Measurement Theory[1]

The term *measurement theory* refers to that part of a physical theory in which the empirical and operational content of the concepts of the theory is determined. Measurements are analyzed both as operational procedures defining the → *observables* of the theory and as physical processes which are themselves subject to the laws of physics.

In classical physics, measurements are performed in order to determine the values of one or several observables of the physical system under consideration. Classical physics allowed the idealized notion that every physical quantity has a definite value at any time, and that this value can be determined with certainty by measurement without influencing the object system in a significant way. By contrast, in quantum mechanics both features fail to hold without strong qualifications. Accordingly, in their seminal paper of 1935 [1], Einstein, Podolsky and Rosen used elements of this description as a sufficient criterion of physical reality, applicable both in classical and quantum mechanics:

> "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to that physical quantity."

As far as *observable* elements of reality represented by quantum mechanics are concerned, this condition must also be regarded as necessary. Hence, an observable is understood to have a definite value if the probability that a measurement indicates a particular value of the observable is equal to one. In quantum mechanics, this can only be satisfied if the system is in an eigenstate of the observable associated with the value in question. Moreover, it turns out that in quantum mechanics the interaction between a measuring apparatus and the measured system is generally not negligible. This leads to the necessity of reconsidering what it means that a measurement determines the value of an observable. Here this question is discussed for the case of an observable represented by a selfadjoint operator A (acting on a complex separable Hilbert space $\mathcal{H}$) with nondegenerate discrete spectrum $\{a_1, a_2, \dots\}$, associated orthonormal basis of eigenvectors $\{\varphi_1, \varphi_2, \dots\}$, and → *spectral decomposition* $A = \sum_i a_i P_i$, where $P_i = |\varphi_i\rangle\langle\varphi_i|$ denotes the projection onto the one-dimensional subspace spanned by φ_i.

A minimal requirement for a physical interaction process between an object system and an apparatus to qualify as a measurement of A is the so-called *calibration condition*: whenever the system is in an eigenstate, the apparatus should indicate the corresponding eigenvalue unambiguously after the interaction has ceased. In quantum mechanics, a measurement is modeled by representing the apparatus by a

[1]In: *Compendium of Quantum Physics*, eds. F. Weinert, K. Hentschel and D. Greenberger, Springer-Verlag, to appear.

Hilbert space $\mathcal{H}_A$, the pointer observable as a selfadjoint operator Z acting on $\mathcal{H}_A$ and the coupling between object and apparatus as a unitary operator U acting on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}_A$ of the total system. Together with the initial apparatus → *state* T_A, these elements, collected into a quadruple $\langle \mathcal{H}_A, T_A, U, Z \rangle$, constitute a *measurement scheme*.

Assuming, for simplicity, that the apparatus initially is in a pure state, described by a unit vector ϕ, the calibration condition can be formalized as follows: the measurement scheme has to be such that for any eigenstate φ_i of A there is an associated (normalized) eigenstate ϕ_i of the pointer Z so that U effects the following transition:

$$\varphi_i \otimes \phi \to U(\varphi_i \otimes \phi) = \psi_i \otimes \phi_i. \tag{1}$$

Here ψ_i is some normalized vector state in $\mathcal{H}$, and the ϕ_i are mutually orthogonal. Thus, if the observable A initially has a definite value a_i, the pointer observable of the apparatus will indicate this value with probability equal to one, in accordance with the → *Born* probability *rule*. If condition (1) is satisfied for all φ_i, the given measurement scheme is called a *premeasurement* of A.

If the system is initially in a vector state φ which is not an eigenstate of A, then φ is a superposition of eigenstates of A, that is, $\varphi = \sum_i c_i \varphi_i$ with more than one of the c_i nonzero. Together with the linearity of U, the rule (1) still determines unambiguously the final state of the total system:

$$\varphi \otimes \phi = \sum_i c_i \varphi_i \otimes \phi \to U \varphi \otimes \phi = \sum_i c_i \psi_i \otimes \phi_i. \tag{2}$$

The final state is a superposition of mutually orthogonal states, and the probability for the pointer to indicate a value a_i is equal to $|c_i|^2 = |\langle \varphi | P_i \varphi \rangle|^2$, thus justifying the Born probability interpretation of the latter expression.

This simplified description also highlights the fundamental dilemma of quantum measurement theory known as the quantum measurement problem, the problem of objectification, or the collapse problem: if an observable A does not have a definite value, then according to quantum mechanics, a premeasurement of A will leave the object-plus-apparatus system in an → *entangled* state in which the pointer observable does not have a definite value – in stark contrast to the fact that every real measurement ends with a definite pointer position. This leaves one with the following alternative: on the one hand, if one requires that quantum mechanics should include an account of its measuring processes – that is, this theory should be semantically complete – then it turns out that the occurrence of definite measurement outcomes contradicts the quantum mechanical account of the measurement dynamics – that is, this theory is semantically inconsistent; on the other hand, if one requires semantical consistency, then quantum mechanics cannot be semantically complete [8]. In the first case, a modification of the axioms of quantum mechanics is required. In the second case, there is no consistent quantum measurement theory, unless an appropriate reinterpretation of what it means for an observable to have a definite value can be found.

There is an enormous amount of literature dealing with the quantum measurement problem, and as yet there is no generally accepted resolution. Rigorous technical presentations of the problem and the spectrum of interpretational options are found, for example, in [9] and [10], whereas philosophical aspects are discussed in [11]. A valuable cross-section of the older literature until 1980 is reprinted in the volume [12]. Interestingly, the founders of quantum mechanics (e.g., [2, 3]) identified the reality of the collapse of the wave function or state vector but did not regard it as a conceptual problem. It was von Neumann in 1932 [4] who pointed out the tension between the collapse process as a random event and the deterministic (unitary, linear) Schrödinger dynamics of a closed system. Somewhat later, Schrödinger [5] conceived his infamous cat paradox to highlight the apparent absurdity of the possibility, suggested by quantum mechanics, of observing macroscopic systems in superpositions of states corresponding to such discernible situations as a cat being dead or alive.

Adopting the collapse postulate has since been taken by many as a pragmatic way of suspending the measurement problem. Following this route, there remains the task for quantum measurement theory to show that quantum mechanics entails the possibility in principle of measuring any of its observables. For an observable represented as a POVM (→ *observable*), the above calibration condition is generally not applicable. However, whenever that condition does apply, it implies the reproduction of probabilities for the object observable in terms of the pointer statistics. This latter condition, called *probability reproducibility condition* [9], can always be taken as the defining criterion for a measurement scheme to constitute a measurement of a given observable. This characterization of the measurements of an observable implements the Born interpretation (→ *Born rule*) of the quantum mechanical probabilities and the idea that any observable is identified by the totality of its statistics. The formal implementation of these ideas, which constitute the mathematical framework of quantum measurement theory, are briefly summarized in the text box below.

Tools of Quantum Measurement Theory

Every measurement scheme $\langle \mathcal{H}_A, T_A, U, Z\rangle$ *defines a unique observable of the object system. If the pointer observable* Z *is represented as a POVM on the (Borel) sets of* $\mathbb{R}$ *(say), then for each state* T *of the object system, the following defines a probability measure on the real line (*X *denotes any Borel subset of* $\mathbb{R}$ *and* I *is the identity operator):*

$$X \mapsto \mathrm{tr}[UT \otimes T_A U^* I \otimes Z(X)] \equiv \mathrm{tr}[TE(X)]. \tag{3}$$

This equation, valid for all states T*, entails the existence of a positive operator* $E(X)$ *associated with each set* X*; moreover, the fact that* $X \mapsto \mathrm{tr}[TE(X)]$ *is a probability measure for each* T *ensures that* $E : X \mapsto E(X)$ *is a POVM on the (Borel) subsets of* $\mathbb{R}$*.*

It is a fundamental theorem of the quantum theory of measurement that for every observable there are measurement schemes (in fact, infinitely many) such that (3) is fulfilled for all object states T [6].

With the existence of premeasurements for any observable thus secured, another task of quantum measurement theory is the description of the effect of a measurement on the object system. Given a measurement scheme for an observable E*, one can ask for the probabilities of the outcomes of any subsequent measurement. If* F *is another POVM on the (Borel) subsets of* $\mathbb{R}$*, to be measured immediately after the* E *measurement, the sequential joint probability for obtaining a value of* E *in a set* X *and a value of* F *in a set* Y *is*

$$\mathrm{tr}[UT \otimes T_A U^* F(Y) \otimes Z(X)] \equiv \mathrm{tr}[\mathcal{I}_X(T)F(Y)]. \tag{4}$$

This relation, valid for all states T*, all observables* F *and all* X, Y*, determines a unique non-normalized object state* $\mathcal{I}_X(T)$*; substituting for* $F(Y)$ *the identity operator, it is seen that* $\mathrm{tr}[\mathcal{I}_X(T)] = \mathrm{tr}[TE(X)]$*. Dividing the joint probability in (4) by the latter probability gives the conditional probability for the occurrence of an outcome in* Y *given that the first measurement led to an outcome in* X*. Thus* $\mathcal{I}_X(T)$ *can be taken to play the role of the final object state in accordance with the collapse postulate. The map* $T \mapsto \mathcal{I}_X(T)$ *is known as a* (quantum) operation*, and* $X \mapsto \mathcal{I}_X$ *is an operation-valued measure called the* instrument *induced by the given measurement scheme* [7].

Any instrument arising from a measurement scheme has the property of complete positivity*: that is, for any operation* $\mathcal{I}_X$*, if extended to a linear map* $I_n \otimes \mathcal{I}_X$ *acting on the trace class operators of the Hilbert spaces* $\mathbb{C}^n \otimes \mathcal{H}$*, the extended map is positive for each* n*. It is another fundamental theorem of quantum measurement theory that every completely positive instrument can be realized by some (in fact, infinitely many) measurement schemes* [6].

With the conceptual tools of measurement theory outlined in the above box, it has become possible to eliminate some long-standing myths and corroborate a number of equally long-standing folk truths. For example, it has long been held without questioning that any measurement collapses the object system into an eigenstate of the measured observable. Measurements with that property are called repeatable. In the example leading to (1), repeatability is achieved by putting $\psi_i = \varphi_i$; but it is by no means necessary to assume that every measurement has this property. Moreover, according to a theorem due to Ozawa [6], in order for an observable to admit a repeatable measurement, this observable must be discrete, that is, have a countable set of values.

The realization that measurements necessarily disturb the object system was made early on in the history of quantum mechanics. However, the nature of that "disturbance" and its quantification have remained the subject of much debate until recently, when it was realized that the notion of instrument allows a rigorous and effective description of the state changes due to measurements. Yet another fundamental theorem of quantum measurement theory is given by the statement that there is no measurement which does not change at least some of the states of the system under investigation: a measurement scheme that leaves unchanged all states of the object defines a trivial observable, that is one whose probability measures do not depend on the state. Thus, there is no information gain in quantum measurements without some disturbance.

The trade-off between information gain and disturbance in quantum measurements has been recognized as a resource for novel applications of quantum measurements, particularly in quantum cryptography, a sub-field of the new area of → *quantum information* science. This is one example for the importance of quantum measurement theory as an applied discipline besides its foundational role.

Applications of quantum measurement theory ranging from nondemolition measurements and analyses of basic experiments to open quantum systems and quantum tomography are covered, for instance, by the monographs [13, 14, 15, 16, 17].

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