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Plato on Why Mathematics is Good for the Soul

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1. The question

ANYONE WHO HAS READ Plato's *Republic* knows it has a lot to say about mathematics. But why? I shall not be satisfied with the answer that the future rulers of the ideal city are to be educated in mathematics, so Plato is bound to give some space to the subject. I want to know why the rulers are to be educated in mathematics. More pointedly, why are they required to study so much mathematics, for so long?

They start in infancy, learning through play (536d–537a). At 18 they take a break for two years' military training. But then they have another ten years of mathematics to occupy them between the ages of 20 and 30 (537bd). And we are not talking baby maths: in the case of stereometry (solid as opposed to plane geometry), Plato has Socrates make plans for it to develop more energetically in the future (528bd), because it only came into existence (thanks especially to Theaetetus) well after the dramatic date of the discussion in the *Republic*. Those ten years will take the Guards into the most advanced mathematical thinking of the day. At the same time they are supposed to work towards a systematic, unified understanding of subjects previously learned in no particular order (χύδην). They will gather them together to form a synoptic view of all the mathematical disciplines 'in their kinship with each other and with the nature of what is' (537c). I shall come back to this enigmatic statement later. Call it, for the time being, Enigma A.

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The extent of mathematical training these people are to undergo is astounding. They are not preparing to be professional mathematicians; nothing is said about their making creative contributions to the subject. Their ten years will take them to the synoptic view, but then they switch to dialectic and philosophy. They are being educated for a life of philosophy and government. How, we may ask, will knowing how to construct an icosahedron (Figure 1) help them when it comes to regulating the ideal market or understanding the Platonic Theory of Forms?

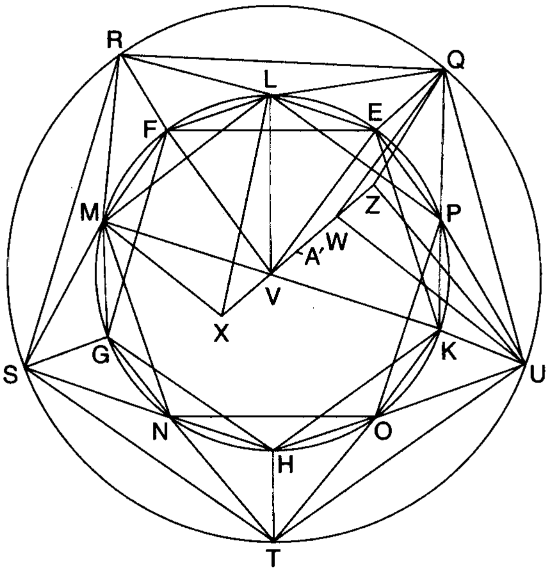


Figure 1

The question is reminiscent of debates in the not so distant past about the value of a classical education. Why should the study of Greek and Latin syntax be advocated, as once it was, as the ideal preparation for entering the Civil Service or the world of business? No doubt, any rigorous discipline helps train the mind and imparts 'transferable skills'. But that is no reason to make Latin and Greek compulsory when other disciplines claim to provide equal rigour, e.g. mathematics. Conversely, readers of the *Republic* are entitled to put the question to Plato: why so much mathematics, rather than something else?

All too few scholars put this question, and when they do, they tend to answer by stressing the way mathematics trains the mind. Plato 'is proposing a curriculum for mental discipline and the development of abstract thought'; he believes no one can become 'a moral hero or saint' without 'discipline in sheer hard thinking'; he advocates mathematics 'not simply because it involves turning away from sense perception but because it is constructive reasoning pursued without reference to immediate instrumental usefulness'.[1] Like the dry-as-dust classicists for whom the value of learning Greek had nothing to do with the value of reading Plato or Homer, this type of answer implies that the *content* of the mathematical curriculum is irrelevant to its goal. At best, if the chief point of mathematics is to encourage the mind in abstract reasoning, the curriculum may help rulers to reason abstractly about non-mathematical problems in ethics and politics.

One ancient writer who did think that mind-training is the point was Plato's arch-rival, the rhetorician Isocrates (*Antidosis* 261–9, *Panathenaicus* 26–8). Speaking of the educational value of mathematics and dialectic, he said it is not the knowledge you gain that is beneficial, but the process of acquiring it, which demands hard thought and precision.[2] From this he concluded, quite reasonably, that young men should not spend too much time on mathematics and dialectic. Having sharpened up their minds, they should turn to more important subjects like public speaking and government. Isocrates was not trying to elucidate Plato's thought. He was sketching a commonsensical *alternative* role for mathematics and

1 Quoted from, respectively, Paul Shorey, *What Plato Said* (Chicago & London, 1933), p. 236; A. E. Taylor, *Plato: The Man and his Work* (London & New York, 1926), p. 283; Terence Irwin, *Plato's Ethics* (New York & Oxford, 1995), p. 301.

2 Quintilian, *Institutio oratoria* I 10.34, describes this as the common view (*vulgaris opinio*) of the educational value of mathematics, and goes on to assemble more substantive (but still instrumental) reasons why an orator needs a mathematical training. Galen, *περὶ ψυχῆς ἁμαρτημάτων* 49.24–50.1 Marquardt, mentions a variety of disciplines by which the soul is sharpened (*θήγεται*) so that it will judge well on practical issues of good and bad: logic, geometry, arithmetic, calculation (*λογιστική*), *architecture*, and astronomy. If architecture (as a form of technical drawing), why not engineering? And what could beat librarianship for encouraging a calm, orderly mind?

dialectic, to counteract the excessive claims coming from the Academy. Mathematics and dialectic would hone young minds for an education in rhetoric.

Isocrates presents himself as taking a conciliatory approach on a controversial issue. Most people, he says, think that mathematics is quite useless for the important affairs of life, even harmful. No one would say that now, because we live in a world which in one way or another has been transformed by mathematics. No one now reads Sir William Hamilton on the bad effects of learning mathematics, so no one needs John Stuart Mill's vigorous and moving riposte.[3] In those days, however, a sophist like Protagoras could openly boast about saving his pupils the bother of learning the 'quadrivium' (arithmetic, geometry, astronomy, and harmonics), which his rival Hippias insisted on teaching; instead of spoiling his pupils' minds with mathematics, Protagoras would proceed at once to what they really wanted to learn, the skills needed to do well in private and public life (Plato, *Protagoras* 318de). At a more philosophical level, Aristippus of Cyrene, who like Plato had been a pupil of Socrates, could lambast mathematics because it teaches nothing about good and bad (Aristotle, *Metaphysics* B 2, 996a 32–b 1). Xenophon's Socrates contradicts Plato's by setting narrowly practical limits to the mathematics required for a good education: enough geometry to measure land, enough astronomy to choose the right season for a journey. Anything more complicated, he says, is a waste of time and effort, while it is impious for astronomers to try to understand how God contrives the phenomena of the heavens (*Memorabilia* IV 7.1–8).

These ancient controversies show that the task of persuasion Plato set himself was still harder then than it would be today. Even Isocrates' mind-sharpening recommendation could not be taken for granted.

A very different account of the mind-sharpening value of mathematics can be found in a later Platonist (uncertain date AD)

[3] Sir William Hamilton, 'On the Study of Mathematics, as an Exercise of Mind', in his *Discussions on Philosophy and Literature, Education and University Reform* (Edinburgh & London, 1852), pp. 257–327; John Stuart Mill, *An Examination of Sir William Hamilton's Philosophy* (London, 1865), chap. 27.

called Alcinous, who says that mathematics provides the precision needed to focus on real beings, meaning abstract, non-sensible beings (*Didaskalikos* 161.10–13 ff.).[4] As we shall see, mathematical objects can only be grasped through precise definition, not otherwise, so there is good sense in the idea that precision is the essential epistemic route to a new realm of beings.[5] In that spirit, more enlightened classicists promote Greek and Latin as a means of access to a whole new realm of poetry and prose which you cannot fully appreciate in translation.

This seems to me a more satisfactory version of the mind-sharpening view than we find in Isocrates, who thinks of mathematics as providing a content-neutral ability you can apply to any field. But I shall argue that Alcinous still does not go far enough. My comparison would be with a classicist who dared claim that embodied in the great works of antiquity is an important part of the truth about reality and the moral life.[6]

The goal of the mathematical curriculum is repeatedly said to be knowledge of the Good (526de, 530e, 531c, 532c). That ten-year immersion in mathematics is the propaedeutic prelude (531d, 536d) to five years' concentrated training in dialectical discussion (539de), which will eventually lead the students to knowledge of the Good. I say 'eventually', because at the age of 35 they break off for 15 years' practical experience in a variety of military and administrative offices (539e–540a). Only when they reach 50 do they resume dialectic for the final ascent to see the Good, the *telos* for which their entire education has been designed (540ab). Knowledge of the Good is obviously relevant to government and to philosophy. So

[4] Alcinous' phrase is θήγουσα τὴν ψυχήν, as in Galen (above, n. 2). The Latin equivalent is *acuere*: Quintilian, *Institutio oratoria* I 10.34, Cicero, *De Republica* I 30.

[5] For a comparable approach today, see Julia Annas, *An Introduction to Plato's Republic* (Oxford, 1981), pp. 238–9, 250–1, 272–3.

[6] In this approach my closest ally is J. C. B. Gosling, *Plato* (London, Boston, Melbourne, & Henley, 1973), chap. 7, but see also, briefly, John Cooper, 'The Psychology of Justice in Plato', *American Philosophical Quarterly*, 14 (1977), 155, repr. in his *Reason and Emotion: Essays on Ancient Moral Psychology and Ethical Theory* (Princeton, 1999), p. 144, and James G. Lennox, 'Plato's Unnatural Teleology', in Dominic J. O'Meara (ed.), *Platonic Investigations* (Studies in Philosophy and the History of Philosophy Vol. 13, Washington, DC, *c.* 1985), n. 30, pp. 215–18.

my question can be put like this: Is the study of mathematics merely instrumental to knowledge of the Good, in Plato's view, or is the content of mathematics a constitutive part of ethical understanding? I shall argue for the latter.[7]

2. Outline of the answer

To launch this idea, and to help make it, if not palatable, at least more intelligible than it is likely to be at first hearing, I shall take a modern foil — a tough-minded logical empiricist of the twentieth century, whose argument I find both strikingly reminiscent of Plato's *Republic* and revealingly different:

> We walk through the world as the spectator walks through a great factory: he does not see the details of machines and working operations, or the comprehensive connections between the different departments which determine the working processes on a large scale. He sees only the features which are of a scale commensurable with his observational capacities: machines, workingmen, motor trucks, offices. In the same way, we see the world in the scale of our sense capacities: we see houses, trees, men, tools, tables, solids, liquids, waves, fields, woods, and the whole covered by the vault of the heavens. This perspective, however, is not only one-sided; it is false, in a certain sense. Even . . . the things which we believe we see as they are, are objectively of shapes other than we see them. We see the polished surface of our table as a smooth plane; but we know that it is a network of atoms with interstices much larger than the mass particles, and the microscope already shows not the atoms but the fact that the apparent smoothness is not better than the 'smoothness' of the peel of a shriveled apple. We see the iron stove before us as a model of rigidity, solidity, immovability; but we know that its particles perform a violent dance, and that it resembles a swarm of dancing gnats more than the picture of solidity we attribute to it. We see the moon as a silvery disk in the celestial vault, but we know it is

[7] This will involve revisiting a number of themes I discussed in 'Platonism and Mathematics: A Prelude to Discussion', in Andreas Graeser (ed.), *Mathematics and Metaphysics in Aristotle* (Xth Symposium Aristotelicum, Bern & Stuttgart, 1987), pp. 213–40. But here they will receive a more expansive treatment, with fewer references to the scholarly literature than was appropriate to the earlier Symposium. Naturally, I cannot promise to be entirely consistent now with what I wrote then.

> an enormous ball suspended in open space. We hear the voice coming from the mouth of a singing girl as a soft and continuous tone, but we know that this sound is composed of hundreds of impacts a second bombarding our ears like a machine gun. The [objects] as we see them have as much similarity to the objects as they are as the little man with the caftan seen in the moor [at dusk from afar] has to the juniper bush [it turns out to be], or as the lion seen in the cinema has to the dark and bright spots on the screen. We do not see the things . . . as they are but in a distorted form; we see a *substitute world*— not the world as it is, objectively speaking.

So wrote Hans Reichenbach in 1938.[8] The idea he formulates of the world as it is objectively speaking is the idea of what the world is discovered to be when one filters out the cognitive effects of our human perspective. More fully, it is the idea of the world described in a way that takes account of all the aspects we miss from our usual perspective, so as to explain why we experience it as we do: the moon is both a silvery disk and an enormous ball far away, and it is the one because it is the other. This idea, I claim, received its first full-scale formulation and defence in the central Books of Plato's *Republic*. Reichenbach's cinema is a twentieth-century version of Plato's famous simile of the cave. Plato is the better poet, but his philosophy is no less tough-minded. Both cinema and cave make us look at our ordinary experience of the world from the outside, as it were, to see how inadequate it is by comparison with the view we would have from the standpoint of a scientific account of the world as it is objectively speaking. The cinema analogy, like the Cave, expresses the idea that human experience is just a particular, parochial perspective which we must transcend in order to achieve a full, accurate, and properly explanatory view of things.

So much for the similarity. But of course there are also differences. Reichenbach can put across his version of the idea in a couple of pages, because his readers grew up in an age already familiar with the contrast between the world as humans experience

[8] *Experience and Prediction: An Analysis of the Foundations and the Structure of Knowledge* (Chicago, 1938), pp. 219–20, omitting three occurrences of his technical term 'concreta'; the example of the little man with the caftan was introduced at p. 198. In his Preface Reichenbach aligns himself with philosophical movements which share 'a strict disavowal of the metaphor language of metaphysics'!

it and the world as science explains it. In Plato's time the idea was a novelty, harder to get across. Moreover, Plato was addressing a wider readership than a technical book of modern philosophy can hope to reach. His readers have further to travel from where they start to where he wants them to end up. They need the imagery and the panoply of persuasive devices that enliven the long argument of *Republic* Books V–VII.

Another difference is that Reichenbach can rest on the *authority* that science enjoys in the modern world. In Plato's day no system of thought or explanation had such authority. Everything was contested, every scheme of explanation had to compete with rivals. Modern logic is a further resource that Reichenbach can take for granted. In Plato's day logic was not yet invented, let alone established. Methods of reasoning and analysis were as contested as the content they were applied to.

But the really big difference between Reichenbach's and Plato's version of the idea of the world as it is objectively speaking is the following. For Reichenbach in the twentieth century the world as it is objectively speaking is the world as described by modern science, above all mathematical physics, and in that description there is no room for values. The world as it is objectively speaking, seen from the standpoint of our most favoured science, is a 'disenchanted' world without goodness in it. For Plato, by contrast, the most favoured science — in his case, mathematics — is precisely what enables us to understand goodness. The mathematical sciences are the ones that tell us how things are objectively speaking, and they are themselves sciences of value. Or so I shall argue. If I am right, understanding the varieties of goodness is for Plato a large part of what it means to understand the world as it is objectively speaking, through mathematics. Plato, like Aristotle and the Stoics after him, really did believe there is value in the world as it is objectively speaking, that values are part of what modern philosophers like to call 'the furniture of the world'.

This is not the place or the time to consider how and why the world became 'disenchanted'. Let it be enough that an understanding of impersonal, objective goodness is for Plato the climax and *telos* of an education in mathematics. It is this concept of impersonal, objective goodness that links the epistemology and metaphysics of the *Republic* to its politics. Plato's vision of the world as it is objectively speaking is the basis, as Reichenbach's could never be, for a political project of the most radical kind. The moral of the Cave is that Utopia can be founded on the rulers' knowledge of the world as it is objectively speaking, because that includes the Good and the whole realm of value.

3. *By-products*

It is relatively easy to prove the negative point that Socrates in the *Republic* does not recommend mathematics solely for its mind-training, instrumental value. He says so himself.

We may start with arithmetic. Socrates gives three reasons why this is a 'must' (ἀναγκαῖον — 526a 8) for the further education of future rulers. His chief reason, expounded at length, is that arithmetic forces the soul towards an understanding of what numbers are in themselves, and thereby focuses thought on a realm of unqualified truth and being (526b, summing up the result of 524d–526b). More about that later. Then he adds two further reasons, each stated briefly. First, arithmetic makes you quicker at other studies, all of which involve number in some way (526b with 522c); this sounds like what we call transferable skills. Second, the subject is extremely demanding to learn and practise (526c); as such, it is a good test of intellectual and moral calibre (cf. 503ce, 535a–537d).

Thus far the relative ranking of intrinsic and instrumental benefits is left implicit. The next section, on (plane) geometry, should leave an attentive reader in no doubt where Plato's priorities lie. Having recommended that geometry be studied for the sake of knowing what everlastingly is, not for the sake of action in the here and now (527ab),[9] Socrates acknowledges that, besides its capacity to drag the soul upwards towards truth, geometry has certain by-products (πάρεργα) which are, he says, 'not small', namely, 'its uses in war, which you mentioned just now, and besides, for the

[9] So too arithmetic should be studied for the sake of knowledge, not trade (525d).

better reception of all studies we know there will be an immeasurable difference between a student who has been imbued with geometry and one who has not'[10] (527c). The term 'by-products' should be decisive. Both the practical application of geometry in war (e.g. for troop formation and the laying out of camp sites — 526d) and transferable skills are relegated to second rank in comparison to pure theoretical knowledge. Plato would hardly write in such terms if he valued geometry for content-neutral skills that the Guards can later apply when ruling or trying to understand the Good. This conclusion is reinforced when we see that the passage belongs to a sequence of episodes which climax in a strong denunciation of any demand for the curriculum to be determined by its practical pay-off.

At the start of the discussion Socrates made a point of saying that any studies chosen for the curriculum must not be useless (note the double negative) for warriors. This is because he and Glaucon are planning the further education of people who have been trained so far to be 'athletes in war' (521d). Arithmetic satisfies that condition, he argues, because a warrior must be able to count and calculate (522e). True, but that is hardly adequate justification for ten years' immersion in number theory. Notice, however, that the justification is introduced by a joke: how ridiculous Agamemnon is made to look in the tragedies which retail the myth that Palamedes was the discoverer of number, the one who marshalled the troops at Troy and counted the ships. As if until then Agamemnon did not even know how many feet he had (522d)! Glaucon agrees. The ability to count and calculate is indeed a 'must' for a warrior, if he is to understand anything about marshalling troops — or rather, Glaucon adds, if he is to be a human being (522e). This last is the give-away. Plato is not serious about

[10] Translations from the *Republic* are my own, but I always start from Shorey's Loeb edition (Cambridge, Mass., 1930–35), so his phrases are interwoven with mine. For passages dealing with music theory, I have borrowed freely from the excellent rendering (with useful explanatory notes) given by Andrew Barker, *Greek Musical Writings*, Vol. II: *Harmonic and Acoustic Theory* (Cambridge, 1989), hereafter cited as *GMW* II. It will become clear how much, as a beginner in mathematical harmonics, I owe to Barker's work.

justifying the study of arithmetic on grounds of its practical utility. His real position becomes clear later (525bc): while it is true that a warrior needs the arithmetical competence to marshal troops in the world of becoming, a *philosopher* needs to study arithmetic for the quite different reason that it turns the soul away from the world where battles are fought. The Guards will continue to be warriors as well as philosophers, but it is their philosophical education that is top of the agenda now.[11]

Glaucon is slow to grasp the point. When the discussion turns to (plane) geometry, it is he who enthuses about the importance of geometry for laying out camp sites, occupying territory, closing up or deploying an army, and manœuvring in battle or on the march (526d). Socrates drily responds that you do not need much geometry (or calculation) for things like that. What we should be thinking about, he says, is whether geometry — geometry at an advanced level[12] — will help one come to know the Good (526de).

Plato did not write these exchanges just to have some fun at his brother's expense. He is preparing a surprise for his readers. The surprise comes when we reach astronomy. Glaucon duly commends the study on the grounds that generals, like sailors and farmers, need to be good at telling the seasons (527d). (Invading armies should beware of Russia in the winter months.)[13] This time Glaucon is on to something worthwhile. Weather-prediction is indeed as important for generals as it is for sailors and farmers, and in the ancient world one of the tasks of astronomy was to construct tables (παραπήγματα) which correlated each day of the

[11] The distinction of roles (warrior *vs* philosopher) provides the context for the claim at 525c that arithmetic should be studied 'both for the sake of war and to attain ease in turning the soul itself from the world of becoming to truth and reality' (525c 4–7), about which Annas, *Introduction to Plato's Republic*, 275, unfairly remarks, 'This utterly grotesque statement may sum up quite well the philosophy behind a lot of NATO research funding.' It would be more apt to wonder how the distinction of roles squares with the 'one man–one job' principle on which the ideal city was founded in Book II.

[12] So too arithmetic should be taken to an advanced level (525c: μὴ ἰδιωτικῶς).

[13] Note that he does not cite Nicias' disastrously superstitious response to the eclipse that occurred when he was one of the generals in charge of the Athenian forces at Syracuse (Thucydides VII 50). To understand eclipses, you need more theory than Glaucon thinks to recommend.

month with the risings and settings of different stars and likely weather patterns:

> Day 6: the Pleiades set in the morning; it is winter and rainy.
> Day 26: summer solstice; Orion rises in the morning; a south wind blows.[14]

An ancient reader would feel that something of real practical utility was under attack when Socrates laughs at Glaucon's justification of astronomy:

> 'You sweet fellow', I said, 'You seem to be afraid of the general public (τοὺς πολλούς), worried you will be thought to recommend studies that have no practical use. The fact is, it's far from easy, it's difficult to hold fast to the belief that there is an instrument (ὄργανον) in the soul which is purged and rekindled in these studies after being ruined and blinded by other pursuits — an instrument more worth saving than a thousand eyes, for only by this can the truth be seen.' (527de)

This leads on to another and bigger surprise, which has shocked modern readers as well. In the ideal city a new kind of astronomy is to be taught, an astronomy that will 'leave the things in the sky alone' in order to concentrate, as geometry does, on 'problems' (530b).[15] The new astronomy will be a purely mathematical study of geometrical solids (spheres) in rotation (528a, e), a sort of abstract kinematics; for only a study of *invisible* being will turn the soul's gaze upwards in the sense that interests Socrates (529b). The idea of an astronomy of the invisible is another topic I shall return to later (call it Enigma B), for, odd as it may seem at first reading, the astronomy section of the *Republic* stands at the origin of the great tradition of Greek mathematical astronomy which culminated in the cosmological system of Claudius Ptolemy. At present I am interested in those impressive-sounding words about the instrument of the soul. How exactly will abstract kinematics enlighten this instrument and prepare it for knowledge of the Good?

[14] Sample entries from D. R. Dicks, *Early Greek Astronomy to Aristotle* (Ithaca, 1970), p. 84.

[15] On the meaning of the term 'problems', see below, n. 18.

What I claim to have shown so far is that the answer has nothing to do with practical utility or transferable skills. These have been faintly praised as 'not small', and set aside. The discussion continues: first stereometry, then back to astronomy, and finally a purely mathematical version of harmonics — but practical utility and transferable skills are not mentioned again. The instrumental benefits of studying mathematics remain πάρεργα, mere by-products of the first two disciplines on the curriculum.

4. *Formal rigour*

To this a further negative point can be added. The benefit of mathematics does not reside in its rigorous procedures. Greek mathematics typically involves deduction from hypotheses, the use of diagrams, and various forms of abstraction to make empirical objects susceptible to mathematical treatment. These formal features (illustrated below) are responsible for the impressive rigour of so much ancient mathematics. But some of the mathematics Plato knows is deliberately excluded from the curriculum of the ideal city. I infer that the ticket for admission is not formal rigour as such.

One significant exclusion is Pythagorean harmonics. This is described as a mathematical analysis of the ratios that structure the scales used in actual music: 'They seek the numbers in these heard concords (συμφωνίαις) and do not ascend to problems to consider which numbers are concordant, which are not, and why each are so' (531c). In Pythagorean music theory,[16] the basic concords are the

[16] By which I mean the theory Plato could study in written works by Philolaus (second half of the fifth century) and Archytas (first half of the fourth century), some fragments of which remain for us to study too: the best source to use is Barker, *GMW* II, chap. 1. Before Walter Burkert's great work, *Lore and Science in Ancient Pythagoreanism* (Cambridge, Mass., 1972, translated by Edwin L. Minar from the German edition of 1962), it was universally believed that the mathematical analysis of the concords goes back to Pythagoras himself (sixth century BC). Now, even that bit of the mathematics for which the Pythagoreans were once celebrated is lost in clouds of mythology.

octave, represented by the ratio 2:1, the fourth (4:3) and the fifth (3:2). We may be surprised at the idea of a ratio being concordant in its own right, because it is the ratio it is, irrespective of the acoustic properties of the notes produced by plucking strings whose lengths have that ratio to each other (call this Enigma C). But the idea is on a par with the carefully prepared idea of astronomy as abstract kinematics (Enigma B). These Pythagorean musical theorists go wrong, on Socrates' view (531b 8–c 1), in just the same way as astronomers go wrong if they focus on the observed phenomena and try to explain, in terms of whole-number ratios (*συμμετρίαι*), the relation of night to day, of these to the month, and of the month to the year (530ab).

The allusion is probably to the project of devising an intercalation cycle to reconcile lunar and solar calendars. Because of discrepancies between the lunar month and the solar year, a harvest festival scheduled for full moon in a certain month of autumn will 'drift' to summer, spring, and winter unless adjustments are made to the calendar. The solution, if you want the festival to take place when the crops are in, rather than before they are sown, is to find an extended period of time which is a common multiple of the lunar and solar cycles, and to intercalate months as necessary to keep the calendars in synch. The best-known authors of such a scheme, Meton and Euctemon in the late fifth century BC, were not Pythagorean,[17] but that does not undermine the *Republic*'s emphatic parallel between harmonics and astronomy. Both should be approached in a way that lifts the mind out of and away from the

[17] Accordingly, the phrase *τοῖς ἐν ἀστρονομίᾳ* (531b 8–c 1) does not specify Pythagoreans. Since these are astronomers who look for *συμμετρίαι* in the observed motions of the heavenly bodies (529d–530b), more is involved than an observational record of risings and settings, etc. Nothing so mathematically detailed as the work of Meton and Euctemon (on which see Dicks, *Early Greek Astronomy*, pp. 85–9) is recorded for Philolaus, as can be verified by consulting Carl A. Huffman, *Philolaus of Croton: Pythagorean and Presocratic, A Commentary on the Fragments and Testimonia with Interpretive Essays* (Cambridge, 1993), Part III 4. As for Archytas, all we have is his description of astronomy in frag. 1 (quoted below): it has achieved 'a clear understanding of the speed of the heavenly bodies and their risings and settings'. In the context he means a mathematical understanding, but he does not claim to have contributed to this himself.

sensible world. They should adopt the 'problem'-oriented style characteristic of arithmetic and geometry.[18]

We now have two examples of Socrates denying a place on the curriculum to a current branch of mathematics. Besides these, he hints at other branches of Pythagorean mathematics,[19] warning Glaucon that they must guard against any study that lacks purpose or completion (ἀτελές); by this he means any study that does not lead to the goal the curriculum is designed for, which is to make 'the naturally intelligent part of the soul useful instead of useless' (530e, recalling 530bc). The naturally intelligent part of the soul is presumably the same as the instrument Socrates spoke of earlier as needing to be purged and rekindled to see the Good. Socrates does not name these other mathematical studies, but we can make a guess. For he starts his discussion of harmonics by quoting, from (as he puts it) 'the Pythagoreans' a remark to the effect that astronomy and harmonics are 'sister sciences' (530d: ἀδελφαὶ ἐπιστῆμαι). We can identify the author of that saying. It was a Pythagorean closer in age to Plato than to Socrates: the philoso-

[18] The word 'problem' here and at 530b (cited above) has often been interpreted (e.g. Burkert, *Lore and Science*, 372 n. 11, p. 424, Alexander P. D. Mourelatos, 'Plato's "Real Astronomy": *Republic* 527d–531d', in John P. Anton (ed.), *Science and the Sciences in Plato* [New York, 1980], pp. 60–2) in the light of a distinction between 'theorems' and 'problems' which, according to Proclus, *Commentary on the First Book of Euclid's Elements*, 77.7–81.22, was a subject of debate in the Academy and earlier. Theorems are assertions, the proof of which ends 'Which was to be demonstrated (Q.E.D.)'. Problems are constructions (e.g. Euclid, *Elements* I 1: 'On a given finite straight line to construct an equilateral triangle'), divisions of a figure, and other activities that end with the words 'Which was to be done'. But neither Glaucon nor the reader could be expected to latch on to this technical meaning without further guidance. The only guidance in the text is the comparison with geometry (530b), which obviously includes 'theorems' as well as 'problems'. At *Theaetetus* 180c (the closest parallel in Plato) the word 'problem' certainly suggests geometry, but equally certainly it suggests a 'theorem' rather than a construction of some sort. Accordingly, I agree with Ian Mueller, 'Ascending to Problems: Astronomy and Harmonics in *Republic* VII', in Anton, *Science and the Sciences*, 10 n. 13, that we should not translate in such a way as to *confine* Platonic astronomy and harmonics to problems in the technical sense. But of course astronomical constructions may be included, and mathematical harmonics will certainly involve dividing the scale.

[19] See the quotation below, with n. 24.

pher, statesman, general, and mathematician of genius, Archytas of Tarentum.

The phrase 'sister sciences' comes from the opening of a work on harmonics, where Archytas sums up the progress of mathematics to date:

> Those who are concerned with the sciences (μαθήματα) seem to me to be men of excellent discernment, and it is not strange that they conceive particular things correctly, as they really are. For since they exercised good discrimination about the nature of the wholes, they were likely also to get a good view of the way things really are taken part by part. They have handed down to us a clear understanding of the speed of the heavenly bodies and their risings and settings, of geometry, of numbers, and not least of music (μωσικᾶς). *For these sciences seem to be sisters.* (Archytas frag. 1 Diels-Kranz; emphasis mine)[20]

Not only is Archytas the one and only Pythagorean to whom history (as opposed to mythology) credits important mathematical discoveries. He is also the founder of a discipline in which Archimedes was later to excel, mathematical mechanics.[21] Plato would certainly not want that on the curriculum.[22] In addition, I think I can show, though not on this occasion, that Archytas was the founder of mathematical optics, such as we find it in Euclid. I

[20] Tr. Barker, *GMW* II, 39–40 with nn. 42–4; text as defended against Burkert's suspicions (*Lore and Science*, pp. 379–80 n. 46: it is a later forgery, designed to match Plato's quotation) by Carl A. Huffman, 'The authenticity of Archytas fr. 1', *Classical Quarterly*, 35 (1985), 344–8. In the more empirically minded Ptolemy, *Harmonics* III 3, p. 93.20–94.20, the sisters are sight and hearing, while their offspring, astronomy and harmonics, are cousins. Archytas, of course, describes all four mathematical sciences as sisters. In so doing he is disagreeing with his predecessor Philolaus, who singled out geometry as 'the mother-city' (μητρόπολις) of the others (Plutarch, *Moralia* 718e; discussion in Huffman, *Philolaus*, 193–9).

[21] Diogenes Laertius VIII 83.

[22] Plutarch's story that Plato censured Archytas, Eudoxus, and Menaechmus for using mechanical devices to find the two mean proportionals needed to double a cube (*Moralia* 718ef; cf. *Marcellus* 14.6) is surely fiction (derived from Eratosthenes' *Platonicus*), but, as Plutarch himself has just remarked of the story that Plato said 'God is always doing geometry' (718c), it has an authentically Platonic ring to it. Plato would agree that the good of geometry (τὸ γεωμετρίας ἀγαθόν) is lost by 'running back to sensible things'.

conclude that, when Plato wrote the *Republic*, there was quite a lot of mathematics in existence which he did *not* want on the curriculum to be studied by the future rulers of the ideal city. His black list includes Pythagorean harmonics, contemporary mathematical astronomy, mathematical mechanics and, I believe, mathematical optics. However subtle and rigorous the mathematics, these studies would all keep the mind focused on sensible things. They do not abstract from sensible features as much as Plato requires.[23]

We should look at the way Socrates introduces Archytas' dictum. He has just said that astronomy should be pursued in the same way as geometry. The visible patterns of motion in the heavens should be treated like the diagrams in geometry, as an aid to thinking about purely abstract mathematical problems (529d–530c). He then continues:

> 'Motion . . . presents not just one but several forms, as it seems to me. A wise man, perhaps, will be able to name them all, but two are quite obvious even to us.'
> 'What kinds are they?'
> 'In addition to the one we have discussed [the motion studied by astronomy]', I said, 'there is its counterpart.'
> 'What sort is that?'

[23] For optics and mechanics in particular as 'subordinate sciences', hence 'more physical' than the abstract mathematics they are subordinate to, see Aristotle, *Posterior Analytics* I 13, 78b 34–79a 16, *Physics* II 2, 194a 7–12, *Metaphysics* XIII 3, 1078a 14–17, and James G. Lennox, 'Aristotle, Galileo, and "Mixed Sciences"', in William A. Wallace (ed.), *Reinterpreting Galileo* (Studies in Philosophy and the History of Philosophy Vol. 15, Washington, DC, 1985), pp. 29–51. Enigma C is Plato's determination to rescue harmonics from being classified, as Aristotle does classify it, on the same level as optics and mechanics. Note that if I am right about Plato's deliberately excluding optics and mechanics from the curriculum, this is quite compatible with the evidence provided by Philodemus, *Academicorum Philosophorum Index Herculanensis* Col. Y, 15–17, as printed in François Lasserre, *De Léodamas de Thasos à Philippe d'Opunte: Témoignages et fragments* (Naples, 1987), p. 221, that both optics and mechanics were cultivated by mathematicians associated with the Academy. Even if this activity postdates the *Republic*, Plato was never in a position to tell grown-up mathematicians what to do or not do (compare *Rep.* 528b 9–c 1), any more than he could (or would) tell grown-up philosophers what to believe: Speusippus, his nephew and successor as head of the school, rejected the Theory of Forms entirely. The educational curriculum of the *Republic* is designed to produce future rulers in an ideal city, not to confine research in real-life Athens to subjects that will lead to knowledge of the Good.

> 'It is probable', I said, 'that as the eyes are framed for astronomy, so the ears are framed for harmonic motion, and that these two sciences are sisters of one another, as the Pythagoreans say — *and we agree*, Glaucon, do we not?'
> 'We do', he said.
> 'Then', I said, 'since the task is so great, shall we not inquire of them [the Pythagoreans] how they speak of these [sciences] and whether they have any other [science] to add?[24] And in all this we will be on the watch for what concerns us.'
> 'What is that?'
> 'To prevent our fosterlings trying to learn anything incomplete (ἀτελές), anything that does not come out at the destination which, as we were saying just now about astronomy, ought to be the goal of it all.' (530ce)

Socrates has already taken astronomy up to the same abstract level as geometry. He will now preserve the 'sisterhood' of astronomy and harmonics by redirecting the latter to the same abstract level as arithmetic.[25] Any science that does not lend itself to such redirection is to be excluded altogether. In other words, Socrates agrees with Archytas' coupling of astronomy and harmonics, but condemns his empirical approach, which seeks numbers in the observed phenomena. Both astronomy and harmonics should be relocated to the mathematics section of the Divided Line. Then the five mathematical disciplines on the curriculum will all be sister sciences. An alert reader may recall that in the Divided Line passage (511b 1–2) Plato put into Glaucon's mouth the phrase 'geometry and its sister

[24] On my translation of πῶς λέγουσι περὶ αὐτῶν καὶ εἴ τι ἄλλο πρὸς τούτοις, the pronouns refer to the closest antecedent, the two sister sciences. Socrates proposes to ask the Pythagoreans how they conceive astronomy and harmonics and whether they have other sciences to recommend besides these two. (The interrogation does not happen in the pages of the *Republic*, for at 531b 7–8 it still lies in the future.) Most translators retreat into vagueness. Bloom (1968) translates as I do, but without specifying the reference. Reeve (1992) refers the pronouns to the more distant ἐναρμόνιον φοράν: 'shouldn't we ask them what they have to say about harmonic motions and whether there is anything else besides them?' To the unproblematic shift from feminine to neuter (common to both versions), this adds a puzzling shift from singular to plural, and it remains unclear what the second question is asking.

[25] Mourelatos, 'Plato's "Real Astronomy"', gives an excellent account of the parallelism between geometry and Plato's redirected astronomy and harmonics; the parallels extend even to the syntax of the sentences describing these sciences.

arts (ἀδελφαῖς τέχναις)'. A nice case of the author making his character anticipate a conclusion which, to his surprise, he will be led to accept.

It is immediately after this discussion of astronomy and harmonics that we first meet Enigma A:

> 'Furthermore', I said, 'if the study of the sciences we have gone through is carried far enough to bring out their community (κοινωνίαν) with each other and their affinity (συγγένειαν), and to demonstrate the ways they are akin (ᾗ ἔστιν ἀλλήλοις οἰκεῖα), the practice will contribute to our desired end and the effort will not be wasted; otherwise it will be labour in vain.' (531cd)

The passage I quoted earlier[26] was a subsequent restatement (537c) which adds to the mystery by speaking of the five mathematical sciences having a 'kinship (οἰκειότητος) with each other *and* with the nature of what is'. But at least we can now say that, if they are sisters in the sense Socrates intends, their kinship with each other will include the methodology familiar from arithmetic and geometry, as described in the Divided Line: deduction from hypotheses and the use of diagrams to represent non-sensible objects which only thought can grasp. The challenge of Enigmas B and C is to explain how this methodology can apply to astronomy and harmonics.

5. *Unqualified being*

The results so far are largely negative. The great value of mathematics is not practical utility, not transferable skills, not the rigorous procedures of mathematical proof; all these are available from the excluded branches of mathematics. Still, in the course of gathering these negative results some positive contrasts have emerged. Epistemologically, Socrates keeps harping on the naturally intelligent instrument in the soul which will remain useless unless it is redirected upwards, away from sensible things. Metaphysically, he keeps saying that, when studied the right way, mathematics aims at knowledge or understanding of unqualified

[26] Above, p. 1.

being, what everlastingly is, or (more simply) truth.[27] These are the phrases, I claim, through which Plato presents his version of the idea of the world as it is objectively speaking.

The idea of unqualified being is first launched in Book V's discussion of the distinction between knowledge and opinion. That discussion is simultaneously our first introduction to the idea of an instrument or power of the soul innately adapted to the acquisition of knowledge as opposed to opinion. *τὸ μὲν παντελῶς ὂν παντελῶς γνωστόν*, we are told at the start of the discussion: 'That which unqualifiedly is is unqualifiedly knowable' (477a). We do not begin to see what this grandiloquent assertion amounts to until we are taken through a series of examples of things which are *not* unqualifiedly what we say they are. A good illustration for present purposes is truth-telling or the obligation to return what one has borrowed. We say (I hope) that these actions are just or right. But, as Socrates pointed out to Cephalus in Book I, if you have borrowed a knife from a friend who has since gone mad, it would not be just or right to give it back, nor to tell him the truth about where it is stored (331c). What is just or right in one set of circumstances is wrong in others. Truth-telling, therefore, is not unqualifiedly just. It is just in many contexts, not so in others.

We can infer that for something to be unqualifiedly just it would have to be just or right in all contexts. If we had a rule or definition of justice valid for any and every context, that would show us an example — one example — of unqualified being. Such is Socrates' revolutionary principle that we should never return wrong for wrong, evil for evil, no matter what is done to us (*Crito* 49c). Unqualified being is something being the case regardless of context. Let us try this on some mathematical examples.

Regardless of context, the sum of two odd numbers is an even number. It is not the case that in some circumstances the square on the hypotenuse of a right-angled triangle is equal, while in other circumstances it is unequal, to the sum of the squares on the other two sides. Pythagoras' theorem, whoever discovered it, is context-invariant. It is important here that Plato does not have the concept

[27] Unqualified being: 521d *et passim*. Truth: 525bc, 526b, 527e. What everlastingly is: 527b.

of necessary truth. Unlike Aristotle, he never speaks of mathematical truths as necessary; he never contrasts them with contingent states of affairs.[28] Invariance across context is the feature he emphasises, and this is a weaker requirement than necessity; or at least, it is weaker than the necessity which modern philosophers associate with mathematical truth. This should make it easier for us to understand how, for Plato, unqualified being is exemplified in the realm of value no less than in mathematics. It is not that we should aim to discover necessary truths in both domains, but that we should aim in both to find truths that are invariant across context, truths that hold unconditionally.

To get from context-invariance to the idea of the world as it is objectively speaking, we need to broaden the scope of context-relativity far beyond the introductory examples of Book V. Instead of pairs of opposite predicates like 'just' and 'unjust', 'beautiful' and 'ugly', 'light' and 'heavy', 'double' and 'half', where it depends on the context which of them is true, we need to get ourselves into a mood to regard all our ordinary, sense-based experience of the world as perspectival and context-dependent, the context in this case being set by the cognitive apparatus we use in ordinary life. For the purposes of ordinary life, the instrument of the soul is directed downwards and manifests itself as the power that Book V calls Opinion as opposed to Knowledge. Opinion is the best you can achieve when dealing with qualified or perspectival being, something that is the case in one context but not in another. Much scholarly ink has gone into controversies about how, in detail, the scope of context-relativity is broadened and whether Plato has arguments to justify the move to a picture of the whole sensible world as the realm of Opinion. This is not the place for those controversies, and in any case my view is that Plato did not think it a matter for argument. What he presents in the Cave simile is the story of a conversion, not a process of argument, and the key agent

[28] This becomes palpable at *Laws* 818ae, a long passage about the 'divine necessities' of mathematics, which turns out to mean that mathematics *must* be learned by any god, daemon or hero who is to be competent at supervising human beings. The necessity that 'even God cannot fight against' is hypothetical necessity, not the necessity of mathematical truth.

of conversion is mathematics.[29] As you get deeper and deeper into (the approved) mathematical studies, you come to think that the non-sensible things they deal with are not only context-invariant. They are also more real than anything you encounter in the fluctuating perspectives of ordinary life in the sensible world (515de). Admittedly, for a Platonist the Forms are yet more real and still more fundamental to explaining the scheme of things than the objects of mathematics. But already with mathematics we can see that abstract reasoning, understood in Plato's way as reasoning about a realm of abstract, non-sensible things, is reasoning about things which are themselves more real and more fundamental to explaining everything else. Mathematics provides the lowest-level articulation of the world as it is objectively speaking.

6. *Abstract objects*

What are these abstract, non-sensible items that mathematics reasons about? The question may be asked, and answered, at two levels: internal and external. By 'internal' I mean internal to the practice of mathematics itself. When you study arithmetic or geometry, what conception do you need of the objects (numbers, figures, etc.) you are dealing with? The external question is metaphysical: Where do these objects belong in the final scheme of things? What is their exact ontological status? We shall see that the *Republic* leaves the external question tantalisingly open. But readers are expected to find the internal question easy to answer. The chief clue is what Glaucon is supposed to know already, from his previous familiarity with mathematics.[30]

Consider this famous passage (emphases mine):

'You will understand better after this preamble (τούτων προειρ-

[29] A similar view in Annas, *Introduction to Plato's Republic*, 238–9.

[30] The passages of Isocrates cited earlier show that plenty of Plato's readers would know as much as Glaucon knows. There is little indication that Glaucon has kept up an interest in the subject since the days when, like other young Athenian aristocrats, he took it as part of his education. To form an idea of the kind of education Plato can assume in his readers, consult H. I. Marrou's wonderful book, *Histoire de l'Éducation dans l'Antiquité* (Paris, 1948; Eng. tr., Madison, 1982).

ημένων):[31] *I think you know* that the practitioners of geometry and arithmetic and such subjects start by hypothesising the odd and the even and the various figures and three kinds of angle and other things of the same family (ἀδελφά) as these in each discipline. They make hypotheses of them as if they knew them to be true.[32] They do not expect to give an account of them to themselves or to others, but proceed as if they were clear to everyone. From these starting points they go through the subsequent steps by agreement (ὁμολογουμένως),[33] until they reach the conclusion they were aiming for.'

'*Certainly I know that much*', he said.

'*Then you also know* that they make use of visible forms and argue about them, though they are not thinking about these forms, but

[31] A typical Platonic self-exemplification: Socrates will deliver a preamble about preambles in mathematics (I owe the observation to Reviel Netz). To my mind, this increases the probability that Plato has in mind a procedure at least nearly as formal as the illustrations from Euclid cited below.

[32] In the phrase ποιησάμενοι ὑποθέσεις αὐτά the accusative αὐτά refers to the three kinds of angle, etc., but this does not mean that mathematicians hypothesise things *as opposed to* propositions: see the survey of ὑποτίθεσθαι plus accusative in C. C. W. Taylor, 'Plato and the Mathematicians: An Examination of Mr Hare's Views', *Philosophical Quarterly*, 17 (1967), 193–203.

[33] Shorey translates 'consistently' here, but at 533c 5 he renders ὁμολογίαν by 'assent' or 'admission' and writes a note on how 'Plato thinks of even geometrical reasoning as a Socratic dialogue'. Most translators accept the desirability of using the same expression in both passages, but they divide into those who think that the point at 533c is that consistency is not enough for knowledge (so, most influentially, Robinson, *Plato's Earlier Dialectic* [2nd edn, Oxford, 1953], pp. 148 and 150) and those, like myself, who think the point is that knowledge or understanding should not depend on an interlocutor's agreement; all relevant objections should have been rebutted. The issue is too large to discuss here (it would involve a full investigation of the tasks of dialectic), but nothing in the present essay will depend on my preferred solution. Notice that in Book IV the principle of opposites, key premise for the proof that the soul has three parts, is accepted as a hypothesis for the discussion to proceed without dealing with all the objections that clever people might make, subject to the agreement that, if it is ever challenged by a successful counter-example, the consequences drawn from it will be 'lost', i.e. they must be regarded as unproven (437a). The parallel with the hypotheses of mathematics is quite close. All the other seven occurrences of ὁμολογουμένως in Plato require to be translated in terms of agreement: *Laches* 186b 4, *Laws* 797b 7, *Menexenus* 243c 4, 245a 7, *Symposium* 186b 5, 196a 6, *Theaetetus* 157e 5. Proclus, *Commentary on Plato's Republic* I 291.20 Kroll, writes of the soul being forced to investigate what follows from hypotheses taken as agreed starting-points (ὡς ἀρχαῖς ὁμολογουμέναις).

> about those they are like. Their arguments are pursued for the sake of the square itself (τοῦ τετραγώνου αὐτοῦ ἕνεκα) and the diagonal itself (διαμέτρου αὐτῆς), not the diagonal they draw, and so it is with everything. The things they mould and draw — things that have shadows and images of themselves in water — these they now use as images in their turn, in order to get sight of those forms themselves, which one can only see by thought.'
> '*What you say is true*', he said. (510ce)

There is a lot here that Glaucon knows and we do not.

The mathematics of Plato's day is largely lost, superseded by Euclid (*c.* 300 BC) and other treatises from the second half of the fourth century onwards. (The *Republic* was written in the first half of the fourth century.) However, Euclid's *Elements* incorporates much previous work, from two main sources: first, earlier *Elements* by Leon and Theudius, both fourth-century mathematicians who spent time in the Academy; and second, the works of Theaetetus and Eudoxus, two outstanding mathematicians with whom Plato had significant contact. If we could read the mathematics available at the time Plato wrote the *Republic*, a good deal of it would look like an early draft of Euclid's *Elements*. This does not quite get us back to the time when Glaucon studied mathematics, but the first *Elements* is credited to Hippocrates of Chios (*c.* 470–400 BC).[34] (The dramatic date of the *Republic* is in the second half of the fifth century, no earlier than 432.) In any case, where stereometry and astronomy are concerned, Plato is obviously thinking of contemporary developments, not harking back to the fifth century; the same may well be true of the other mathematical disciplines. All in all, Euclid is now our best guide for contextualising the passage quoted. With due caution, therefore, let me present some Euclidean starting-points which seem to illustrate what Socrates says about mathematical hypotheses.[35]

[34] The evidence for earlier *Elements* and their authors is Proclus, *Commentary on the First Book of Euclid's Elements*, 66.20–68.10 Friedlein, relying (it is commonly agreed) on a history of mathematics by Aristotle's pupil, Eudemus of Rhodes (second half of the fourth century). Plato died in 347 BC, so the time-gap is relatively small.

[35] Lasserre, *De Léodamas de Thasos à Philippe d'Opunte*, pp. 191–214 (Greek text), pp. 397–423 (translation), gives an impressive array of Euclidean starting-points already familiar to Plato and the Academy.

First, some of the geometrical definitions at the start of *Elements* I:

> 8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie on a straight line.
> 9. And when the lines containing the angle are straight, the angle is called **rectilineal**.
> 10. When a straight line set up on a straight line makes the adjacent angles equal, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.
> 11. An **obtuse angle** is an angle greater than a right angle.
> 12. An **acute angle** is an angle less than a right angle.
> 13. A **boundary** is that which is an extremity of anything.
> 14. A **figure** is that which is contained by any boundary or boundaries.
> 15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point lying within the figure are equal to one another.[36]

And so on for semicircle and the varieties of rectilineal figure (*Elements* I Defs 18–22). No elucidation, no account given of what these definitions mean or why they are true. The learner is expected to accept that these *are* the three kinds of angle and the various figures.

The presentation becomes still more abrupt if we subtract the neatly numbered tabulation of modern editions and translations. In the original, the arithmetical definitions that open Book VII would have looked more like this (without the bold type, spacing between words, and punctuation, which I keep as an aid to modern readers):

> An **unit** is that in accordance with which (καθ' ἥν)[37] each of the things that exist is called one, and a **number** is a multitude composed of units. A number is **a part** of a number, the less of the greater, when it measures the greater, and **parts** when it does not measure it, and

[36] I quote the *Elements* from Sir Thomas Heath, *The Thirteen Books of Euclid's Elements*, translated with introduction and commentary (2nd edn, Cambridge, 1926).

[37] Here I follow Paul Pritchard, *Plato's Philosophy of Mathematics* (Sankt Augustin, 1995), pp. 13–14, in rejecting Heath's translation 'that in virtue of which', on the grounds that this suggests the unit is what *makes* something one, the cause of its unity. Aristotle in *Metaphysics* X inquires into what makes each of the things that exist one. Euclid merely presupposes they are each one.

> the greater number is a **multiple** of the less when it is measured by the less. An **even number** is that which is divisible into two equal parts, and an **odd number** is that which is not divisible into two equal parts, or that which differs by an unit from an even number. An **even-times even number** is that which is measured by an even number according to an even number.[38]

And so on for even-times odd number, odd-times odd number, prime number, numbers prime to one another, composite number and numbers composite to one another, etc., and finally perfect number (*Elements* VII Defs 9–22). Once again, Socrates' description is vindicated to a T. We may fairly hope that Euclid can also tell us something about what Glaucon knows about the mathematicians' use of visible forms.

In one respect, however, Euclid is likely to be misleading. The *Elements* is a book, and a long one at that. Diagrams can be included in a book, but not the moulded figures Socrates also mentions.[39] We will shortly hear of mathematical 'experts' laughing away an objection. That implies an oral presentation, which would be less formal than Euclid and would not include more initial hypotheses than were needed for the occasion. Much may be presupposed without explicit statement.

We should not exaggerate the difference this makes. Greek school-teaching was not child-oriented or kind. It included lots of dictation and rote-learning.[40] When Plato in the *Republic* has Socrates urge that play, not force, is the way to bring children into mathematics (536d–537a), he goes knowingly against the grain of the culture; in the *Laws* (819ac) the idea is presented as an import from Egypt. Equally innovating is the famous remark that sums up the message of the Cave. Education is not, as some people say, a

[38] Heath's translation still, but with 'and' inserted to mark each occurrence of the connective δέ and the full stops indicating asyndeton in the sequel. In Book VII none of the MSS number the definitions; in Book I most do not. (I owe thanks to Reviel Netz for calling my attention to this fact, which can be verified by looking at the *apparatus criticus* of Heiberg's edition of the *Elements* [Leipzig: Teubner, 1883–8].)

[39] Natural as it is to suppose the reference is to three-dimensional figures used in solid geometry, *Timaeus* 50ab speaks of moulding a piece of soft gold into a triangle and other (plane) figures.

[40] Marrou, *Histoire*, Part II, chaps 6–8.

matter of putting knowledge into souls that lack it, like putting sight into blind eyes. The soul already possesses the 'instrument with which each person learns'. What is needed is to turn it around, as if it were an eye enfeebled by darkness, so that it can see invariant being instead of perspectival becoming (518bd). Part of the point of the mathematical scene in Plato's *Meno* is to contrast ordinary didactic instruction with the way Socrates gets the slave to see how to double the given square 'without teaching him', simply by his usual method of question and answer. And even Socrates starts out by asking whether the slave knows what a square is, namely, a figure like the one drawn which has all four sides equal (*Meno* 82bc).

I conclude that the oral teaching Glaucon is familiar with would reflect the formality of Euclid's procedure more closely than the education we are used to. In any case, the future rulers will not go on to their five years' dialectic until they have achieved a synoptic view of all the mathematical disciplines (Enigma A), and dialectic will centre on explaining the hypotheses of mathematics in a way that mathematics does not, and cannot, do (510b, 511b, 533c). For this purpose, not only the hypotheses of arithmetic and geometry, but also those of astronomy and harmonics, will need explicit formulation — all of them. In the long run, there will be no significant difference between oral and written mathematics.

It is the hypotheses that make it possible to use 'visible forms' (diagrams) to think about abstract, non-sensible objects. Socrates says that mathematicians argue about visible forms in order to reach results about something else. Without a more or less explicit idea of what that something else is, the procedure would be aimless. The visible forms mentioned are square and diagonal. Ancient readers would probably think at once of a geometer demonstrating the well-known proposition that the diagonal of a square is incommensurable with its side — no unit, however small, will measure both without remainder.[41] This example, a favourite

[41] An alternative, proposed by R. M. Hare, 'Plato and the Mathematicians', in Renford Bambrough (ed.), *New Essays on Plato and Aristotle* (London, 1965), p. 25, is the square and diagonal drawn by Socrates in the *Meno* (82b–85a) to help the slave discover how to double the given square. But incommensurability lurks

with Aristotle too,[42] makes good sense of Socrates' observations, because the proposition is simply not true of the diagonal and side drawn in the diagram for the proof; to borrow a phrase from Ian Mueller, it is a proposition that 'is always disconfirmed by careful measurement'.[43] The geometer is well aware of that. He is using the diagram to prove something that holds for the square *as defined* in his initial hypotheses: 'Of quadrilateral figures, a **square** is that which is both equilateral and right-angled' (*Elements* I Def. 22). It has all four sides and all four angles *exactly* equal. That is what Socrates calls 'the square itself', the square represent*ed* (more or less accurately) by the diagram. He is right, moreover, that it can only be seen in thought. The diagram represent*ing* this square is drawn 'for the sake of', as an aid to reasoning about, a square that the eyes do not see.

So far Socrates has said nothing that should surprise, nothing metaphysical, nothing with which Aristotle would disagree. His remarks articulate a conception of geometrical practice that any student of the subject must internalise. To an educated person like

there too, as becomes clear when Socrates allows the slave to *point* to the line that will do the trick if he prefers not to specify its length in feet (83e 11–84a 1).

[42] At *Prior Analytics* I 23, 41a 26–7, Aristotle outlines a *reductio* proof which supposes that side and diagonal are commensurable and then shows how, in consequence, the same number will be both odd and even, which is impossible. Briefer allusions to the theorem at *De Anima* III 6, 430a 31 and other places listed in Bonitz, *Index Aristotelicus* (Berlin, 1870), 185a 7–16, with the comment 'saepissime pro exemplo affertur'. The *reductio* proof is usually taken to be the one we read at *Elements* X, Appendix 27.

[43] Ian Mueller, 'Ascending to Problems', p. 115. This is the place to acknowledge a wider debt over the years to the sanity and good judgement of Mueller's writings on Greek mathematics. Particularly relevant to the present discussion, besides the paper just cited, are 'Mathematics and Education: Some Notes on the Platonist Programme', in Ian Mueller (ed.), *ΠΕΡΙ ΤΩΝ ΜΑΘΗΜΑΤΩΝ*: *Essays on Greek Mathematics and its Later Development*, *Apeiron*, 24 (1991), 85–104; 'Mathematical Method and Philosophical Truth', in Richard Kraut (ed.), *The Cambridge Companion to Plato* (Cambridge, 1992), pp. 170–99; 'Greek arithmetic, geometry and harmonics: Thales to Plato', in C. C. W. Taylor (ed.), *Routledge History of Philosophy* Vol. I: *From the Beginning to Plato* (London & New York, 1997), pp. 271–322; 'Euclid's *Elements* from a philosophical point of view', forthcoming. Without his work and Barker's (n. 10 above) this essay could not have been written.

Glaucon, it is familiar stuff.[44] What is more, it is a conception of geometrical practice which supports Alcinous' claim that the precision of mathematics is the essential epistemic route to a new realm of objects. Without a definition of square we would never be able to demonstrate a property such as incommensurability, which cannot be detected by the senses.

Visible forms were also used to diagram numbers. Here is the first proposition of Euclid, *Elements* VII:

> *Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.*
>
> For, the less of two unequal numbers *AB*, *CD* being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;
> I say that *AB*, *CD* are prime to one another, that is, that an unit alone measures *AB*, *CD*.
>
> For, if *AB*, *CD* are not prime to one another, some number will measure them.
>
> Let a number measure them, and let it be *E*; let *CD*, measuring *BF*, leave *FA* less than itself,
> let *AF*, measuring *DG*, leave *GC* less than itself,
> and let *GC*, measuring *FH*, leave an unit *HA*.
>
> Since, then, *E* measures *CD*, and *CD* measures *BF*, therefore *E* also measures *BF*.
>
> But it also measures the whole *BA*;
> therefore it will also measure the remainder *AF*.
>
> But *AF* measures *DG*;
> therefore *E* also measures *DG*.
>
> But it also measures the whole *DC*;
> therefore it will also measure the remainder *CG*.
>
> But *CG* measures *FH*;
> therefore *E* also measures *FH*.
>
> But it also measures the whole *FA*;
> therefore it will also measure the remainder, the unit *AH*, though *E* is a number: which is impossible.
>
> Therefore no number will measure the numbers *AB*, *CD*;
> therefore *AB*, *CD* are prime to one another. Q.E.D.

[44] That is why it helps him understand what Socrates was getting at in his first, densely compressed account of the upper two parts of the Divided Line (510b 4–9), to which Glaucon reasonably responded, 'I don't understand quite what you mean.'

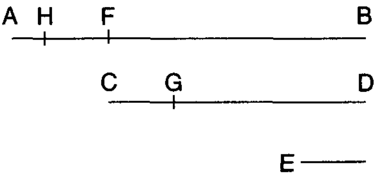


Figure 2

Notice that the unit is represented in Figure 2 by the line *AH*, not by a point. This may shed light on a passage in Book VII where Socrates speaks of the educational value of arithmetic. Provided arithmetic is studied for the sake of knowledge (τοῦ γνωρίζειν ἕνεκα), he says, not trade,

> 'It strongly leads the soul upwards and compels it to discourse about the numbers themselves. If someone proposes to discuss visible or tangible bodies having number, this is not allowed. For *you know*, I take it, what experts in these matters do if someone tries by argument to divide the one itself (αὐτὸ τὸ ἕν) [i.e. argues that the one itself can be divided]. They laugh at him and won't allow it. If you cut it up, they multiply it, always on guard lest the one should turn out to be not one, but a multiplicity of parts.'
> '*You are absolutely right*', he said.
> 'Suppose then, Glaucon, someone were to ask them, "You wonderful people, what kind of numbers are these you are talking about, in which the one (τὸ ἕν) is such as you demand (ἀξιοῦτε), each of them equal to every other without the slightest difference and containing no part within itself?" What do you think they would reply?'
> 'This, I think — that they are speaking of those numbers which can only be thought, and which you cannot handle in any other way.' (525d–526a)

Imagine someone refusing to accept the visible line *AH* in Figure 2 as a unit, on the grounds that it can be divided into parts in the same way as the other lines in the diagram, which were progressively divided in the course of the proof. The experts do not deny that the line *AH* can be divided; Glaucon has already agreed with Socrates that any visible unit will appear both one and indefinitely many (524e–525a). Instead, they laugh. They laugh, I take it, because to suppose that the divisibility of the line *AH* has significance in an arithmetical context, where it is *stipulated* that *AH* represents a unit, is to confuse arithmetical with geometrical division in the most laughable way.[45] Of course, we could take as unit a smaller line — say, a fourth part of *AH*. But now *AH* is four units instead of one ('If you cut it up, they multiply it').[46] The theorem is not falsified, merely inapplicable.

When Socrates speaks of 'the one itself' (cf. also 524e 6), he refers to something there are many of ('each of them equal to every other'), something that can be multiplied to compose a number.[47] His 'one' is just like Euclid's 'unit', not a number but a component of number. Recall the first two definitions of *Elements* VII: a number is a multitude composed of units, where a unit (μονάς) is 'that in accordance with which each of the things that exist is called one'. I understand this as follows.

Take anything that exists and think away all its features save that it is one thing. That 'abstracted' one thing is a Euclidean unit. Combine (in thought, of course — how else?) three such units, all absolutely alike (for there is nothing left by which they could differ), and you have a number — a three. Ancient arithmetic knows no such thing as *the* number three, only many sets of three units — many abstract triplets. It follows that, for a Greek mathematician, numerical equality is equinumerosity, not identity: '$3 + 3 = 6$' does not mean that *the* number 6 is identical with *the* number which results from adding 3 to *itself*, but that a pair of triplets contains exactly as many units as a sextet. For a more general illustration, consider *Elements* IX 35, where Euclid writes,

[45] A similar interpretation in Jowett & Campbell's commentary (Oxford, 1894), ad loc., except that they imagine a schoolmaster gently laughing at a pupil's 'natural mistake' where I imagine the learner as more contentious and the laughter as derisive. The learner is certainly not thinking of fractions, since at this period mathematicians studied (what we treat as) fractions as ratios between positive integers. Even Greek traders used only 2/3 and unit fractions of the form $1/n$.

[46] Cf. Theon of Smyrna, *The mathematics which is useful for reading Plato*, 18.18–21 Hiller: 'When the unit is divided in the domain of visible things, it is certainly reduced as a body and divided into parts which are smaller than the body itself, but it is increased in numbers, because many things take the place of one' (tr. Van Der Waerden).

[47] The same idea at *Philebus* 56ce: whereas in practical arithmetic people count unequal units (two armies, two cows, etc.), theoretical arithmetic requires that one posit (θήσει) a unit (μόνας) which is absolutely the same as every other of the myriad units.

> Let there be as many numbers as we please in continued proportion, *A*, *BC*, *D*, *EF*, beginning from *A* as least, and let there be subtracted from *BC* and *EF* the *numbers* *BG*, *FH*, each equal to *A*;
> I say that, as *GC* is to *A*, so is *EH* to *A*, *BC*, *D*.

Note the plural I have italicised: *A*, *BG*, and *FH* are three different number*s*, all equal to each other and each diagrammed separately in Figure 3. By contrast, Heath's algebraic paraphrase is

$$(a_{n+1} - a_1) : (a_1 + a_2 \ldots + a_n) = (a_2 - a_1) : a_1,$$

where the repeated use of a single symbol a_1 presupposes in the modern manner that equal numbers are identical — a nice illustration for the thesis that it was the incorporation of algebra into mainstream mathematics during the Renaissance that created the modern concept of number.[48]

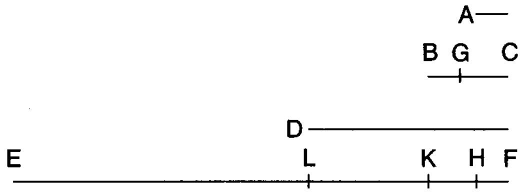


Figure 3

The Euclidean conception of units and numbers makes good sense of what Socrates and Glaucon say in the last passage quoted. It is obviously true that Euclid's numbers can only be thought and cannot be handled in any other way. For the units that compose them require a deliberate act of abstraction: in each case your thought must set aside or ignore the many parts/features of the line (or pebble, bead on an abacus, or any other sensible object that might be to hand) in order to consider it as just: one thing. Once again, this would be conception of unit and number that any

[48] The classic statement of this thesis is Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra* (Cambridge, Mass. & London, 1968; translated from the German of 1934–36). But the ancient conception did not disappear at once. Euclid was still studied, while Diophantus was rediscovered and interpreted algebraically. Frege's task in *The Foundations of Arithmetic* (1884) was to clear up the resulting confusion about what numbers are.

student would internalise. Glaucon already knows how experts answer the laughable suggestion. He can supply for himself (and for us) the mathematicians' answer to the question what kind of numbers they are talking about. To educated readers of the *Republic* it should all be familiar stuff.

What is more, Euclid's way of doing arithmetic is guaranteed to be virtually useless to traders (and modern accountants). He talks only of numbers that satisfy some general condition, never of 7, 123, or 1076; he never does what schoolchildren today call 'sums' or 'exercises'. 'Two unequal numbers being set out': they could be *any* unequal numbers whatsoever. That quest for generality marks the mathematician's desire for context-invariance.

7. *The metaphysics of mathematical objects*

But what, you may ask, *are* these units, numbers, and figures? Do they really exist, or are they just convenient posits to help us reason about objects still more rarefied and abstract, such as the Forms? That question — the external question — was certainly debated in the Academy, as we can tell from the last two Books of Aristotle's *Metaphysics*. There we learn that Plato and his associates, Speusippus and Xenocrates, each had their own answer, while Aristotle disagreed with the lot. But the question is not discussed in the *Republic*. In the two passages quoted in the previous section, Socrates is reporting what practising mathematicians do and say, not offering his own philosophical account of the ontological status of mathematical objects. In the next passage he says that such an account would be too much for the project in hand. After setting out the famous proportion between the various cognitive states represented in the Divided Line, 'As being (οὐσία) is to becoming (γένεσις), so is understanding (νόησις) to opinion (δόξα), and as understanding (νόησις) is to opinion (δόξα), so knowledge (ἐπιστήμη) is to confidence (πίστις) and thought (διάνοια) to conjecture (εἰκασία)', he adds: 'Let us leave aside the proportion exhibited by the *objects* of these states when the opinable (δοξαστόν) and the intelligible (νοητόν) are each divided into two.

Let us leave this aside, Glaucon, lest it fill us up with many times more arguments/ratios[49] than we have had already' (534a).

To refuse to contemplate the result of dividing the objects on the intelligible section of the Line is to refuse to go into the distinction between the objects of mathematical thought (διάνοια) and Forms. Pythagoras' theorem (Euclid, *Elements* I 47), 'In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle', refers to three squares each of which, unlike the squares in Figure 4, has all four sides and all four angles exactly equal, as laid down in *Elements* I Def. 22. A theorem about three squares different in area cannot be straightforwardly construed as dealing with the (necessarily unique) Platonic Form Square, any more than the three equal numbers of Figure 3 can be construed as the (necessarily unique) Form of some number. The *Republic* tells us that practising mathematicians talk about plural, idealised entities which are not Forms. To judge by Euclid, this is true — a plain fact, which readers should be familiar with. About Forms the mathematicians need neither know nor care. Plato *may* have thought that the mathematicians' multiple non-sensible particular numbers and figures (the 'intermediates' as they have been called in the scholarly literature since Aristotle) could ultimately be derived from Forms, so that in the end mathematics would turn out to be an indirect way of talking about Forms.[50] Perhaps mathematical entities are the 'divine reflections' outside the cave (532c 1), dependent on the 'real things' they image. But whatever Plato thought, or hoped to show, Greek mathematics is quite certainly not a direct way of talking about Forms. If Plato has Socrates decline further clarification of the matter, we may safely infer that he supposed his message about

[49] The phrase πολλαπλασίων λόγων plays on the mathematical and dialectical meanings of λόγος.

[50] The evidence is slim: an objection by Aristotle (*Metaphysics* XIV 3, 1090b 32–1091a 3; cf. I 9, 991b 29–30, III 6, 1002b 12ff.) that for mathematical numbers Plato never provided metaphysical principles at their own (intermediate) level. If, as so often, Aristotle is here using a point *of* Plato's philosophy as a point *against* it, this might suggest that Plato did not in fact wish to claim ultimate metaphysical reality for intermediates.

mathematics and the Good could be conveyed without settling the exact ontological status of mathematical entities.

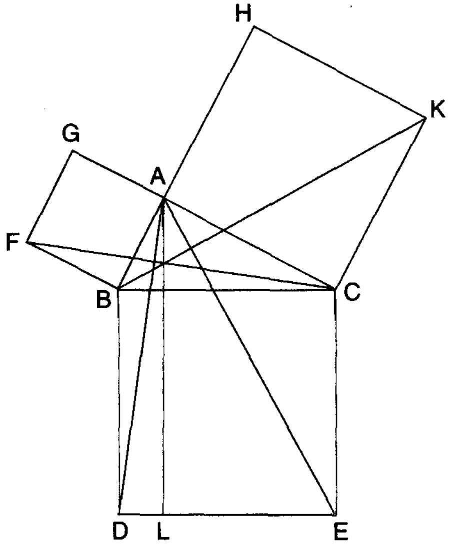


Figure 4

8. Controversial interlude

In denying that Plato thinks mathematics is directly about Forms, I am taking a controversial line. I should say something to pacify scholars who suppose otherwise. Two sentences have been influential in encouraging the interpretation I reject:

> (1) 'Their arguments are pursued for the sake of the square itself (τοῦ τετραγώνου αὐτοῦ ἕνεκα) and the diagonal itself (διαμέτρου αὐτῆς), not the diagonal they draw.' (510de, p. 24 above)
>
> (2) 'If someone tries by argument to divide the one itself (αὐτὸ τὸ ἕν), they laugh at him and won't allow it.' (525de, p. 30 above; cf. also αὐτὸ τὸ ἕν at 524e 6)

The issue is whether that little word 'itself' signals reference to a Platonic Form, as in phrases like 'justice itself' (517e 1–2), 'beautiful itself' (507b 5), or 'the equal itself' (*Phaedo* 74a 11–12).

The word 'itself' is certainly not decisive on its own, otherwise a

Form of thirst would intrude into Book IV's analysis of the divided soul. When Socrates there speaks of 'thirst itself' (437e 4: αὐτὸ τὸ διψῆν), he means to pick out a type of appetite in the soul, not a Form; in context, the phrase is equivalent to his earlier locution 'thirst qua thirst' (437d 8: καθ' ὅσον δίψα ἐστί). Even the intensified expression 'itself by itself' (αὐτὸ καθ' αὑτό), which often signals a Platonic Form (e.g. 476b 10–11, *Phaedo* 100b 6, *Symposium* 211b 1, *Parmenides* 130b 8, 133a 9, c 4), does not always do so. Otherwise, when Socrates in the *Phaedo* recommends using 'pure thought itself by itself to try to hunt down each pure being itself by itself' (66a 1–3), he would be telling one Form to study another. In Plato 'itself' and 'itself by itself' standardly serve to remove some qualification or relation mentioned in the context. Their impact is negative. Only the larger context will determine what remains when the qualification or relation is thought away. When the *Phaedo* (74a) distinguishes 'the equal itself' from 'equal *sticks and stones*', what remains is indeed a Form. But when Adeimantus in the *Republic* (363a) complains that parents and educators of the young do not praise justice itself (αὐτὸ δικαιοσύνην), only the good reputation you get from it, 'justice itself' does not yet signify a transcendent Platonic Form.[51] And when in the *Theaetetus* the well-known fallacious argument against the possibility of judging what is not is framed within a distinction between 'what is not itself by itself' and 'what is not *about something that is*' (188d, 189b), it is definitely not the *Sophist*'s Form of Not-Being that remains; it is a blank nothing, which no one could judge.

Now in (1) 'the diagonal itself' is opposed to 'the diagonal *they draw*', in (2) 'the one itself' contrasts with a one composed of many parts. In both cases the larger context is mathematics, not metaphysics. It is to mathematics, then, that we should look to judge the

[51] Nor does it even at 472c, where justice itself, the virtue they have been trying to define, is contrasted with the perfectly just man of Glaucon's challenge in Book II (360e–361d): see Adam's commentary (Cambridge, 1902), ad loc. The Theory of Forms makes its first appearance in the *Republic*, complete with the *Phaedo*'s technical terminology of participation, at 475e–476d. Socrates starts by saying it would not be easy to explain to someone other than Glaucon. That marks the context as more metaphysical than the earlier ones. In such a context, a phrase like 'the beautiful itself' does indicate a transcendent Platonic Form.

effect of the word 'itself'. In (1) it tells us to ignore the wobbles in the drawing and the fact that the line has breadth, in (2) to abstract from the many parts of the item we take as unit.[52] Any page of Euclid shows that that is how mathematicians proceed. What remains when they do so is not a Form, but an ideal exemplification of the relevant definition.

Another standard view I reject is that Socrates means to criticise the mathematicians for the procedures he describes.[53] 'Plato's criticism of the mathematicians' is a staple of the scholarly literature. The most influential sentence here is

> (3) 'They make hypotheses of them as if they knew them to be true. They do not expect to give an account of them to themselves or to others, but proceed as if they were clear to everyone.' (510c, p. 23 above)

Now mathematical thought (διάνοια) is twice characterised as a state in which the soul is *forced* (ἀναγκάζεται) to make use of hypotheses (510b 5, 511a 4; cf. 511c 7). It would seem harsh to pillory the mathematicians for doing something they are forced to do.

Why are they forced to use hypotheses? Plato's answer, I suggest, is that hypotheses are intrinsic to the nature of mathematical thought. There is no other way of doing deductive mathematics than by deriving theorems and constructions from what is laid down at the beginning. The very idea of an *Elements* is to find the simplest and most primitive starting-points from which the rest can be derived; that is what the title Στοιχεῖα means.[54] To demand

[52] Compare 'five and seven themselves (αὐτὰ πέντα καὶ ἑπτά)' *vs* 'seven men and five men' at *Theaetetus* 195e–196a. The latter are objects of perception, the former can only be grasped in thought, yet in the context they cannot be Forms.

[53] A leading exponent of this view was Richard Robinson, *Plato's Earlier Dialectic*, pp. 146–56, according to whom Plato criticises the mathematicians for *failing* to treat their starting-points as hypotheses: they take them as evident and known when they should regard them as tentative hypotheses. Robinson's account is echoed in Annas, *Introduction to Plato's Republic*, pp. 277–9, and many others.

[54] See Walter Burkert, 'Στοιχεῖον — Eine semasiologische Studie', *Philologus*, 103 (1959), 167–97, where the theory that στοιχεῖα originally meant the letters of the alphabet is finally laid to rest. What Euclid was admired for was not original mathematical results, but his skill at systematising the results of creative mathematicians like Theaetetus and Eudoxus: see the introductory scholia to *Elements* V and XIII (282.13–20 and 654.1–10 Heiberg-Menge).

that the mathematicians give an account of their initial hypotheses, to themselves and others, would be to make them stop doing mathematics and do something else instead. The best and brightest of the Guards will indeed do that later. They will stop treating mathematical hypotheses as starting-points (511b 5: ἀρχάς) and try to account for them in terms of Forms (511bc, 533c). But this activity is dialectic, not mathematics reformed to meet a criticism. Socrates expressly says that *only* dialectic can do the job (533c), the soul engaged in mathematical thought *cannot* (511a 5–6); and Glaucon knows very few professional mathematicians who are also skilled in dialectic (531de). It is thus no criticism to say that mathematicians give no account of their hypotheses. It is simply to say that mathematics is what they are doing, not dialectic.[55]

Another influential passage is where Socrates mocks the language of geometry:

> 'This at least', I said, 'will not be disputed by those who have even a slight acquaintance with geometry, that this science is in direct contradiction with the language its practitioners use in their arguments.'
> 'How so?' he said.
> 'They talk in a way that is both quite ludicrous and unavoidable (μάλα γελοίως τε καὶ ἀναγκαίως). They speak as if they were doing something and developing all their arguments for the sake of action. They use words like "to square", "to apply", "to add", and so on, whereas in fact the entire study is pursued for the sake of knowledge.'
> 'That is so', he said.
> 'Then must we not agree on a further point?'
> 'What?'
> 'That this knowledge at which the study of geometry aims is

[55] Compare Aristotle, *Eudemian Ethics* II 11, 1227b 28–30: 'Just as in the theoretical branches of knowledge the hypotheses are starting points, so in the productive ones the end is the starting point and hypothesis.' His examples are reasoning from the hypothesis that the angles in a triangle equal two right angles and reasoning from the goal of making something healthy. In the context of this parallel between ethical deliberation and mathematical thought, the analogue to the statement 'those who do not lay down some end are not deliberators' (*EE* II 10, 1226b 29–30) is that, if you do not lay down hypotheses, you opt out of mathematics.

knowledge of what always is,[56] not of what at a particular time comes to be and perishes.'
'That is readily admitted', he said. 'Geometry is knowledge of what always is.'[57] (527ab)

A good illustration for these remarks is the way Euclid sets about proving Pythagoras' theorem (*Elements* I 47) with the aid of Figure 4 (square-bracketed references are to earlier results used on the way):

Let *ABC* be a right-angled triangle having the angle *BAC* right.
I say that the square on *BC* is equal to the squares on *BA*, *AC*.
For let there be *described* on *BC* the square *BDEC*, and on *BA*, *AC* the squares *GB*, *HC*; [I 46]
through *A* let *AL* be *drawn* parallel to either *BD* or *CE*, and let *AD*, *FC* be *joined*.
Then, since each of the angles *BAC*, *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it, the two straight lines *AC*, *AG* not lying on the same side make the adjacent angles equal to two right angles;
therefore *CA* is in a straight line with *AG*. [I 14]
For the same reason
BA is also in a straight line with *AH*.
And, since the angle *DBC* is equal to the angle *FBA*: for each is right:
let the angle *ABC* be *added* to each:
therefore the whole angle *DBA* is equal to the whole angle *FBC*. [Common Notion 2]
And, since *DB* is equal to *BC*, and *FB* to *BA*,
the two sides *AB*, *BD* are equal to the two sides *FB*, *BC* respectively,
and the angle *ABD* is equal to the angle *FBC*;
therefore the base *AD* is equal to the base *FC*,
and the triangle *ABD* is equal to the triangle *FBC*. [I 4]
Now the parallelogram *BL* is double of the triangle *ABD*,
for they have the same base *BD* and are in the same parallels *BD*, *AL*. [I 41]
And the square *GB* is double of the triangle *FBC*,

56 Shorey and some other translators miss the point that this clause is governed by the preceding ἕνεκα.

57 I doubt Plato means Glaucon to do more here than affirm the first of Socrates' alternatives. Glaucon grasped at 511cd that mathematics without dialectic is not knowledge in the fullest sense, but Socrates has just spoken of geometry as a science (527a 2: ἐπιστήμη).

for they again have the same base *FB* and are in the same parallels *FB*, *GC*. [I 41]

Therefore the parallelogram *BL* is also equal to the square *GB*.

Similarly, if *AE*, *BK* be *joined*,

the parallelogram *CL* can also be proved equal to the square *HC*;

therefore the whole square *BDEC* is equal to the two squares *GB*, *HC*. [Common Notion 2]

And the square *BDEC* is described on *BC*,

and the squares *GB*, *HC* on *BA*, *AC*.

Therefore the square on the side *BC* is equal to the squares on the sides *BA*, *AC*.

Therefore *in right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.* Q.E.D.

Notice that the diagram is constructed step by step before the argument begins at 'Then, since . . .'; only *AE* and *BK* are added later. Euclid starts by asking us to accept that *ABC* is a right-angled triangle (as defined at *Elements* I Def. 21) and then asks us to agree to his describing squares on each of its three sides (an operation licensed by the immediately preceding *Elements* I 46). Finally, he asks to draw various lines (licensed by *Elements* I Postulate 1, 'To draw a straight line from any point to any point'). These lines are not mentioned in the proposition, which asserts a relationship between the squares on the sides of a right-angled triangle. But they are crucial to the proof, for they create the triangles (*ABD*, *FBC*) and parallelograms (*BL*, *CL*) on which the argument will turn. Not to accept them would be to deny the reality of the continuum, which is a presupposition of every proof in the book. Without the activity of drawing them, the proof could not get started. Likewise, without the (non-physical) action of adding the angle *ABC* to each of the angles *DBC* and *FBA*, the proof could not be continued. Socrates is right to say that the verbs of action ('let *AL* be drawn', 'let *AD*, *FC* be joined', 'let the angle *ABC* be added') are unavoidable. Banishing them would be the death of (Greek) geometry.

But he is having fun when he says they are ludicrously at odds with the aim of the subject, which is to gain knowledge of invariant being. The theorem proved is an eternal, context-invariant truth.

What takes place in time is only the process of coming to know it is true by drawing the lines and conducting the proof.[58] And it is typical of human learning in general, not peculiar to geometry, that it takes time and effort. Even the arithmetical proof at *Elements* VII 1 (quoted above) involves the operation of continual subtraction. We should not mistake a joke for serious criticism.

Admittedly, while the hypotheses remain unaccounted for, mathematics does not rank as knowledge or understanding in the fullest sense (511cd, 533c). By providing such accounts, in the light of a first principle (the Good), dialectic will give the subject-matter of mathematics an intelligibility that mathematics on its own cannot achieve: mathematics studies things that are 'intelligible with the aid of a (first) principle' (511d 2: νοητὰ μετὰ ἀρχῆς). But all that follows from this is that the mathematicians *would* be open to criticism if they claimed to know that their hypotheses are true. In (3) Socrates does not suggest that they do claim this, only that they proceed *as if* they knew them to be true and *as if* they were clear to everyone. To judge by the quotations I gave earlier from the opening of Euclid's *Elements* I and VII, Socrates has it exactly right. Euclid never claims to know, but proceeds as if he did. He does not claim that his definitions are clear to everyone, but he proceeds as if they were. That is how (Greek) mathematics is done. Criticism is beside the point. Still less should anyone call upon Euclid to reform his mathematics. What Socrates is asking Glaucon to do (and through him, readers of the *Republic*) is something quite different: to agree that his description of mathematical procedures is plain fact, familiar stuff, and to reflect on the epistemological peculiarity of mathematics *as such*.

Glaucon understands this pretty well:

> 'I understand', he said, 'though not adequately, for it is no slight task you appear to have in mind. You mean to say that the region of intelligible being which is contemplated by dialectical knowledge is *clearer* than the part studied by the arts (so called) which use hypotheses as starting points. Mathematicians are forced to contemplate their objects by thought (διανοίᾳ) rather than perception,

[58] A point insisted upon by Speusippus: Proclus, *Commentary on the First Book of Euclid's Elements*, 77.15–78.8 Friedlein; Aristotle, *De Caelo* I 10, 279b 32–280a 2.

> but because they study them from hypotheses, without having gone back to a (first) principle,[59] you do not think they have understanding (νοῦν) of them, even though they are intelligible with the aid of a principle. And I think you call the cognitive state (ἕξιν) of the geometers and other mathematicians thought (διάνοιαν), not understanding (νοῦν), because you take it to be *intermediate* between opinion (δόξης) and understanding.'
> 'You have got the point', I said, 'quite adequately.' (511cd)

That is the main result of the Divided Line passage: the introduction of a new intermediate epistemic state, which turns out to have an intermediate degree of clarity when it is compared, on the one side with the ordinary person's opinion about sensibles, and on the other side with the dialectician's understanding of Forms. Socrates can then correlate this intermediate degree of cognitive clarity with the intermediate degree of truth or reality which belongs to the non-sensible objects that mathematicians talk about (511de). In sum, mathematics is not criticised but *placed.* Its intermediate placing in the larger epistemological and ontological scheme of the *Republic* will enable it to play a pivotal, and highly positive, role in the education of future rulers.

9. *Values in the Cave*

This brings me back to mathematics as the lowest-level articulation of the world as it is objectively speaking. The next step is to bring value into the picture. For that we must return to the Cave.

The prisoners, remember, are immobilised by chains which stop them seeing anything but the shadows on the back of the cave. The shadows are cast by firelight playing on a series of objects and puppet-like figures (human and animal) which are carried, unseen by the prisoners, along the top of a low wall behind them. The story starts when one of these prisoners is untied and forced to turn around to answer questions about the objects on the wall. That

[59] Note the aorist ἀνελθόντες (511d 1). My warrant for inserting '(first)', where the Greek speaks simply of a principle or starting-point, is the larger context. Socrates has just sketched dialectic's ascent to the first principle (starting-point) of everything (τὴν τοῦ παντὸς ἀρχήν — 511b 7). That is the ἀρχή (singular) which contrasts with the mathematicians' plural ἀρχαί, their hypotheses.

turning around (περιαγωγή — 515c 7, 518d 4, e 4, 521c 6), or conversion (μεταστροφή — 518d 5, 525a 1, 526e 3, 532b 7), is the first stage of a long arduous journey which takes the freed prisoner up out of the cave into the brightly lit world outside, where their eyes gradually adjust, first to seeing shadows, then reflections, then the actual people and things reflected, then stars and moon (by night, of course), and finally the sun itself (515c–516b).

The story continues with an account of what happens to people who return to the cave (516e–518b). It is during this second phase that Socrates tells us to apply the whole Cave image to the two preceding images, the Sun and Divided Line (517ac). Specifically, we should give the sun outside the cave the same role as it had earlier in the analogy of the Sun: in both Sun and Cave the sun represents the Form of the Good (508e 2–3, 517b 8-c 1: ἡ τοῦ ἀγαθοῦ ἰδέα). We have long been aware that the Form of the Good is the 'greatest study (505a 2: μέγιστον μάθημα)', because knowledge of it is the only sure guide to living well and enjoying the benefits of justice (505ab, repeated here at 517c). That much is clear (at least in outline): the goal and climax of the education that Socrates and Glaucon are planning for the rulers of the ideal city is knowledge of the Good.

Less clear is how we should understand the objects seen by the freed prisoner on the way up past the low wall to the world outside. What do the puppets on the wall represent, or the reflections outside the cave? And what is the significance of the fact that both puppets and reflections are likenesses of the animals themselves and other originals in the upper world? Socrates instructs us to apply the prisoner's upward journey to the soul's ascent into the intelligible region of the Divided Line (517b 4–5). This at once suggests mathematics and dialectic, with their respective objects, as described at the end of Book VI. In that case, the animals and other originals will represent the Forms studied by dialectic, while both reflections and puppets will be mathematical objects (perhaps conceived at different levels of abstraction). But the surrounding narrative, about the journey back to the cave, would suggest a different solution.

For the examples mentioned in the story are values. When someone goes down into the cave again, to begin with, before their eyes have adjusted to the semi-darkness, they will appear ridiculous if they have to dispute in court or elsewhere about the shadows of *the just*, or about the puppets those shadows derive from, in terms intelligible to people who have not seen justice itself (517de). Later, however, after getting used to the poor light, they will do much better than the prisoners at knowing what the shadows are[60] and knowing what they are shadows of. They will do better at this precisely because they have seen the truth about what is *beautiful, just, and good* (520c). Thus at least some of the shadows, hence at least some of the puppet-like figures carried along the wall, represent values like justice. If so, the same must be true of the corresponding reflections and their originals outside the cave. The conversion and ascent is progress towards an understanding of true values.

One important difference between the puppets inside the cave and the reflections outside is that the puppets are manipulated by people behind the wall. Some of these puppeteers speak, but their voices echo off the back of the cave in such a way that the prisoners suppose they come from the shadows in front of them. Other puppeteers remain silent: the effects they produce are purely visual (514b–515b). This distinction suggests to me that among the puppeteers are the poets and painters who transmit the values of the community.[61] The idea will be that the prisoners' experience of those values is mediated by the culture they grow up in. To get outside the cave is to transcend one's culture and achieve a more objective understanding of justice, beauty, and goodness.

But that does not settle the question how, by what studies, this progress is achieved. The story indicates that innate to every prisoner is an instrument (*ὄργανον*) capable of understanding true values, but that to activate this capacity the whole personality

[60] This could be ambiguous between knowing their metaphysical status ('They are but shadows') and knowing their ethical value ('This is something just, that unjust'). The context ensures that the latter is the meaning intended.

[61] I elaborate this suggestion in 'Culture and Society in Plato's *Republic*', *The Tanner Lectures on Human Values*, 22 (1999), to which the present essay is a sort of sequel.

must be turned around, away from the world of becoming, so as to redirect the 'eye of the soul' towards the realm of true being (518b–519b). Socrates then asks (521c), 'What studies will have that effect?' The question is open. The answer, of course, is mathematics, but Socrates has to argue it at length (522b–531d). In effect, he is arguing that an education in advanced mathematics is progress towards understanding true values. At the end, after five mathematical disciplines have been selected for the curriculum, he sums up: these are the studies that will effect the conversion *and* the ascent to the objects on the wall *and* the journey up out of the cave as far as the reflections outside (532bd). Only the last stage, represented in the simile by looking at the people and other real things outside, is reserved for dialectic. I conclude that mathematics provides the lowest-level articulation of objective value.

It will not do to object that values need not enter the story until the rulers-to-be reach dialectic. If there are puppets representing justice, and mathematics is what takes the freed prisoners to the objects on the wall, then mathematics already gives them a better understanding of justice than they had before, even if they do not realise this until they come back down again. In the poetic narrative of the Cave, the first thing that happens after the prisoners are released from their chains is that they are shown the puppets one by one and forced to answer the question 'What is it?' (515d). In the retrospective prose of the mathematical curriculum, the first question they are forced to confront is 'What kind of numbers are the mathematicians talking about?' (525d–526b, quoted on p. 30 above). It would surely take lots of mathematics and much philosophising to convince one that pure numbers are the key to debates about justice in court or assembly. That insight should be reserved for prisoners who have made the ascent and then returned. It is important here that even the higher level of dialectic turns out to have a strongly mathematical content.

Dialectic is not deductive proof, but philosophical discussion aimed at testing and securing definitions (533ab), and we have already seen that what the future rulers are to discuss in this way is the hypotheses they relied on when doing mathematics (511b, 533c). It is these that will lead to the unhypothetical first principle of everything, the Good (511b, 533bd). Just how they will lead up to knowledge of the Good is a difficult and debated question in the scholarly literature. For present purposes, it is enough that dialectic is described in terms that suggest what we might call a meta-mathematical inquiry. The education of the rulers is mathematical, in one sense or another, all the way to the top. The famous image of dialectic as the coping stone (*θριγκός*) of the curriculum (534e) implies the completion of a single, unified building, not a transfer to different subjects in a different building.

Yet the education of the rulers is also, from beginning to end, about value. At the beginning, as we have seen, they meet puppets of the just (note the plural *ἀγάλματα* at 517d 9); at the end the Good, which Socrates describes both as the cause of all things right and beautiful, and as that which anyone who is going to act wisely either in private or in public life must know (517c). They return having seen 'justice itself' (517e). But when? Plato could easily have made Socrates say that dialectic involves *both* trying to account for mathematical hypotheses in terms of Forms *and* discussing the Form of Justice.[62] Instead, he leaves us to infer that dialectical debate about the conceptual foundations of mathematics is itself, at a very abstract level, a debate about values like justice. I think the inference is correct. The mathematics and meta-mathematics prescribed for the future rulers is much more than instrumental training for the mind. They are somehow supposed to bring an enlargement of ethical understanding. My final question is, How could that be?

[62] F. M. Cornford, 'Mathematics and Dialectic in the *Republic* VI–VII', *Mind*, 41 (1932), 37–52 and 173–90, cited from R. E. Allen, *Studies in Plato's Metaphysics* (London & New York, 1965), chap. 5, 80 ff., argued that Plato divides the description of dialectic into two parts, one about mathematical dialectic and mathematical Forms (533a–534b), the second about moral dialectic and moral Forms (534bd), each part having its own distinctive methodology. His argument has not won acceptance, and in any case Socrates implies that a dialectical account of the Good will be of the same type, subject to the same tests, as the dialectical account of anything else (534b 8: *ὡσαύτως*).

10. Harmonics

The place to start looking for an answer, I suggest, is the discussion of harmonics. When Socrates insists that mathematical harmonics should 'ascend to problems to consider which numbers are concordant, which are not, and why each are so', Glaucon exclaims, 'You are speaking of a task which is superhuman (δαιμόνιον πρᾶγμα)'. Socrates corrects him: 'Say rather, a task which is useful if directed towards investigating the beautiful and good, but useless if otherwise pursued' (531c). Both Pythagorean harmonics and Plato's are concerned with concord (συμφωνία). The difference is whether they seek concords in heard sounds or at a more abstract level. Socrates implies that moving to the more abstract level is a prerequisite for harmonics to help us understand values like beauty and goodness. At that level, the answer to the question 'Why are these numbers, unlike others, concordant?' cannot be that they determine intervals which *sound* good to the ear. So what kind of explanation can Plato have in mind? That was Enigma C.[63]

As before, the best guide is Euclid. In the preamble to his *Sectio Canonis* we find this:

> Among notes we recognize some as concordant, others as discordant, the concordant making a single blend out of the two, while the discordant do not. In view of this it is reasonable (εἰκός) that the concordant notes, since they make a single blend of sound out of the two, are among those numbers which are spoken of under a single name in relation to each other, being either multiple or epimoric. (149.17–24 Jan)[64]

This preliminary remark relies on a feature of the vocabulary the Greeks used to speak of ratios. For the multiple ratios 2:1, 3:1, 4:1, etc., they had one-word expressions, ending in -πλασιος, just like our 'double', 'triple', 'quadruple', and so on. Unlike us, they also

[63] P. 14 above.

[64] Tr. Barker, *GMW* II, p. 193, but with Mueller's rendering of εἰκός as 'reasonable' substituted for Barker's 'to be expected' (Mueller, 'Ascending to Problems', p. 113). It is kinder to Euclid to have him talk of what ought to be, rather than of what can in fact be expected in advance. Kinder still, and linguistically permissible, would be 'appropriate'.

had a series of one-word expressions for epimoric ratios, which are ratios of the form $n+1:n$. Thus 3:2 (the ratio of the fifth) is ἡμιόλιος, meaning 'half-and-whole'; 4:3 (the ratio of the fourth) is ἐπίτριτος, meaning 'third-in-addition'; 5:4 is ἐπιτέταρτος ('fourth-in-addition'), and so on. Other ratios, by contrast, collectively called 'epimeric', had no such expression assigned to them in the language, but were specified long-windedly as, e.g., 'seven to four'. Euclid's idea, then, is that Greek gives apt recognition to the unity of sound in a concord by assigning a single expression to the corresponding mathematical ratio.[65]

Whatever we think of this linguistic observation, it is clearly not an *explanation* of which numbers are concordant and which are not, and there is no reason to think that Euclid meant it as an explanation. For lots of multiple and epimoric ratios produce discordant intervals. But in the *Sectio Canonis* he does assume, when the mathematics gets going after the preamble, that concordant ratios are all either multiple or epimoric. That assumption was devotedly maintained in the tradition of mathematical harmonics to which Euclid belongs, despite a notorious difficulty caused by the interval of octave plus fourth. This is concordant to the ear, but its ratio is 8:3, which is neither multiple nor epimoric. The choice before a theorist was either to modify their mathematics or to say, in Platonic style, 'So much the worse for empirical perception'. Euclid deftly escapes the dilemma by not mentioning this interval anywhere. But he is useful for our purposes in two ways. First, the *Sectio Canonis* is an example of the *sort* of mathematics Plato will have had in mind when he called for an investigation, by means of problems, of which numbers are concordant and which are not. Second, the assumption that concordant ratios are all either multiple or epimoric may provide at least a glimpse of the *sort* of

[65] This is not the only place where Euclid shows an interest in names. At *Elements* VII 37 he proves the trivial-seeming proposition, 'If a number be measured by any number, the number which is measured will have a part called by the same name (ὁμωνύμου) as the measuring number', and at VII 38 that 'If a number have any part whatever, it will be measured by a number called by the same name as the part'. Note once again the non-modern idea that the part and the measuring number are distinct, not one and the same number.

explanation he wanted of why certain numbers are intrinsically concordant.

But Euclid is around half a century later than Plato.[66] If we track back to the time when the *Republic* was written, it seems that Glaucon, knowledgeable though he is about music (398e; cf. 548de), is not familiar with any *mathematical* treatment of the subject. For when Socrates refers to Pythagorean harmonics as an approach to music which goes wrong in the same way as the calendaric astronomy he castigated earlier, Glaucon does not recognise the allusion. He supposes that Socrates means an empirical, string-torturing approach which gives the ear primacy over reason, not a mathematics which seeks *numbers* in heard concords. Socrates has to explain that he means 'the Pythagoreans' he mentioned earlier (530e–531c), i.e. Archytas. I infer that readers of the *Republic* are not expected to be familiar with Archytas' mathematical harmonics; it is recherché stuff.

But it was known to Euclid. Proposition 3 of the *Sectio Canonis*, 'In the case of an epimoric interval, no mean number, neither one nor more than one, will fall within it proportionally', was first proved by Archytas.[67] We can hope that Archytas may offer further help with Enigma C. Here, then, is some more Archytas:

> There are three means in music. One is arithmetic, the second geometric, the third subcontrary, which they call 'harmonic'. There is an **arithmetic** mean when there are three terms, proportional in that they exceed one another in the following way: the second exceeds the third by the same amount as that by which the first exceeds the second. In this proportion it turns out that the interval [*sc.* the musical interval] between the greater terms is less, and that between the lesser terms is greater. There is a **geometric** mean when

[66] Assuming the *Sectio Canonis* is by Euclid. But the attribution is debated: see André Barbera, *The Euclidean Division of the Canon: Greek and Latin Sources* (University of Nebraska Press, 1991), pp. 3–36. If the treatise is not by Euclid, its date becomes uncertain, but it is still the best means to contextualise Plato's discussion of concord.

[67] Boethius, *Institutio Musica* III 11; for a discussion of differences between Archytas' proof and Euclid's, see Wilbur Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht & Boston, 1975), pp. 212–25.

> they are such that as the first is to the second, so is the second to the third. With these the interval made by the greater terms is equal to that made by the lesser. There is a subcontrary mean, which we call '**harmonic**', when they are such that the part of the third by which the middle term exceeds the third is the same as the part of the first by which the first exceeds the second. In this proportion the interval between the greater terms is greater, and that between the lesser terms is less. (Archytas frag. 2 Diels-Kranz)[68]

The musical significance of these means may be illustrated as follows.

(i) The three numbers 12, 9, 6 are in arithmetical proportion, and 9 is the arithmetical mean between 12 and 6, because 12 exceeds 9 by the same amount as 9 exceeds 6. The ratio of 9 to 6 is 3:2, that of the musical fifth. The ratio of 12 to 9 is 4:3, that of the fourth. The fifth being a larger span than the fourth, the latter is what Archytas speaks of as the lesser interval determined by the ratio of the greater numbers (12 and 9).

(ii) The three numbers 6, 12, 24 are in geometrical proportion, and 12 is the geometrical mean between 6 and 24, because the ratio of 24 to12 is 12:6, which in turn is 2:1, the ratio of the octave. So, as Archytas puts it, the interval made by the greater terms (24 and 12) is equal to the interval made by the lesser (12 and 6) — an octave in both cases.

(iii) The three numbers 12, 8, 6 are in harmonic proportion, and 8 is the harmonic mean between 12 and 6, because $8-6 = 2$ and $12-8 = 4$: the difference in each case is a third part of the relevant extreme term, since 2 is a third part of 6 and 4 is a third part of 12 (in modern fractional notation, $2 = 6/3$ and $4 = 12/3$). Here the greater terms (12 and 8) make the greater interval, because the ratio of 12 to 8 is 3:2, the fifth, while the interval represented by 8:6 is 4:3, the fourth.

To explain how the three means were put to use in Greek music theory, I call on Andrew Barker (square bracketed additions mine):

> The series 6, 12, 24 etc., in geometric proportion, represents a sequence of notes an octave apart. If we take the first two numbers

[68] Tr. Barker, *GMW* II, p. 42, from whose notes ad loc. I borrow the illustrations that follow.

> and insert the arithmetic mean, we get 6, 9, 12, the octave being divided into a fifth [because 9:6 is 3:2] followed by a fourth [because 12:9 is 4:3]. A harmonic mean inserted between the original terms gives 6, 8, 12, dividing the octave into a fourth [because 8:6 is 4:3] followed by a fifth [because 12:8 is 3:2]. When the two sequences are combined, 6, 8, 9, 12, they yield two fourths [8:6 is 4:3 and 12:9 is 4:3] separated by the 'tone' of ratio 9:8, and can represent the fixed notes bounding a pair of disjoined tetrachords. [A tetrachord is a fourth, the upper and lower notes of which are fixed, but not the notes inserted in between. By varying the latter — in particular the distance of the highest from the upper bound — different musical 'genera' were produced: the enharmonic, the chromatic, the diatonic. Thus the tetrachord is a basic unit of scalar organization.][69] These are the fundamental relations on which all the complex structures of Pythagorean and Platonist harmonics are built.[70]

Finally, an excerpt from the passage in Plato's *Timaeus* where the Divine Craftsman constructs the soul of the world as an elaborate scale or attunement of 27 notes, starting from two sequences in geometric proportion (1, 2, 4, 8 and 1, 3, 9, 27):

> 'Next he filled out the double and triple intervals, once again cutting off parts from the mixture[71] and placing them in the intervening gaps, so that in each interval there were two means, the one exceeding [one extreme] and exceeded [by the other extreme] by the same part of the extremes themselves, the other exceeding [one extreme] and exceeded [by the other] by an equal number.' (35c–36a)[72]

This is Archytas' language for the harmonic and arithmetic means, but redirected to elucidate the harmonious structure of a non-

[69] For this and further details, see *GMW* II, pp. 11–13.

[70] *GMW* II, pp. 42–3, n. 59.

[71] The recipe for the mixture is given at 35ab: (i) take the indivisible Being that is always unchangingly the same and mix with the divisible being that comes to be in bodies, (ii) likewise, mix indivisible Sameness with its divisible counterpart, and (iii) indivisible Difference with divisible difference, then (iv) blend all three ingredients into a unity. For present purposes, all we need to understand of this is that the 'stuff' from which soul is made has some sort of intermediate status between Forms and sensibles; in this respect it is comparable to the objects of mathematics. But it is not *soul*, properly speaking, until the appropriate musico-mathematical organisation has been imposed upon it.

[72] Tr. Barker, *GMW* II, p. 59.

sensible entity, the soul.[73] The World Soul in the first instance, but the Divine Craftsman will later give the same structure to the less pure mixture from which he makes human souls (41d, 43d). Glaucon's exclamation, 'You are speaking of a task which is superhuman (δαιμόνιον πρᾶγμα)', may be pregnant with more meaning than he realises.

What I propose we should take from all this is the idea that the concords can be derived by operations with what Archytas called 'the three means in music'. Concord is explained by proportion. And these operations can be redirected to the analysis of structures which have little or nothing to do with sound. Soul provides the non-sensible subject-matter for a harmonics of the inaudible.[74]

If this seems too general to explain why certain numbers are concordant, not others, let me add a further conjecture, suggesting that Plato may owe more to Archytas' language than appears from the *Timaeus* passage just quoted.[75] In the *Harmonics* of Ptolemy (second century AD) we find a discussion of 'the principles adopted by the Pythagoreans in their postulates about the concords', which offers strictly *mathematical* reasons for the thesis that multiple and epimoric ratios are a *better* (ἀμείνων) kind of ratio than epimerics. They are 'better' because of the *simplicity* of the comparison between the two terms of the ratio. In the case of epimorics like 3:2, the excess [of the greater over the smaller term] is a simple part [integral factor, namely 1] of each of the terms. Multiples like 2:1 are even finer because the smaller term is itself a simple part of the greater. No such straightforward comparison of the terms is possible with an epimeric like 7:3. This result can then be used to explain why notes in the 'better' ratios sound better to the ear (Ptolemy, *Harmonics* I 5, 11.1–12.7 Düring). If Andrew Barker is right in maintaining that Archytas is the only Pythagorean we can identify as a plausible source for Ptolemy's report, then here is a

[73] Archytas' language, but the scale itself is Philolaus' diatonic. Archytas' own scalar divisions are more complicated, because designed to account for actual musical practice: Barker, *GMW* II, pp. 46–52.

[74] So Burkert, *Lore and Science*, pp. 372–3.

[75] What follows is inspired by Andrew Barker, 'Ptolemy's Pythagoreans, Archytas, and Plato's conception of mathematics', *Phronesis*, 39 (1994), 113–35.

striking precedent for Plato to embrace the idea of a mathematics which makes direct use of evaluative concepts like 'better', and musical concepts like 'concordant', without first deriving them from auditory experience. From Plato's standpoint, Archytas' fault would be his developing such a mathematics merely in order to explain, from above as it were, the auditory experience we enjoy.

Indeed, Plato's own account of why concordant intervals sound good to the ear is a strictly physical explanation given in a much later section of the *Timaeus* (80ab). Following Archytas (frag. 1 Diels-Kranz), Timaeus states that pitch depends on the velocity with which air is driven to the ear by the source of the sound: the faster the transmission, the higher the pitch (*Timaeus* 67ac). When the slower and the faster of two motions have a certain 'similarity', they are heard as a single 'blend' of high and low: 'Hence they provide pleasure (ἡδονή) to people of poor understanding, and delight (εὐφροσύνη) to those of good understanding, *because of the imitation of the divine attunement* that comes into being in mortal movements' (80b).[76] Pleasure as such is merely the perception of restoration processes in the body (64c–65b). But what delights a listener familiar with the harmonics of the World Soul is that the agreeable stimulation of concordant sounds is a sensuous realisation of the non-sensible concords in the divine attunement. As the poet said, 'Heard melodies are sweet, but those unheard are sweeter'.

11. The ethical value of concord and attunement

This is the point at which to notice that concord has long been a value important to the overall argument of the *Republic*.

Way back in Book III, for example, Socrates laid down a rule that the material environment of the ideal city should be so designed that the young grow up surrounded by works of grace and beauty, whose impact on eye and ear will imperceptibly, from childhood on, guide them to likeness, to friendship, to concord (συμφωνία) with the beauty of reason (401cd). Their musical and

[76] For the details, see Barker, *GMW* II, pp. 61–2, whose translation I have borrowed; Cornford goes badly wrong by applying 'because . . .' to both types of person.

gymnastic training will harmonise (ἡρμόσθαι) the two elements in their soul, the spirited and the philosophical (as if they were strings on a lyre), relaxing and tightening them as necessary to 'tune' the soul to be both brave and temperate (410a–412b; cf. 441e–442a). In Book IV temperance is first said to be more like a sort of concord and attunement (συμφωνίᾳ τινὶ καὶ ἁρμονίᾳ) than the virtues of wisdom and courage are (430e; cf. 431e), and then defined as agreement (ὁμόνοια) or concord (συμφωνία) between the naturally inferior and naturally superior elements as to which should rule, both in the city and in the individual soul; the result of such concord is that the strongest elements and the weakest and those in between all sing together to the same melody (432ab; cf. 442cd). Later, in Book IV, he defines justice as an attunement (ἁρμονία) which harmonises the three parts of the soul as if they were the highest, lowest and middle notes of a scale (443de). Later still, in Book VII, he finds good attunement (εὐαρμοστία) and good rhythm (εὐρυθμία) in the souls of the future rulers, the result of their habituation to the attunements and rhythms of the music and stories prescribed for their elementary education (522a). Many more examples could be collected. The musical terms concord and attunement are significant leitmotifs in the discussion of the non-mathematical education of the future rulers of the ideal city.

Equally significant is that the definition of temperance in terms of concord turns up in Aristotle's training manual for dialectical debate, the *Topics*, as an example of a definition to which objection can be made on the grounds that the supposed genus is a term used metaphorically, not in its proper meaning. Properly speaking, 'every concord is in sounds' (*Topics* IV 3, 123a 33–7).[77] How wrong can you be? If readers of the *Republic* start out with the

[77] Compare Jowett's introduction to his translation (2nd edn, Oxford, 1875), p. 82: 'When divested of metaphor, a straight line or square has no more to do with right and justice than a crooked line with vice.' Note that Aristotle's stand does not stop him entertaining the view that heard concords are to be explained by mathematical ratios: *Posterior Analytics* II 2, 90a 18–23, *De Anima* III 2, 426a 27–b 7 (with the text and interpretation of Andrew Barker, 'Aristotle on Perception and Ratios', *Phronesis*, 26 [1981], 248–66), *De Sensu* 3, 439b 19–440a 3 and 7, 448a 8–13. Nor does it make him better than anyone else at explaining mathematically why some ratios are concordant, others not.

impression that Plato's talk about concord and attunement in the soul is meant as metaphor, they should have second thoughts when they come to the passage about mathematical harmonics, which expressly denies that concord has to be a relation between sounds. In Plato's view, concord can also be a relation between pure numbers. In which case there is no reason to cry 'Metaphor!' when Plato has Socrates speak of concord between the different parts of city and soul. For a Platonist, much that we lesser mortals take as metaphor comes to be seen as a further instantiation of a concept which is more abstract and wide-ranging than ordinary folk suppose.[78] The *Timaeus* account of the musico-mathematical structure of the soul may be hard for us to grasp,[79] but to call it metaphorical would be absurd.

Let me offer a partial analogy from modern music. Bach wrote *The Art of Fugue* as an open score and did not designate the instrumentation. It has been played on keyboards of different sorts, by string quartets, by whole orchestras; I have heard it played by a brass quintet. What Bach really created, one might say, is the abstract structure represented by the notation (all those little diagrams) on the page. To undergo a Platonic conversion with respect to *The Art of Fugue* would be to come to think that, while it may be realised audibly in different sound-media, none of these performances (token or type) is as real, as beautiful, or as valuable, as the abstract structure which is '*The Art of Fugue* itself'. Plato finds concord and attunement in many different media. Not only in music, but also in the social order of the ideal city, in the psychic structure of a virtuous individual, and more broadly still, when he is doing physics in the *Timaeus*, throughout the cosmos.

In short, wherever Plato can find some quantitative dimension (see *Republic* 432a, 462bc), he can speak literally of concord, attunement, ratio, and proportion.[80] It follows that, by studying

[78] Example: for Aristotle, *Rhetoric* III 2, 1405a 26–7, it is metaphor to call your crime a mistake, or his mistake a crime; for Plato, any crime *is* at bottom a mistake.

[79] For help, see Barker, *GMW* II, pp. 58–60.

[80] Compare the intimate association of goodness and beauty with measure and proportion (*συμμετρία* at *Philebus* 64d–65a. It is in the *Philebus* that Plato develops in most detail the idea that measure and proportion require a quantitative dimension (the *ἄπειρον*) on which to impose their order (24a–26d).

mathematical harmonics, the rulers will gain an abstract, principled understanding of structures they will want to create and sustain when they return to the cave to rule. For Plato, the important task of ruling is not day-to-day decision-making, but establishing and maintaining good structures, both institutional and psychological. In both city and soul, dispositions and structures are prior to their expression in action (433d–434c, 443b–444a); the *Republic* combines virtue ethics with virtue politics. Thus knowing what numbers are concordant, and why, has a very great deal to do with the tasks of government, because concord is an important structural value at the lower level of ethics and politics.

12. An astronomy of the invisible

We can now dispatch Enigma B of Section 4. In the most literal meaning of the word, an attunement (ἁρμονία) is a way of tuning the instrument to certain intervals which, like our musical scales, lends a particular character or 'colour' to the subsequent melodies. The attunement is one thing, the melodies played with it another. Just so, temperance conceived as concord or attunement is the virtuous disposition, not the actions it leads to, when the three parts sing together in unison about which of them should decide what to do.

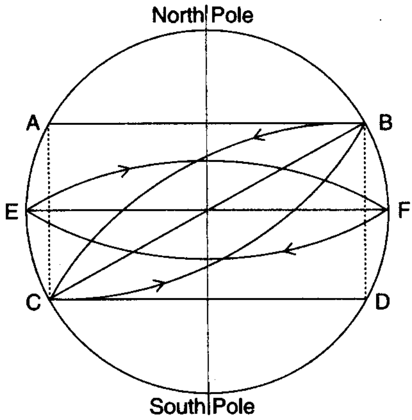
The astronomical relevance of this distinction may be illustrated by the long-lived fancy of the so-called 'harmony of the spheres', which makes its first recorded appearance in the Myth of Er at the end of Plato's *Republic* (616c–617d). The myth depicts eight hemispherical whorls nested inside each other revolving around the Spindle of Necessity (a column of light running from top to bottom of the universe). The rim of the outermost whorl, as seen from above, represents the circle of the fixed stars; the other rims correspond to the circles of the sun, the moon and the five planets known to the Greeks. Each circle carries a siren who emits a single note: 'And from these sounds, eight in all, is made the concord of a single ἁρμονία' (617b).

Imagine hearing the eight notes of an octave sounded together. A cosmic cacophony! Think instead of the eight notes constituting an octave scale or attunement — not the melody but a framework for various melodies — and all becomes clear. The celestial music heard by Er does not come from the sirens, but from Lachesis, Clotho and Atropos, who 'sing (ὑμνεῖν) to the ἁρμονία of the Sirens, Lachesis of what has been, Clotho of what is, and Atropos of what will be'.[81] So too with the World Soul in the *Timaeus*: the attunement (ἁρμονία) described in the previous section is its structure, not the motions it is designed for. We must now ask: What are the motions of the World Soul?

Believe it or not, they are the motions that produce, on the one hand, the diurnal rotation of the heaven from East to West, on the other, the annual journey of the sun along the ecliptic between the winter and summer solstices. The first is called the motion of the Same, because it is the principle of regularity. The second, at an oblique angle to the motion of the Same and in the reverse sense (see Figure 5 overleaf), is the motion of the Different; this is the principle of variation. The two motions together produce the regular variation of the seasons. We have passed, in two sentences (*Timaeus* 36b 6–c 5), from harmonics to astronomy.

This is not the place to try to elucidate the astronomical system of the *Timaeus*.[82] Rather, we must struggle with the fact that the motions of the Same and the Different are not the observable motions they cause, but movements of *thought* in the intelligence of the World Soul. Later we find the same two movements in the human soul, initially deformed by the trauma of birth but stabilising as the child gradually becomes more rational (43a–44c). The puzzle is that both movements, that of the Same and that of the Different, are described as circular. How can thought move in circles? Or a soul revolve? Aristotle protested, 'It is quite wrong (οὐ καλῶς) to say the soul is a magnitude' (*De Anima* I 3, 407a 2–3).

[81] Translation and elucidation due to Barker, *GMW* II, pp. 57–8.
[82] A helpful commentary may be found in F. M. Cornford, *Plato's Cosmology* (London, 1937), pp. 72–93, sceptically reviewed by Dicks, *Early Greek Astronomy*, chap. 5. Figure 5 is taken and adapted from Cornford, p. 73.



AB is a diameter of the summer tropic, *CS* a diameter of the winter tropic. *CB*, the diagonal of the rectangle obtained by joining *AC*, *BD*, is a diameter of the ecliptic, a great circle touching the summer tropic at *B* and the winter tropic at *C*. The motion of the Same makes the whole sphere of the cosmos revolve from East to West in the plane of the quarter *EF*. The motion of the Different gives sun, moon, and planets an additional movement in the reverse sense in the plane of the diagonal *CB*.

Figure 5

As before, I take Aristotle's reaction as confirming that Plato meant exactly what Timaeus said:

> 'When the whole fabric of the soul had been finished to the satisfaction of its maker's mind, he next began to fashion *within* the soul all that is corporeal, and he brought the two together and fitted (προσήρμοττεν) them *centre to centre*. And the soul, being *everywhere* inwoven from the centre to the outermost heaven and enveloping the heaven *all round* on the outside, *revolving* within its own limit, made a divine beginning of ceaseless and intelligent life for all time.' (36de; tr. Cornford, slightly changed)

The spatial language is unmistakable. Soul, both human and divine, has extension in three dimensions.

This does not make it corporeal. The soul–body contrast remains as strong in the *Timaeus* as in other dialogues. But the distinguishing marks of corporeality for Plato are visibility and tangibility (*Timaeus* 31b); in more modern terms, corporeal things must have secondary qualities. Soul, then, as a non-corporeal thing, must be invisible and intangible, without secondary qualities. But this is compatible with its having extension in three dimensions and primary qualities such as size or shape — just like the abstract, non-sensible objects of solid geometry.[83] In which case, there is no reason why it cannot also move in ways that will provide a challenging study for the purely mathematical astronomy projected in *Republic* VII:

> 'These patterns in the heaven, since they are embroidered in the visible realm, we should regard as the most beautiful and the most exact of visible designs, yet we should hold that they fall far short of the true patterns of movement achieved by invariant swiftness (τὸ ὂν τάχος) and invariant slowness (ἡ οὖσα βραδυτής) in true number and all true figures [i.e. the motions trace out geometrically perfect figures[84] at speeds measured by exact numbers[85]] in relation to each other as they carry round the things contained in them [i.e. the heavenly bodies visible in the sky]. All this is to be grasped by reason and thought, not by sight.'[86] (529cd)

Think of a series of still photographs of the heaven, each

[83] Here I am indebted to presentations by Sarah Broadie and David Sedley at a Cambridge seminar on the *Timaeus*; see David Sedley, '"Becoming like god" in the *Timaeus* and Aristotle', in T. Calvo and L. Brisson (eds), *Interpreting the Timaeus-Critias* (Sankt Augustin, 1997), pp. 327–39. On circular thought, there is much to be learned from Edward N. Lee, 'Reason and Rotation: Circular Movement as the Model of Mind (Nous) in Later Plato', in W. H. Werkmeister (ed.), *Facets of Plato's Thought* (Assen, 1976), pp. 70–102, even though (as his title reveals) he denies that Plato meant it literally.

[84] Not any old figures such as modern geometry could comprehend, but figures accessible to Greek geometry. In the practice of Plato's day, this means spheres of various diameters.

[85] Whole numbers, thereby establishing at the invisible level the proportions (συμμετρίαι) sought by the empirical astronomers dismissed earlier: see 530a 1, *Timaeus* 36d, 39cd.

[86] Any translation of this passage involves interpretation. (i) The grammatical subject of the passive verb φέρεται and the active φέρει is τὸ ὂν τάχος καὶ ἡ οὖσα βραδυτής, but swiftness and slowness cannot literally carry or be carried along. Hence Adam's idea ad loc. (with Appendix X) that τὸ ὂν τάχος καὶ ἡ οὖσα βραδυτής designates mathematical counterparts of the visible stars, which are moved along; the objection is that these could hardly be called swiftness and slowness. My verb 'achieved' is meant to suggest, what is true, that the patterns we are talking about are made by the swiftness and slowness of the different movements *in relation to each other*. This is the mathematical correlate of the visible patterns of movement in the night sky that result from the different relative speeds

exposed for the whole night on successive dates. Instead of spots of light (one for each star and planet) you see *lines* of light crossing and criss-crossing each other. To study the daytime movement of the sun, the Greeks used a hemispherical dial (πόλος), shaped like the vault of heaven, in which a shadow was cast by the gnomon or pointer.[87] The sun's shadow moves in a circular path which shifts through the year between the circles marked on the dial to represent the tropics. All these visible patterns are patterns of *movement.* They are the visible embroideries that Socrates began from,[88] which he says should be treated like the diagrams in geometry, except that what we see in the heaven are diagrams made by a craftsman like Daedalus, legendary maker of *moving* statues (529de). Socrates goes on to a quite different set of motions, the 'true patterns of movement' which produce the motions we observe. But we have to go to the *Timaeus* to learn what these 'true' motions might be.

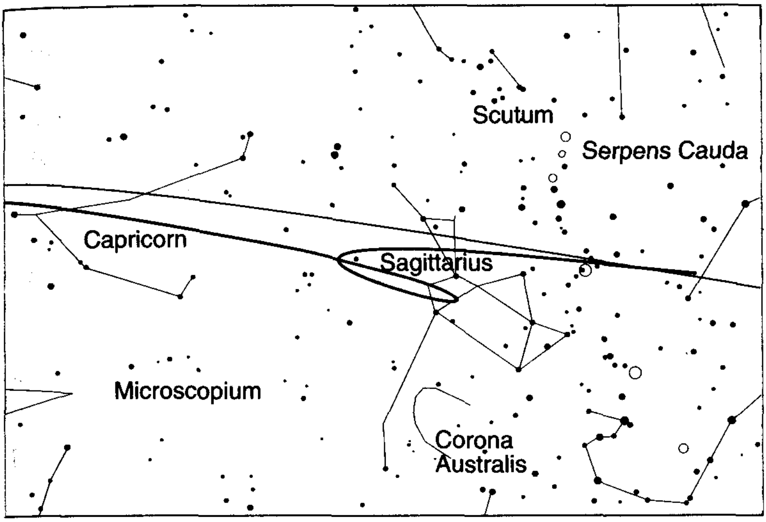
The irregularity of the observed motions was well known. Besides the incommensurability of the periodic revolutions of sun and moon, already mentioned,[89] the very term 'planet' means 'wanderer'. Mercury, Venus, Mars, Jupiter, and Saturn, the five known planets, all exhibit the phenomenon of retrogradation. Like the sun, they have periodic revolutions eastwards through the zodiac, some shorter and some longer than the solar year, but in their case there is an added complication. From time to time they appear to stop, to reverse their journey, and then resume as before;

of the fixed stars, sun, moon and planets. (ii) In translating τὸ ὂν τάχος καὶ ἡ οὖσα βραδυτής by 'invariant swiftness and invariant slowness', I rely on the notion of unqualified being that Socrates has been gradually developing since the argument with the lovers of sights and sounds in Book V; cf. n. 27 above. (iii) The visible bodies are 'contained' in the movements, I take it, in the sense that they always appear in the position determined for them at the time by the movements responsible for their travel.

87 First attested by Herodotus II 109, as commonly interpreted.

88 Not constellations of stars, as proposed by Ivor Bulmer-Thomas, 'Plato's Astronomy', *Classical Quarterly*, 34 (1984), 107–12. However beautiful they may be, the constellations group fixed stars which all move round together at the same speed (the motion of the Same), without changing position relative to each other.

89 P. 14 above.



Mars in Sagittarius, 3/86–11/86; loop dimensions: 11° 47′×1° 11′.

Figure 6

at the same time they exhibit changes in latitude.[90] Figure 6 shows an example: the path traced out by Mars between March and November 1986.[91] The first reasonable explanation of this phenomenon was given by the mathematician Eudoxus of Cnidus, who postulated that a planet's 'wandering' was due to four homocentric spheres (the innermost carrying the planet) revolving about the earth in different directions at different speeds. According to Simplicius (sixth century AD, but relying on earlier sources), the resultant of the several motions to which each planet was thereby subjected took the form of a 'hippopede' or 'horse-fetter', a figure-of-eight on its side like the modern mathematical symbol for infinity.[92]

At this point it is worth recalling the discussion of diagrams in the Divided Line passage. A diagram is a visible form used as an image to aid thinking about something abstract and non-sensible.

[90] For beginners like myself, Dicks, *Early Greek Astronomy*, chap. 1 is a helpful explanation of the various astronomical phenomena relevant to the astronomy of Plato's time.

[91] Reproduced from a fascinating article by Ido Yavetz, 'On the Homocentric Spheres of Eudoxus', *Archive for History of Exact Sciences*, 52 (1998), 221–78.

[92] Simplicius, *Commentary on Aristotle's De Caelo*, 496.29–497.5 Heiberg; Yavetz, 'Homocentric Spheres', queries Simplicius' reliability on this point.

Socrates supposed a relation of likeness between image and the imaged, but he did not say *how* like they have to be. The triangles and squares in Figure 4 are very like the figures they represent, but it takes an effort of thought to treat the line *AH* in Figure 2 as the likeness of an arithmetical unit. Again, it is hard to see the three-dimensionality of the icosahedron in Figure 1, and there is no more than a circle to represent the sphere in which it is inscribed. More entertaining examples include the diagram of an infinite line at Euclid, *Elements* I 12, and the squashed circle at *Elements* III 10, which represents a case that Euclid immediately proves to be impossible, where one (regular) circle intersects another at more than two points. The greater the complexity of the item represented, the more thought is summoned to supplement the visual data. The same holds, in a different way, of the simple lines in *Elements* V, which presents Eudoxus' general theory of proportions: the lines diagram magnitudes as such — any magnitudes whatsoever, be they lines, figures, solids, or times. A Greek mathematician confronted with contemporary observational records showing a pattern of movement even roughly like the path traced by Mars in Figure 6 might well be persuaded to look on it as an aid to thinking about hippopedes.

Eudoxus' system of homocentric spheres was certainly crucial to Aristotle's cosmology. But in Plato all we clearly find is the problem Eudoxus tried to solve. At the end of the *Republic*, the Myth of Er vaguely postulates that sun, moon and planets each have an additional motion contrary to the daily rotation of the whole heaven (617ab). The *Timaeus* marks a small advance in that the seven contrary motions into which the Different is split (one each for sun, moon and five planets) are at an oblique angle to the motion of the Same (36d). Both passages make vague claims about differences in speed between the various motions. The challenge is to replace vagueness by precision, and to address the problem of retrogradation. This is referred to in the *Timaeus* (40cd; perhaps also 38d), but the contrary motions of the Same and the Different cannot begin to make sense of it. Their resultant is a spiral motion in a continuously forward direction (*Timaeus* 39a: ἕλικα), not a hippopede that turns back on itself.

I would emphasise that the challenge is there *de facto* in the texts regardless of whether we believe the popular story retailed by Simplicius, that Plato set this problem to the mathematicians of his day: 'By hypothesizing what uniform, circular, ordered motions will it be possible to save the appearances relating to planetary motion?' (*Commentary on Aristotle's De Caelo*, 492.31–493.5 Heiberg, referring back to 488.18–24).[93] Even if Plato said nothing of the sort, the *de facto* challenge remains. Eudoxus responded to it with mathematical brilliance, although we do not know whether he produced his theory in time to influence Plato.[94] But that chronological uncertainty is irrelevant to the main contrast between Platonic and Aristotelian astronomy, which is as follows.

Whatever the full mathematical story turns out to be, Aristotle in the *De Caelo* wants to construe it in terms of the unstoppable circular movement natural to his diaphanous, imperishable fifth element, the aether, which he added to earth, air, fire, and water in order to give Eudoxus' spheres a material realisation. For Plato in the *Timaeus*, by contrast, the phenomena of the heavens are due to the perfectly circular, perfectly regulated movements of thought in the intelligence of the god (the World Soul) who guides the cosmos. Who now is to say which philosopher made the more reasonable choice at the time? And who can deny the relevance of Plato's choice to our understanding of the *Republic*'s sketch of an astronomy of the invisible?

[93] On the dubious credentials of the story, see now Leonid Zhmud, 'Plato as "Architect of Science"', *Phronesis*, 43 (1998), 211–44.

[94] See the careful discussion by Hans-Joachim Waschkies, *Von Eudoxus zu Aristoteles: Das Fortwirken der Eudoxischen Proportionstheorie in der Aristotelischen Lehre vom Kontinuum* (Amsterdam, 1977), pp. 34–58, who concludes that Eudoxus lived from *c.* 391 to *c.* 338, having moved his school to Athens *c.* 361, where, according to Proclus, *Commentary on the First Book of Euclid's Elements*, 67.2–3 Friedlein, he became an associate (rather than a member) of the circle of people gathered around Plato in the Academy. This dating makes Eudoxus too young to influence the *Republic*, but not too young to influence the *Timaeus*. Whether he did influence the *Timaeus* in some way is a further question, not to be discussed here.

13. The relationship of the Republic *and* Timaeus

In appealing to the *Timaeus* for help with Enigmas B and C, I am going beyond the *Republic*, not interpreting it. The *Republic* gives no more than a sketch of the redirected astronomy and harmonics, a sketch which might be filled out in different ways. But a programmatic sketch is all the *Republic* needs for its immediate purpose of persuading Glaucon (and through him the reader) that the ideal city is a Utopia that could in practice be realised. Socrates faced up to the question of practicability towards the end of Book V (472a). Answering it takes him to the end of Book VII (541ab). The mathematical curriculum is part of a long, unitary argument to establish that, if talented men and women with a passion for knowledge are educated in the right studies, they will rule both reluctantly (hence without being corrupted in the manner of the rulers we are familiar with) and wisely (hence to the benefit of the whole community).[95] The crux of the argument is the claim that true ethical insight presupposes an intense mathematical training, which neither Glaucon nor the reader has had. Plato's task in Books V–VII is to persuade us, through Glaucon, that the most important kind of knowledge is out of our reach, beyond our present capability, so that we would do well, should the day of Utopia come, to give political power to philosophers whose knowledge we do not share. To understand this, Glaucon (the reader) does not need to know the details of the advanced mathematics envisaged for the Guards' further education. Suppose Socrates tried to explain: would he (we) understand? (Eudoxus' system of homocentric spheres is exceedingly difficult to understand.)

Besides, how much mathematics does Socrates know? More than Glaucon, to be sure, but he does not claim to have covered the ten-year curriculum himself. Rather, he has a vision of how

[95] For more on the issue of practicability, see my 'Utopia and Fantasy: The Practicability of Plato's Ideal City', in Jim Hopkins and Anthony Savile (eds), *Psychoanalysis, Mind and Art: Perspectives on Richard Wollheim* (Oxford, 1992), pp. 175–87. [N.B. at p. 177, 5 lines from the bottom, after 'a way to overcome', insert 'the metaphysical obstacles to the realization of perfection, but for a way to overcome'.]

mathematics should be pursued in the ideal city, and it is the optimism of this vision that he aims to communicate. He is equally optimistic about dialectic and the Good, yet on this he has no knowledge, only opinions to share with Glaucon and the reader (506bc, 509c, 533a). A sketch of the subjects that will educate the rulers is just the right thing for his, and Plato's, present purpose.

After hearing how astronomy should be studied, Glaucon replies, 'You prescribe a task that will multiply the labour many times over as compared with the way astronomy is done at present' (530c).[96] After the sketch of a harmonics of pure numbers, he says, 'You are speaking of a task which is superhuman' (531c). That, I take it, is the kind of response Plato would like from readers of his *Republic*: an awesome respect.

The *Timaeus*, by contrast, is addressed to interlocutors (and hence to readers) who have enough mathematics to understand the harmonic structure of the World Soul and the astronomical system it controls, not to mention the stereometrical construction (53c–55b) of the four elements — earth (cube), air (octahedron), fire (pyramid), and water (icosahedron) — out of two kinds of triangle (right-angled isosceles and half-equilateral). The *Republic* does no more than mention the Craftsman who made the heaven (530a). The *Timaeus* is the appropriate place to study his mathematical design. And it is by way of prelude to the Divine Craftsman's construction of the elements that Timaeus says to his interlocutors, 'The account will be unfamiliar; but you are schooled in those branches of learning which my explanations require, and so will follow me' (53c; tr. Cornford). Yet although more advanced mathematically than the *Republic*, the *Timaeus* also presents itself as a sort of sequel to it.

The dialogue begins with a summary (17c–19b) of the institutions of the ideal city, to remind Timaeus, Critias, and Hermocrates of the fuller account Socrates gave them 'yesterday'. From antiquity onwards, many have imagined that yesterday Socrates met with his present interlocutors and began, 'Yesterday I went down to the Piraeus with Glaucon, son of Ariston'; the narrative of the

[96] Is this a Platonic hint at the need to multiply the number of spheres?

Republic, which seems to be addressed directly to the reader on the day after the festival of Bendis, turns out to have been delivered to Timaeus, Critias, and Hermocrates. But the discussion in the *Timaeus* takes place during 'the festival of the goddess [i.e. Athena]' (21a, 26e), which must be either the Greater or the Lesser Panathenaea, and it is now known that both these festivals were months away from the Bendidea.[97] Plato has changed the date to a different month (and for all we can tell, a different year) to stop us imagining that Timaeus and the rest listened to the narrative of the *Republic*. Instead, they were given its political content in a different form.

This has been thought to create a problem about the relation of the two dialogues, but to my mind it is the solution. Imagine Timaeus having to listen while Socrates tells Glaucon that stereometry has not yet been properly developed, or that astronomy and harmonics should be redirected to a realm of invisible and inaudible being. Nothing could be more inappropriate.[98] Nor does Timaeus need the images and other persuasive devices of the *Republic*. As someone of considerable political experience in a well-governed state (20a), he can cope perfectly well with a plain statement of the institutions of the ideal city. Socrates' flat summary is appropriate to his presence, and indicates to readers of the *Timaeus* what *sort* of sequel they are embarking on.

Thus in appealing to the *Timaeus* for help with Enigmas B and C, I am simply tracking the path laid down by Plato for such readers as can follow him into the later dialogue's larger and more detailed vision of the world as it is objectively speaking: a world in which mathematical proportion reigns supreme, because the Divine Craftsman is good and therefore wants the cosmos to be as like himself as material circumstances allow (29de). It is beyond dispute that in the *Timaeus* value is part of 'the furniture of the world'. Value is out there in 'the world as it is objectively speaking' because mathematical proportion is there, and mathematical proportion is the chief expression of the goodness of the Divine Craftsman's

[97] Details in Cornford, *Plato's Cosmology*, pp. 4–5.

[98] This helps to explain why Socrates confines his summary to the basic institutions proposed in *Republic* II–V, saying nothing about the central Books.

beneficent design. A good example is the continued geometric proportion which binds the four main world masses (earth, air, fire, and water) into a single cosmos, where each part is friendly to every other (*Timaeus* 31b–32c). Already in the *Gorgias* (507e–508a) 'geometric equality' (i.e. geometric proportion) is hailed as the greatest power among gods and men and throughout the cosmos. The next question is whether the *Timaeus* can also help with Enigma A.

14. The synoptic view

Enigma A arose from the fact that the Guards who are selected at age 20, after their military training, to spend the next ten years studying mathematics are required to 'bring together all the [mathematical] subjects which previously, during their childhood education [up to 18], they learned in no particular order (χύδην), to form a synoptic view of their kinship (οἰκειότητος) with each other and with the nature of what is' (537c, p. 1 above).[99] Success in this task will be an important test of which Guards are fitted to go on to five years' dialectic, for only someone who can view things synoptically has a truly dialectical nature (537c). This helps to explain why Socrates said earlier that the curriculum will not contribute to the desired end, knowledge of the Good, unless it is carried far enough to bring out the different disciplines' kinship with each other (531cd, p. 19 above). The synoptic view of mathematics anticipates, and prepares you for, the higher synoptic vision of the Forms in the light of the Good, as depicted by the simile of the sun.

It seems clear that part of what it means to achieve the synoptic view is to see the five mathematical disciplines in a particular order. And there can be little doubt about what that order is: arithmetic, plane geometry, stereometry, astronomy, harmonics. Look back over the way Socrates introduced the several subjects of the

[99] This is a subject little studied in the scholarly literature. I have been helped by Konrad Gaiser, 'Platons Zusammenschau der mathematischen Wissenschaften', *Antike und Abendland*, 32 (1986), 89–124, and Ian Robins, 'Mathematics and the Conversion of the Mind: *Republic* vii 522c1–531e3', *Ancient Philosophy*, 15 (1995), 359–91.

curriculum. After satisfying himself that arithmetic would be appropriate, he asks, 'What about the study that comes next (τὸ ἐχόμενον τούτου)? Is that suited to our purpose?' (526c 8–9). He expects Glaucon to be able to recognise, without being told, that 'the study which comes next' is geometry — and Glaucon does. The third discipline they discuss is astronomy, until Socrates pulls up and says that was a mistake (528ab). They went straight from the study of two-dimensional plane figures to three-dimensional figures *in circular motion*. The right way is first to take the third dimension 'itself by itself' (αὐτὸ καθ' αὑτό),[100] before adding the property of motion. And this lesson is repeated later, to make sure the reader does not miss it: astronomy should be fourth, not third (528de).

Thus far we have a steady increase in complexity: from extensionless to extended magnitude, from two to three dimensions, from solid figures as such to spheres in motion. This goes some way to explain the choice of order. In various ways the more complex disciplines presuppose or build upon the simpler.[101] At school we may learn Pythagoras' theorem one week (Euclid, *Elements* I 47), spend the next proving the infinity of prime numbers (*Elements* IX 20), and visit a planetarium at the weekend. That is learning things higgledy-piggledy (χύδην), in no particular order. There is good sense in the idea that a mature understanding of mathematics requires a more systematic approach. Not only should we grasp each mathematical discipline as an orderly body of knowledge developed out of a set of first principles (its hypotheses), but we should understand the several disciplines as themselves forming a unified system, a family (to repeat the image Plato took over from Archytas), in which the prior and simpler provides the basis for a series of more and more elaborate developments.

[100] A nice illustration for my earlier discussion of the phrase 'itself by itself' (p. 36 above).

[101] Compare Aristotle, *Posterior Analytics* I 27, 87a 31–7, *Metaphysics* I 2, 982a 25–8, XIII 3, 1078a 9–13. A clear example is the constant use made of plane geometry in the stereometrical constructions of *Elements* XIII, due originally to Theaetetus. Timaeus introduces the stereometrical construction of the four elements out of two types of triangle by saying, 'Now everything that has bodily form also has depth. Depth, moreover, is of necessity comprehended within surface, and any surface bounded by straight lines is composed of triangles' (*Tim.* 53c, tr. Zeyl).

Once again, a modern foil may help to bring out Plato's point, this time by contrast rather than resemblance. Here is Hegel on the standard Euclidean proof of Pythagoras' theorem (quoted above):

> The real defectiveness of mathematical knowledge, however, concerns both the knowledge itself and its content. Regarding the knowledge, the first point is that the *necessity* of the construction is not apprehended. This does not issue from the Concept of the theorem; rather it is commanded, and one must blindly obey the command to draw precisely these lines instead of an indefinite number of others, not because one knows anything but merely in the good faith that this will turn out to be expedient for the conduct of the demonstration. Afterwards this expediency does indeed become manifest, but it is an *external* expediency because it manifests itself only *after* the demonstration.
>
> Just so, the demonstration follows a path that begins somewhere — one does not yet know in what relation to the result that is to be attained. As it proceeds, these determinations and relations are taken up while others are ignored, although one does not by any means see immediately according to what necessity. An *external* purpose rules this movement.
>
> The evident certainty of this defective knowledge, of which mathematics is proud and of which it also boasts as against philosophy, rests solely on the poverty of its purpose and the defectiveness of its material and is therefore of a kind that philosophy must spurn.[102]

Fair enough if the proof is taken on its own, as an isolated lesson at school. It did indeed, I remember, feel like a conjuring trick. (Not that any Greek mathematician would mind astonishing the audience.) But this is proposition 47 of the first Book of the *Elements*. The proof brings to bear on the new problem several theorems proved earlier, which in turn flow from the hypotheses laid down at the start. The square-bracketed references accompanying my quotation[103] show that the proof uses Common Notion 2 and the results proved

[102] Preface to *The Phenomenology of Spirit* (1807), translated by Walter Kaufman in his *Hegel: Reinterpretation, Texts, and Commentary* (New York, 1965), p. 420. Kaufman compares Schopenhauer's even more vituperative version of the same charge in *The World as Will and Idea* (1819), Vol. I, §15 (denounced as 'ignorant strictures' by Heath *ad* Euc. *Elem.* I 47). I am grateful to G. A. Cohen for bringing these texts to my attention and for discussion of their import.
[103] Above, p. 39.

at I 4, I 14, I 41, and I 46. But I 14, for example, rests on I 13, Postulate 4, and Common Notions 1 and 3; similarly, I 46 on I 34 and I 37, and so on. I 47, as the very last proposition of Book I, is underpinned by a good deal of what precedes. In its turn I 47 enters into the proof of II 9–14. Euclid's careful elaboration of the initial input is an architectural masterpiece. As each proposition is proved, it finds its proper place in the whole — but what that place is may only become clear in the sequel. (Hegel's complaint could be generalised from the steps within a given proof to the succession of propositions within a Book.) A reader who continues as far as *Elements* VI 31 will find there the more general theorem, 'In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.' But this time the proof depends on the theory of proportions established in Book V. There is reason to think that Euclid's aim in I 47 was to show that Pythagoras' theorem, a special case of VI 31, could be proved *without* invoking the theory of proportions.[104] Only from a synoptic view of the *Elements* does this subtlety become apparent.

So too, I suggest, with the several mathematical disciplines. Each has to be grasped as a unified system *and* seen in the appropriate relation to the others. Someone who has achieved that integrated vision has not only assimilated a vast amount of mathematics. They have assimilated it as a structured whole. And for Plato, assimilation means that your soul takes on the structure of the abstract realm you study. This explains that mysterious addition 'their kinship with each other *and with the nature of what is*'. For Plato, as for Aristotle, knowledge and understanding depend on receptivity. You submit your soul to be in-formed by the world as it is objectively speaking. A soul that assimilates the vast abstract system of the mathematics on the curriculum is in turn assimilated to it. *You* come to be like, akin to, of the same family as, the nature of what is (in the sense of unqualified, context-invariant being):

> 'The motions akin (συγγενεῖς) to the divine part in us are the thoughts and revolutions of the universe (αἱ τοῦ παντὸς

[104] Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, Mass. & London, 1981), pp. 172–3.

> διανοήσεις καὶ περιφοραί). These, surely, are the ones which each of us should follow. We should correct the circuits in our head that were thrown off course at our birth, by learning to know the attunements and revolutions of the world (τὰς τοῦ παντὸς ἁρμονίας τε καὶ περιφοράς), and so make our intelligent part like the objects it knows, as it was in its original condition. And when the likeness is complete, we shall have achieved our goal: the best life offered to humankind by the gods, both now and forever.' (*Timaeus* 90cd)[105]

The system we internalise and become assimilated to is the articulation, at the level of mathematical thought (διάνοια), of the world as it is objectively speaking.

Contrast Reichenbach:

> It is true that our substitute world is one-sided; but at least it shows us some essential features of the world. Scientific investigation adds many new features; we look through the microscope and the telescope, construct models of atoms and planetary systems, and penetrate by X-rays into the interior of living bodies. Our task is to organize all the different pictures obtained in this way into one superior whole. Though this whole is not, in itself, a picture in the sense of a direct perspective, it may be called intuitive in a more indirect sense. We wander through the world, from perspective to perspective, carrying our own subjective horizons with us; it is by a kind of intellectual integration of subjective views that we succeed in constructing a total view of the world, the consistent expansion of which entitles us to ever increasing claims of objectivity.[106]

Reichenbach's problem is the characteristically modern one of working outwards to the world from within. Plato would agree that we always begin from a perspective conditioned by our physical make-up and our historical, cultural circumstances. But he would not be satisfied with Reichenbach's solution: to collect all the perspectives we can and organise them into an explanatory whole. For Reichenbach, objectivity is a goal we can only aim at. It lies tantalisingly beyond even the best and most coherent 'total view of the world'.

105 My translation borrows from both Cornford and Zeyl (1997). For the trauma of birth, see 43a–44c, cited p. 57 above.

106 *Experience and Prediction*, p. 225.

Plato, like Aristotle, tells a different story. Both Platonic Forms and Aristotelian forms will impress themselves accurately on our minds if only we allow them to do so. This is the point of the language of assimilation that both philosophers use. The world as it is objectively speaking will help us become assimilated to it. But we must cooperate by trying to clear away the one-sided preconceptions we grew up with, so that our concepts are entirely determined by what they are concepts of. The guarantee that this is possible is that *naturally* intelligent instrument, the 'eye of the soul', to which Socrates keeps referring (508d, 518ce, 519ab, 527b, de, 530c, 532c, 533d).[107] In Aristotle it is the potential intellect, the capacity we are born with to use thought and reasoning to reach correct concepts, where the standard of correctness is not the rules of our linguistic community but the world as it is objectively speaking. The other side of this coin, for Plato (not Aristotle), is that someone whose soul has become assimilated to objective being can take it as a model for reorganising the social world:

> 'Do you think there is any difference between blind people and people who lack knowledge of any real being, who consequently have no clear pattern in their souls and who cannot, as if they were painters, proceed either to set norms for what is beautiful and just and good in human life here, or to guard and preserve them once they have been established, by looking to what is most true, constantly referring to it, and contemplating it as accurately as they can?'
> 'No, by Zeus', he said, 'there isn't much difference between them.' (*Rep.* 484cd; cf. 500b–502a)

In its immediate context this is about the rulers' knowledge of the Forms. But one cannot reproduce Forms on earth. What one can reproduce, at least approximately, are structures that exemplify Forms like Justice and Temperance. If, as I have been arguing,

[107] A word on Plato's famous theory of recollection, which appears only in the *Meno*, *Phaedo*, and *Phaedrus*, not in the *Republic*: this should be regarded as an account of how the 'eye of the soul' can attain the knowledge it is naturally capable of, namely, by uncovering knowledge that is already present to it, as part of the original constitution of the soul. The *Republic* makes do with the more modest thesis, shared with Aristotle, that the soul has the capacity to attain knowledge of the world as it is objectively speaking.

mathematics is the route to knowledge of the Good because it is a constitutive part of ethical understanding, the corollary is that, when they return to the cave, the philosophers will think of the mathematical structures they internalised on the way up as abstract schemata for applying their knowledge of the Good in the social world. According to Plato's *Laws* (967e–968a), no one is fit to govern unless they have understood the community (*κοινωνίας*) of the mathematical disciplines; that understanding will enable them to design a well-tuned system (*συναρμοττόντως*) of character-shaping norms and practices for a human community.

No interpretation of the synoptic view can claim to be more than an imaginative projection of what might be. But sympathetic imaginative projection is precisely the effort Plato is asking from his readers here, because most of us have not studied enough mathematics to be able to share the synoptic view. As usual, Glaucon's response is telling: 'It is a huge task you describe' (531d). The least we can do is imagine a task that would indeed take many years to complete.

The snag is discipline No. 5, mathematical harmonics. That seems to presuppose and build upon arithmetic rather than astronomy, its immediate predecessor in the preferred order. To bring harmonics into line as the climax of the sequence, note two things. First, the all-pervasive role of ratio in Greek mathematics. From arithmetic through plane and solid geometry to astronomy, ratio and proportion keep turning up in the proofs. Harmonics, though mathematically simpler than advanced geometry and astronomy, is the first discipline to take ratio itself as the primary object of study.[108]

Next we should ask what harmonic ratios are ratios of. On Archytas' theory (frag. 1 Diels-Kranz) they will be ratios of the velocities with which air is moved by the sources of different sounds. In the *Republic* this appears as the reason why astronomy and harmonics are sister sciences: as astronomy studies motion visible to the eyes, so harmonics studies musical motion (530d: *ἐναρμόνιον φοράν*) audible to the ears. But Socrates then rejects

[108] So Robins, 'Mathematics', p. 388.

the Pythagorean idea of seeking numbers, i.e. ratios, in heard concords (531bc). His redirected harmonics, like his redirected astronomy, will need some non-sensible kind of motion to focus on. And what could this be but the movements of thought in the World Soul which the *Timaeus* casts as the objects of Platonic astronomy? Archytas' harmonics does not presuppose the corresponding type of astronomy, but Platonic harmonics does. For Platonic harmonics explains the good structure of the World Soul, which is expressed in the movements of thought studied by Platonic astronomy.

15. Unity

The reason why concord, attunement, and proportion are valued in Plato's *Republic* is that they create and sustain unity. Both in city and in soul a plurality of elements is unified into a well-functioning whole. It is not too much to say that in the ethical–political Books of the *Republic* unity is the highest value, which explains the more specific values of concord and attunement: 'Can we think of a greater evil for a city than that which pulls it apart and makes it many instead of one? Or of a greater good than that which binds it together and makes it one?' (462ab). This is Socrates specifying the final end (σκόπος) to which all legislation should be referred. Existing cities like Athens fail the test. Since they are split between rich and poor, who are at enmity with each other, none of them should be spoken of as 'a' city; they are rather two or more cities (cf. 551d). Only the ideal city is really one, not only in the sense that limits are put on its size and geographical spread, but also in the more important sense that it is a unified community. Likewise, its citizens, unlike those of other cities, are each one because they stick to the one job for which their nature is best fitted (422e–423d). The same principle holds within the individual soul: injustice is a kind of civil war between the different elements of your personality, while justice harmonises them together and makes you one instead of many (443e–444b; cf. 554de). Similarly in the cosmos at large: 'Of all bonds the best is that which makes itself and the terms it connects a unity in the fullest sense; and it is of the nature of proportion (ἀναλογία) to effect this most perfectly' (*Timaeus* 31c; tr. after Cornford). It is mathematical proportion that finally fulfils the longing Socrates expressed in the *Phaedo* (99c) for a new kind of scientific explanation, designed to show that the good is what binds things together.

But unity is also the first principle of number. Euclid spoke for Greek arithmetic generally when he defined number as a multitude of units, where a unit is anything considered as one (*Elements* VII Defs 1 and 2). Despite Frege's justly famous critique of this conception,[109] it served as the basis for some high-grade mathematics. Socrates and Glaucon are well aware that an object considered as one can also be considered as many (525e). One cow is many cuts of beef. The number you come up with depends on the description under which you count, the unit you choose to count with. No mathematician denies that a visible or tangible unit (such as the lines standardly used to diagram numbers) is divisible into parts. But, as we saw earlier, they laugh at you if you say that makes the unit many instead of one. For the unit they are talking about is a unit accessible only to thought, not to sight (524d–526b). It is grasped by a deliberate act of thought, by setting aside or abstracting from the presence of many parts.

Now this passage is the *Republic*'s first example of what is meant by the power of mathematics to effect the conversion of the soul. It is the most elementary example of the intellect (the instrument of the soul) being forced to turn towards something non-sensible and abstract. The next step is to go beyond counting and calculating to begin a systematic study of what Socrates calls 'the nature of the numbers' (525c) or 'the numbers themselves' (525d). Note the plural. This is number theory as we find it in Books VII–IX of Euclid's *Elements*. Recall the variety of kinds of number that Euclid sets out for study: even-times even, even-times odd, odd-times odd, prime number, numbers prime to one another, composite number, numbers composite to each other, perfect number.[110] As you leave

[109] Which begins by quoting *Elements* VII Def. 1 (in Greek): G. Frege, *The Foundations of Arithmetic*, translated by J. L. Austin (Oxford, 1950), §29.
[110] P. 26 above.

behind the everyday practice of counting and calculating (whether for trade or for military purposes), a whole new realm of abstract objects opens to the eye of the soul. To the right type of mind, it is a paradise to explore, even before you go on to the extended paradise of geometry and other branches of mathematics. Infinitely more attractive than the mundane tasks of government. All the same, the concept on which that number theory, in all its ramifications, is founded — the concept of unity — is simultaneously, as we have seen, the key value concept of Plato's ethics and politics.

In the cultural climate of the time it was not idiosyncratic to regard concord, attunement, proportion, order, and unity as important values. They are values that crop up constantly when Greeks talk about art and beauty, and about the things and people they admire. Towards the end of the fifth century, the sculptor Polycleitus of Argos wrote a book called the *Canon*, or *Rule*, which set out the ideal proportions (συμμετρίαι) for relating the parts of the human body to each other. He illustrated the scheme by his famous sculpture of a spear-carrier, the Doryphoros. It was in this book, which became well known, that he said, 'Perfection comes about little by little through many numbers' (frag. 2 Diels-Kranz).[111] If that suggests an attempt to mathematicise art, Plato's proposal is far more ambitious: to mathematicise ethics and politics and, simultaneously, to moralise mathematics. What is distinctive about Plato is his systematic exploitation of the fact that Greek value-concepts like concord, proportion, and order are also central to contemporary mathematics. The fundamental concepts of mathematics are the fundamental concepts of ethics and aesthetics as well, so that to study mathematics is simultaneously to study, at a very abstract level, the principles of value. Your understanding of value is enlarged as you come to see that such principles have applications in quite unexpected domains, some of them beyond the limits of human life in society. Conversely, your understanding of mathematics is perfected when you see it as the abstract articulation

[111] For a sane introduction to the problems of interpreting this dictum, I recommend A. F. Stewart, 'The canon of Polycleitus: a question of evidence', *Journal of Hellenic Studies*, 98 (1978), 122–31.

of value. The realm of mathematics is 'intelligible with the aid of a first principle' (511d), because in the light of the Good you see mathematics for what it really is.

Consider now this passage from the closing pages of *Republic* Book IX:

> 'Then throughout their life a person of understanding (*ὅ γε νοῦν ἔχων*) will direct all their powers to this one end [that their soul may possess temperance and justice together with wisdom]. First, they will prize the studies (*μαθήματα*) that fashion these qualities in their soul, disprizing others.'
> 'That is clear', he said.
> 'Second,' I said, 'so far from entrusting the condition and care of their body to the irrational pleasures of the beast within and bending their life in that direction, they will not even make health their chief aim, nor give primacy to the ways of becoming strong or healthy or beautiful [i.e. physical training], except in so far as such things help them be temperate. Always you will find them adjusting the attunement of their body to maintain the concord in their soul.'
> 'That's exactly what they will do', he said, 'if they are to be true musicians.' (591cd)

No one would dare to translate *μαθήματα* here as 'mathematical studies', although the *Republic* was influential in the process by which the word acquired its specialised meaning 'mathematics'.[112] Yet there can be no doubt that the studies in question are those which were selected in Book VII to lead potential philosophers to knowledge of the Good: mathematics and meta-mathematical dialectic. Mathematics and dialectic are good for the soul, not only because they give you understanding of objective value, but also because in so doing they fashion justice and temperance with wisdom in your soul. They make all the difference to the way you think about values in practice.

This Book IX passage is about the individual philosopher living in a non-ideal city. Socrates goes on to speak of the individual mentioned as maintaining order and concord (*σύνταξίν τε καὶ*

[112] Behind the *Republic* stands the use of *μαθήματα* in Archytas, frag. 1, as quoted above, p. 16.

συμφωνίαν) in their acquisition of wealth (591d 6–7), which philosophers in the ideal city do not have. He also speaks of a 'providential conjuncture' which would enable the individual to take part in the politics of the city of their birth (592a). If that did come about, the philosopher would accomplish much greater good and would 'grow in stature' (497a). Then the mathematics and meta-mathematics would be brought to bear on the life of a whole community instead of the life of a single individual. A modern reader is likely to feel thoroughly alienated by this idea. We shudder at the prospect of anyone laying claim to scientific knowledge of values. An alternative response is to join the prisoners who scoff at a philosopher forced into a debate about justice in court or assembly before their eyesight has had time to adjust to the darkness of the cave (516e–517a). Like a games theorist who lands in a real prison, the philosopher's mind is still too full of diagrams and formulae to be able to explain what is just in terms that ordinary people understand.

One of those who scoffed was Aristotle:

> They ought in fact to demonstrate <the nature of> the Good itself in the opposite way to the way they do it now. At present, they begin with things that are *not* agreed to have goodness and proceed to show the goodness of things which *are* agreed to be goods. For example, starting from numbers they show that justice and health are goods, on the grounds that justice and health are types of order and numbers [i.e. justice is determined by ratios of gain and loss, health by ratios of heat and cold in the body], while numbers and units possess goodness because unity is the Good itself. They ought rather to start from agreed goods like health, strength, temperance, and argue that the beautiful is present even more in unchanging things (ἐν τοίς ἀκινήτοις), which are all examples of order and stability. Then, if the former are goods, *a fortiori* the latter must be goods, because they have order and stability to a greater degree. (*Eudemian Ethics* I 8, 1218a 15–24)

Aristotle goes on to complain about the reckless (i.e. metaphorical) language the Platonists use to show that the Good is unity. What does it mean to say that numbers strive for unity? 'They ought to take more trouble over this, and not accept without argument things that are not easy to believe even with an argument' (1218a 28–30?).[113]

Aristotle's point, I take it, is that the value of unity and harmony in their psychic and social realisations is made intelligible from below, as it were. The earlier Books of the *Republic* give us richly detailed descriptions of human life which make it easy to see that, and why, psychic harmony and political unity are good things to aim at. It does not obviously follow that the very same unifying, harmonious relationship, abstractly considered, will be equally or more valuable in a different realisation; still less does it follow that the abstract relationship is itself a thing of value. But when one is trying to understand Plato, Aristotle's objections are often a good guide to his meaning. Often, what Aristotle does is take a point *of* Plato's philosophy and turn it into a point *against* him. That, I suggest, is what he is doing in the passage just quoted. Like many objections brought against Platonism from the side of so-called common sense (or what modern philosophers call 'our intuitions'), Aristotle's criticism just begs the question at issue.

It is important, however, that Aristotle's scoffing is restricted to the Platonist attempt at a mathematical explanation, from above, of 'thicker' values like justice and health. He himself analyses distributive and rectificatory justice in terms of geometric and arithmetic proportion respectively (*Nicomachean Ethics* V 3–5), while in the passage just quoted it is *in propria persona* that he says, 'the beautiful is present even more in unchanging things, which are all examples of order and stability'. And he is happy to allow that mathematics does teach us, at its own abstract level, about order and beauty:

> Now since the good and the beautiful are different (for the former is always found in action, whereas the beautiful is present also in

[113] I have translated a crabbed, condensed text in a manner that brings out what I take to be the meaning. In doing so I have been helped by the translation and commentary of Michael Woods (Oxford, 1982), who is in turn indebted to Jacques Brunschwig's pioneering article, '*E.E.* I 8 et le περὶ τἀγαθοῦ', in Paul Moraux and Dieter Harlfinger, *Untersuchungen zur Eudemischen Ethik* (Berlin, 1971), pp. 197–222.

> unchanging things),[114] those who assert that the mathematical sciences say nothing about the beautiful or the good are wrong. For these sciences say and demonstrate the most about them. Just because they do not speak of them by name, but demonstrate their effects and ratios (λόγους), that does not mean they say nothing about them. The chief forms of beauty are order (τάξις) and proportion (συμμετρία) and definiteness (τὸ ὡρισμένον), which the mathematical sciences demonstrate most of all. (*Metaphysics* XIII 3, 1078a 31–b 2)

This is his reply to Aristippus' extremist view that mathematics is useless because it teaches nothing about good and bad.[115] It is a reply that distances him from the mind-sharpening vindication as well. In the ancient debate about the benefits of learning mathematics, Aristotle is closer to Plato than to Isocrates, because he agrees that the *content* of mathematics is relevant to understanding value as an aspect of the world as it is objectively speaking.

16. On the Good

I close with the story Aristotle liked to tell when beginning a course of lectures, about what happened when Plato announced a public lecture on the Good:

> Everyone came expecting they would acquire one of the sorts of thing people normally regard as good, on a par with wealth, good health, or strength. In sum, they came looking for some wonderful kind of happiness. But when the discussion turned out to be about mathematics, about numbers and geometry and astronomy, and then, to cap it all, he claimed that Good is One [i.e. that Goodness is Unity — καὶ τὸ πέρας ὅτι ἀγαθόν ἐστιν ἕν], it seemed to them, I imagine, something utterly paradoxical (παντελῶς . . . παράδοξόν τι). The result was that some of them sneered at the lecture, and others were full of reproaches. (Aristoxenus, *Elementa Harmonica* II 1, p. 30.20–31.2 Meibom)

[114] Aristotle does not always confine 'good' to the sphere of action in this way. In the *Eudemian Ethics* passage unchanging things are good because they are beautiful, but he has just warned that this kind of good is not an end you can realise in action (1218b 4–7). In *Metaphysics* XII 7 the unchanging Prime Mover is both the most beautiful and the best.

[115] P. 4 above.

Appropriately, our source for this story is an anti-mathematical, anti-Pythagorean, treatise on harmonics by Aristoxenus of Tarentum, who agreed with Aristotle that concord resides only in sound.[116] Aristoxenus himself draws a moral from the story that would be approved by the quality control inspectors who currently tyrannise British universities: the audience should know in advance what kind of discussion to expect, so lecturers should start (as Aristotle used to do) with a clear outline of what they are going to say. But the moral I think we should draw in the Academy is that Platonism is a philosophy which is paradoxical by deliberate intent.[117] It goes knowingly *παρὰ δόξαν*, against the common opinion of humankind.[118]

[116] For a good, balanced introduction to the problems and controversies connected with the Aristoxenus passage, see Konrad Gaiser, 'Plato's Enigmatic Lecture "On the Good"', *Phronesis*, 25 (1980), 5–37.

[117] The extremely paradoxical nature of the proposal that philosophers should rule is thrice emphasised: 472a 7 (*οὕτω παράδοξον λόγον*), 473e 4 (*πολὺ παρὰ δόξαν*), 490a 5 (*σφόδρα παρὰ δόξαν*). The reason why it is paradoxical is the opinion people have of what philosophers are like. The entire argument down to the end of Book VII is designed to overcome that opinion by displaying the true philosopher as someone whose passion for knowledge and truth enables them to overcome the power of opinion within their own soul.

[118] In writing this essay I have learned much from the discussion of successive versions, first at the original Symposium at the British Academy, later at meetings in Oxford and the University of Illinois at Chicago, finally at the annual Princeton Colloquium on Ancient Philosophy in 1998, where my commentator was Charles Kahn. Special gratitude is due to the members of a term-long seminar in Pittsburgh on the central books of the *Republic*. Individuals who have been helpful include Julia Annas, David Fowler, Carl Huffman, Dan Jacobson, Constance Meinwald, Reviel Netz, Ruth Padel, Michael Rohr, Heda Segvic, Leonid Zhmud.