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Type Logics and Pregroups

Abstract. We discuss the logic of pregroups, introduced by Lambek [34], and its connections with other type logics and formal grammars. The paper contains some new ideas and results: the cut-elimination theorem and a normalization theorem for an extended system of this logic, its P-TIME decidability, its interpretation in **L1**, and a general construction of (quasi-ordered) bilinear algebras and pregroups whose universe is an arbitrary monoid.

Keywords: bilinear algebra, pregroup, residuation, Lambek calculus, sequent system

1. Type logics

The Lambek calculus **L**, introduced in Lambek [31] under the name Syntactic Calculus, is a standard type logic for categorial grammars. *Types* (formulas) are formed out of atoms (variables or constants) by means of operation symbols $\backslash, /, \otimes$. (The residuals $\backslash, /$ may also be denoted by $\rightarrow$ and $\leftarrow$, respectively.) A Gentzen-style system for **L** admits the following axioms and rules:

$$\text{(Id) } A \Rightarrow A,$$

$$(\backslash\text{L})\ \frac{\Gamma, B, \Delta \Rightarrow C;\ \Phi \Rightarrow A}{\Gamma, \Phi, A\backslash B, \Delta \Rightarrow C}\quad (\backslash\text{R})\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}$$

$$(/\text{L})\ \frac{\Gamma, B, \Delta \Rightarrow C;\ \Phi \Rightarrow A}{\Gamma, B/A, \Phi, \Delta \Rightarrow C}\quad (/\text{R})\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}$$

$$(\otimes\text{L})\ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A \otimes B, \Delta \Rightarrow C}\quad (\otimes\text{R})\ \frac{\Gamma \Rightarrow A;\ \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}$$

for any types A, B, C and type sequences Γ, Δ, Φ provided that $\Gamma \neq \epsilon$ in rules ($\backslash$R) and (/R). Lambek [31] proves the cut-elimination theorem for **L**: the following rule is admissible:

$$\text{(CUT) } \frac{\Phi \Rightarrow A;\ \Gamma, A, \Delta \Rightarrow B}{\Gamma, \Phi, \Delta \Rightarrow B}.$$

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L extends the rewriting system of classical categorial grammars, based on $(\backslash, /)-$types and rewriting rules:

$$\text{(A1) } A, A\backslash B \Rightarrow B, \ \text{(A2) } A/B, B \Rightarrow A,$$

which we call logic **AB**, after Ajdukiewicz [2] and Bar-Hillel [3]. **AB** is equivalent to the fragment of **L**, restricted to $(\backslash, /)-$types and rules (\L), (/L). (CUT) is admissible in this fragment.

Different variants of the Lambek calculus appear in literature. Admitting $\Gamma = \epsilon$ in (\R), (/R) yields the Lambek calculus with (possibly) empty antecedents (**L***). It is a conservative fragment of **L1**; the latter admits one new constant 1, one new rule and one new axiom:

$$\text{(1L) } \frac{\Gamma, \Delta \Rightarrow A}{\Gamma, 1, \Delta \Rightarrow A} \ \text{(1R) } \Rightarrow 1.$$

By affixing $\vee$ (join) and $\wedge$ (meet) one obtains Full Lambek Calculus (**FL**) (see Ono [40]) with the additional rules:

$$(\vee\text{L}) \ \frac{\Gamma, A, \Delta \Rightarrow C; \ \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \vee B, \Delta \Rightarrow C} \ (\vee\text{R}) \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$$

$$(\wedge\text{L}) \ \frac{\Gamma, A_i, \Delta \Rightarrow B}{\Gamma, A_1 \wedge A_2, \Delta \Rightarrow B} \ (\wedge\text{R}) \ \frac{\Gamma \Rightarrow A; \ \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

for $i = 1, 2$. It can also contain 0, with or without the axiom:

$$\text{(0L) } \Gamma, 0, \Delta \Rightarrow A,$$

which expresses the interpretation of 0 as the lower bound.

All systems mentioned above admit cut elimination and are decidable. The consequence relation is undecidable for each of them, since one can reduce general word problems for semigroups to this relation [10, 17].

Non-associative variants of these systems are obtained by replacing sequences of types by bracketed sequences of types and a slight modification of rules; for instance, the premise of (/R) will take the form $[\Gamma, A] \Rightarrow B$. The non-associative Lambek calculus (**NL**) was proposed in Lambek [32].

The consequence relation for **NL** is P-TIME [17], and similarly for **NL1** [9]. The consequence relations for **NL** with $\wedge$ and **NL** with $\wedge, \vee$, assuming the distribution laws for $\wedge, \vee$, are decidable; this follows from Finite Embeddability Property of residuated groupoids with $\vee$ or $\vee, \wedge$ plus distribution [21]. The provability relations for **L** and **L*** are NP-complete [43].

Type logics are substructural logics; their sequent systems avoid structural rules (Weakening, Contraction, Exchange) and **NL**-like systems also Associativity. They can be strengthened by adding some of these rules. **L*** with Exchange:

$$\text{(EXC)}\ \frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C}$$

is a conservative fragment of Linear Logic from [23], and without (EXC) of Bilinear Logic from [33]. **FL** with all structural rules amounts to Propositional Intuitionistic Logic.

Among other interesting extensions, let us mention modal type logics, e.g. Propositional Linear Logic (with exponentials), modal versions of **NL**, **L**, designed for linguistics [38], and **FL** with Kleene star [26, 18].

In linguistic interpretations, types denote categories of expressions. Categories can refer to ontology (semantic categories) or syntax (syntactic categories). Categorial grammars are intended to join the two levels [39]. Type-forming constructions can be interpreted in both ways. For instance, $A\backslash B$ can denote the set of functions (morphisms) from A to B, on the semantical level, and a residual operation on languages (sets of strings), on the syntactic level. The Curry-Howard correspondence between (Natural Deduction) proofs in Intuitionistic Logic and typed $\lambda-$calculus rises a possibility of reading provable sequents as schemes of semantic transformations [5, 6, 7]. Since axioms and rules of type logics are sound in algebras of languages, these logics also yield correct inferences about syntactic types. In some cases, they are (strongly) complete with respect to algebras of languages, e.g. the $\otimes-$free **L*** (resp. **L**) with $\wedge$ with respect to (resp. $\epsilon-$free) languages (strong completeness [11]), and **L*** (resp. **L**) with respect to (resp. $\epsilon-$free) languages (weak completeness [42]).

Abstract algebraic models for **L** are *residuated semigroups*: structures $\mathcal{M} = (M, \leq, \cdot, \backslash, /)$ such that $\leq$ is a partial ordering on M, $\cdot$ is an asociative, binary operation on M (product), and $\backslash, /$ are binary operations on M (residual operations), which satisfy *the residuation law*:

$$\text{(RES)}\ a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b$$

for all $a, b, c \in M$. In these models, $\otimes$ is interpreted as $\cdot$, and $\Rightarrow$ as $\leq$. From (RES) it follows that $(M, \leq, \cdot)$ is a partially ordered semigroup, this means, $\cdot$ is monotone in both arguments:

$$\text{(MON1) if } a \leq b \text{ then } ca \leq cb \text{ and } ac \leq bc.$$

Also $\backslash, /$ are isotone in one argument and antitone in the other:

$$\text{(MON2) if } a \leq b \text{ then } c\backslash a \leq c\backslash b,\ b\backslash c \leq a\backslash c,\ a/c \leq b/c,\ c/b \leq c/a.$$

Models of **NL** are residuated groupoids, defined as above except $\cdot$ need not be associative; (MON1), (MON2) also hold for them. For **L***, **L1** (resp. **NL***, **NL1**), one employs *residuated monoids* (resp. *residuated groupoids with identity*), this means, residuated semigroups (resp. groupoids) with an element 1, satisfying: $1a = a = a1$, for all elements a. For systems with (EXC), one assumes that $\cdot$ is commutative; then $a\backslash b = b/a$, for all a, b. For **FL**, $(M, \leq)$ must be a lattice, possibly with the lower bound 0 ($\cdot$ is associative).

In models of Bilinear Logic (**BL**), called *bilinear algebras*, 0 need not be the lower bound. It is *a dualizing element*, satisfying: $a = 0/(a\backslash 0) = (0/a)\backslash 0$, for all elements a. One defines two *negations*: $a^r = a\backslash 0$, $a^l = 0/a$, and proves:

$$a^{rl} = a = a^{lr},\ aa^r \leq 0,\ a^l a \leq 0,\ (a^r b^r)^l = (a^l b^l)^r.$$

One also defines the operation *par*: $a \oplus b = (b^r a^r)^l$, which is associative, and together with $\cdot$ it satisfies De Morgan laws:

$$\text{(DM)}\ (ab)^l = b^l \oplus a^l,\ (ab)^r = b^r \oplus a^r,\ (a \oplus b)^l = b^l a^l,\ (a \oplus b)^r = b^r a^r.$$

Also $a \oplus 0 = a = 0 \oplus a$, $a\backslash b = a^r \oplus b$, $a/b = a \oplus b^l$. If $\cdot$ is commutative, this construction yields general algebraic models of Multiplicative Linear Logic (**MLL**).

Residuated semigroups and monoids can be formed as powersets of semigroups and monoids, respectively. Let $\mathcal{M} = (M, \cdot)$ be a semigroup. For sets $X, Y \subseteq M$, one defines:

$$X \cdot Y = \{ab : a \in X,\ b \in Y\},$$

$$X\backslash Y = \{a \in M : X \cdot \{a\} \subseteq Y\},\ Y/X = \{a \in M : \{a\} \cdot X \subseteq Y\}.$$

Then, $(P(M), \subseteq, \cdot, \backslash, /)$ is a residuated semigroup. We call this algebra *the powerset algebra* over $\mathcal{M}$ and denote $P(\mathcal{M})$. It is a complete lattice (of sets). If 1 is the identity element of $\mathcal{M}$, then $\{1\}$ is the identity element in $P(\mathcal{M})$.

Language frames are powerset algebras $P(\Sigma^*)$. Σ^* is the set of finite strings on the alphabet Σ (a free monoid with concatenation and the identity ϵ). An analogous construction over free groupoids yields tree language frames (intended models for **NL**).

Since distribution (for $\wedge, \vee$) is not provable in **FL**, then this logic is not complete with respect to powerset algebras. To get a fully adequate semantics one considers a closure operation on $P(M)$ and a model consisting of closed subsets of M [40]. For closed sets $X, Y \subseteq M$, define $X \otimes Y$ as the closure of $X \cdot Y$. For Linear Logic, one takes a commutative monoid $\mathcal{M}$ and an arbitrary set $0 \subseteq M$. Then, $C(X) = (X\backslash 0)\backslash 0$ is a closure operation, and the closed sets (facts) form a model of this logic [23]. Similar constructions can be applied to **BL** and other Non-Commuative Linear Logics [1, 45].

Systems which are sound with respect to language frames, e.g. **L** (algebras $P(\Sigma^+)$), **L***, **FL**, cannot yield linguistically wrong inferences provided that the initial assumptions are correct. Certain 'paradoxes' discussed in literature are simply examples of an incorrect initial type assignment. For instance, one assigns n/n to adjectives (n to common nouns) and $(n/n)/(n/n)$ to *very*. In **L***, these two types are equivalent, which motivates some authors to claim the linguistic inadequacy of **L***. Yet, this example merely shows that $(n/n)/(n/n)$ is not a correct type of *very* (on the basis of **L***). In $P(\Sigma^*)$, $\epsilon \in X/X$, for any set $X \subseteq \Sigma^*$, and consequently, ϵ is of type n/n, but the concatenation of *very* and ϵ is not. Lambek [36] (this issue) proposes a different typing.

The situation is more complicated for systems which are not sound with respect to language frames. Commutative systems overgenerate, and systems like **BL** refer to closed sets and operations on closed sets, admitting no reasonable interpretation in language frames. The linguistic adequacy of (some of) these systems can be justified on the basis of finer representations of language expressions, usually involving type-theoretic semantics [39, 6, 7].

The logic of pregroups, being the major focus of this paper, continues the line of **BL**. Actually, it simplifies it by identifying operations $\cdot$ and $\oplus$, whence also 1 and 0. On the one hand, this move leads to a nice computation technique, based on simple reductions $a^l a \leq 1$, $aa^r \leq 1$ (^{l}, ^{r} are called the adjoint operations). On the other hand, it rises problems of linguistic adequacy. These problems are here even more urging than for the case of **BL**: the latter logic is conservative over **L***, whence it safely processes types in the language of **L***, whereas the logic of pregroups is not conservative over **L***. For instance, $(ab)/c \leq a(b/c)$ and $(a/((b/b)/c))/c \leq a$ are valid in pregroups, but (the corresponding sequents are) not provable in **L***, so they are not valid in language frames. These sequents are also unprovable in Intuitionistic Logic, and consequently, intractable by type-theoretic semantics. In the author's opinion, to find a linguistically natural, adequate semantics for the logic of pregroups is one of the most important challenges in this area.

Let $\mathcal{L}$ be a type logic (in the form of a sequent system). *A categorial grammar* based on $\mathcal{L}$ (shortly: $\mathcal{L}-$grammar) is defined as a triple $G = (\Sigma, I, s)$ such that Σ is a finite alphabet, I is a finite relation between elements of Σ and formulas of $\mathcal{L}$, and s is a designated atomic formula. We say that G *assigns* type A to a string $a_1 \cdots a_n$ ($a_i \in \Sigma$), if there exist types

A_i such that $a_i I A_i$, for $i = 1, \ldots, n$, and $A_1, \ldots, A_n \Rightarrow A$ is provable in $\mathcal{L}$. *The language* of G ($L(G)$) consists of all strings $x \in \Sigma^+$ such that G assigns s to x.

The languages of **AB**-grammars are precisely the $\epsilon-$free context-free (CF) languages [3], and the same holds for **L**-grammars, **L***-grammars and **BL**-grammars [41]. This is also true for **NL**-grammars [12, 28] and for grammars based on finitely axiomatizable theories on **NL** [17]. The equivalence of pregroup grammars, i.e. grammars based on the logic of pregroups, and $\epsilon-$free context-free grammars is proved in [14]. For type logics with (EXC), non-associative systems remain CF [29, 25, 14], but **L** with (EXC) generates all permutation closures of $\epsilon-$free context-free languages [13], whence it goes beyond CF. **FL** generates non-CF languages, e.g. the meet of any two CF languages [27]. Non-associative variants of **FL** (even with assumptions) remain CF [21].

The paper is organized as follows. In section 2, we present the logic of pregroups in two forms: (1) in the original form due to Lambek [34], where (iterated) adjoints are applied to atoms only, (2) in an extended form, where adjoints are applied to arbitrary formulas. For the second form we prove the cut-elimination theorem and the Lambek normalization theorem (proved by Lambek for the first form). As a consequence, we get the P-TIME decidability of both forms. We show that the logic of pregroups is faithfully interpretable in **L1**. The results concerning (2) are new. We also briefly discuss some extensions of this logic. In section 3, we survey the author's earlier results on models of this logic (i.e. pregroups). As a new topic, we provide a general construction of (quasi-ordered) bilinear algebras and pregroups whose universe is an arbitrary monoid, with a bijection defined on it.

2. The logic of pregroups

A pregroup is a structure $\mathcal{M} = (M, \leq, \cdot, ^l, ^r, 1)$ such that $(M, \leq, \cdot, 1)$ is a partially ordered (p.o.) monoid, and ^{l}, ^{r} are unary operations on M, satisfying the adjoint laws:

$$(\mathrm{A}l)\ a^l a \leq 1 \leq a a^l,\ (\mathrm{A}r)\ a a^r \leq 1 \leq a^r a,$$

for all $a \in M$. Pregroups were introduced by Lambek [34] as a generalization of p.o. groups; some ideas appeared in earlier articles of Lambek. a^l (resp. a^r) is called *the left* (resp. *right*) *adjoint* of a. This terminology refers to category theory (adjoints of functors).

If $\cdot$ is commutative, then $a^l a = 1 = aa^l$, $aa^r = 1 = a^r a$, whence $a^l = a^{-1} = a^r$. Commutative pregroups are simply p.o. Abelian groups. They are not adequate for language description, so Lambek focuses on non-commutative pregroups, and even free pregroups, defined below.

First, we note that the following laws easily follow from (Al), (Ar):

$$1^l = 1 = 1^r,\ a^{lr} = a = a^{rl},\ (ab)^l = b^l a^l,\ (ab)^r = b^r a^r,$$

$$a \leq b \text{ iff } b^l \leq a^l \text{ iff } b^r \leq a^r.$$

For any pregroup, one defines $a \backslash b = a^r b$, $a/b = ab^l$ and easily proves (RES). Accordingly, every pregroup is a residuated monoid with these residual operations. It follows that all sequents provable in **L1** are valid in pregroups under this translation of residuals. The converse is not true; some counterexamples have been mentioned in section 1.

For an integer $n \geq 0$, one defines $a^{(n)} = a^{rr\ldots r}$ and $a^{(-n)} = a^{ll\ldots l}$, where adjoints are iterated n times. The following laws are provable:

$$a^{(n)} a^{(n+1)} \leq 1 \leq a^{(n+1)} a^{(n)}, \tag{1}$$

$$(ab)^{(2n)} = a^{(2n)} b^{(2n)},\ (ab)^{(2n+1)} = b^{(2n+1)} a^{(2n+1)}, \tag{2}$$

$$a \leq b \text{ iff } a^{(2n)} \leq b^{(2n)} \text{ iff } b^{(2n+1)} \leq a^{(2n+1)}, \tag{3}$$

for all $n \in \mathcal{Z}$ ($\mathcal{Z}$ denotes the set of integers).

Let $(P, \leq)$ be a (finite) poset. Elements of P are denoted by *atoms*: $p, q, r, \ldots$. *Simple terms* are of the form $p^{(n)}$, for any $p \in P$ and $n \in \mathcal{Z}$. *A term* is a finite sequence (string) of simple terms. Terms are also called *types*. Greek capitals range on terms, and t denotes a simple term. If $p \leq q$, then we write $p^{(n)} \leq q^{(n)}$, if n is even, and $q^{(n)} \leq p^{(n)}$, if n is odd. One defines a binary relation $\Rightarrow$ on the set of terms as the least reflexive and transitive relation, satisfying the clauses:

(CON) $\Gamma, p^{(n)}, p^{(n+1)}, \Delta \Rightarrow \Gamma, \Delta$,

(EXP) $\Gamma, \Delta \Rightarrow \Gamma, p^{(n+1)}, p^{(n)}, \Delta$,

(IND) $\Gamma, p^{(n)}, \Delta \Rightarrow \Gamma, q^{(n)}, \Delta$, if $p^{(n)} \leq q^{(n)}$.

(CON), (EXP), (IND) are called Contraction, Expansion and Induced Step, respectively. They can be treated as rules of a term rewriting system. $\Gamma \Rightarrow \Delta$ is true iff Γ can be transformed to Δ by a finite number of applications of these rules. This rewriting system is Lambek's original form of the logic of pregroups. This logic is also called Compact Bilinear Logic (**CBL**).

One defines Γ^l and Γ^r as follows:

$$\epsilon^l = \epsilon^r = \epsilon,\ (p^{(n)})^l = p^{(n-1)},\ (p^{(n)})^r = p^{(n+1)},$$

$$(t_1, \ldots, t_n)^l = ((t_n)^l, \ldots, (t_1)^l),\ (t_1, \ldots, t_n)^r = ((t_n)^r, \ldots, (t_1)^r).$$

The set of all terms with the relation $\Rightarrow$, operations $\cdot$ (concatenation), ^{l}, ^{r}, and the identity ϵ constitutes a quasi-pregroup. A quasi-pregroup is defined like a pregroup except that $\leq$ can be a quasi-order, and in monoid equations $=$ is replaced by $\sim$, where: $a \sim b$ iff $a \leq b$ and $b \leq a$.

So, $\Gamma \sim \Delta$ iff $\Gamma \Rightarrow \Delta$ and $\Delta \Rightarrow \Gamma$. The relation $\sim$ is nontrivial also for a trivial poset $(P, =)$; for instance $p, p^{(1)}, p \sim p$; $p, p^{(-1)}, p \sim p$. (In pregroups, $aa^r a = a$ and $aa^l a = a$.) It is a congruence on the quasi-pregroup, defined above. The quotient-structure with the ordering defined by:

$$[\Gamma] \leq [\Delta] \text{ iff } \Gamma \Rightarrow \Delta$$

is a pregroup, called the free pregroup generated by $(P, \leq)$, and denoted $F(P, \leq)$ or $F(P)$. One easily proves that it is free in the sense that every function f from P to a pregroup $\mathcal{M}$ which preserves the order ($p \leq q$ entails $f(p) \leq f(q)$) can uniquely be extended to an (order-preserving) homomorphism from $F(P)$ to $\mathcal{M}$. This yields the completeness theorem: $\Gamma \Rightarrow \Delta$ iff $f([\Gamma]) \leq f([\Delta])$, for any order-preserving function f from P to a pregroup $\mathcal{M}$.

Lambek [34] proves the following normalization theorem for **CBL** (also called the Lambek switching lemma): if $\Gamma \Rightarrow \Delta$ then there exists Φ such that $\Gamma \Rightarrow \Phi$ without (EXP) and $\Phi \Rightarrow \Delta$ without (CON). Consequently, if $\Gamma \Rightarrow t$, where t is a simple term or ϵ, then Γ can be reduced to t by (CON) and (IND) only. Such reductions are easily computable and can be simulated by a context-free grammar. This yields the P-TIME decidability of **CBL** ($O(n^3)$). (Notice that $\Gamma \Rightarrow \Delta$ iff $\Delta^l, \Gamma \Rightarrow \epsilon$.) The Lambek normalization theorem is equivalent to the cut-elimination theorem for a sequent system for **CBL** [16].

Here we prove these facts for a different formalization of **CBL**. Our system admits formulas built from atoms (variables and 1) by means of connectives $\otimes, ^l, ^r$. A, B, C range on formulas, and Γ, Δ, Φ on finite sequences of formulas. Sequents are of the form $\Gamma \Rightarrow \Delta$. This presentation of **CBL** is more akin to standard logical formalisms and can naturally be enriched by other connectives and modalities. In the metalanguage, we use the notation $A^{(n)}$, defined as $a^{(n)}$ for pregroups. The axiom system looks as follows:

$$\text{(PId) } \Gamma \Rightarrow \Gamma$$

$$(\mathrm{P1L})\ \frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, 1^{(n)}, \Gamma' \Rightarrow \Delta} \quad (\mathrm{P1R})\ \frac{\Gamma \Rightarrow \Delta, \Delta'}{\Gamma \Rightarrow \Delta, 1^{(n)}, \Delta'}$$

$$(\mathrm{P}\otimes\mathrm{L0})\ \frac{\Gamma, A^{(2n)}, B^{(2n)}, \Gamma' \Rightarrow \Delta}{\Gamma, (A \otimes B)^{(2n)}, \Gamma' \Rightarrow \Delta} \quad (\mathrm{P}\otimes\mathrm{R0})\ \frac{\Gamma \Rightarrow \Delta, A^{(2n)}, B^{(2n)}, \Delta'}{\Gamma \Rightarrow \Delta, (A \otimes B)^{(2n)}, \Delta'}$$

$$(\mathrm{P}\otimes\mathrm{L1})\ \frac{\Gamma, B^{(2n+1)}, A^{(2n+1)}, \Gamma' \Rightarrow \Delta}{\Gamma, (A \otimes B)^{(2n+1)}, \Gamma' \Rightarrow \Delta} \quad (\mathrm{P}\otimes\mathrm{R1})\ \frac{\Gamma \Rightarrow \Delta, B^{(2n+1)}, A^{(2n+1)}, \Delta'}{\Gamma \Rightarrow \Delta, (A \otimes B)^{(2n+1)}, \Delta'}$$

$$(\mathrm{P}l\mathrm{L})\ \frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, A^l, A, \Gamma' \Rightarrow \Delta} \quad (\mathrm{P}l\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \Delta'}{\Gamma \Rightarrow \Delta, A, A^l, \Delta'}$$

$$(\mathrm{P}r\mathrm{L})\ \frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, A, A^r, \Gamma' \Rightarrow \Delta} \quad (\mathrm{P}r\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, \Delta'}{\Gamma \Rightarrow \Delta, A^r, A, \Delta'}$$

$$(\mathrm{AIL})\ \frac{\Gamma, A^{(m+n)}, \Gamma' \Rightarrow \Delta}{\Gamma, (A^{(m)})^{(n)}, \Gamma' \Rightarrow \Delta} \quad (\mathrm{AIR})\ \frac{\Gamma \Rightarrow \Delta, A^{(m+n)}, \Delta'}{\Gamma \Rightarrow \Delta, (A^{(m)})^{(n)}, \Delta'}$$

The role of 'P' in '(P1L)', '(P1R)' etc. is to distinguish these rules from analogous rules for **L1**. (AIL) and (AIR) are called adjoint insertion rules. Notice that they do nothing, if $m \cdot n \geq 0$. If $m \cdot n < 0$, they introduce some new adjoint symbols. For instance, from $A^{rr} = A^{((-1)+3)}$ one gets A^{lrrr}. The rules for $\otimes$ allow to introduce $\otimes$ in the scope of adjoint symbols, and similarly for 1.

One could also consider a different axiomatization, related to axiom systems for **BL**, in which the rules for ^{l} would be:

$$(\mathrm{P}l\mathrm{L'})\ \frac{\Gamma \Rightarrow A, \Delta}{A^l, \Gamma \Rightarrow \Delta} \quad (\mathrm{P}l\mathrm{R'})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A^l}$$

and similarly for ^{r} [20]. Then, one could drop (n) in $\otimes-$rules and $1-$rules. Although this axiom system is even more compatible with other logical formalisms than ours, it has some disadvantages from our point of view. First, its derivations are not parallel to those in the rewriting system (they are parallel in our system). Second, it needs a two-premise (structural) rule:

$$(\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta;\ \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

whereas our system admits one-premise rules only, which essentially simplifies its deductive structure. For instance, cut elimination for our system can be proved in a more elegant way, and it is closely related to the Lambek normalization theorem.

An expedient simplification is to admit adjoints in contexts $p^{(n)}$ only and to define A^l, A^r in metalanguage: $(p^{(n)})^l = p^{(n-1)}$, $(p^{(n)})^r = p^{(n+1)}$,

$1^l = 1^r = 1$, $(A \otimes B)^l = B^l \otimes A^l$, $(A \otimes B)^r = B^r \otimes A^r$. The corresponding sequent system admits (PId), (1L), (1R) (without (n)), (P$\otimes$L0), (P$\otimes$R0) (without $(2n)$)), (PlL), (PlR), (PrL), (PrR). This system will be denoted by **CBL'**; it has been studied in [16]. **CBL'** is equivalent to the above sequent system for **CBL** with respect to provability. Its virtue is simplicity, its sin is laziness: some essential parts of logical arguments are not formalized in it, but left to metatheory.

The above full system is appropriate for the pure **CBL**, corresponding to the trivial poset. For a nontrivial poset $(P, \leq)$, we identify variables with elements of P (so, they become constants) and add the following rules, for all $p, q \in P$ such that $p^{(n)} \leq q^{(n)}$:

$$\text{(INDL)}\ \frac{\Gamma, q^{(n)}, \Gamma' \Rightarrow \Delta}{\Gamma, p^{(n)}, \Gamma' \Rightarrow \Delta}\quad \text{(INDR)}\ \frac{\Gamma \Rightarrow \Delta, p^{(n)}, \Delta'}{\Gamma \Rightarrow \Delta, q^{(n)}, \Delta'}$$

We sketch the proof of the cut-elimination theorem. First, we consider the fragment, restricted to sequents $\Rightarrow \Delta$. Clearly, this is a conservative fragment of our system, and it employs neither (PId) with $\Gamma \neq \epsilon$, nor left-introduction rules. We prove that the rules:

$$\text{(1-CUT}l\text{)}\ \frac{\Rightarrow \Gamma, A^l;\ \Rightarrow A, \Delta}{\Rightarrow \Gamma, \Delta}\quad \text{(1-CUT}r\text{)}\ \frac{\Rightarrow \Gamma, A;\ A^r, \Delta}{\Rightarrow \Gamma, \Delta}$$

are admissible in the system (this means, the set of provable sequents is closed under these rules). We need the following lemmas. We write $\vdash\Rightarrow \Delta$ for '$\Rightarrow \Delta$ is provable'.

(L1) Rules (PlR) and (PrR) can be restricted to the scheme: from $\Rightarrow \Gamma, \Delta$ infer $\Rightarrow \Gamma, p^{(n+1)}, p^{(n)}, \Delta$.

(L2) If $\vdash\Rightarrow \Gamma, \Delta$ and $\vdash\Rightarrow \Phi$, then $\vdash\Rightarrow \Gamma, \Phi, \Delta$.

(L3) If $\vdash\Rightarrow \Gamma$ and $\vdash\Rightarrow \Delta$, then $\vdash\Rightarrow \Gamma, \Delta$.

(L4) Rules (P1R), (P$\otimes$R0), (P$\otimes$R1) and (AIR) are reversible (this means, if the conclusion is provable, then the premise is provable).

(L5) If $\vdash\Rightarrow \Gamma, A^l, A, \Delta$ or $\vdash\Rightarrow \Gamma, A, A^r, \Delta$, then $\vdash\Rightarrow \Gamma, \Delta$.

The admissibility of (1-CUTl) and (1-CUTr) follows from (L3) and (L5).

To prove (L1), we show that the rule:

$$\frac{\Rightarrow \Gamma, \Delta}{\Rightarrow \Gamma, A^{(n+1)}, A^{(n)}, \Delta} \tag{4}$$

is derivable in the restricted system (this rule subsumes (PlR) and (PrR)). Assume $\Rightarrow \Gamma, \Delta$. We prove $\Rightarrow \Gamma, A^{(n+1)}, A^{(n)}, \Delta$ by induction on A. For $A = p$, the rule is given. For $A = 1$, we use (P1R). Let $A = B^{(m)}$, where $m \neq 0$. By the induction hypothesis, we obtain $\Rightarrow \Gamma, B^{(m+n+1)}, B^{(m+n)}, \Delta$, whence $\Rightarrow \Gamma, A^{(n+1)}, A^{(n)}, \Delta$, by (AIR). Let $A = B \otimes C$. Let n be even. By the induction hypothesis, we get $\Rightarrow \Gamma, C^{(n+1)}, B^{(n+1)}, B^{(n)}, C^{(n)}, \Delta$, whence $\Rightarrow \Gamma, A^{(n+1)}, A^{(n)}, \Delta$, by (P⊗R1), (P⊗R0). If n is odd, the reasoning is similar.

(L2) can easily be proved by induction on the derivation of $\Rightarrow \Phi$. (L3) is a consequence of (L2). The proof of (L4) proceeds by induction on the derivation of the conclusion in the system restricted as in (L1); it is easy, since the designated formula of the conclusion can be introduced by the same rule only ((AIL), (AIR) are restricted to $m \cdot n < 0$).

To prove (L5), we show that the rule:

$$\frac{\Rightarrow \Gamma, A^{(n)}, A^{(n+1)}, \Delta}{\Rightarrow \Gamma, \Delta} \tag{5}$$

is admissible (not derivable) in the system. We proceed by induction on A.

The case $A = p$ is the most involved. We prove a more general claim: if $\Rightarrow \Gamma, p^{(n)}, q^{(n+1)}, \Delta$ is provable and $p^{(n)} \leq q^{(n)}$, then $\Rightarrow \Gamma, \Delta$ is provable. We proceed by induction on the derivation of $\Rightarrow \Gamma, p^{(n)}, q^{(n+1)}, \Delta$ in the system restricted as in (L1) (this part of the proof is essentially due to Lambek [34]). There are two interesting cases: (i) the sequent arises by the scheme from (L1), introducing $p^{(n)}$ or $q^{(n+1)}$, (ii) it arises by (INDR), introducing $p^{(n)}$ or $q^{(n+1)}$. For (i), $\Rightarrow \Gamma, \Delta$ results from the premise, by (INDR). For (ii), we apply the induction hypothesis to the premise.

The remainder of the proof is routine. For $A = 1$, we use (L4). For $A = B^{(m)}$, where $m \neq 0$, and $A = B \otimes C$, we use (L4) and apply the induction hypothesis. The proof of (L5) is finished.

We define Γ^l and Γ^r as for the rewriting system (replace t_i by A_i, and remember that the adjoint symbols are connectives, not metalanguage operations).

(L6) $\vdash \Rightarrow \Gamma, \Delta^l$ iff $\vdash \Rightarrow \Delta^r, \Gamma$.

We prove the 'only if' part of (L6). Assume $\vdash \Rightarrow \Gamma, \Delta^l$. By (P$r$R), $\vdash \Rightarrow \Delta^r, \Delta$, whence $\vdash \Rightarrow \Delta^r, \Gamma, \Delta^l, \Delta$, by (L2). Then, $\vdash \Rightarrow \Delta^r, \Gamma$, by (L5). The proof of the 'if' part is similar.

We define: $\vdash_1 \Gamma \Rightarrow \Delta$ iff $\vdash \Rightarrow \Gamma^r, \Delta$. By (L6), $\vdash_1 \Gamma \Rightarrow \Delta$ iff $\vdash \Rightarrow \Delta, \Gamma^l$. We also write $\vdash_2 \Gamma \Rightarrow \Delta$, if $\Gamma \Rightarrow \Delta$ is provable in the full system, presented above.

(L7) $\vdash_1 \Gamma \Rightarrow \Delta$ iff $\vdash_2 \Gamma \Rightarrow \Delta$.

The 'if' part of (L7) can easily be proved by induction on derivations in the full system. For the 'only if' part, one first shows that (R) is admissible in the full system (induction on derivations). Then, one proceeds by induction on the derivation of $\Rightarrow \Gamma^r, \Delta$ in the one-sided system. We consider two cases. Assume that $\Rightarrow \Gamma^r, \Delta$ arises by (PrR), introducing the last formula in Γ^r and the first formula in Δ. Then, $\Gamma = (A, \Gamma')$, $\Delta = (A, \Delta')$, and the premise is $\Rightarrow (\Gamma')^r, \Delta'$. By the induction hypothesis, $\vdash_2 \Gamma' \Rightarrow \Delta'$, whence $\vdash_2 \Gamma \Rightarrow \Delta$, by (R). Assume that Γ^r, Δ arises by (PlR), with the introduced formulas as above. Then, $\Gamma = (A, \Gamma')$, $\Delta = (A^{rl}, \Delta')$, and the premise is as above. By the induction hypothesis, $\vdash_2 \Gamma' \Rightarrow \Delta'$. We have $\vdash_2 A \Rightarrow A^{rl}$, by (PId) and (AIR), whence $\vdash_2 \Gamma \Rightarrow \Delta$, by (R).

Accordingly, we can prove basic facts on the full system, using properties of the one-sided system. By (L7), (L3), (L5),(R) the full system admits the following forms of cut.

$$(\text{TRAN}) \ \frac{\Gamma \Rightarrow \Phi; \ \Phi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\text{CUT1}) \ \frac{\Phi \Rightarrow A; \ \Gamma, A, \Gamma' \Rightarrow \Delta}{\Gamma, \Phi, \Gamma' \Rightarrow \Delta} \quad (\text{CUT2}) \ \frac{\Gamma \Rightarrow \Delta, A, \Delta'; \ A \Rightarrow \Phi}{\Gamma \Rightarrow \Delta, \Phi, \Delta'}$$

(PlL') ,(PlR') and analogous rules for ^{r} are admissible and reversible in the full system (the proof is left to the reader).

The full system is complete with respect to pregroups (under assignments preserving the order in $(P, \leq)$), which can be proved like for the Lambek system (see above).

The system is equivalent to a rewriting system of Lambek style. The rewriting rules look as follows.

(R1) $\Gamma, 1^{(n)}, \Gamma' \Rightarrow \Gamma, \Gamma'$, (R2) $\Gamma, A^{(n)}, A^{(n+1)}, \Gamma' \Rightarrow \Gamma, \Gamma'$,

(R3) $\Gamma, (A \otimes B)^{(2n)}, \Gamma' \Rightarrow \Gamma, A^{(2n)}, B^{(2n)}, \Gamma'$,

(R4) $\Gamma, (A \otimes B)^{(2n+1)}, \Gamma' \Rightarrow \Gamma, B^{(2n+1)}, A^{(2n+1)}, \Gamma'$,

(R5) $\Gamma, (A^{(m)})^{(n)}, \Gamma' \Rightarrow \Gamma, A^{(m+n)}, \Gamma'$,

(E1) $\Gamma, \Gamma' \Rightarrow \Gamma, 1^{(n)}, \Gamma'$, (E2) $\Gamma, \Gamma' \Rightarrow \Gamma, A^{(n+1)}, A^{(n)}, \Gamma'$,

(E3) $\Gamma, A^{(2n)}, B^{(2n)}, \Gamma' \Rightarrow \Gamma, (A \otimes B)^{(2n)}, \Gamma'$,

(E4) $\Gamma, B^{(2n+1)}, A^{(2n+1)}, \Gamma' \Rightarrow \Gamma, (A \otimes B)^{(2n+1)}, \Gamma'$,

(E5) $\Gamma, A^{(m+n)}, \Gamma' \Rightarrow \Gamma, (A^{(m)})^{(n)}, \Gamma'$,

(IND) (same as for the Lambek rewriting system).

One easily proves: $\vdash_2 \Gamma \Rightarrow \Delta$ iff $\Gamma \Rightarrow \Delta$ in the sense of the rewriting system. (TRAN) is essential in the proof of the 'if' part. R-rules of the rewriting system correspond to left-introduction rules of the sequent system, and E-rules to right-introduction rules.

The sequent system employs no interaction between antecedents and consequents of sequents. Then, every provable sequent has a proof in which all left-introduction rules precede all right-introduction rules. This yields a Lambek-style normalization theorem for the rewriting system: if $\Gamma \Rightarrow \Delta$, then there exists Φ such that $\Gamma \Rightarrow \Phi$ without E-rules and $\Phi \Rightarrow \Delta$ without R-rules. Let $\Delta = p^{(n)}$ or $\Delta = \epsilon$. Then, Δ is not the right-hand side of any E-rule, nor the left-hand side of any R-rule. Consequently, if $\Gamma \Rightarrow \Delta$, then there exists a reduction of Γ to Δ, using R-rules and (IND) only. Similarly, if $\Delta \Rightarrow \Gamma$, then there exists an expansion of Δ to Γ, using E-rules and (IND) only.

Assuming the normalization theorem, one easily shows that the rewriting system is equivalent to the full sequent system, without applying (TRAN). Since the rewriting system admits (TRAN) by definition, this yields the admissibility of (TRAN) in the sequent system (a cut elimination theorem). Above we have noted that, using the admissibility of (TRAN) in the sequent system, the equivalence of both systems can be shown without referring to the normalization theorem, and it implies this theorem. In this sense, cut elimination for the sequent system is equivalent to the normalization theorem.

In the rewriting system, $\Gamma \Rightarrow \Delta$ iff $1 \Rightarrow \Gamma^r, \Delta$ (use (L7)). By the normalization theorem, the latter holds iff ϵ can be expanded to Γ^r, Δ by E-rules and (IND). The rewriting process, based on E-rules and (IND), can be simulated by a context-free grammar. This yields a P-TIME decision procedure for **CBL**. By an easy adaptation of the proof from [14], one shows that **CBL**-grammars are equivalent to $\epsilon-$free context-free grammars.

We note above that $(a/((b/b)/c))/c \leq a$ is valid in pregroups, but not in residuated monoids, whence the corresponding sequent is provable in **CBL**, but not in **L1** (or **L***, which is a conservative subsystem of **L1**). In the language of **L1**, one takes the sequent $(p/((q/q)/r))/r \Rightarrow p$, while the translation in **CBL** is $(p \otimes ((q \otimes q^l) \otimes r^l)^l) \otimes r^l \Rightarrow p$, according to the rules: $A \backslash B = A^r \otimes B$, $A/B = A \otimes B^l$. Then, **CBL** is essentially stronger than **L1**.

On the other hand, there exists a faithful interpretation of **CBL** in **L1**. This interpretation is based on the normalization theorem. We define a syntactic map I from formulas of **CBL** to formulas of **L1**.

$$I(p) = p,\ I(1^{(n)}) = 1,$$

$$I(p^{(n+1)}) = I(p^{(n)})\backslash 1, \text{ for } n \geq 0, \; I(p^{(n-1)}) = 1/I(p^{(n)}), \text{ for } n \leq 0,$$

$$I((A^{(m)})^{(n)}) = I(A^{(m+n)}), \; I((A \otimes B)^{(2n)}) = I(A^{(2n)}) \otimes I(B^{(2n)}),$$

$$I((A \otimes B)^{(2n+1)}) = I(B^{(2n+1)}) \otimes I(A^{(2n+1)}).$$

We set $I(\Gamma) = (I(A_1), \ldots, I(A_n))$, for $\Gamma = (A_1, \ldots, A_n)$, and $I(\epsilon) = \epsilon$.

This interpretation involves a reduction of Γ to a sequence Γ', in which adjoints appear in contexts $p^{(n)}$ only, preserves p, $\otimes$ and 1 and translates $p^{(n)}$, for $n > 0$, into $(\ldots(p\backslash 1)\backslash \cdots \backslash 1)\backslash 1$ (n copies of 1), and for $n < 0$, into $1/(1/\cdots/(1/p)\ldots)$. The following theorem is formulated for the pure **CBL** (for the trivial poset $(P, =)$).

(I) $\Gamma \Rightarrow 1$ is provable in **CBL** iff $I(\Gamma) \Rightarrow 1$ is provable in **L1**.

Notice that $\Gamma \Rightarrow \Delta$ is provable in **CBL** iff $\Delta^l, \Gamma \Rightarrow 1$ is so. Then, in a sense, (I) interprets the full **CBL** in **L1**.

We prove the 'if' part. Assume that $I(\Gamma) \Rightarrow 1$ is provable in **L1**. Then, it is provable in **CBL**, since **CBL** is stronger than **L1** (residuals are translated as above). In **CBL**, $I(A) \Leftrightarrow A$ is provable (induction on A), whence $\Gamma \Rightarrow 1$ is provable in **CBL**, by (CUT1).

We prove the 'only if' part. Assume that $\Gamma \Rightarrow 1$ is provable in **CBL**. Since $1 \Rightarrow \epsilon$ is provable, then $\Gamma \Rightarrow \epsilon$ is provable, by (TRAN). By the normalization theorem, there exists a reduction of Γ to ϵ, by R-rules and (IND). Equivalently, there exists a proof of $\Gamma \Rightarrow \epsilon$ in the sequent system; this proof employs (PID) $\epsilon \Rightarrow \epsilon$ and left-introduction rules only. Like in (L1), rules (PlL) and (PrL) can be restricted to the scheme: from $\Gamma, \Gamma' \Rightarrow \Delta$ infer $\Gamma, p^{(n)}, p^{(n+1)}, \Gamma' \Rightarrow \Delta$. We show that $I(\Gamma) \Rightarrow 1$ is provable in **L1**, by induction on such proofs in the sequent system.

The only interesting case is the restricted form of (PlL) and (PrL). By the induction hypothesis, $I(\Gamma), I(\Gamma') \Rightarrow 1$ is provable in **L1**. For $n < 0$, $I(p^{(n)}) = 1/I(p^{(n+1)})$, and for $n \geq 0$, $I(p^{(n+1)}) = I(p^{(n)})\backslash 1$. By (1L), $I(\Gamma), 1, I(\Gamma') \Rightarrow 1$ is provable in **L1**, whence $I(\Gamma), I(p^{(n)}), I(p^{(n+1)}), I(\Gamma') \Rightarrow 1$ is provable in **L1**, by (Id), (/L), (\L).

(I) can be generalized to nontrivial posets $(P, \leq)$; one must add to **L1** new axioms $p \Rightarrow q$ such that $p \leq q$ in the poset. Since this set of new axioms is closed under (CUT), cut elimination holds for the extended system [18].

An analogous theorem can be proved with $p^{(n)}$ instead of 1 on the right-hand side of the sequent:

(I') $\Gamma \Rightarrow p^{(n)}$ is provable in **CBL** iff $I(\Gamma) \Rightarrow I(p^{(n)})$ is provable in **L1**.

Let $n > 0$. Assume that $\Gamma \Rightarrow p^{(n)}$ is provable in **CBL**. Then, $p^{(n-1)}, \Gamma \Rightarrow 1$ is provable in **CBL** (use (CUT1)), whence $I(p^{(n-1)}), I(\Gamma) \Rightarrow 1$ is provable in **L1**, by (I). Consequently, $I(\Gamma) \Rightarrow I(p^{(n)})$ is provable in **L1**, by (\R) and the definition of (I). This argument can be reversed, since (\R) is reversible in **L1**. For $n < 0$, the assumption implies $\Gamma, p^{(n+1)} \Rightarrow 1$ in **CBL**. By (I), $I(\Gamma), I(p^{(n+1)}) \Rightarrow 1$ is provable in **L1**, whence $I(\Gamma) \Rightarrow I(p^{(n)})$ is so, by (/R). Again, the argument can be reversed, since (/R) is reversible in **L1**. Let $n = 0$. Assume that $\Gamma \Rightarrow p$ is provable in **CBL**. Then, $\Gamma, p^r \Rightarrow \epsilon$ is provable in **CBL**. By induction on proofs in the full sequent system for **CBL**, restricted as in (L1), we show that $I(\Gamma) \Rightarrow p$ is provable in **L1**. The only interesting case is: p^r is introduced by (PrL). Then, $\Gamma = (\Gamma', p)$, and the premise is $\Gamma' \Rightarrow \epsilon$. By (I), $I(\Gamma') \Rightarrow 1$ is provable in **L1**. By (Id) and (\L), $I(\Gamma'), 1\backslash p \Rightarrow p$ is provable in **L1**. Since $p \Rightarrow 1\backslash p$ is provable in **L1**, by (Id), (1L), (\R), then $I(\Gamma) \Rightarrow p$ is so, by (CUT). The converse implication can be proved as for the case of (I). Clearly, (I') also holds for non-trivial posets $(P, \leq)$, if **L1** is extended as above.

In (I) and (I'), the right-hand sequent is provable in the fragment of **L1**, restricted to (Id), (1R) and left-introduction rules, which is a variant of **AB** with $\otimes$ and 1.

Let us return to the sequent $(p/((q/q)/r))/r \Rightarrow p$. The type on the right-hand side is denoted by A. We add the clauses $I(A\backslash B) = I(A^r \otimes B)$, $I(A/B) = I(A \otimes B^l)$ to the definition of I. We have:

$$I(A) = I((p \otimes ((q \otimes q^l) \otimes r^l)^l) \otimes r^l)$$

$$= (I(p) \otimes I(((q \otimes q^l) \otimes r^l)^l)) \otimes I(r^l) = (I(p) \otimes (I(r^{ll}) \otimes I((q \otimes q^l)^l))) \otimes I(r^l)$$

$$= (I(p) \otimes (I(r^{ll}) \otimes (I(q^{ll}) \otimes I(q^l)))) \otimes I(r^l)$$

$$= (p \otimes ((1/(1/r)) \otimes ((1/(1/q)) \otimes (1/q)))) \otimes (1/r) = B.$$

$B \Rightarrow p$ is provable in **L1**, by (Id), (1L), (/L), ($\otimes$L).

The interpretation I is implicitly employed by Lambek [34, 35] and other authors, when they consider pregroup typing of natural languages. More precisely, they proceed as above except for the last step: they do not apply $I(p^{(n)})$. This procedure can also be described as applying translation rules for residuals and R-rules (R3)-(R5) as far, as possible. After replacing $\otimes$ by ',', the resulting sequence is reduced to a simple type by (R2), (IND). As we have seen, the final part of the procedure can be performed in **L1** enriched with poset axioms.

Since **L1** is compatible with type-theoretic semantics and sound (not complete) with respect to language frames, then this final part cannot lead to linguistically ill inferences. The first part, based on translation rules and (R3)-(R5), is more problematic. In the above example, from $B \Rightarrow p$ one infers $A \Rightarrow p$, although $A \Rightarrow B$ is not provable in **L1** (notice that $B \Rightarrow A$ is provable in **L1**, and $A \Leftrightarrow B$ in **CBL**).

The above remarks are not intended to object the plausibility of pregroup typing. They only show that it is based on some algebras different than language frames. In section 3, we shall discuss these differences in more detail.

Like **L** and related systems, **CBL** can be enriched with new connectives and modalities.

A pregroup $\mathcal{M}$ is called *a lattice ordered pregroup* ($l-$pregroup), if $(M, \leq)$ is a lattice. In $l-$pregroups the following equations are true:

$$(a \vee b)^l = a^l \wedge b^l,\ (a \vee b)^r = a^r \wedge b^r, \tag{6}$$

$$(a \wedge b)^l = a^l \vee b^l,\ (a \wedge b)^r = a^r \vee b^r, \tag{7}$$

$$a(b \vee c) = ab \vee ac,\ (a \vee b)c = ac \vee bc, \tag{8}$$

$$a(b \wedge c) = ab \wedge ac,\ (a \wedge b)c = ac \wedge bc, \tag{9}$$

for all elements a, b, c. (8) hold in all residuated lattices, but (9) are peculiar to $l-$pregroups.

The simplified sequent system **CBL'** can be extended to the language of $l-$pregroups by the following rules:

$$(\mathrm{P}\vee\mathrm{L})\ \frac{\Gamma, A, \Gamma' \Rightarrow \Delta;\ \Gamma, B, \Gamma' \Rightarrow \Delta}{\Gamma, A \vee B, \Gamma' \Rightarrow \Delta},\ (\mathrm{P}\vee\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A_i, \Delta'}{\Gamma \Rightarrow \Delta, A_1 \vee A_2, \Delta'},$$

$$(\mathrm{P}\wedge\mathrm{L})\ \frac{\Gamma, A_i, \Gamma' \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2, \Gamma' \Rightarrow \Delta},\ (\mathrm{P}\wedge\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A, \Delta';\ \Gamma \Rightarrow \Delta, B, \Delta'}{\Gamma \Rightarrow \Delta, A \wedge B, \Delta'}.$$

Recall that A^l, A^r are defined in metalanguage; one must add obvious clauses for $(A \circ B)^l$, $(A \circ B)^r$, where $\circ$ is $\vee$ or $\wedge$.

For the full system **CBL**, one needs $(n)-$variants of these rules. Unfortunately, cut elimination fails: (TRAN), (CUT1), (CUT2) are not admissible. By (Al), (7), (8):

$$1 \leq (a \vee b)(a \vee b)^l = a(a \vee b)^l \vee b(a \vee b)^l,$$

is true in $l-$pregroups, whence $\Rightarrow (p \otimes (p \vee q)^l) \vee (q \otimes (p \vee q)^l)$ is valid. This sequent is not provable in **CBL'** without cut, since each possible proof must introduce the external $\vee$, but sequents $\Rightarrow p \otimes (p \vee q)^l$, $\Rightarrow q \otimes (p \vee q)^l$ are not valid (this example comes from [16]). With (TRAN), the system is complete with respect to $l-$pregroups. Its decidability remains open.

CBL can be supplied with modalities [20]. It is also reasonable to admit partial commutation: $p\otimes q = q\otimes p$, for some atoms p, q. We do not elaborate on these topics, since they are discussed in other papers, included in this issue [22, 30].

3. Pregroups

Pregroups are models of **CBL**. They have been defined in section 2. Here we summarize basic facts about them; for more details, see [14, 15].

Let $\mathcal{M} = (M, \leq, \cdot, ^l, ^r, 1)$ be a pregroup. If $\backslash, /$ are defined by $a\backslash b = a^r b$, $a/b = ab^l$, then $(M, \leq, \cdot, \backslash, /, 1)$ is a residuated monoid. It satisfies:

$$a\backslash(bc) \leq (a\backslash b)c,\ (ab)/c \leq a(b/c), \tag{10}$$

for all $a, b, c \in M$. Since $\geq$ are true in any residuated semigroup, (10) yields equalities.

Conversely, any residuated monoid, satisfying (10), is a pregroup with adjoints defined as follows: $a^l = 1/a$, $a^r = a\backslash 1$. This is the only possible pregroup structure on this monoid, since residuals and adjoints are uniquely determined by the p.o. monoid structure:

$$a\backslash b = \max\{z \in M : az \leq b\},\ b/a = \max\{z \in M : za \leq b\}, \tag{11}$$

$$a^l \text{ (resp. } a^r\text{) is the only } b \text{ such that } ba \leq 1 \leq ab \text{ (resp. } ab \leq 1 \leq ba\text{).} \tag{12}$$

Of course, if $(M, \leq, \cdot, 1)$ is an arbitrary p.o. monoid, then the above operations need not be defined for all elements. This means that (11), (12) define partial operations on arbitrary p.o. monoids.

Every p.o. monoid $\mathcal{M}$ contains a largest pregroup. It consists of all elements $a \in M$ such that $a^{(n)}$ exists, for all $n \in \mathcal{Z}$. We denote this pregroup by $\mathrm{Pg}(\mathcal{M})$.

For the language frame $P(\Sigma^*)$, ordered by inclusion, this largest pregroup is the trivial group $\{\epsilon\}$, since $L_1L_2 \subseteq \{\epsilon\} \subseteq L_2L_1$ implies $L_1 = L_2 = \{\epsilon\}$. For the relational frame $\mathrm{REL}(U) = P(U^2)$, with inclusion, relational product and the identity relation I_U, $\mathrm{Pg}(\mathrm{REL}(U))$ consists of bijective relations, i.e. bijective functions from U into U, and the induced ordering is the identity. So, $\mathrm{Pg}(\mathrm{REL}(U))$ is a group [15].

Let $(P, \leq)$ be a poset. We consider the p.o. monoid $\mathcal{F}(P, \leq)$, shortly $\mathcal{F}(P)$, of all order-preserving functions from P into P, with function composition and pointwise ordering: $f \leq g$ iff, for all $x \in P$, $f(x) \leq g(x)$. For

$f \in \mathcal{F}(P)$, functions f^l, f^r (if exist) satisfy the Galois correspondences:

$$x \leq f(y) \text{ iff } f^l(x) \leq y,\ f(x) \leq y \text{ iff } x \leq f^r(y), \tag{13}$$

which are equivalent to:

$$f^l(x) = \min\{y \in P : x \leq f(y)\},\ f^r(x) = \max\{y \in P : f(y) \leq x\}. \tag{14}$$

Since f^l, f^r must be defined on all $x \in P$, then, for any $x \in P$, there exist $y, z \in P$ such that $f(y) \leq x \leq f(z)$. Consequently, if f^l, f^r exist, then f must be downward and upward unbounded on P (we use this terminology for functions defined on a poset, although it sounds naturally for functions defined on a totally ordered set).

Lambek [34] shows that the set of all downward and upward unbounded, order-preserving functions on $(\mathcal{Z}, \leq)$ is a pregroup (not a group). It is the only possible pregroup of that kind; if $(P, \leq)$ is an at least 3-element, totally ordered set, and $\mathrm{Pg}(\mathcal{F}(P))$ consists of all downward and upward unbounded functions, then $(P, \leq)$ is isomorphic to $(\mathcal{Z}, \leq)$ [14]. If $(P, \leq)$ is a totally ordered set, then $\mathrm{Pg}(\mathcal{F}(P))$ consists, in general, of some (not all) downward and upward unbounded functions, and similarly for posets $(P, \leq)$.

Pregroups are closed under products, substructures and order-preserving homomorphisms. For instance, the product of the Lambek pregroup (from the above paragraph) by itself is a new pregroup, isomorphic to the pregroup of all order-preserving functions on $\{0,1\} \times \mathcal{Z}$ with the lexicographic ordering, which are downward and upward unbounded on both $\{0\} \times \mathcal{Z}$ and $\{1\} \times \mathcal{Z}$.

Every pregroup $\mathcal{M}$ can be embedded in $\mathrm{Pg}(\mathcal{F}(M, \leq))$, where $(M, \leq)$ is the poset underlying $\mathcal{M}$. The embedding is $(a \mapsto f_a)_{a \in M}$, where $f(a)(x) = ax$.

If $(P, \leq)$ is a lattice, then $\mathcal{F}(P)$ is also a lattice with $f \vee g$, $f \wedge g$ defined pointwise, but $\mathrm{Pg}(\mathcal{F}(P))$ need not be an $l-$pregroup. It is an $l-$pregroup, if $(P, \leq)$ is a totally ordered set. To prove this fact assume that $f, g \in \mathcal{F}(P)$ and $f^{(n)}, g^{(n)}$ exist, for all $n \in \mathcal{Z}$. We show that $(f \vee g)^{(n)}$ and $(f \wedge g)^{(n)}$ exist, for all $n \in \mathcal{Z}$. We have to prove:

$$(f \vee g)^{(2n)} = f^{(2n)} \vee g^{(2n)},\ (f \wedge g)^{(2n)} = f^{(2n)} \wedge g^{(2n)}, \tag{15}$$

$$(f \vee g)^{(2n+1)} = f^{(2n+1)} \wedge g^{(2n+1)},\ (f \wedge g)^{(2n+1)} = f^{(2n+1)} \vee g^{(2n+1)}. \tag{16}$$

Of course, these equalities are true in $l-$pregroups, but we have not shown that $\mathrm{Pg}(\mathcal{F}(P))$ is an $l-$pregroup. We argue as follows. By the assumption, the elements on the right-hand side of (15), (16) exist, whence it follows from (15), (16) that $(f \vee g)^{(n)}$, $(f \wedge g)^{(n)}$ exist, for all $n \in \mathcal{Z}$.

To prove (15), (16) we need special cases of (16):

$$(f \vee g)^r = f^r \wedge g^r,\ (f \wedge g)^r = f^r \vee g^r, \tag{17}$$

$$(f \vee g)^l = f^l \wedge g^l,\ (f \wedge g)^l = f^l \vee g^l. \tag{18}$$

If the functions, defined by the right-hand sides of equations (14), exist, then they equal f^l, f^r, respectively. For the first equation (17), we have: $y \leq f^r(x) \wedge g^r(x)$ iff $y \leq f^r(x)$ and $y \leq g^r(x)$ iff $f(y) \leq x$ and $g(y) \leq x$ iff $(f \vee g)(y) \leq x$, which yields the equation. This argument is valid for any lattice P. For the second equation (17), we have: $y \leq f^r(x) \vee g^r(x)$ iff $y \leq f^r(x)$ or $y \leq g^r(x)$ iff $f(y) \leq x$ or $g(y) \leq x$ iff $(f \wedge g)(y) \leq x$, which yields the equation. This argument is valid, if P is a totally ordered set; then, $x \leq y \vee z$ iff $x \leq y$ or $x \leq z$, and $x \wedge y \leq z$ iff $x \leq z$ or $y \leq z$. The proof of (18) is dual. Now, (15), (16) can easily be proved by induction on n (first, for $n \geq 0$, second, for $n \leq 0$).

Let $\mathcal{M}$ be a pregroup. An element a is said to be *injective*, if: $ab = ac$ implies $b = c$, and *surjective*, if: $ba = ca$ implies $b = c$. In pregroups, a is injective iff $a^l a = 1$ iff $a^r a = 1$, and a is surjective iff $aa^l = 1$ iff $aa^r = 1$. (To prove ($\Rightarrow$), use $aa^l a = a$, $aa^r a = a$.) Clearly, a is injective (resp. surjective) iff a^l is surjective (resp. injective); a^l can be replaced by a^r. If ab is injective, then b is injective; if ab is surjective, then a is surjective. An element is said to be *bijective*, if it is both injective and surjective. The bijective elements of $\mathcal{M}$ form the largest group, contained in $\mathcal{M}$.

If every element of $\mathcal{M}$ is injective or surjective, then every element of $\mathcal{M}$ is bijective [14]. Observe that $a^l a = 1$ iff $a^r \leq a^l$, and $aa^l = 1$ iff $a^l \leq a^r$. In a totally ordered pregroup, $a^r \leq a^l$ or $a^l \leq a^r$, for all elements a, whence every element is bijective. Then, every totally ordered pregroup is a totally ordered group (an $o-$group). Every finite pregroup is a group ($\leq$ is the identity relation) [14]. If $(P, \leq)$ is a dense, totally ordered set, then $\mathrm{Pg}(\mathcal{F}(P))$ is an $l-$group [14]. If a pregroup contains the least or the greatest element, then it is the trivial (one-element) group [15].

In free pregroups $F(P, \leq)$, defined in section 2, no element $a \neq 1$ is injective or surjective [15]. It suffices to notice that elements $[p^{(n)}]$ are not injective, since $\epsilon \Rightarrow p^{(n-1)}, p^{(n)}$ does not hold. Consequently, the Lambek pregroup $\mathrm{Pg}(\mathcal{F}(\mathcal{Z}, \leq))$ is not free.

Analogous results can be obtained for left (resp. right) pregroups, which are defined as pregroups except omitting ^{r} and (Ar) (resp. l and (Al)) [15].

As we have seen, all pregroups (up to isomorphism) can be obtained as pregroups of some order-preserving functions on a poset. This characterization is not very useful, in practice, since functions are higher-order notions, and one does not precisely see the resulting structures. Also, algebras of bilinear and linear logic, defined by means of a closure operation on the powerset of a monoid (see section 1), refer to higher-order notions and are not easy to study. Below we outline another construction of such algebras, which is essentially first-order (though it employs an inductively defined subset of a monoid). We believe that this construction is more expedient than the ones, mentioned above, for producing and studying algebras of bilinear and linear logic as well as pregroups.

Recall that a bilinear algebra is a residuated monoid with a dualizing element 0, satisfying $a = 0/(a\backslash 0) = (0/a)\backslash 0$, for any element a. We present a general construction of bilinear algebras.

We start from an arbitrary monoid $\mathcal{M} = (M, \oplus, 0)$, supplied with an arbitrary bijection $l : M \mapsto M$. (Now, we prefer an additive notation for the initial monoid.) We write a^l for $l(a)$ and a^r for $l^{-1}(a)$. Then, the following equations are true:

$$a^{lr} = a = a^{rl}, \text{ for all } a \in M. \tag{19}$$

We define: $a\backslash b = a^r \oplus b$, $a/b = a \oplus b^l$. This yields the following equations:

$$(a\backslash b)/c = a\backslash(b/c), \ a^r = a\backslash 0, \ a^l = 0/a, \tag{20}$$

$$a = 0/(a\backslash 0) = (0/a)\backslash 0, \ a \oplus b = a^l\backslash b = a/b^r, \ a = 0^l\backslash a = a/0^r. \tag{21}$$

We also define a new binary operation $a \cdot b = (b^l \oplus a^l)^r$. Then:

$$(a \cdot b)^l = b^l \oplus a^l, \ (a \oplus b)^r = b^r \cdot a^r. \tag{22}$$

Using the first equation (22) and (19), we prove associativity: $(a \cdot b) \cdot c = (c^l \oplus (a \cdot b)^l)^r = (c^l \oplus (b^l \oplus a^l))^r = ((c^l \oplus b^l) \oplus a^l)^r = a \cdot (b \cdot c)$. We also get:

$$0^r \cdot a = a = a \cdot 0^r. \tag{23}$$

We define $1 = 0^r$. We need an ordering on M. We shall define a quasi-ordering by means of its positive cone. A set $F \subseteq M$ is called *a positive cone* (for 1), if it satisfies the following conditions:

(F1) $(a\backslash b) \in F$ iff $(b/a) \in F$,

(F2) $(a/a) \in F$,

(F3) if $(a/b) \in F$ and $(b/c) \in F$, then $(a/c) \in F$,

for all $a, b, c \in M$. Since (F1) can be replaced by two implications, positive cones are closed under arbitrary meets, and consequently, every monoid $\mathcal{M}$

with l contains a least positive cone (for 1), which will be denoted by $F_{(\mathcal{M},l)}$. Since $1 = 1/1$, by (21), then $1 \in F_{(\mathcal{M},l)}$, by (F2). Also $0^l \in F_{(\mathcal{M},l)}$, by (21), (F2), (F1).

We define: $a \leq_F b$ iff $(b/a) \in F$. By (F2), (F3), $\leq_F$ is a reflexive and transitive relation on M. We define: $a \sim_F b$ iff $a \leq_F b$ and $b \leq_F a$. Then, $\sim_F$ is an equivalence relation on M. We prove:

(P1) $0^l \sim_F 1$,

(P2) if $a \in F$ and $a \leq_F b$, then $b \in F$,

(P3) $a \in F$ iff $1 \leq_F a$,

(P4) $a \leq_F c/b$ iff $b \leq_F a\backslash c$,

for all $a, b, c \in M$. We have $0^l = 0^l/1$, and $0^l \in F$, so $1 \leq_F 0^l$. Also, $1 = 0^l\backslash 1$, and $1 \in F$, so $0^l \leq_F 1$. This yields (P1). Assume $a \in F$ and $a \leq_F b$. Then, $(b/a) \in F$ and $(a/1) \in F$, so $(b/1) \in F$, by (F3), whence $b \in F$. This yields (P2). (P3) follows from $a = a/1$. (P4) follows from (F1) and (20).

(P5) $a \leq_F b$ iff $b^l \leq_F a^l$ iff $b^r \leq_F a^r$,

for all $a, b \in M$. Notice $b = b^{lr} = b^l\backslash 0$. Then, $(b/a) \in F$ iff $((b^l\backslash 0)/a) \in F$ iff $(b^l\backslash(0/a)) \in F$ (use (20)) iff $(b^l\backslash a^l) \in F$. This yields the first equivalence. The second one follows from the first one, by (19).

We are ready to prove (RES) for the structure $(M, \leq_F, \cdot, \backslash, /)$. We have: $a \cdot b \leq_F c$ iff $(b^l \oplus a^l)^r \leq_F c$ iff $c^l \leq_F b^l \oplus a^l$ iff $c^l \leq_F b^l/a$ iff $a \leq_F c^l\backslash b^l$ iff $a \leq_F c \oplus b^l$ iff $a \leq_F c/b$. The equivalence for $\backslash$ follows from the one for $/$, by (P4).

From (RES), one easily infers monotonicity laws for $\cdot, \backslash, /$, which also yields: $a \leq_F b$ implies $c \oplus a \leq_F c \oplus b$ and $a \oplus c \leq_F b \oplus c$ (use the middle equation (21)). Consequently, $\sim_F$ is a congruence in $(M, \oplus, \cdot, \backslash, /, ^l, ^r)$. The quotient-structure is a residuated monoid with the dualizing element $[0]$. Accordingly, it is a bilinear algebra.

The quotient-operation $\oplus$ is the par operation in this algebra. Since $(b^l \oplus a^l)^r = (b^r \oplus a^r)^l$ is true in bilinear algebras, we get $(b^l \oplus a^l)^r \sim_F (b^r \oplus a^r)^l$. A direct proof looks as follows. We denote the right-hand term by $a \circ b$ and prove that $a \circ b \leq_F c$ iff $b \leq_F a\backslash c$, as in the above proof of (RES). By (RES), $a \circ b \leq_F c$ iff $a \cdot b \leq_F c$, for all c, whence $a \circ b \sim_F a \cdot b$.

Before factorization, the structure, constructed above, is a bilinear quasi-algebra, in which $\sim_F$ takes the part of equality. Evidently, all bilinear algebras and quasi-algebras can be constructed in this way; one starts form the underlying monoid or quasi-monoid $(M, \oplus, 0)$ with negations $^l, ^r$ and defines $F = \{a \in M : 0^r \leq a\}$.

Let $\mathcal{M} = (M, \oplus, 0)$ and l be fixed. $(M, \leq_F, \cdot, \backslash, /, 1, 0)$ is a bilinear algebra if, and only if, $\sim_F$ is the identity relation, which is equivalent to the condition:

$$\text{for all } a, b \in M, \text{ if } (a/b) \in F \text{ and } (b/a) \in F \text{ then } a = b. \tag{24}$$

Clearly, (24) holds for some positive cone $F \subseteq M$ iff it holds for $F_{(\mathcal{M},l)}$. In general, $F_{(\mathcal{M},l)}$ determines the finest bilinear quasi-algebra on $(\mathcal{M}, l)$.

For commutative monoids, this construction yields algebras and quasi-algebras of **MLL**. For an arbitrary monoid and an involution l (satisfying $a^{ll} = a$, and consequently, $a^l = a^r$), it yields models of Cyclic **MLL** [45].

In pregroups $a \cdot b = a \oplus b$. It holds in the quotient-structure iff $(a \oplus b)^l \sim_F b^l \oplus a^l$. By (19), this condition is equivalent to $(a \oplus b)^r \sim_F b^r \oplus a^r$. One can guarantee it by affixing a new clause to (F1)-(F3). One can also begin with a pair $(\mathcal{M}, l)$, satisfying:

$$(a \oplus b)^l = b^l \oplus a^l, \text{ for all } a, b \in M; \tag{25}$$

equivalently $(a \oplus b)^r = b^r \oplus a^r$. This implies $0^l = 0^r = 0$. Then, for any positive cone F, the quotient-structure is a pregroup, and all pregroups can be constructed in this way.

We shall characterize the finest quasi-pregroups which can be constructed in this way on the basis of a free monoid $(\Sigma^*, \cdot, \epsilon)$ and a bijection $l : \Sigma^* \mapsto \Sigma^*$; here Σ can be infinite (even uncountable). Now $0 = \epsilon$.

By (25), for $x \in \Sigma^*$, $x = a_1 \cdots a_n$, where $a_i \in \Sigma$, we have $x^l = (a_n)^l \cdots (a_1)^l$. Also $(a_i)^l \in \Sigma$. Otherwise, $(a_i)^l = xy$, for some $x, y \neq \epsilon$, whence $a_i = y^r x^r$; then, $x^r = \epsilon$ or $y^r = \epsilon$, which yields $x = \epsilon$ or $y = \epsilon$. Similarly, $(a_i)^r \in \Sigma$. Accordingly, l restricted to Σ is a bijection of Σ onto Σ. This restricted mapping is denoted by π. It naturally extends to a homomorphism $\pi : \Sigma^* \mapsto \Sigma^*$ (it is an isomorphism). We get $x^l = (\pi(x))^R$, where y^R denotes the reversal of y.

For $a, b \in \Sigma$, we define: $a \equiv b$ iff $a = \pi^n(b)$ or $b = \pi^n(a)$, for some integer $n \geq 0$ (here π^n denotes the iteration of π n times). $\equiv$ is an equivalence relation on Σ; the equivalence classes of $\equiv$ are called *orbits*. Let P be a selector of the family of all orbits. For $p \in P$, $n \geq 0$, by $p^{(n)}$ we denote the element $(\pi^{-1})^n(p)$, where π^{-1} is the converse of π, and by $p^{(-n)}$ the element $\pi^n(p)$. The orbit determined by p consists of all $p^{(n)}$, for $n \in Z$. If the orbit is infinite; then, $p^{(m)} \neq p^{(n)}$, for $m \neq n$. If it is finite (of cardinality $k > 0$), then $p^{(0)}, p^{(1)}, \ldots, p^{(k-1)}$ are different, and $p^{(n)} = p^{(n+k)}$, for all $n \in \mathcal{Z}$. Clearly, $(p^{(n)})^l = p^{(n-1)}$ and $(p^{(n)})^r = p^{(n+1)}$.

Every string from Σ^* is uniquely represented as $(p_1)^{(k_1)} \cdots (p_n)^{(k_n)}$, where $p_i \in P$, and $k_i \in \mathcal{Z}$, if the orbit of p_i is infinite, $0 \leq k_i < k$, if the orbit of p_i is of cardinality k. The least positive cone $F_{(\Sigma^*,l)}$ determines the smallest quasi-ordering $\leq$ on Σ^* such that the resulting structure is a quasi-pregroup. This quasi-ordering must satisfy (CON) and (EXP) from section 2 (with $\leq$ in the place of $\Rightarrow$); if the orbit of p is of finite cardinality k, then $n+1$ is counted modulo k. Obviously, the smallest quasi-ordering is the reflexive and transitive closure of the relation defined by (CON), (EXP). The quotient-construction yields the corresponding pregroup.

We have seen that the finest quasi-pregroups and pregroups, determined by free monoids with a bijection, satisfying (25), are closely related to free pregroups on a trivial poset. The only difference is that, for some p, one may count $p^{(n)}$ modulo a positive integer. The resulting pregroup is a p.o. group if, and only if, π is an involution. (Then, each orbit is of cardinality at most 2.)

This approach rises many interesting problems which are deferred to further research. One of them is to characterize those algebras $(M, \cdot, l, r, 1)$ ($(M, \cdot, 1)$ is a monoid, l is a bijection on M, and r is the converse of l) which can be expanded to a pregroup $(M, \leq, \cdot, ^l, ^r, 1)$. One easily shows that they form an elementary class, recursively axiomatizable by quasi-equations. Can the axiomatization be finite, or equational, or both? Does it consist of monoid axioms, (19), (25) (write $(ab)^l = b^l a^l$), and $aa^l a = a$, $aa^r a = a$?

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