v(\alpha) = 0$ or $v(\neg\alpha) = 0$;
$(Biv.6)$	$v(\neg\neg\alpha) = 1$	implies	$v(\alpha) = 1$;
$(Biv.7)$	$v(\neg \circ \alpha) = 1$	implies	$v(\alpha) = 1$ and $v(\neg\alpha) = 1$.

A **PI**-*valuation* is a bi-valuation function subject to the clauses (Biv1)–(Biv4); analogously an **mbC**-*valuation*, a **bC**-*valuation* and a **Ci**-*valuation* is a bi-valuation function subject, respectively, to clauses (Biv1)–(Biv5), (Biv1)–(Biv6) and (Biv1)–(Biv7).

Such conditions on valuations permit us to give a completeness result w.r.t. bi-valuations for each paraconsistent system referred above (details can be found in [7]). We denote the bi-valuation semantics by `Biv`, for short.

Another semantic characterization for the paraconsistent systems **PI, mbC, bC** and **Ci** can be attained w.r.t. possible-translations semantics, as discussed in Sect. 5.1.

The systems **PI, mbC, bC** and **Ci** will be used to define the classes of cathodic systems, as shown in the next section.

3. Cathodic Modalities

From the viewpoint of combination of logics, the cathodic systems could be seen as a result of fusion (a particular case of fibring) between modal logics and non-modal logics. Several results about preservation of completeness in fibring have been obtained (see [6]), but in all cases the classical negation is involved, instead of a paraconsistent negation. As the method treated here to obtain cathodic systems involve weak (paraconsistent) negation, such preservation results cannot be directly applied, which justifies the approach developed here.

Each cathodic system can be obtained by extending an element of the anodic class $\mathbf{K}^{k,l,m,n}$ by expanding the language (adding $\neg$ and $\circ$) and by

adding specific paraconsistent axioms, as done in [5]. For simplicity we start from paraconsistent systems and define cathodic systems more directly. It is not difficult to see that both presentations are equivalent.[2]

The *minimal normal*[3] cathodic modal systems $\mathbf{PI}^{\Box,\Diamond}$, $\mathbf{mbC}^\Box$, $\mathbf{bC}^\Box$ and $\mathbf{Ci}^\Box$ are obtained, respectively, from $\mathbf{PI}$, $\mathbf{mbC}$, $\mathbf{bC}$ and $\mathbf{Ci}$ by adding the following modal axioms and modal rule:

(**K**) $\Box(p \supset q) \supset (\Box p \supset \Box q)$
(**K1**) $\Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$
(**K2**) $\Diamond(p \vee q) \supset \Diamond p \vee \Diamond q$
(**K3**) $(\Diamond p \supset \Box q) \supset \Box(p \supset q)$
(**Nec**) $\vdash \alpha$ implies $\vdash \Box\alpha$

It is to be remarked that the Deduction Theorem holds for all cathodic systems; the proof is virtually the same as for the classical modal logics, as cathodic systems do not require any new rules other than (**MP**), (**US**) and (**Nec**).

Since from the system $\mathbf{mbC}$ it is possible to define a classical negation $\sim$, then the possibility operator $\Diamond$ can be defined from the necessity operator $\Box$ as usual in modal logic.

$$\Diamond\alpha \stackrel{\mathrm{Def}}{=} \sim\Box\sim\alpha$$

In this way, the axioms (**K1**)–(**K3**) are innocuous in $\mathbf{mbC}^\Box$, $\mathbf{bC}^\Box$ and $\mathbf{Ci}^\Box$, as the reader can verify. Thus, only the system $\mathbf{PI}^{\Box,\Diamond}$ is indeed a bi-modal system.

The main interest in this paper is to consider classes of cathodic systems, and the specific modal axiom schema $\Diamond^k\Box^l\alpha \supset \Box^m\Diamond^n\alpha$, introduced by Lemmon and Scott in [16], and denoted by $\mathbf{G}^{k,l,m,n}$, will be used. The systems $\mathbf{PI}^{\Box,\Diamond}$, $\mathbf{mbC}^\Box$, $\mathbf{bC}^\Box$ and $\mathbf{Ci}^\Box$ will be extended with $\mathbf{G}^{k,l,m,n}$ and the classes $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ will be defined.

Observe that when the indexes k, l, m, n in $\mathbf{G}^{k,l,m,n}$ are zero, we have the systems $\mathbf{PI}^{\Box,\Diamond}$, $\mathbf{mbC}^\Box$, $\mathbf{bC}^\Box$ and $\mathbf{Ci}^\Box$ as particular cases.

As $\mathbf{PI}^{\Box,\Diamond}$ is a bi-modal system, then the class $\mathbf{PI}^{k,l,m,n}$ requires also the dual instance of the axiom schema $\mathbf{G}^{k,l,m,n}$, i.e., the axiom $\Diamond^m\Box^n\alpha \supset \Box^k\Diamond^l\alpha$, denoted by $\mathbf{G}^{m,n,k,l}$, because in $\mathbf{PI}$ no form of classical negation can be defined.

One of the interests of cathodic systems is the potentiality in avoiding some modal paradoxes, as for instance Urmson's paradox. Consider the following sentence:

---

[2] A different approach was taken in [5], where the cathodic systems are defined from anodic ones, by adding the axioms:

(**BP1**) $\Box\sim\alpha \supset \sim\Diamond\alpha$
(**BP2**) $\sim\Diamond\alpha \supset \Box\sim\alpha$

From this it is possible to show that $\Box\alpha$ is equivalent to $\sim\Diamond\sim\alpha$.

[3] A modal system $\mathbf{S}$ is classified as *normal* if it contains the Distribution Axiom (**K**) and the Necessitation Rule (**Nec**) among its axioms and rules, and as *minimal* if it has only (**K**) as a modal axiom and only (**Nec**) as a modal rule.

"It is optional that you attend or not to my talk, but your choice is not indifferent".

It is clear that the notions "optional" (Opt) and "indifferent" (Ind) are distinct in such sentence. If we interpret $\Diamond$ as "permitted" and $\Box$ as "obligatory", then it is natural in modal logic to formalize Opt and Ind as:

$$Opt(q) = \Diamond q \wedge \Diamond \sim q$$
$$Ind(q) = \sim\Box q \wedge \sim\Box \sim q$$

In the standard modal systems a contradiction occurs because $\sim$ is a classical negation and $\Box\alpha$ is equivalent to $\sim\Diamond\sim\alpha$. Consequently, Opt and Ind are equivalent, which entails the paradox.

As the cathodic modalities have degrees of negation, this leads to a powerful expressivity gain in the language, permitting to express some aspects of common sense reasoning. Of course we can use *weak negation* ($\neg_w$) to express the notions Opt and Ind. In this no paradox occurs because

$$\Box\alpha \neq \neg_w \Diamond \neg_w \alpha$$

In the study of cathodic systems the alternative characterization, obtained w.r.t. *modal possible-translations semantics* (see Sect. 5), permits us to give an interesting interpretation to certain philosophical problems, as for example to the theory of "impossible worlds" of [18], which seems to be new and promising.

4. Kripke-Style Semantic Completeness

In this section the main points behind the completeness w.r.t. Kripke semantics for cathodic systems are underlined. The formal treatment for the systems in the class $\mathbf{PI}^{k,l,m,n}$, in particular, is similar to the anodic case, as in [4], with some modifications concerning negation. The other cases, namely $\mathbf{mbC}^{k,l,m,n}, \mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ will be treated in more details.

We will concentrate here on the effects of negation on the completeness proof. Since negation in the class $\mathbf{PI}^{k,l,m,n}$ is very weak, it has to be treated in a somewhat particular way. On the other hand, the classes $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ can be dealt with almost simultaneously, due to the possibility of defining a form of classical negation within them.

One of our aims is also to obtain a semantic characterization by means of modal possible-translations semantics (which will be done in Sect. 5.2). We reserve the notation $\mathcal{L}$ to denote any paraconsistent system among $\mathbf{PI}, \mathbf{mbC}$, $\mathbf{bC}$ and $\mathbf{Ci}$, and $\mathcal{L}^{k,l,m,n}$ to denote any cathodic class among $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$.

A *cathodic frame* is a relational structure $\mathfrak{F} = \langle W, R \rangle$, where $W \neq \varnothing$ is a universe and R is a binary relation on W.

Since each class of cathodic systems $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}, \mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ contains the axiom schema $\mathbf{G}^{k,l,m,n}$, the accessibility relations in the

respective frame for each class of systems will satisfy the condition $\mathbf{C}^{k,l,m,n}$ (as for classical modal logics), which is expressed in first-order logic by:

$$\forall w_1 \forall w_2 \forall w_3 ((w_1 R^k w_2 \wedge w_1 R^m w_3) \supset \exists w_4 (w_2 R^l w_4 \wedge w_3 R^n w_4))$$

where wR^0w' means that $w = w'$. The condition $\mathbf{C}^{k,l,m,n}$ specifies r-step accessibility relations between worlds w and w' for $r \in \{k,l,m,n\}$. Note that the condition $\mathbf{C}^{0,0,0,0}$ imposes no restriction on the accessibility relations.

The following definition specifies what is a bi-valued relational model (or bi-valued Kripke model) for the cathodic systems $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$.

Definition 4.1. A *bi-valued relational model* for a cathodic system $\mathcal{L}^{k,l,m,n}$ is a pair $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}} = \langle \mathfrak{F}, v \rangle$ where $\mathfrak{F}$ is a frame and $v : Var \times W \longrightarrow \{0,1\}$ is a two-valued modal assignment satisfying the conditions:

- For $\mathcal{L}^{k,l,m,n} = \mathbf{PI}^{k,l,m,n}$:
    - (i)   v(p,w)=1 or v(p,w)=0;
    - (ii) $v(\alpha \supset \beta, w) = 1$ iff $v(\alpha, w) = 0$ or $v(\beta, w) = 1$;
    - (iii) $v(\alpha \wedge \beta, w) = 1$ iff $v(\alpha, w) = 1$ and $v(\beta, w) = 1$;
    - (iv) $v(\alpha, w) = 0$ implies $v(\neg \alpha, w) = 1$;
    - (v) $v(\Box \alpha, w) = 1$ iff $v(\alpha, w') = 1$, $\forall w'(wRw')$;
    - (vi) $v(\Diamond \alpha, w) = 1$ iff $v(\alpha, w') = 1$, $\exists w'(wRw')$.
- For $\mathcal{L}^{k,l,m,n} = \mathbf{mbC}^{k,l,m,n}$ add:
    - (vii) $v(\circ \alpha, w) = 1$ implies $v(\alpha, w) = 0$ or $v(\neg \alpha, w) = 0$;
- For $\mathcal{L}^{k,l,m,n} = \mathbf{bC}^{k,l,m,n}$ add:
    - (viii) $v(\neg\neg \alpha, w) = 1$ implies $v(\alpha, w) = 1$.
- For $\mathcal{L}^{k,l,m,n} = \mathbf{Ci}^{k,l,m,n}$ add:
    - (ix) $v(\neg \circ \alpha, w) = 1$ implies $v(\alpha, w) = 1$ and $v(\neg \alpha, w) = 1$.

When there is no need to specify a particular class, we shall write simply $\mathfrak{M}_{\mathtt{Biv}}$ instead of $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}}$.

Note that the sentence $\Box^2 \alpha$ is an abbreviation for $\Box\Box\alpha$. By an iterated process, we have that, in a bi-valued relational model, a valuation v applied to sentences of the form $\Box^r \alpha$ and $\Diamond^r \alpha$ is expressed in the following way:

- $v(\Box^r \alpha, w) = 1$ iff $v(\alpha, w') = 1$ for all $w' \in W$ such that wR^rw'.
- $v(\Diamond^r \alpha, w) = 1$ iff $v(\alpha, w') = 1$ for some $w' \in W$ such that wR^rw'.

A sentence α is said to be *satisfied in a bi-valued relational model* $\mathfrak{M}_{\mathtt{Biv}}$, if there is a $w \in W$ such that $v(\alpha, w) = 1$ (notation: $\mathfrak{M}_{\mathtt{Biv}}, w \vDash \alpha$). A sentence α is said to be *valid in a bi-valued relational model* $\mathfrak{M}_{\mathtt{Biv}}$, if $v(\alpha, w) = 1$ for all $w \in W$ (notation: $\mathfrak{M}_{\mathtt{Biv}} \vDash \alpha$). A sentence α is said to be *valid on a frame* $\mathfrak{F}$, if α is valid in all bi-valued relational models based on $\mathfrak{F}$ (notation: $\mathfrak{F} \vDash \alpha$). $\mathfrak{F}$ is said to be a *frame for* an arbitrary system $\mathbf{S}$ if every theorem of $\mathbf{S}$ is valid on $\mathfrak{F}$.

All these definitions will be opportunely extended to three-valued relational models (cf. Definition 5.4).

A sentence α is a *semantical consequence* of Γ with respect to a class $\mathcal{F}$ of frames if $\mathfrak{F} \vDash \Gamma$ implies $\mathfrak{F} \vDash \alpha$, for each $\mathfrak{F} \in \mathcal{F}$, where $\mathfrak{F} \vDash \Gamma$ means that $\mathfrak{F} \vDash \gamma$ for all $\gamma \in \Gamma$.

Theorem 4.2. (Soundness for $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$) *Each theorem of $\mathcal{L}^{k,l,m,n}$ is valid in the class of frames $\mathfrak{F} = \langle W, R \rangle$, where R satisfies the condition $\mathbf{C}^{k,l,m,n}$.*

Proof. It is sufficient to check that all axioms are sound, and that all rules preserve validity. For the positive and modal axioms and rules the argument is routine. For the specific axioms, the argument is as follows:

For the axiom ($\mathbf{PI}$) suppose, by *Reductio*, that there exists a bi-valued relational model $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}}$ based on $\mathfrak{F}$ such that $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}} \nvDash p \vee \neg p$. By using the definition of $\vee$ and definition of validity, there exists $w \in W$ such that $v((p \supset \neg p) \supset \neg p, w) = 0$. By Definition 4.1 (ii) we have that: $v(p \supset \neg p, w) = 1$ and $v(\neg p, w) = 0$ so, there are two possibilities:

1. $v(p, w) = 0$ and $v(\neg p, w) = 0$. Since $v(p, w) = 0$ then, by Definition 4.1 (iv), follows that $v(\neg p, w) = 1$. Absurd.
2. $v(\neg p, w) = 1$ and $v(\neg p, w) = 0$. Absurd.

Therefore, $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}} \vDash p \vee \neg p$ for any model $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}^{k,l,m,n}}$ based on $\mathfrak{F} \in \mathcal{F}$.

For the axiom ($\mathbf{mbC}$) the argument is analogous, by *Reductio*, by using Definition 4.1 (vii); for the axiom ($\mathbf{bC}$) use Definition 4.1 (viii) and for the axiom ($\mathbf{Ci}$) use Definition 4.1 (ix). $\square$

Given a system $\mathbf{S}$, a set Δ of sentences is said to be *non-trivial* if $\Delta \nvdash_{\mathbf{S}} \alpha$ for some $\alpha \in For_{\mathbf{S}}$; otherwise, Δ is *trivial* (w.r.t. $\mathbf{S}$).

To deal with cathodic systems, the usual notion of a saturated set (or maximal non-trivial set with respect to a given sentence) is generalized in the sense that the saturation is defined with respect to a collection of sentences instead of a single sentence. This is adequate to handle the so called prime theories (see definition below), used to deal with the specific class $\mathbf{PI}^{k,l,m,n}$.

Definition 4.3. Let $\mathbf{S}$ be a system and Δ and Λ be non-trivial subsets of For such that $\Delta \cap \Lambda = \varnothing$. Δ is *non-trivial Λ-maximal* if:

(i) $\Delta \nvdash \lambda$, for all $\lambda \in \Lambda$;
(ii) For each $\alpha \in For$ such that $\alpha \notin \Delta, \Delta \cup \{\alpha\} \vdash \lambda$, for some $\lambda \in \Lambda$.

If the set Λ is not specified, the non-trivial Λ-maximal set Δ will be referred to as a maximal non-trivial set, for short.

A discussion about saturated sets versus maximal non-trivial sets in abstract terms (for classical logic) is given in [2]. The author shows that such notions are equivalent in classical logic and shows that the semantics of saturated sets are minimal. In our case such notions are not coincident, and this permits us to deal with systems with weak negations (and even with without negation, see [4]) in a very natural way.

Let Δ be a set of $\mathbf{S}$-sentences; Δ is called an $\mathbf{S}$-*theory* if it satisfies: $\Delta \vdash \delta$ implies $\delta \in \Delta$; an $\mathbf{S}$-*theory* Δ is called *prime* if it is non-trivial and satisfies: $\Delta \vdash \alpha \vee \beta$ implies $\Delta \vdash \alpha$ or $\Delta \vdash \beta$.

Lemma 4.4. *Let Δ be a non-trivial Λ-maximal set. If Λ is a singleton, then Δ is a prime set.*

Proof. A simple argument by *Reductio.*   □

Now, consider the definition of the following particular sets: $Den(\Delta)$ (the *denecessitation set* of Δ) and $Pos(\Delta)$ (the *possibilitation set* of Δ) defined as:

$$Den(\Delta) = \{\alpha : \Box\alpha \in \Delta\} \quad \text{and} \quad Pos(\Delta) = \{\Diamond\alpha : \alpha \in \Delta\}$$

Fact 4.5. *If Δ is a $\mathcal{L}^{k,l,m,n}$-theory, then $Den(\Delta)$ is also an $\mathcal{L}^{k,l,m,n}$-theory.*

Proof. Suppose $Den(\Delta) \vdash \alpha$, then there exists a set $\{\beta_1, \ldots, \beta_n\} \subseteq Den(\Delta)$ such that $\{\beta_1, \ldots, \beta_n\} \vdash \alpha$ so, by the Deduction Theorem, it follows that $\vdash \beta_1 \supset (\beta_2 \supset \ldots (\beta_n \supset \alpha)\ldots)$. By (**Nec**), (**K**) and (**MP**) n-times it follows that $\vdash \Box\beta_1 \supset (\Box\beta_2 \supset \ldots (\Box\beta_n \supset \Box\alpha)\ldots)$, and so $\Delta \vdash \Box\beta_1 \supset (\Box\beta_2 \supset \ldots (\Box\beta_n \supset \Box\alpha)\ldots)$. On the other hand, each $\beta_i \in Den(\Delta)$, for $1 \le i \le n$, is such that $\Box\beta_i \in \Delta$. Hence, applying (**MP**) n-times it follows that $\Delta \vdash \Box\alpha$. As Δ is an $\mathcal{L}^{k,l,m,n}$-theory, then $\Box\alpha \in \Delta$. Therefore, $\alpha \in Den(\Delta)$.   □

Definition 4.6. The *cathodic canonical frame* for $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ is the pair $\widehat{\mathfrak{F}} = \langle \widehat{W}, \widehat{R} \rangle$ where the universe $\widehat{W}$ is formed by the Λ-maximal non-trivial sets in $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$, for all $\Lambda \subset For$, and for $\Delta, \Delta' \in \widehat{W}$, $\Delta\widehat{R}\Delta'$ iff $Den(\Delta) \subseteq \Delta'$.

The canonical frame for the class $\mathbf{PI}^{k,l,m,n}$ is more subtle and is defined separately (in the end of the section).

Definition 4.7. The *cathodic canonical model* $\widehat{\mathfrak{M}}$ for $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ based on the cathodic canonical frame $\widehat{\mathfrak{F}}$ is the pair $\langle \widehat{\mathfrak{F}}, \widehat{V} \rangle$ where $\widehat{V}$ is the collection of all modal valuations such that each $v_\Delta \in \widehat{V}$ is defined from some $\Delta \in \widehat{W}$ as:

$$v_\Delta(p) = \begin{cases} 1 & \text{if } p \in \Delta \\ 0 & \text{if } p \notin \Delta \end{cases}$$

The following properties of maximal non-trivial prime sets hold in the cathodic systems.

Lemma 4.8. *Let $\mathcal{L}^{k,l,m,n} = \langle For, \vdash \rangle$ be a cathodic system, and $\Delta, \Lambda \in For$. If Δ is a non-trivial Λ-maximal set of sentences in $\mathcal{L}^{k,l,m,n}$, then:*

- *For $\mathcal{L}^{k,l,m,n} = \mathbf{mbC}^{k,l,m,n}$.*
  - (i)   *Δ is a $\mathcal{L}^{k,l,m,n}$-theory;*
  - (ii)   *If $\alpha \in \Delta$ and $\alpha \supset \beta \in \Delta$, then $\beta \in \Delta$;*
  - (iii)   *If Δ is prime: $\alpha \supset \beta \in \Delta$ iff $\alpha \notin \Delta$ or $\beta \in \Delta$;*
  - (iv)   *If Δ is prime, then for each $\alpha \in For$ and each $\lambda \in \Lambda$, we have that $\alpha \in \Delta$ or $\alpha \supset \lambda \in \Delta$, but not both;*
  - (v)   *$\alpha \wedge \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$;*
  - (vi)   *If Δ is prime, then: $\alpha \notin \Delta$ implies $\neg\alpha \in \Delta$;*
  - (vii)   *$\circ\alpha \in \Delta$ implies $\alpha \notin \Delta$ or $\neg\alpha \notin \Delta$;*

- *For $\mathcal{L}^{k,l,m,n} = \mathbf{bC}^{k,l,m,n}$ add*:
  (viii) $\neg\neg\alpha \in \Delta$ *implies* $\alpha \in \Delta$;
- *For $\mathcal{L}^{k,l,m,n} = \mathbf{Ci}^{k,l,m,n}$ add*:
  (ix) $\neg \circ \alpha \in \Delta$ *implies* $\alpha \in \Delta$ *and* $\neg\alpha \in \Delta$.

Proof. (i) The argument follows by *Reductio*.

(ii) Consequence of reflexivity of $\vdash$ and item (i) above.

(iii) From left to right, it follows immediately by *Reductio*; from right to left: suppose that $\alpha \notin \Delta$ or $\beta \in \Delta$:
- Case 1: consider $\alpha \notin \Delta$, and suppose by *Reductio* that $\alpha \supset \beta \notin \Delta$. Since Δ is a non-trivial Λ-maximal set, $\Delta \cup \{\alpha\} \vdash \lambda_1$, for some $\lambda_1 \in \Lambda$ so, $\Delta \vdash \alpha \supset \lambda_1$. As $\lambda_1 \supset (\lambda_1 \vee \lambda_2)$ is a theorem of $\mathcal{L}^{k,l,m,n}$ then, by transitivity, $\Delta \vdash \alpha \supset (\lambda_1 \vee \lambda_2)$. Analogously, since $\alpha \supset \beta \notin \Delta$ then $\Delta \vdash (\alpha \supset \beta) \supset (\lambda_1 \vee \lambda_2)$. From (**A3**) it follows that $\Delta \vdash \lambda_1 \vee \lambda_2$, and as Δ is a prime set, $\Delta \vdash \lambda_1$ or $\Delta \vdash \lambda_2$ for $\lambda_1, \lambda_2 \in \Lambda$. Absurd.
- Case 2: the argument follows from (**A1**) and item (i).

(iv) Suppose that Δ is a prime set. The first case follows by *Reductio*; the second case is also by *Reductio*, using an analogous argument as in Case 1 of (iii).

(v) The right-to-left direction follows from (**A4**) and item (i); the other direction follows from (**A5**), (**A6**) and item (i).

(vi) Suppose by *Reductio* that $\alpha \notin \Delta$ and $\neg\alpha \notin \Delta$. By using an analogous argument as in Case 1 of (iii) we have that $\Delta \vdash \alpha \supset (\lambda_1 \vee \lambda_2)$ and $\Delta \vdash \neg\alpha \supset (\lambda_1 \vee \lambda_2)$ thus, $\Delta \vdash (\alpha \vee \neg\alpha) \supset (\lambda_1 \vee \lambda_2)$ for some $\lambda_1, \lambda_2 \in \Lambda$. From (**PI**) if follows that $\Delta \vdash \lambda_1 \vee \lambda_2$. As Δ is a prime set an absurd follows.

(vii) The argument follows by *Reductio* by using the axiom (**mbC**).

(viii) Immediate by using the axiom (**bC**).

(ix) The argument uses the axiom (**Ci**) and Lemma 4.8 (i) and (v). $\square$

The following result is a generalization of the well-known Lindenbaum's argument.

Theorem 4.9. *Let* **S** *be a system, Δ and Λ be non-trivial subsets of For such that $\Delta \cap \Lambda = \varnothing$. So, there is a prime set Π non-trivial Λ-maximal such that $\Delta \subset \Pi$ and $\Pi \cap \Lambda = \varnothing$.*

Proof. By starting from an enumeration of *For*, define the following extensions of Γ.

$$\Gamma_0 = \Delta$$
$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\alpha_n\} & \text{if } \Gamma_n \cup \{\alpha_n\} \nvdash \lambda, \text{ for all } \lambda \in \Lambda \\ \Gamma_n & \text{if } \Gamma_n \cup \{\alpha_n\} \vdash \lambda, \text{ for some } \lambda \in \Lambda \end{cases}$$

Consider:

$$\Pi = \bigcup_{i < \aleph_0} \Gamma_i.$$

It remains to be shown that Π is a non-trivial Λ-maximal prime set. The result can in fact be easily obtained from a standard argument. $\qquad\square$

In [5] the formal treatment of anodic systems required the concept of *factual sets*, that is, sets that are $\Diamond$-non-trivial in the sense of not containing all sentences of the kind $\Diamond\alpha$ ($\Diamond$-trivial sets are called *hypothetical*). In this way, the concept of "it is possible" can be expressed in the anodic system independently of the concept "it is necessary" even without any *Falsum* particle.

The notion of factual sets is inherent in systems containing a form of classical negation, as proven below.

Lemma 4.10. *Let $\mathcal{L}^{k,l,m,n}$ be a cathodic system that permit us to define a classical negation $\sim$. If Δ is a non-trivial maximal extension of $\mathcal{L}^{k,l,m,n}$, then Δ is factual.*

Proof. Indeed, for each axiom α of $\mathcal{L}^{k,l,m,n}$ we have $\Delta \vdash \alpha$ so, by (**Nec**), $\Delta \vdash \Box\alpha$. By definition, $\Delta \vdash \sim\Diamond\sim\alpha$, i.e., $\sim\Diamond\sim\alpha \in \Delta$. As $\sim$ is a form of classical negation, then $\Diamond\sim\alpha \notin \Delta$. Therefore Δ is factual. $\qquad\square$

Lemma 4.11. *Let $\mathcal{L}^{k,l,m,n}$ be $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ or $\mathbf{Ci}^{k,l,m,n}$. If Δ is a Λ-maximal non-trivial set of sentences of $\mathcal{L}^{k,l,m,n}$ such that $\Box\alpha \notin \Delta$, then $Den(\Delta) \cup \{\sim\alpha\}$ is a non-trivial set.*

Proof. Suppose by *Reductio* that $Den(\Delta) \cup \{\sim\alpha\}$ is trivial; then $Den(\Delta) \cup \{\sim\alpha\} \vdash \alpha$, for any α. Thus, $Den(\Delta) \vdash \sim\alpha \supset \alpha$. On the other hand, we have that $Den(\Delta) \vdash \alpha \supset \alpha$ so, $Den(\Delta) \vdash (\alpha \wedge \sim\alpha) \supset \alpha$. As $\sim$ is a classical negation, then it holds in $\mathcal{L}^{k,l,m,n}$ that $\vdash \alpha \vee \sim\alpha$ so, $Den(\Delta) \vdash \alpha$. By Fact 4.5 we have that $\alpha \in Den(\Delta)$, therefore $\Box\alpha \in \Delta$. Absurd. $\qquad\square$

Theorem 4.12. (Fundamental Theorem of Canonical Models) *Let $\mathcal{L}^{k,l,m,n}$ be $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ or $\mathbf{Ci}^{k,l,m,n}$. The valuation v_Δ in the canonical model for $\mathcal{L}^{k,l,m,n}$ defines an $\mathcal{L}^{k,l,m,n}$-valuation.*

Proof. The non-trivial cases are for valuations of $\mathcal{L}^{k,l,m,n}$ involving $\Box$, $\neg$ and $\circ$.

- For $\mathcal{L}^{k,l,m,n} = \mathbf{mbC}^{k,l,m,n}$ we need to show that:
    - (i) $v_\Delta(\Box\alpha) = 1$ iff $v_\Gamma(\alpha) = 1$, for each $\Gamma \in \widehat{W}$ such that $\Delta\widehat{R}\,\Gamma$.
      ($\Longrightarrow$) If $v_\Delta(\Box\alpha) = 1$ then $\Box\alpha \in \Delta$ so, $\alpha \in Den(\Delta)$. For each $\Gamma \in \widehat{W}$ such that $Den(\Delta) \subseteq \Gamma$ we have that $\Delta\widehat{R}\,\Gamma$ and $\alpha \in \Gamma$. As v_Γ is the characteristic function of Γ it follows that $v_\Gamma(\alpha) = 1$ for each $\Gamma \in \widehat{W}$ such that $\Delta\widehat{R}\,\Gamma$.
      ($\Longleftarrow$) If $v_\Delta(\Box\alpha) = 0$ then $\Box\alpha \notin \Delta$ so, by Lemma 4.11, $Den(\Delta) \cup \{\sim\alpha\}$ is a non-trivial set. By Theorem 4.9, there exists $\Gamma \in \widehat{W}$ such that $Den(\Delta) \cup \{\sim\alpha\} \subseteq \Gamma$. As $\sim$ is a classical negation, then $\alpha \notin \Gamma$ and so, $v_\Gamma(\alpha) = 0$. Therefore, $v_\Gamma(\alpha) = 0$ for some $\Gamma \in \widehat{W}$ such that $\Delta\widehat{R}\,\Gamma$.
    - (ii) $v_\Delta(\alpha) = 0$ implies $v_\Delta(\neg\alpha) = 1$.
      If $v_\Delta(\alpha) = 0$ then $\alpha \notin \Delta$ so, by Lemma 4.8 (vi) it follows that $\neg\alpha \in \Delta$. Therefore, $v_\Delta(\neg\alpha) = 1$.

(iii) $v_\Delta(\circ\alpha) = 1$ implies $v_\Delta(\alpha) = 0$ or $v_\Delta(\neg\alpha) = 0$.
Immediate from Lemma 4.8 (vii).

- For $\mathcal{L}^{k,l,m,n} = \mathbf{bC}^{k,l,m,n}$ add:
 (iv) $v_\Delta(\neg\neg\alpha) = 1$ implies $v_\Delta(\alpha) = 1$.
 Immediate from Lemma 4.8 (viii).
- For $\mathcal{L}^{k,l,m,n} = \mathbf{Ci}^{k,l,m,n}$ add:
 (v) $v_\Delta(\neg\circ\alpha) = 1$ implies $v_\Delta(\alpha) = 1$ and $v_\Delta(\neg\alpha) = 1$.
 Immediate from Lemma 4.8 (ix).

Therefore, the function v_Δ is an $\mathcal{L}^{k,l,m,n}$-valuation. $\qquad\square$

Corollary 4.13. *Let Δ and Δ' be non-trivial Λ-maximal extensions of $\mathcal{L}^{k,l,m,n}$, where $\mathcal{L}^{k,l,m,n}$ is $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ or $\mathbf{Ci}^{k,l,m,n}$. Then:*

$$Den^n(\Delta) \subseteq \Delta' \quad iff \quad Pos^n(\Delta') \subseteq \Delta.$$

Proof. ($\Longrightarrow$) Consider $\Diamond^n\alpha \in Pos^n(\Delta')$ then $\alpha \in \Delta'$. As a form of classical negation $\sim$ is definable in $\mathcal{L}^{k,l,m,n}$, then $\sim\alpha \notin \Delta'$. From the hypothesis $Den^n(\Delta) \subseteq \Delta'$ it follows that $\sim\alpha \notin Den^n(\Delta)$ so, $\Box^n\sim\alpha \notin \Delta$ and thus, $\sim\Box^n\sim\alpha \in \Delta$. Therefore, as $\sim\Box^n\sim\alpha$ is equivalent to $\Diamond^n\alpha$ it follows that $\Diamond^n\alpha \in \Delta$.

($\Longleftarrow$) Analogous, using the fact that $\sim\Box^n\alpha$ is equivalent to $\Diamond^n\sim\alpha$. $\quad\square$

We denote by $\overline{\Gamma}$ the *set-theoretical complement* of Γ, i.e., the set $\overline{\Gamma} = \{\gamma : \gamma \notin \Gamma\}$. A theory T is said to be *generated* by Δ if $T = \{\alpha : \Delta \vdash \alpha\}$. Now, it remains to be shown that the accessibility relations of the canonical model for $\mathbf{mbC}^{k,l,m,n}, \mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$ satisfy the condition $\mathbf{C}^{k,l,m,n}$.

Theorem 4.14. *Let $\mathcal{L}^{k,l,m,n} = \langle For, \vdash \rangle$ be $\mathbf{mbC}^{k,l,m,n}, \mathbf{bC}^{k,l,m,n}$ or $\mathbf{Ci}^{k,l,m,n}$. The accessibility relation of the canonical frame for $\mathcal{L}^{k,l,m,n}$ satisfies the condition $\mathbf{C}^{k,l,m,n}$.*

Proof. Suppose $\Delta_1 \widehat{R^k} \Delta_2$ and $\Delta_1 \widehat{R^m} \Delta_3$. By Corollary 4.13, we have:

(Hyp.1) $\quad Den^k(\Delta_1) \subseteq \Delta_2$ iff $Pos^k(\Delta_2) \subseteq \Delta_1$

(Hyp.2) $\quad Den^m(\Delta_1) \subseteq \Delta_3$ iff $Pos^m(\Delta_3) \subseteq \Delta_1$

Consider $\Lambda = \overline{Dep^l(\Delta_2)} \cup \overline{Dep^n(\Delta_3)}$, where $Dep^k(\Delta) = \{\alpha : \Diamond^k\alpha \in \Delta\}$ and let T be the $\mathcal{L}^{k,l,m,n}$-theory generated by the set:

$$\mathcal{H} = \{\alpha : \Box^l\alpha \in \Delta_2\} \cup \{\beta : \Box^n\beta \in \Delta_3\}$$

First, notes that $\Lambda \neq \varnothing$. Indeed, if not, we would have that $\{\varphi : \Diamond^l\varphi \notin \Delta_2\} = \varnothing$ and $\{\psi : \Diamond^n\psi \notin \Delta_3\} = \varnothing$. In this case, $Dep^l(\Delta_2)$ and $Dep^n(\Delta_3)$ would be trivial. Absurd, since by Lemma 4.10, Δ_2 and Δ_3 are factual sets. Clearly $T \neq \varnothing$ since Δ_2 and Δ_3 are non-trivial maximal extension of $\mathcal{L}^{k,l,m,n}$. Now, we show that:

$$T \cap \Lambda = \varnothing \tag{1}$$

Suppose by *Reductio* that $\alpha \in T$ and $\alpha \in \Lambda$ so: ($\Box^l\alpha \in \Delta_2$ or $\Box^n\alpha \in \Delta_3$) and ($\Diamond^l\alpha \notin \Delta_2$ or $\Diamond^n\alpha \notin \Delta_3$). Hence, we have to show that in each of the four possibilities a contradiction is derived.

(a) $\Box^l\alpha \in \Delta_2$ and $\Diamond^l\alpha \notin \Delta_2$;
(b) $\Box^l\alpha \in \Delta_2$ and $\Diamond^n\alpha \notin \Delta_3$;
(c) $\Box^n\alpha \in \Delta_3$ and $\Diamond^l\alpha \notin \Delta_2$;
(d) $\Box^n\alpha \in \Delta_3$ and $\Diamond^n\alpha \notin \Delta_3$.

Case (a) If $\Diamond^l\alpha \notin \Delta_2$ then, as $\sim$ is a classical negation $\sim\Diamond^l\alpha \in \Delta_2$. So $\Box^l\sim\alpha \in \Delta_2$, hence $\sim\alpha \in Den^l(\Delta_2)$. As $\Box^l\alpha \in \Delta_2$ then $\alpha \in Den^l(\Delta_2)$. Absurd, since $Den^l(\Delta_2)$ is an $\mathcal{L}^{k,l,m,n}$-theory.

Case (b) Suppose that $\Diamond^n\alpha \notin \Delta_3$ then, by (Hyp.2), $\Diamond^n\alpha \notin Den^m(\Delta_1)$, so $\Box^m\Diamond^n\alpha \notin \Delta_1$. From $\mathbf{G}^{k,l,m,n}$ it follows that $\Diamond^k\Box^l\alpha \notin \Delta_1$. On the other hand, if $\Box^l\alpha \in \Delta_2$ then, $\Diamond^k\Box^l\alpha \in Pos^k(\Delta_2)$. By (Hyp.1) it follows that $\Diamond^k\Box^l\alpha \in \Delta_1$. Contradiction.

Case (c) Analogous to case (b).

Case (d) Analogous to case (a).

As T and Λ are sets of formulas in $\mathcal{L}^{k,l,m,n}$, and $T \cap \Lambda = \varnothing$, by Theorem 4.9 it follows that there exists a maximal prime theory Δ_4 of $\mathcal{L}^{k,l,m,n}$ such that $T \subseteq \Delta_4$ and $\Delta_4 \cap \Lambda = \varnothing$. Since T is generated by $\mathcal{H}$, then $Den^l(\Delta_2) \cup Den^n(\Delta_3) \subseteq \Delta_4$. Hence, $Den^l(\Delta_2) \subseteq \Delta_4$ and $Den^n(\Delta_3) \subseteq \Delta_4$. Therefore, there exists Δ_4 such that $\Delta_2\widehat{R^l}\Delta_4$ and $\Delta_3\widehat{R^n}\Delta_4$. $\qquad\square$

Corollary 4.15. (Completeness for $\mathbf{mbC}^{\,k,l,m,n}, \mathbf{bC}^{\,k,l,m,n}$ and $\mathbf{Ci}^{\,k,l,m,n}$) *Let $\mathcal{L}^{k,l,m,n} = \langle For, \vdash \rangle$ be $\mathbf{mbC}^{\,k,l,m,n}, \mathbf{bC}^{\,k,l,m,n}$ or $\mathbf{Ci}^{\,k,l,m,n}$. If $\Gamma \vDash \alpha$, then $\Gamma \vdash \alpha$.*

Proof. Suppose that $\Gamma \nvdash \alpha$, for some $\alpha \in For$. Clearly $\alpha \notin \Gamma$ and $\Gamma \cap \{\alpha\} = \varnothing$. By Theorem 4.9, we can extend Γ to a non-trivial $\{\alpha\}$-maximal set Δ such that $\Delta \nvdash \alpha$, so $\alpha \notin \Delta$. Therefore $v_\Delta(\alpha) = 0$. As $\Gamma \subseteq \Delta$ then, $v_\Delta(\gamma) = 1$ for each $\gamma \in \Gamma$. Hence, α is falsified in the canonical model for $\mathcal{L}^{k,l,m,n}$ that validates all element of Γ, so $\Gamma \nvDash \alpha$. $\qquad\square$

Since the negation of $\mathbf{PI}$ is too weak, the canonical frame for the class $\mathbf{PI}^{k,l,m,n}$ has to be defined in a special way.

The *canonical frame* for $\mathbf{PI}^{k,l,m,n}$ is the pair $\widehat{\mathfrak{F}} = \langle \widehat{W}, \widehat{R} \rangle$, where the universe $\widehat{W}$ is formed by the maximal non-trivial prime sets of $\mathbf{PI}^{k,l,m,n}$ and for each $\Delta, \Delta' \in \widehat{W}$, $\Delta\widehat{R}\Delta'$ iff $Den(\Delta) \subseteq \Delta' \subseteq Dep(\Delta)$, where $Dep(\Delta) = \{\alpha : \Diamond\alpha \in \Delta\}$.

Justifying the intuition that $\mathbf{PI}^{k,l,m,n}$ is "almost positive" (in the sense that the only property of its underling negation is $p \vee \neg p$) the completeness proof for $\mathbf{PI}^{k,l,m,n}$ w.r.t. bi-valued relational model, is basically the same of the anodic (i.e., positive systems studied in [4]), but details are omitted here.

We have shown is that the cathodic systems are characterized w.r.t. a Kripke-style semantics. This result in spite of its own interest, is also crucial to give a second semantical approach to cathodic systems, by means of *modal possible-translations semantics*, as shown in the next section. This alternative semantics permit us to explain how the cathodic systems can support contradictions, albeit avoiding deductive triviality.

5. A New Semantics for Cathodic Systems

Since the connection between modalities and logics of formal inconsistency (LFIs) is the constituent ingredient of cathodic systems, and considering how expressive the possible-translations semantics (PTS) for paraconsistent logics are, as explained below, it seems just natural to extend the possible-translations semantics to modal paraconsistent logics. Our purpose in this section is to define possible-translations semantics for cathodic systems, which will be referred to as modal possible-translations semantics.

The reason we start from logics such as **PI**, **mbC**, **bC** and **Ci** is that those logics already dispose of the semantical machinery of possible-translations semantics. This kind of semantics permits the understanding of the role of contradictory, but non-trivial situations in argumentation.

In the sequel the formal definition of a *modal possible-translations semantics structure* will be presented for cathodic systems. We start from the standard definition of possible-translations semantics given for the paraconsistent systems treated here, as in [7].

5.1. Possible-Translations Semantics

This section surveys the formal definition of possible-translations structure (PTS) adequate for **PI**, **mbC**, **bC** and **Ci**, what attend our aims.

It is convenient, firstly, to exhibit the matrices that will be used to define the possible-translations semantics for **PI, mbC, bC** and **Ci**. These matrices are also used to define the *three-valued relational models* $\mathfrak{M}_{\mathrm{Thv}}^{\mathcal{L}^{k,l,m,n}}$, used to give the modal possible-translations semantics for the classes **PI**^{k,l,m,n}, **mbC**^{k,l,m,n}, **bC**^{k,l,m,n} and **Ci**^{k,l,m,n} of cathodic systems.

Consider $\mathscr{M}$ a family of three-valued matrices[4] with truth-values $\{F, t, T\}$, where the distinguished values are t and T, and whose language contains the connectives $\sqcap, \sqcup, \sqsupset, \neg_1, \neg_2, \circ_1, \circ_2, \circ_3$ such that the connectives are governed by the tables given in Table 4:

The *translations* from a paraconsistent system $\mathcal{L}$ into $\mathscr{M}$ consist of all functions in the set $\mathrm{TR}_{\mathcal{L}}$ of mappings $\mathtt{t} : For_{\mathcal{L}} \longrightarrow For_{\mathscr{M}}$ subject to the clauses in Table 3:

To each specific case we need to choose adequate translations to $\mathcal{L}$. The specific choices in each case are given in Table 2 below

From the restrictions on translations for $\mathcal{L}$, one can determine the set $\mathscr{M}_{\mathcal{L}} \subseteq \mathscr{M}$ of the specific three-valued matrices given in the Table 1 able to characterize $\mathcal{L}$ in terms of possible-translations semantics.

Definition 5.1. *A possible-translations semantical structure for* $\mathcal{L}$ *is a pair* $\mathrm{PT} = \langle \mathscr{M}_{\mathcal{L}}, \mathrm{TR}_{\mathcal{L}} \rangle$, *where* :

(i) $\mathscr{M}_{\mathcal{L}}$ is the collection of three-valued matrices for $\mathcal{L}$ described in Table 1;
(ii) $\mathrm{TR}_{\mathcal{L}} = \{\mathtt{t} : For_{\mathcal{L}} \longrightarrow For_{\mathscr{M}_{\mathcal{L}}}\}$ is a family of translations following the restrictions given in Table 2.

---

[4] Of course $\mathscr{M}$ can be also legitimately seen as a collection of three-valued logics.

TABLE 1. Tables of $\mathscr{M}_\mathcal{L}$

$\mathcal{L}$	$\mathscr{M}_\mathcal{L}$ is composed by the following tables
PI	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$ and $\neg_2$
mbC	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_2$, $\circ_2$ and $\circ_3$
bC	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_3$, $\circ_2$ and $\circ_3$
Ci	$\sqsupset$, $\sqcap$, $\sqcup$, $\neg_1$, $\neg_3$ and $\circ_1$

TABLE 2. Restrictions on translations of $\mathcal{L}$

Logic	Restrictions on translations
PI	(Tr.1), (Tr.2), (Tr.3), (Tr.4) e (Tr.5)
mbC	(Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.5) e (Tr.7)
bC	(Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.6) e (Tr.7)
Ci	(Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.6), (Tr.8) e (Tr.9)

The notion of (semantical) consequence relation in PTS is defined as follows.

Definition 5.2. Let $\Gamma \cup \{\alpha\}$ be a set of $\mathcal{L}$-formulas and $\vDash_{\mathscr{M}_\mathcal{L}}$ the consequence relation determined by $\mathscr{M}_\mathcal{L}$. The *consequence relation* in PTS, denoted by $\vDash_{\mathsf{PT}}$, is defined as:

$$\Gamma \vDash_{\mathsf{PT}} \alpha \text{ iff } \mathtt{t}(\Gamma) \vDash_{\mathscr{M}_\mathcal{L}} \mathtt{t}(\alpha), \text{ for each translation } \mathtt{t} \in \mathrm{Tr}_\mathcal{L}.$$

As standard in PTS, soundness is obtained by means of translations in the following sense.

Theorem 5.3. (Soundness for **PI, mbC, bC** and **Ci**) *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* $\mathcal{L}$. *Then:* $\Gamma \vdash_\mathcal{L} \alpha$ *implies* $\Gamma \vDash_{\mathsf{PT}} \alpha$.

Proof. The proof is routine, and consists in showing that the translations of each axiom of $\mathcal{L}$ are tautology in $\mathscr{M}_\mathcal{L}$, and that the translations of the rules preserve such tautologies. □

The strategy to obtain a characterization via possible-translations semantics for a paraconsistent system $\mathcal{L}$, is to show that the bi-valuation semantics (denoted by $\mathtt{Biv}_\mathcal{L}$) can be simulated by means of a translation $\mathtt{t}$ and a three-valued valuation $\overline{v}$. This procedure is known as *Representability Lemma* (cf. [17]). In a few words, what this shows is that for each bi-valuation v in $\mathcal{L}$, there exist a translation $\mathtt{t}$ and a three-valued valuation $\overline{v}$ satisfying:

$$\overline{v}(\mathtt{t}(\alpha)) \in \{T, t\} \text{ iff } v(\alpha) = 1$$

The Representability Lemma for the systems **PI, mbC, bC** and **Ci**, among others, is detailed in [17], and is the key ingredient in obtaining completeness w.r.t. possible-translations semantics.

TABLE 3. Clauses on translations

(Tr.1)	$\mathsf{t}(p) = p$
(Tr.2)	$\mathsf{t}(\alpha \supset \beta) = \mathsf{t}(\alpha) \sqsupset \mathsf{t}(\beta)$
(Tr.3)	$\mathsf{t}(\alpha \wedge \beta) = \mathsf{t}(\alpha) \sqcap \mathsf{t}(\beta)$
(Tr.4)	$\mathsf{t}(\alpha \vee \beta) = \mathsf{t}(\alpha) \sqcup \mathsf{t}(\beta)$
(Tr.5)	$\mathsf{t}(\neg\alpha) \in \{\neg_1\mathsf{t}(\alpha),\ \neg_2\mathsf{t}(\alpha)\}$
(Tr.6)	$\mathsf{t}(\neg\alpha) \in \{\neg_1\mathsf{t}(\alpha),\ \neg_3\mathsf{t}(\alpha)\}$
(Tr.7)	$\mathsf{t}(\circ\alpha) \in \{\circ_2\mathsf{t}(\alpha),\ \circ_3\mathsf{t}(\alpha),\ \circ_2\mathsf{t}(\neg\alpha),\ \circ_3\mathsf{t}(\neg\alpha)\}$
(Tr.8)	$\mathsf{t}(\circ\alpha) \in \{\circ_1\mathsf{t}(\alpha),\ \circ_1\mathsf{t}(\neg\alpha)\}$
(Tr.9)	$\mathsf{t}(\neg\alpha) = \neg_1\mathsf{t}(\alpha)$ implies $\mathsf{t}(\circ\alpha) = \circ_1\mathsf{t}(\neg\alpha)$

5.2. Modal Possible-Translations Semantics

The same strategy used above defines *modal possible-translations semantics* for the cathodic systems. The goal here is to extend this method to cathodic systems in order to obtain an alternative characterization for them w.r.t. modal possible-translations semantics. The strategy is to extended the Representability Lemma to the modal cases, proving in each case that valuation in $\mathfrak{M}_{\mathtt{Biv}}$ can be represented by modal possible-translations semantics in $\mathfrak{M}_{\mathtt{Thv}}$.

Definition 5.4. A *three-valued relational model* $\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}$ for a cathodic system $\mathcal{L}^{k,l,m,n}$ is a pair $\langle \mathfrak{F}, \overline{v} \rangle$ where $\mathfrak{F}$ is a frame and $\overline{v} : Var \times W \longrightarrow \{T, t, F\}$ is a three-valued modal assignment (determined by $\mathscr{M}_{\mathcal{L}}$) satisfying the conditions:

(i) $\overline{v}(p, w) \in \{T, t, F\}$, *for* $p \in Var$;
(ii) $\overline{v}(\alpha \bowtie \beta, w) = \overline{v}(\alpha, w) \bowtie \overline{v}(\beta, w)$, *for* $\bowtie \in \{\sqsupset, \sqcap, \sqcup\}$;
(iii) $\overline{v}(\neg_i\alpha, w) = \neg_i\overline{v}(\alpha, w)$ *for* $1 \leq i \leq 3$;
(iv) $\overline{v}(\circ_i\alpha, w) = \circ_i\overline{v}(\alpha, w)$ *for* $1 \leq i \leq 3$;
(v) $\overline{v}(\Box\alpha, w) = \begin{cases} t & \text{if } \overline{v}(\alpha, w') \in \{T, t\}, \quad \forall w'(wRw') \\ F & \text{if } \overline{v}(\alpha, w') = F, \quad\ \ \exists w'(wRw') \end{cases}$
(vi) $\overline{v}(\Diamond\alpha, w) = \begin{cases} t & \text{if } \overline{v}(\alpha, w') = \{T, t\}, \quad \exists w'(wRw'); \\ F & \text{if } \overline{v}(\alpha, w') = F, \quad\ \ \forall w'(wRw'). \end{cases}$

In order to define the notion of modal possible-translation structure it is necessary to extend the conditions on translations, given in the Table 3, to the modal case.

The *translations* from the cathodic systems $\mathcal{L}^{k,l,m,n}$ into $\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}$ consist of all functions in the set $\mathrm{TR}_{\mathcal{L}^{k,l,m,n}}$ of mappings $\mathsf{t} : For_{\mathcal{L}^{k,l,m,n}} \longrightarrow For_{\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}}$ subject to the clauses given in Table 3 plus the following:

(Tr.10) $\mathsf{t}(\Box\alpha) = \Box\mathsf{t}(\alpha)$
(Tr.11) $\mathsf{t}(\Diamond\alpha) = \Diamond\mathsf{t}(\alpha)$

Definition 5.5. A *modal possible-translations structure* for a cathodic system $\mathcal{L}^{k,l,m,n}$ is a triple $\mathtt{MPT} = \langle \mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}, \mathrm{TR}_{\mathcal{L}^{k,l,m,n}}, \mathfrak{F} \rangle$ such that:

(i) $\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}$ is a three-valued relational model;
(ii) $\mathfrak{F}$ is a frame for $\mathcal{L}^{k,l,m,n}$;
(iii) $\mathrm{TR}_{\mathcal{L}^{k,l,m,n}} \subseteq \mathrm{TR}_{\mathcal{L}}$ such that $\langle \mathscr{M}_{\mathcal{L}}, \mathrm{TR}_{\mathcal{L}} \rangle$ is a *PT*-structure for $\mathcal{L}$.

Table 4. Tables of $\mathscr{M}$

$\sqcap$	T	t	F
T	t	t	F
t	t	t	F
F	F	F	F

$\sqcup$	T	t	F
T	t	t	t
t	t	t	t
F	t	t	F

$\sqsupset$	T	t	F
T	t	t	F
t	t	t	F
F	t	t	t

	$\neg_1$	$\neg_2$	$\neg_3$
T	F	F	F
t	F	t	t
F	T	t	T

	$\circ_1$	$\circ_2$	$\circ_3$
T	T	t	F
t	F	F	F
F	T	t	F

The notions of satisfiability and validity in a three-valued relational model $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{k,l,m,n}}$, and validity on a frame $\mathfrak{F}$, are similar to those for the two-valued case (given at page 10). The notion of validity on a frame, of course, forms a *consequence relation on a frame*, denoted by $\vDash_{\mathfrak{F}}$.

Definition 5.6. Let $\Gamma \cup \{\alpha\}$ be a set of $\mathcal{L}^{k,l,m,n}$-formulas and $\vDash_{\mathfrak{F}}$ be the consequence relation on $\mathfrak{F}$. *The consequence relation* in MPT, denoted by $\vDash_{\text{MPT}}$, is defined as:

$$\Gamma \vDash_{\text{MPT}} \alpha \ \text{ iff } \ \mathbf{t}(\Gamma) \vDash_{\mathfrak{F}} \mathbf{t}(\alpha)$$

for all translations $\mathbf{t} \in \text{Tr}_{\mathcal{L}^{k,l,m,n}}$.

We are now in position to prove the following:

Theorem 5.7. (Soundness w.r.t. MPT) *Let $\Gamma \cup \{\alpha\}$ be a set of $\mathcal{L}^{k,l,m,n}$-formulas and $\mathcal{F}$ be a class of frames for $\mathcal{L}^{k,l,m,n}$. Then:*

$$\Gamma \vdash_{\mathcal{L}^{k,l,m,n}} \alpha \ \text{ implies } \ \Gamma \vDash_{\text{MPT}} \alpha$$

Proof. We need to show that the translations $\mathbf{t}$ applied to the axioms of $\mathcal{L}^{k,l,m,n}$ are valid in $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{k,l,m,n}}$. The non-modal portion of the proof is analogous to Theorem 5.3, since the valuations in this case are independent of the accessibility relation among worlds. It remains to be shown that the translations of (**K**), (**K1**), (**K2**), (**K3**) and $\mathbf{G}^{k,l,m,n}$ are valid in $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{k,l,m,n}}$ an the rule (**Nec**) preserve validity.

- Axiom (**K**)

  By (Tr.1), (Tr.2) and (Tr.10) we have that:

  $$\mathbf{t}(\Box(p \supset q) \supset (\Box p \supset \Box q)) = \boxdot(p \sqsupset q) \sqsupset (\boxdot p \sqsupset \boxdot q).$$

  Suppose, by *Reductio*, that there exists a model $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{k,l,m,n}}$ based on a frame $\mathfrak{F}$ such that $\mathfrak{M}_{\text{Thv}}^{\mathcal{L}^{k,l,m,n}} \nvDash \boxdot(p \sqsupset q) \sqsupset (\boxdot p \sqsupset \boxdot q)$. By using the definition of validity in a three-valued relational model the absurd can be easily obtained.

- The argument to the axioms (**K1**), (**K2**) and (**K3**) is similar.

- Axiom schema $\mathbf{G}^{k,l,m,n}$

  By (Tr.1), (Tr.2), (Tr.10) and (Tr.11) we have that:

  $$\mathbf{t}(\Diamond^k \Box^l \alpha \supset \Box^m \Diamond^n \alpha) = \lozenge^k \boxdot^l \alpha \sqsupset \boxdot^m \lozenge^n \alpha.$$

Suppose, by *Reductio*, that there exists a model $\mathfrak{M}_{\texttt{Thv}}^{\mathcal{L}^{k,l,m,n}}$ based on a frame $\mathfrak{F}$ such that $\mathfrak{M}_{\texttt{Thv}}^{\mathcal{L}^{k,l,m,n}} \not\models \diamondsuit^k \Box^l \alpha \sqsupset \Box^m \diamondsuit^n \alpha$. Then there is a world $w_1 \in W$ such that:

(a) $\overline{v}(\diamondsuit^k \Box^l \alpha, w_1) \in \{T, t\}$
(b) $\overline{v}(\Box^m \diamondsuit^n \alpha, w_1) = F$

By using an analogous argument as in classical modal case for the Lemmon-Scott axiom, see proposition 4.2.1 of [11], a contradiction can be obtained.

- Rule (**Nec**)

  By (Tr.10) we have to show that $\mathcal{F} \models \texttt{t}(\alpha)$ implies $\mathcal{F} \models \Box\texttt{t}(\alpha)$.

  Suppose, by *Reductio*, that $\mathcal{F} \models \texttt{t}(\alpha)$ and that there exists $\mathfrak{M}_{\texttt{Thv}}^{\mathcal{L}^{k,l,m,n}}$ based on some $\mathfrak{F} \in \mathcal{F}$, such that $\mathfrak{M}_{\texttt{Thv}}^{\mathcal{L}^{k,l,m,n}}, w \not\models \Box\texttt{t}(\alpha)$ for some $w \in W$. This means that $\overline{v}(\Box\texttt{t}(\alpha), w) = F$. By Definition 5.4 (v), it follows that there exists $w' \in W$ such that wRw' and $\overline{v}(\texttt{t}(\alpha), w') = F$. Hence, $\mathfrak{M}_{\texttt{Thv}}^{\mathcal{L}^{k,l,m,n}}, w' \not\models \texttt{t}(\alpha)$ and so $\mathfrak{F} \not\models \texttt{t}(\alpha)$, for some $\mathfrak{F} \in \mathcal{F}$. Absurd.   □

Our next step will be establish completeness. Taking into account that valuations of the consistency connective $\circ$ take the valuation of negation $\neg$ into account, it is convenient to define a non-canonical measure of complexity ℓ of formulas including the modal ones as follows:

Definition 5.8. Let **S** be a system, and *For* be a set of sentences of **S**. The function $\ell : For \longrightarrow \mathbb{N}$ denote the *complexity length* of sentences, and is defined as:

(i) $\ell(p) = 0$, for $p \in Var$;
(ii) $\ell(\neg\alpha) = \ell(\alpha) + 1$;
(iii) $\ell(\alpha \bowtie \beta) = \texttt{max}\{\ell(\alpha), \ell(\beta)\} + 1$, for $\bowtie \in \{\supset, \wedge, \vee\}$;
(iv) $\ell(\circ\alpha) = \ell(\alpha) + 2$;
(v) $\ell(\Box\alpha) = \ell(\alpha) + 1$.

The next four lemmas prove the representability of the bi-valued relational models by means of appropriate translations and three-valued relational models for $\textbf{PI}^{k,l,m,n}$, $\textbf{mbC}^{k,l,m,n}$, $\textbf{bC}^{k,l,m,n}$ and $\textbf{Ci}^{k,l,m,n}$.

Lemma 5.9. ($\textbf{PI}^{k,l,m,n}$-Representability) *Given a* $\textbf{PI}^{k,l,m,n}$-*valuation* v *in* $\mathfrak{M}_{\texttt{Biv}}^{PI^{k,l,m,n}}$ *and a frame* $\mathfrak{F} = \langle W, R \rangle$ *for* $\textbf{PI}^{k,l,m,n}$ *it is possible to define a translation* $\texttt{t}$ *in* $\texttt{Tr}_{PI^{k,l,m,n}}$, *a valuation* $\overline{v}$ *and a three-valued relational model* $\mathfrak{M}_{\texttt{Thv}}^{PI^{k,l,m,n}}$ *such that, for all formula* α *in* $\textbf{PI}^{k,l,m,n}$ *and all* $w \in W$:

(i) $\overline{v}(\texttt{t}(\alpha), w) = t$ *iff* $v(\alpha, w) = 1$
(ii) $\overline{v}(\texttt{t}(\alpha), w) = F$ *iff* $v(\alpha, w) = 0$

Proof. Consider $\overline{v} : Var \times W \longrightarrow \{T, t, F\}$ defined as:

(Val) $\overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$

Clearly $\overline{v}$ can be extended homomorphically to all formulas in the matrix $\mathcal{M}_{PI^{k,l,m,n}}$. Now define the intended translation in the following way:

$(T1)$ $\mathtt{t}(p) = p$
$(T2)$ $\mathtt{t}(\alpha \supset \beta) = \mathtt{t}(\alpha) \sqsupset \mathtt{t}(\beta)$
$(T3)$ $\mathtt{t}(\alpha \wedge \beta) = \mathtt{t}(\alpha) \sqcap \mathtt{t}(\beta)$
$(T4)$ $\mathtt{t}(\alpha \vee \beta) = \mathtt{t}(\alpha) \sqcup \mathtt{t}(\beta)$
$(T5)$ $\mathtt{t}(\neg\alpha) = \begin{cases} \neg_1\mathtt{t}(\alpha) & \text{if } v(\neg\alpha, w) = 0 \\ \neg_2\mathtt{t}(\alpha) & \text{if } v(\neg\alpha, w) = 1 \end{cases}$
$(T6)$ $\mathtt{t}(\Box\alpha) = \boxdot\mathtt{t}(\alpha)$
$(T7)$ $\mathtt{t}(\Diamond\alpha) = \Diamond\mathtt{t}(\alpha)$

Note that the choices are allowed by restrictions (Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.5), (Tr.10) and (Tr.11) that characterize translations of $\mathbf{PI}^{k,l,m,n}$. The model $\mathfrak{M}_{\mathrm{Thv}}^{PI\,k,l,m,n}$ is obtained by extending $\mathfrak{F}$ with the valuation $\overline{v}$ defined above. The result is proven by induction on ℓ.

1. The atomic case follows from (Val) and (T1).
2. Consider that the induction hypothesis is valid for all formula α with $\ell(\alpha) \leq k$, for some k:
    (IHa) $\overline{v}(\mathtt{t}(\alpha), w) = t$ iff $v(\alpha, w) = 1$
    (IHb) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$
3. The cases where $\alpha = \beta \supset \gamma, \alpha = \beta \wedge \gamma$ and $\alpha = \beta \vee \gamma$ the result follows easily by induction hypothesis.
4. Consider $\alpha = \neg\beta$:
    - Part A: $\overline{v}(\mathtt{t}(\neg\beta), w) = t$ iff $v(\neg\beta, w) = 1$.
      ($\Longrightarrow$) Suppose $\overline{v}(\mathtt{t}(\neg\beta), w) = t$. By (T5), we have that $\overline{v}(\neg_1\mathtt{t}(\beta), w) = t$ or $\overline{v}(\neg_2\mathtt{t}(\beta), w) = t$. By the tables of $\neg_1$ and $\neg_2$, it is easy to see that the negation in question must be $\overline{v}(\neg_2\mathtt{t}(\beta), w) = t$. In this case, by (T5), $v(\neg\beta, w) = 1$.
      ($\Longleftarrow$) Suppose $v(\neg\beta, w) = 1$. By (T5), we have that $\mathtt{t}(\neg\beta) = \neg_2\mathtt{t}(\beta)$ so, $\overline{v}(\mathtt{t}(\neg\beta), w) = \overline{v}(\neg_2\mathtt{t}(\beta), w)$, and then, by Definition 5.4 (iii), it follows that $\overline{v}(\mathtt{t}(\neg\beta), w) = \neg_2\overline{v}(\mathtt{t}(\beta), w)$. The hypothesis (IHa) and (IHb) imply that there are just two possibilities of valuation for $\overline{v}(\mathtt{t}(\beta), w)$:
        - $\overline{v}(\mathtt{t}(\beta), w) = t$. From table of $\neg_2$, it follows that $\neg_2\overline{v}(\mathtt{t}(\beta), w) = t$.
        - $\overline{v}(\mathtt{t}(\beta), w) = F$. From table of $\neg_2$, it follows that $\neg_2\overline{v}(\mathtt{t}(\beta), w) = t$.
      Therefore, $\overline{v}(\mathtt{t}(\neg\beta), w) = t$.
    - Part B: $\overline{v}(\mathtt{t}(\neg\beta), w) = F$ iff $v(\neg\beta, w) = 0$.
      ($\Longrightarrow$) Suppose, by *Reductio*, $v(\neg\beta, w) = 1$. Again, by (T5) and Definition 5.4 (iii) it follows that $\overline{v}(\mathtt{t}(\neg\beta), w) = \neg_2\overline{v}(\mathtt{t}(\beta), w)$. Therefore, in both $\overline{v}(\mathtt{t}(\beta), w) = t$ or $\overline{v}(\mathtt{t}(\beta), w) = F$ it follows that $\overline{v}(\mathtt{t}(\neg\beta), w) = t$. Absurd.
      ($\Longleftarrow$) Suppose $v(\neg\beta, w) = 0$. From $\mathbf{PI}^{k,l,m,n}$-valuation it follows that $v(\beta, w) = 1$ so, by (IHa), $\overline{v}(\mathtt{t}(\beta), w) = t$. Table of the negation $\neg_1$ implies that $\neg_1\overline{v}(\mathtt{t}(\beta), w) = F$; thus, by Definition 5.4 (iii), it follows that $\overline{v}(\neg_1\mathtt{t}(\beta), w) = F$. Therefore $\overline{v}(\mathtt{t}(\neg\beta), w) = F$.
5. Consider $\alpha = \Box\beta$:

- Part A: $\overline{v}(\mathtt{t}(\Box\beta), w) = t$ iff $v(\Box\beta, w) = 1$.
  $\overline{v}(\mathtt{t}(\Box\beta), w) = t$ iff$_{(T6)}$ $\overline{v}(\Box\mathtt{t}(\beta), w) = t$ iff$_{\text{Def. 5.4 (v)}}$ $\overline{v}(\mathtt{t}(\beta),$
  $w') = t$ for each $w' \in W$ such that wRw' iff$_{(IHb)}$ $v(\beta, w) = 1$ for
  each $w' \in W$ such that wRw' iff $v(\Box\beta, w) = 1$.
- Part B: $\overline{v}(\mathtt{t}(\Box\beta), w) = F$ iff $v(\Box\beta, w) = 0$.
  Analogous to Part A.
6.   The case where $\alpha = \Diamond\beta$ is analogous to the previous case.   □

Lemma 5.10. ($\mathbf{mbC}^{k,l,m,n}$-Representability) *Given a* $\mathbf{mbC}^{k,l,m,n}$*-valuation v in* $\mathfrak{M}^{mbC^{k,l,m,n}}_{\mathtt{Biv}}$ *and a frame* $\mathfrak{F}$ *for* $\mathbf{mbC}^{\,k,l,m,n}$ *it is possible to find a translation* $\mathtt{t}$ *in* $\mathrm{Tr}_{mbC^{k,l,m,n}}$, *a valuation* $\overline{v}$ *in a three-valued relational model* $\mathfrak{M}^{mbC^{k,l,m,n}}_{\mathtt{Thv}}$ *such that every formula* α *in* $\mathbf{mbC}^{k,l,m,n}$ *and for all* $w \in W$:

(i) $\overline{v}(\mathtt{t}(\alpha), w) = T$ *implies* $v(\neg\alpha, w) = 0$
(ii) $\overline{v}(\mathtt{t}(\alpha), w) = F$ *iff* $v(\alpha, w) = 0$

Proof. The argument is essentially similar to the previous lemma, but here the language includes the connective $\circ$. We will only emphasize the subtleties concerning $\circ$.

Consider $\overline{v} : Var \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val)\quad \overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Clearly $\overline{v}$ can be homomorphicaly extended to all formulas in the matrix $\mathscr{M}_{mbC^{k,l,m,n}}$. The translations differ from the previous lemma in:

$$(T5)\quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_1\mathtt{t}(\alpha) & \text{if } v(\neg\alpha) = 0 \text{ or } v(\alpha) = 0 = v(\neg\neg\alpha) \\ \neg_2\mathtt{t}(\alpha) & \text{if } v(\neg\alpha) = 1 \end{cases}$$

$$(T8)\quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_3\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \\ \circ_2\mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \text{ and } v(\neg\alpha) = 0 \\ \circ_2\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 1 \end{cases}$$

The choices are allowed by restrictions (Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.5), (Tr.7), (Tr.10) and (Tr.11) that characterize the translations of $\mathbf{mbC}^{\,k,l,m,n}$. The model $\mathfrak{M}^{mbC^{k,l,m,n}}_{\mathtt{Thv}}$ is obtained by extending $\mathfrak{F}$ with the valuation $\overline{v}$ defined above. The statement is proven by induction on the length of complexity of ℓ.

1.   The atomic case follows from (Val) and (T1).
2.   Assume the induction hypothesis for all formulas α with $\ell(\alpha) \leq k$, for some k:
     (IHa) $\overline{v}(\mathtt{t}(\alpha), w) = T$ implies $v(\neg\alpha, w) = 0$
     (IHb) $\overline{v}(\mathtt{t}(\alpha), w) = F$ iff $v(\alpha, w) = 0$
3.   For the cases where $\alpha = \beta \supset \gamma$, $\alpha = \beta \wedge \gamma$, $\alpha = \beta \vee \gamma$, $\alpha = \neg\beta$, $\alpha = \Box\beta$ and $\alpha = \Diamond\beta$ is analogous to the previous lemma.
4.   Consider $\alpha = \circ\beta$

- Part A: $\overline{v}(\mathtt{t}(\circ\beta), w) = T$ implies $v(\neg(\circ\beta), w) = 0$
  If $\overline{v}(\mathtt{t}(\circ\beta), w) = T$ then, by (T8), we have three possibilities for the translations: $\overline{v}(\circ_3\mathtt{t}(\beta), w) = T$, $\overline{v}(\circ_2\mathtt{t}(\neg\beta), w) = T$ or $\overline{v}(\circ_2\mathtt{t}(\beta), w) = T$. From the tables of $\circ_3$ and $\circ_2$ we can see that is impossible to obtain such values; in this case the result follows by vacuity.
- Part B: $\overline{v}(\mathtt{t}(\circ\beta), w) = F$ iff $v(\circ\beta, w) = 0$
  ($\Longrightarrow$) Suppose, by *Reductio*, $\overline{v}(\mathtt{t}(\circ\beta), w) = F$ and $v(\circ\beta, w) = 0$. From the hypothesis $v(\circ\beta, w) = 0$ we have, by $\mathbf{mbC}^{k,l,m,n}$-valuation, that $v(\beta, w) = 0$ or $v(\neg\beta, w) = 0$. We will analyze each possibility:
  (a)   If $v(\beta, w) = 0$ then, by (IHb) it follows that $\overline{v}(\mathtt{t}(\beta), w) = F$ so, by (T8), the only possibility of translation is $\mathtt{t}(\beta) = \circ_2\beta$. Therefore, from the truth-table of $\circ_2$, we have that $\overline{v}(\circ_2\mathtt{t}(\beta), w) = t$. Since $\overline{v}(\mathtt{t}(\circ\beta), w) = F$, by (T8), it follows that $\overline{v}(\circ_2\mathtt{t}(\beta), w) = F$. Absurd.
  (b)   If $v(\neg\beta, w) = 0$, as $\ell(\neg\beta) < \ell(\circ\beta)$ then, by (IHb), it follows that $\overline{v}(\mathtt{t}(\neg\beta), w) = F$. On the other hand, recall that, by hypothesis, $\overline{v}(\mathtt{t}(\circ\beta), w) = F$. So, as $v(\circ\beta, w) = 1$, the only possibility for the translation is $\overline{v}(\circ_2\mathtt{t}(\neg\beta), w) = F$. Definition 5.4 (v) implies that $\circ_2\overline{v}(\mathtt{t}(\neg\beta), w) = F$ so, by the table of $\circ_2$, we have that $\overline{v}(\mathtt{t}(\neg\beta), w) = t$. Absurd.
  ($\Longleftarrow$) If $v(\circ\alpha, w) = 0$ then, by (T8), we have $\mathtt{t}(\circ\alpha) = \circ_3\mathtt{t}(\alpha)$, so $\overline{v}(\mathtt{t}(\circ\alpha), w) = \overline{v}(\circ_3\mathtt{t}(\alpha), w)$. Definition 5.4 (v), implies that $\overline{v}(\circ_3\mathtt{t}(\alpha), w) = \circ_3\overline{v}(\mathtt{t}(\alpha), w)$. Therefore, from the table of consistency operator $\circ_3$, we have that $\overline{v}(\mathtt{t}(\circ\alpha), w) = F$.   □

Lemma 5.11. ($\mathbf{bC}^{k,l,m,n}$-Representability) *Given a $\mathbf{bC}^{k,l,m,n}$-valuation v in $\mathfrak{M}^{bC^{k,l,m,n}}_{\mathtt{Biv}}$ and a frame $\mathfrak{F} = \langle W, R \rangle$ for $\mathbf{bC}^{k,l,m,n}$ it is possible to find a translation $\mathtt{t}$ in $\mathrm{Tr}_{bC^{k,l,m,n}}$, a valuation $\overline{v}$ and a model $\mathfrak{M}^{bC^{k,l,m,n}}_{\mathtt{Thv}}$ such that, for each formula α in $\mathbf{bC}^{k,l,m,n}$ and for all $w \in W$:*

(i) $\overline{v}(\mathtt{t}(\alpha), w) = T$ *implies* $v(\neg\alpha, w) = 0$
(ii) $\overline{v}(\mathtt{t}(\alpha), w) = F$ *iff* $v(\alpha, w) = 0$

Proof. Consider $\overline{v} : Var \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val)\quad \overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Again, $\overline{v}$ can be homomorphically extended to all formulas in the matrix $\mathscr{M}_{bC^{k,l,m,n}}$. The translations differ from the previous lemma in:

$$(T5)\quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_3\mathtt{t}(\alpha) & \text{if } v(\alpha) = 1 = v(\neg\alpha) \\ \neg_1\mathtt{t}(\alpha) & \text{if } v(\alpha) = 0 \end{cases}$$

$$(T8)\quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_3\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \\ \circ_2\mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \text{ and } v(\neg\alpha) = 0 \\ \circ_2\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 1 \end{cases}$$

Choices are allowed in this case by restrictions (Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.6), (Tr.7), (Tr.10) and (Tr.11) which characterize translations of $\mathbf{bC}^{k,l,m,n}$. The model $\mathfrak{M}_{\mathtt{Thv}}^{bC\,^{k,l,m,n}}$ is obtained by extending $\mathfrak{F}$ with the valuation $\overline{v}$ defined above. The statement is proven by induction on the length of complexity of ℓ, with the same procedure used in Lemmas 5.9 and 5.10.   □

Lemma 5.12. ($\mathbf{Ci}^{\,k,l,m,n}$-Representability) *Given a $\mathbf{Ci}^{k,l,m,n}$-valuation v in $\mathfrak{M}_{\mathtt{Biv}}^{Ci\,^{k,l,m,n}}$ and a frame $\mathfrak{F} = \langle W, R \rangle$ for $\mathbf{Ci}^{\,k,l,m,n}$ it is possible to define a translation $\mathtt{t}$ in $\mathtt{Tr}_{Ci^{k,l,m,n}}$, a valuation $\overline{v}$ and a Kripke model $\mathfrak{M}_{\mathtt{Thv}}^{Ci\,^{k,l,m,n}}$ such that, for every α in $\mathbf{Ci}^{\,k,l,m,n}$ and for all $w \in W$:*

$$\overline{v}(\mathtt{t}(\alpha), w) = T \quad implies \quad v(\neg\alpha, w) = 0$$
$$\overline{v}(\mathtt{t}(\alpha), w) = F \quad iff \quad v(\alpha, w) = 0$$

Proof. Consider $\overline{v} : Var \times W \longrightarrow \{T, t, F\}$ defined as:

$$(Val) \quad \overline{v}(p, w) = \begin{cases} F & \text{if } v(p, w) = 0 \\ T & \text{if } v(\neg p, w) = 0 \\ t & \text{if } v(p, w) = 1 \end{cases}$$

Once more, $\overline{v}$ can be homomorphically extended to all formulas in the matrix $\mathscr{M}_{Ci^{k,l,m,n}}$. The translations differ from the previous lemma in:

$$(T5) \quad \mathtt{t}(\neg\alpha) = \begin{cases} \neg_3\mathtt{t}(\alpha) & \text{if } v(\alpha) = 1 = v(\neg\alpha) \\ \neg_1\mathtt{t}(\alpha) & \text{if } v(\alpha) = 0 \end{cases}$$

$$(T8) \quad \mathtt{t}(\circ\alpha) = \begin{cases} \circ_1\mathtt{t}(\neg\alpha) & \text{if } v(\circ\alpha) = 1 \\ \circ_1\mathtt{t}(\alpha) & \text{if } v(\circ\alpha) = 0 \end{cases}$$

In this case, choices are allowed by restrictions (Tr.1), (Tr.2), (Tr.3), (Tr.4), (Tr.6), (Tr.8), (Tr.9), (Tr.10) and (Tr.11) that characterize the admissible translations of $\mathbf{Ci}^{k,l,m,n}$. The model $\mathfrak{M}_{\mathtt{Thv}}^{Ci\,^{k,l,m,n}}$ is obtained by extending $\mathfrak{F}$ with the valuation $\overline{v}$ defined above. As expected, the statement is proven by induction on the length of complexity ℓ with the same procedure used in Lemmas 5.9 and 5.10.   □

Now, from the Representability Lemma for each class $\mathbf{PI}^{k,l,m,n}$, $\mathbf{mbC}^{k,l,m,n}$, $\mathbf{bC}^{k,l,m,n}$ and $\mathbf{Ci}^{k,l,m,n}$, showed in Lemmas 5.9–5.12, completeness w.r.t. modal possible translations semantics, can be obtained in each case, as done below.

Corollary 5.13. (Completeness w.r.t. MPT) *Let $\Gamma \cup \{\alpha\}$ a set of formulas of $\mathcal{L}^{k,l,m,n}$. Then:*

$$\Gamma \models_{\mathtt{MPT}} \alpha \quad implies \quad \Gamma \vdash_{\mathcal{L}^{k,l,m,n}} \alpha$$

Proof. Suppose that $\Gamma \nvdash_{\mathcal{L}^{k,l,m,n}} \alpha$; then, since $\mathcal{L}^{k,l,m,n}$ is sound and complete w.r.t. $\mathfrak{M}_{\mathtt{Biv}}^{\mathcal{L}\,^{k,l,m,n}}$ (see Corolary 4.15), there exists $w \in W$ and a $\mathcal{L}$-valuation v such that $v(\gamma, w) = 1$ for all $\gamma \in \Gamma$ and $v(\alpha, w) = 0$. By the Representability Lemma specific for each $\mathcal{L}^{k,l,m,n}$ (Lemmas 5.9–5.12), it is possible to find a translation $\mathtt{t}$ in $\mathtt{TR}_{\mathcal{L}^{k,l,m,n}}$ and a valuation $\overline{v}$ in $\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}\,^{k,l,m,n}}$ such that,

$\overline{v}(\mathtt{t}(\Gamma), w) \in \{T, t\}$ and $\overline{v}(\mathtt{t}(\alpha), w) = F$ so, $\mathtt{t}(\Gamma) \nvDash_{\mathfrak{M}_{\mathtt{Thv}}^{\mathcal{L}^{k,l,m,n}}} \mathtt{t}(\alpha)$. Therefore, by Definition 5.6, $\Gamma \nvDash_{\mathtt{MPT}} \alpha$. $\qquad\square$

As argued in [7], possible-translations semantics offer an immediate decision procedure for any system that is complete with respect to a possible-translations semantical structure $\mathrm{PT} = \langle \mathcal{M}, \mathrm{TR} \rangle$, provided $\mathcal{M}$ is decidable and TR is recursive. In this way, our modal possible-translations semantics immediately provide a decision procedure for the cathodic logics and constitute, in this way, an alternative to the well-known method of finite model property. Although a precise relationship between modal possible-translations semantics and finite model property is still to be clarified, it is very likely that both could be seen as expressions of a more universal construction.

A particularly interesting application of the modal possible-translations semantics is to give a modal interpretation to the theory of "impossible worlds" (cf. [18]) where E. Zalta proposes a metaphysical theory of *genuine impossible worlds* and argues that impossible worlds are coherent and can be a valid alternative for the analysis of philosophical questions. As an impossible world is one that *does not derive* all classical truths, and the paraconsistent worlds in the MPT *do not validate* all classical truths, it seems natural to take such worlds as legitimate impossible worlds, and, moreover, constructed upon sound mathematical foundations (as we have seen). All this, we believe, is universal logic at work.

Acknowledgements

References

[1] Batens, D.: Paraconsistent extensional propositional logics. Logique et analyse **90–91**, 195–234, (1980)

[2] Béziau, J.-Y.: From paraconsistent logic to universal logic. Sorites **12**, 5–32 (2001)

[3] Blok, W.J., Pigozzi, D.: Algebraizable Logics, Vol 396 of Memoirs of the American Mathematical Society. American Mathematical Society (1989)

[4] Bueno-Soler, J.: Completeness and incompleteness for anodic modal logics. J. Appl. Non Class. Logics **19**(3), 291–310 (2009). Pre-print available at: http://www.cle.unicamp.br/e-prints/vol_9,n_5,2009.html

[5] Bueno-Solerm, J.: Multimodalidades anódicas e catódicas: a negação controlada em lógicas multimodais e seu poder expressivo (Anodic and cathodic multimodalities: controled negation in multimodal logics and their expressive power). Ph.D Thesis, in Portuguese, IFCH-Unicamp, Campinas, Brazil (2009)

[6] Carnielli, W.A., Coniglio, M.E., Gabbay, D., Gouveia, P., Sernadas, C.: Analysis and Synthesis of Logics. Sringer, Amsterdam (2007)

[7] Carnielli, W.A., Coniglio, M.E., Marcos, J.: Logics of formal inconsistency. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 14, pp. 1–93. Springer, Amsterdam (2007)

[8] Costa-Leite, A.: Paraconsistência, modalidades e cognoscibilidade. Master's thesis, IFCH-Unicamp, Campinas, SP, Brazil (2003)

[9] Carnielli, W.A., Marcos, J.: A taxonomy of **C**-systems. In: Carnielli, W.A., Coniglio, M.E., D'Ottaviano, I.M.L. (eds.) Paraconsistency—The Logical Way to the Inconsistent, Lecture Notes in Pure and Applied Mathematics, vol. 228, pp. 1–94. New York, Marcel Dekker (2002)

[10] Coniglio, M.E.: Logics of deontic inconsistency. CLE e-Prints **7**(4) (2007). ftp://logica.cle.unicamp.br/pub/e-prints/vol.7,n.4,2007.pdf

[11] Canielli, W.A., Pizzi, C.: Modalities and Multimodalities. Springer, Amsterdam (2008)

[12] Coniglio, M.E., Peron, N.M.: A paraconsistentist approach to Chisholm's paradox. In: Fourth World Congress of Paraconsistency (WCP4): The Fourth World Congress of Paraconsistency, pp. 18–19. Ormond College, Melbourne (2008)

[13] da Costa, N.C.A., Carnielli, W.A.: On paraconsistent deontic logic. Philosophia **16**(3/4), 293–305 (1986)

[14] Henkin, L.: Fragments of the propositional calculus. J. Symb. Logic **14**(1), (1949)

[15] Hansen, J., Pigozzi, G., van der Torre, L.: Ten philosophical problems in deontic logic. In: Boella, G., van der Torre, L., Verhagen, H. (eds.), Normative Multi-agent Systems, number 07122 in Dagstuhl Seminar Proceedings, Dagstuhl, Germany, 2007. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany

[16] Lemmon, E.J., Scott, D.: An introduction to modal logic. Blackwell, Oxford (1977)

[17] Marcos, J.: Possible-translations semantics for some weak classically based paraconsistent logics. J. Appl. Non Class. Logics **18**(1):07–28 (2008)

[18] Zalta, E.N.: A classically-based theory of impossible worlds. Notre Dame J. Formal Logic **38**(4):640–660 (1997)

Juliana Bueno-Soler
Department of Computer Science
Institute of Mathematics and Statistics
University of São Paulo
São Paulo, SP, Brazil
e-mail: `juliana.bueno@cle.unicamp.br`