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Buss, Samuel R.

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## Pool resolution is NP-hard to recognize

Samuel R. Buss

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**Abstract** A pool resolution proof is a dag-like resolution proof which admits a depth-first traversal tree in which no variable is used as a resolution variable twice on any branch. The problem of determining whether a given dag-like resolution proof is a valid pool resolution proof is shown to be NP-complete.

**Keywords** Resolution · Proof search · Computational complexity · Propositional logic · NP-completeness · Satisfiability

**Mathematics Subject Classification (2000)** 03F20 · 03F07 · 68Q17

Propositional resolution has been the foundational method for reasoning in propositional logic, especially for forming refutations of satisfiability of set of clauses. In recent years, the most successful satisfiability testers have used the DPLL (Davis-Putnam-Logeman-Loveland) algorithm combined with clause learning, backtracking, restarts, and other techniques. (See Beame et al. [4] for an overview of clause learning.) Pool resolution was introduced by Van Gelder [11] as an resolution-based refutation system that provides a good theoretical model for the proofs produced by real-world satisfiability testing algorithms that incorporate clause learning and backtracking. Van Gelder proved that pool resolution is exponentially stronger than regular resolution. Bacchus et al. [3], building on techniques from [4], proved that pool resolution can “effectively p-simulate” full resolution; and Buss et al. [6, Theorem 19] gave an

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S. R. Buss (✉)

Department of Mathematics, University of California, La Jolla, San Diego, CA 92093-0112, USA  
e-mail: sbuss@math.ucsd.edu

effective p-simulation for a system similar to pool resolution. However, it is open whether pool resolution can directly p-simulate full resolution.

Van Gelder defined pool resolution algorithmically; however, we shall use his characterization that a pool resolution proof is a dag-like resolution proof that admits a regular, depth-first traversal. A depth-first traversal defines a tree on the clauses in the proof, which is a subgraph of the dag. The tree is called “regular”, provided that no branch in the tree that contains two clauses that are derived by resolution on the same variable.

Actually, Van Gelder defined pool resolution using an extended form of the resolution that allows any two clauses to be resolved with any resolution variable—regardless of whether the variable occurs appropriately in the clauses. This extended resolution rule was called the *degenerate* resolution rule by [3].

A depth first traversal  $\tau$  of a refutation  $R$  and the associated traversal tree  $T_\tau$  are formally defined as follows. If  $C$  is a non-initial clause in  $R$  and  $D$  is one of the hypotheses of the inference used to derive  $C$ , then we call  $D$  a *child* of  $C$ . We assume w.l.o.g. that  $R$  is rooted, that is, that every clause in  $R$  is a descendent of the empty clause. A depth first traversal  $\tau$  of  $R$  is a sequence  $E_0, E_1, \dots, E_p$  containing the clauses of  $R$ , each clause exactly once, starting with the empty clause. For  $1 \leq i \leq m$ ,  $E_i$  must be a child of an earlier  $E_j$ , where  $j$  must be the maximum value  $< i$  such that not all of  $E_j$ 's children occur among  $E_0, \dots, E_{i-1}$ . In this case,  $E_i$  is also a child of  $E_j$  in the tree  $T_\tau$  induced by the traversal  $\tau$ , and all edges in  $T_\tau$  are obtained in this way.

The traversal  $\tau$  is called *regular* provided  $T_\tau$  has no branch that contains two clauses derived by resolution on the same variable.  $R$  is a pool resolution refutation if and only if it admits a regular depth first traversal.

The POOL RESOLUTION problem is the decision problem of deciding whether a given dag-like resolution proof  $R$  is also a pool resolution refutation. Note that this problem is clearly in NP, since the algorithm can just non-deterministically guess a regular, depth-first traversal.

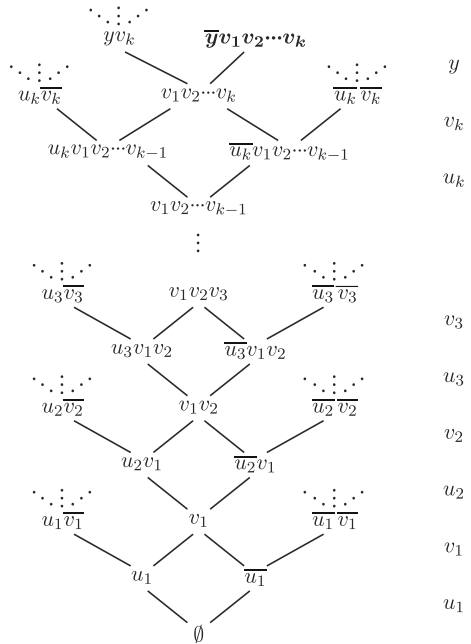
**Theorem 1** *The POOL RESOLUTION problem is NP-complete.*

To fully specify the POOL RESOLUTION problem, we need to say how the dag-like proof  $R$  is presented. Our proof of Theorem 1 will make the strongest possible assumptions: First, we will work only with proofs  $R$  that are refutations in which all resolution inferences are standard. (A “refutation” is a proof that ends with the contradictory clause  $\emptyset$ .) Furthermore, the refutation  $R$  will be specified as a sequence of clauses, and each non-initial clause can be derived in exactly one way from the earlier clauses. Thus,  $R$  will admit a unique dag structure.

There have been a number of results, including [1, 2, 8–10], about the hardness of finding resolution proofs, or of determining whether resolution proofs exist. Theorem 1, however, is more in the spirit of hardness results by Buss and Hoffmann [5] and Hoffmann [7]: these show that, given a particular resolution refutation, it is hard to determine if it satisfies extra conditions.

The rest of the paper gives the proof of the theorem. The main construction for the proof will be a reduction from the NP-complete satisfiability problem SAT to POOL RESOLUTION. An instance  $\Gamma$  of SAT consists of a set of  $m$  clauses  $C_1, \dots, C_m$  involving  $k$  variables  $x_1, \dots, x_k$ .

**Fig. 1** Shows the root portion of the dag refutation  $R$ . The end clause is  $\emptyset$ . The only initial clause, shown in *boldface*, is  $\overline{y}v_1v_2 \cdots v_k$ . The other leaves, decorated with  $\cdot \cdot \cdot \cdot \cdot \cdot$ 's are derived from the proof fragments shown in Figs. 2, 3, and 4. The variables in the *right* column indicate the resolution variable for the corresponding inferences



Given  $\Gamma$ , we will construct another set  $\Pi$  of clauses and a dag-like resolution refutation  $R$  of  $\Pi$ . The propositional variables in  $\Pi$  will be  $u_i$  and  $v_i$  for  $1 \leq i \leq k$ ,  $c_j$  for  $1 \leq j \leq m$ , and one further variable  $y$ . We will prove that  $R$  is a valid pool resolution refutation iff  $\Gamma$  is satisfiable.

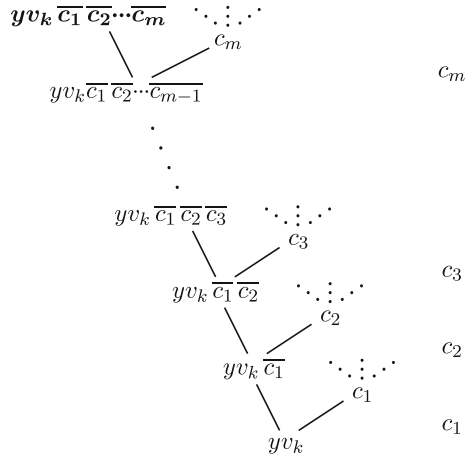
The root portion of the refutation  $R$  is shown in Fig. 1. The figure uses the following conventions. (1) Each node in the dag is labeled with a clause. (2) Each non-initial clause  $C$  has two children (immediate successors)  $D_0$  and  $D_1$ , indicated by edges drawn from  $C$  upward towards  $D_0$  and  $D_1$ , such that  $C$  is inferred from the two children clauses using resolution with respect to some *resolution variable*. (3) The resolution variable is easily determined from  $D_0$  and  $D_1$ , and is also indicated in the column on the right side of the figure. (4) Initial clauses are written in boldface. (5) Other leaves in the figure, decorated with  $\cdot \cdot \cdot \cdot \cdot \cdot$ 's are *not* initial clauses; rather their derivations are shown in other figures.

The remaining portions of  $R$  are shown in Figs. 2, 3, and 4. It should be noted that no clause appears more than once in  $R$ . In particular, the clauses  $c_i$  are used multiple times in Figs. 2 and 3, but these represent multiple uses of the same clause, and each  $c_i$  is derived exactly once as shown in Fig. 4.

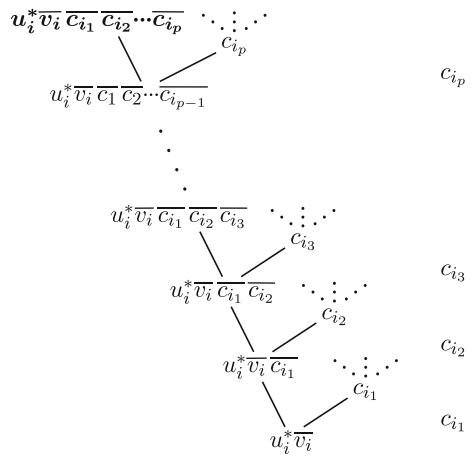
Examining the refutation  $R$  in Figs. 1, 2, 3, and 4 shows that the only way that a traversal  $\tau$  can fail to be regular is for the resolution variable  $y$  to be used twice along some branch of  $T_\tau$ . In fact, the variable  $y$  is the only variable that is used twice along any directed path in  $R$ .

As shown in Fig. 4, the variable  $y$  is the resolution variable used to derive each clause  $c_j$ . It is also used as the resolution variable at the top of Fig. 1. In the traversal

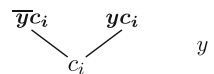
**Fig. 2** The derivation of the clause  $yv_k$



**Fig. 3** Shows the derivation of  $u_i^* \overline{v_i}$ , where  $u_i^*$  is either  $u_i$  or  $\overline{u_i}$ . Letting  $\ell$  be  $x_i$  or  $\overline{x_i}$ , respectively, then  $C_{i_1}, C_{i_2}, \dots, C_{i_p}$  are the clauses that contain  $\ell$



**Fig. 4** The derivation of  $c_i$



tree  $T_\tau$ , the clause  $yv_k$  will be the child of the clause  $v_1 v_2 \dots v_k$  which is derived using  $y$  as the resolution variable. In addition, as shown in Fig. 2,  $c_j$  is in the sub-derivation of  $R$  rooted at  $yv_k$ . Therefore, if there is any clause  $c_j$  which is not visited before  $yv_k$  in the traversal, then there will be a branch in  $T_\tau$  containing two uses of  $y$  as a resolution variable. It follows that any regular traversal must visit every  $c_j$  before visiting  $yv_k$ .

The only way to visit a clause  $c_j$  before  $yv_k$  is by visiting the clauses  $u_i^* \overline{v_i}$  that are derived as shown in Fig. 3, where  $u_i^*$  is either  $u_i$  or  $\overline{u_i}$ . There are  $2k$  such sub-derivations, two for each  $\Gamma$ -variable  $x_i$ . Fixing the value of  $i$ , let the literal  $\ell$  be either  $x_i$  or  $\overline{x_i}$ . In the first case, the variable  $u_i^*$  is  $u_i$ , and in the second case,  $u_i^*$  is  $\overline{u_i}$ . Let  $\mathcal{C}(\ell)$  be the set

of clauses in  $\Gamma$  which contain  $\ell$ , and enumerate this set as  $\mathcal{C}(\ell) = \{C_{i_1}, \dots, C_{i_p}\}$ . Here  $p = p(\ell)$  is the number of clauses that contain  $\ell$ . Then, the clause  $u_i^* \bar{v}_i$  is derived as shown in Fig. 3. Note in particular, that the derivation of  $u_i^* \bar{v}_i$  includes the derivations of the clauses  $c_{i_1}, \dots, c_{i_p}$ .

**Lemma 2** *Let  $\tau$  be a depth-first traversal of  $R$  and  $1 \leq i \leq k$ . Then at most one of the clauses  $u_i \bar{v}_i$  and  $\bar{u}_i v_i$  can appear in  $\tau$  before the clause  $yv_k$ .*

The proof of the lemma is almost obvious. Suppose  $u_i \bar{v}_i$  appears in the traversal before  $\bar{u}_i v_i$ . This means that  $u_i v_1 \dots v_{i-1}$  also appears in the traversal before  $\bar{u}_i v_i$ . Hence, since  $yv_k$  is in the sub-derivation rooted at  $u_i v_1 \dots v_{i-1}$  and  $\bar{u}_i v_i$  is not, it follows that  $yv_k$  precedes  $\bar{u}_i v_i$  in the traversal. A similar argument applies if  $\bar{u}_i v_i$  precedes  $u_i \bar{v}_i$  in the traversal.  $\square$

We define a partial truth assignment  $\alpha_\tau$  as follows.

$$\alpha_\tau(x_i) = \begin{cases} T & \text{if } u_i \bar{v}_i \text{ precedes } yv_k \text{ in } \tau \\ F & \text{if } \bar{u}_i v_i \text{ precedes } yv_k \text{ in } \tau \\ * & \text{otherwise} \end{cases}$$

where  $T$ ,  $F$ , and  $*$  represent the values *True*, *False*, and “undefined”. The third situation arises when neither clause precedes  $yv_k$  in  $\tau$ . The partial assignment  $\alpha_\tau$  induces a (partial) truth assignment on literals in the obvious way, and  $\alpha_\tau$  satisfies  $\Gamma$  provided every  $C_i \in \Gamma$  contains at least one literal that is set to *True* by  $\alpha_\tau$ .

**Lemma 3** *The traversal  $\tau$  is regular if and only if  $\alpha_\tau$  satisfies  $\Gamma$ .*

To prove the lemma, first suppose  $\alpha_\tau$  satisfies  $\Gamma$ . Then each clause  $C_j$  in  $\Gamma$  contains some literal  $\ell$  such that  $\alpha_\tau(\ell) = T$ . Letting,  $u_i^*$  equal  $u_i$  or  $\bar{u}_i$ , respectively, if  $\ell$  is  $x_i$  or  $\bar{x}_i$ , this means  $u_i^* \bar{v}_i$  is traversed in  $\tau$  before  $yv_k$ . Therefore, since  $C_j$  is one of the clauses containing  $\ell$ , the unit clause  $c_j$  is also traversed before  $yv_k$ .

It follows, that if  $\alpha_\tau$  satisfies  $\Gamma$ , then every  $c_j$  is traversed before  $yv_k$ . This suffices to make the traversal  $\tau$  regular.

Now suppose  $\alpha_\tau$  does not satisfy  $\Gamma$ . Let  $C_j$  be a clause in  $\Gamma$  that is not made true by  $\alpha_\tau$ . By Lemma 2, this means that there is no  $u_i^* \bar{v}_i$  which is traversed before  $yv_k$  which has the unit clause  $c_j$  in its sub-derivation. Therefore,  $c_j$  is traversed after  $yv_k$ . This ensures that  $\tau$  is not a regular traversal since  $y$  is used as a resolution variable both to derive the clause  $v_1 v_2 \dots v_k$  from  $yv_k$ , and to derive  $c_j$ , and since  $c_j$  is in the sub-derivation rooted at  $yv_k$ .  $\square$

Lemma 3 shows that if  $R$  has a regular traversal, then  $\Gamma$  is satisfiable. On the other hand, if  $\alpha$  is a satisfying assignment for  $\Gamma$ , then it is straightforward to construct a traversal  $\tau$  such that  $\alpha_\tau = \alpha$ .

That completes the proof of the theorem.

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