Lüders Rule[1]

The *Lüders rule* describes a change of the state of a quantum system under a selective measurement: if an $\rightarrow$ *observable* A, with eigenvalues a_i and associated eigenprojections P_i, $i = 1, 2, \ldots$, is measured on the system in a $\rightarrow$ *state* T, then the state transforms to $\widetilde{T}_k := P_k T P_k / \mathrm{tr}\,[T P_k]$ on the condition that the result a_k was obtained. This rule was formulated by Gerhart Lüders [1] as an elaboration of the work of John von Neumann [2] on the measurement process and it is an expression of the $\rightarrow$ *projection postulate*, or the collapse of the wave function.

From the perspective of quantum $\rightarrow$ *measurement theory*, the Lüders rule characterizes just one (albeit distinguished) form of state change that may occur in appropriately designed measurements of a given observable with a discrete spectrum. In general, the notion of $\rightarrow$ *instrument* is used to describe the state changes of a system under a measurement, whether selective or not. The Lüders instrument $\mathcal{I}^L$ consists of the $\rightarrow$ *operations* $\mathcal{I}^L_X$ of the form $\mathcal{I}^L_X(T) = \sum_{a_i \in X} P_i T P_i$, and it is characterized as a repeatable, ideal, nondegenerate measurement [3, Theorem IV.3.2], see also [7, Theorem 4.7.2]. In such a measurement, with no selection or reading of the result, the state of the system undergoes the transformation $T \mapsto \mathcal{I}^L_{\mathbb{R}}(T) = \sum_i P_i T P_i = \sum_i \mathrm{tr}\,[T P_i] \widetilde{T}_i$, the projection postulate then saying that if a_k is the actual measurement result, this state collapses to $\widetilde{T}_k$.

Lüders measurements offer an important characterization of the compatibility of observables A, B with discrete spectra: A and B commute if and only if the expectation value of B is not changed by a nonselective Lüders operation of A in any state T [1]. This result is the basis for the axiom of local commutativity in relativistic quantum field theory: the mutual commutativity of observables from local algebras associated with two spacelike separated regions of spacetime ensures, and is necessitated by, the impossibility of influencing the outcomes of measurements in one region through nonselective measurements performed in the other region.

The Lüders rule is directly related to the notion of conditional probability in quantum mechanics, conditioning with respect to a single event. According to $\rightarrow$ *Gleason's theorem*, the generalized probability measures μ on the projection lattice $\mathcal{P}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ with dimension $\dim(\mathcal{H}) \geq 3$ are uniquely determined by the state operators through the formula $\mu(P) = \mathrm{tr}\,[TP]$ for all $P \in \mathcal{P}(\mathcal{H})$. For

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any μ and for any P such that $\mu(P) \neq 0$ there is a unique generalized probability measure μ_P with the property: for all $R \in \mathcal{P}(\mathcal{H})$, $R \leq P$, $\mu_P(R) = \mu(R)/\mu(P)$. The state operator defining μ_P is given by the Lüders form: if μ is determined by the state T, then μ_P is determined by the state $PTP/\mathrm{tr}\,[TP]$ [4].

The Lüders rule is also an essential structural element in axiomatic reconstructions of quantum mechanics. As shown in [5], it occurs in various disguised forms as an axiom in → *quantum logic*; for example, it plays a role in the formulation of the covering law; see also [8, Chapter 16], [9].

The Lüders rule has a natural generalization to measurements with a discrete set of outcomes $a_1, a_2, \dots$, represented by a positive operator measure such that each a_i is associated with a positive operator A_i. The generalized Lüders instrument, defined via the operations $T \mapsto \mathcal{I}^L_X(T) = A_i^{1/2} T A_i^{1/2}$, is known to have approximate repeatability and ideality properties [6]. The Lüders theorem extends to generalized measurements under certain additional assumptions [10] but is not valid in general [11].

The Lüders rule is widely used as a practical tool for the effective modeling of experiments with quantum systems undergoing periods of free evolution separated by iterated measurements. It is successfully applied in the quantum jump approach [12]. The single- and → *double-slit experiments* with individual quantum objects are the classic illustrations of the physical relevance of the Lüders rule.

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