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Recursive analysis of singular ordinary differential equations

Peter Buser[a,*], Bruno Scarpellini[b]

[a] Section de Mathématiques, IGAT, SB, GEOM, station 8, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
[b] Mathematisches Institut der Universität, Rheinsprung 21, CH-4051 Basel, Switzerland

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ABSTRACT

We investigate systems of ordinary differential equations with a parameter. We show that under suitable assumptions on the systems the solutions are computable in the sense of recursive analysis. As an application we give a complete characterization of the recursively enumerable sets using Fourier coefficients of recursive analytic functions that are generated by differential equations and elementary operations.

1. Introduction

In this paper we investigate systems of ordinary differential equations related to the question of whether a function generated by an analog machine can exhibit non-recursive phenomena. This problem was addressed e.g. in [3,5,6,19,25–28]. In [3], the concept of an analog machine was dealt with as follows: a certain class $\mathscr{A}$ of smooth complex valued functions (of several variables) was generated by starting with simple functions—like $e^{i\lambda x}$ with $\lambda \in \mathbb{Q}$—from which new functions were obtained by elementary operations of analysis such as addition, multiplication, integration etc., and by solving polynomial ODEs. We then looked at the subset $\mathscr{A}_F \subset \mathscr{A}$ consisting of all real holomorphic 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$ that lie in $\mathscr{A}$. In other words, $\mathscr{A}_F$ is the set of all Fourier series $f(x) = \sum_{m\in\mathbb{Z}} a_m e^{imx}$, $x \in \mathbb{R}$, that can be generated by the "analog machine $\mathscr{A}$".

Any function $f \in \mathscr{A}_F$ gives rise to a set $E_f \subset \mathbb{N}$ defined in the following way:

$$n \in E_f \quad \text{iff} \quad \int_0^{2\pi} f(x)e^{-inx}dx \neq 0. \tag{1.1}$$

The main result in [3] was that given any recursively enumerable set $E \subset \mathbb{N}$, there is a function $f \in \mathscr{A}_F$ such that $E = E_f$. In the present paper we show that conversely, for any $f \in \mathscr{A}_F$ the set E_f is recursively enumerable. Hence, we have an entirely analytic characterization of the recursively enumerable sets. The precise statement of the result is given in Theorem 4.4.

Our approach will be to show that all functions generated in $\mathscr{A}$ are computable in the sense of recursive analysis and then to use that approximations of such functions can be computed by Turing machines. We are thus led to investigate singular

* Corresponding author.
E-mail addresses: peter.buser@epfl.ch (P. Buser), bscarpellini@hotmail.com (B. Scarpellini).

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polynomial ODEs from the point of view of recursive analysis. For simplicity, we restrict ourselves to real valued functions; the extension to complex functions is straightforward (see the remark following Theorem 4.4).

Let us briefly digress into discussing how recursive analysis is used in this paper. It was an important achievement of mathematical logic to make the concept of computable number theoretic function precise. The concept emerged from a series of seemingly different definitions (see e.g. [11,8,23,14]) that all turned out to be equivalent. Later, several authors applied this notion to functions $f(\zeta)$ of a real variable ζ, giving rise to the field of recursive analysis, whose aim is to study topics from classical analysis from the recursive point of view. We refer the reader to [20] for an overview; see also [19]. For our paper the basic objects are the computable numbers and functions which we define here as follows (precise definitions will be given in Section 2). Let $\zeta_q, q \in \mathbb{N}$, be a recursive enumeration of the rational numbers. A real number ζ is computable if there is a recursive function $\sigma(l), l \in \mathbb{N}$, such that

$$|\zeta - \zeta_{\sigma(l)}| \leq \tfrac{1}{l}, \quad l \geq 1. \tag{1.2}$$

Likewise, let $\psi_q(x_1, \dots, x_s), q \in \mathbb{N}$, be a recursive enumeration of the polynomials in s variables with rational coefficients. A function $f(x) = f(x_1, \dots, x_s)$, continuous on $D = [a_1, b_1] \times \cdots \times [a_s, b_s]$, is computable on D if there is a recursive function $\sigma(l), l \in \mathbb{N}$, such that

$$\sup_{x \in D} |f(x) - \psi_{\sigma(l)}(x)| \leq \tfrac{1}{l}, \quad l \geq 1. \tag{1.3}$$

These definitions are easily recognized as being equivalent to the definitions given in the literature. Our choice comes close to the Definitions 1 and 2 that were given in [19]. It is straightforward to extend them to vector valued and complex valued functions. One of the problems that arises is that of showing that the family of computable functions is closed under the typical operations encountered in analysis. For some of these, like addition, multiplication, integration, this is straightforward. Since differentiation may lead from computable to non-computable functions ([21, p. 543], [17]), however, the situation is less clear in the case of differential equations. In fact, as shown in [21], there are solutions $u(t, x_1, \dots, x_n)$ of the wave equation in $\mathbb{R}^n$ $(n \geq 2)$ for which $u(0, x_1, \dots, x_n)$ is computable but $u(1, x_1, \dots, x_n)$ is not. It may also happen that the maximal interval of a computable solution is non-computable ([9, section 6.3]). In contrast to this, positive results are available for ODEs; see e.g. [13, chapter 7], and [9].

In the present context we are given a vector function

$$f(y, \lambda) = (f_1(y, \lambda), \dots, f_n(y, \lambda)), \quad y = (y_1, \dots, y_n),$$

defined for $|y| \leq N + 1$ (for some N) and $\lambda \in [a, b]$, with computable a and b. We assume that each component f_j is computable on $\mathscr{D} = \{y \in \mathbb{R}^n \mid |y| \leq N + 1\} \times [a, b]$ via (1.3) and that each member $\psi^{n+1}_{\sigma_j(l)}, l \geq 1$ (notation as in Section 2), of the approximating sequence of f_j satisfies an l-independent Lipschitz condition, to be specified later. We are also given an $n \times n$-matrix $D(y) = \big(d_{jk}(y)\big)$, whose entries $d_{jk}(y), j, k = 1, \dots, n$, are polynomials in $y = (y_1, \dots, y_n)$ with computable coefficients. In this setting we investigate the parameter dependent system of ODEs

$$D(y)y_t = f(y, \lambda), \quad y = y(t, \lambda), \tag{1.4}$$

as regards the aspect of recursive analysis. To this end we let a solution family $y(t, \lambda), t \in [0, T], \lambda \in [a, b]$ of (1.4) be given and assume the following: (i) the vector function $y(0, \lambda), \lambda \in [a, b]$ is computable, (ii) the determinant $\det(D(y(t, \lambda)))$ is different from zero for $t \in [0, T], \lambda \in [a, b]$. Our main result then is Theorem 3.2: *if the given solution family $y(t, \lambda), t \in [0, T], \lambda \in [a, b]$, satisfies* (i), (ii), *and if $a, b, T \in \mathbb{Q}$, then $y(t, \lambda), t \in [0, T], \lambda \in [a, b]$, is computable.*

Using an approximation argument one can show that Theorem 3.2 also holds if we only assume that a, b, T are computable (Corollary 3.10).

Theorem 3.2 and its corollaries extend previously known results considerably; this holds in particular for the systems investigated in [20].

For further literature on computable functions we refer the reader to [29], where a notion of computability is used that varies from the one used here or in [19,20] in as much as it is more abstract and of greater generality. In [12], Kawamura extends and improves the concept of differential recursion introduced by Moore in [16]. He proves among other things a result [12, Thm. 3.10] asserting that differential recursion preserves abstract oracle-based computability in the sense of Weihrauch [29]. It seems feasible to take this result as a starting point for an alternative proof of our Theorem 3.2 and its corollaries. We have, however, not pursued this approach. Variants of Kawamura's Theorem 3.10 are given by Ruohonen [24, Thms 1–3] and by Collins and Graça [4, Thm. 8].

Graça in his Ph.D. thesis [9] gives a detailed discussion of the relationship between computable real functions and solutions of polynomial systems of type $y' = f(t, y)$ with computable polynomial right hand side $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$. It is a difficult open question whether the class of functions based on the solutions $y(t, \lambda)$ of our system (4.1) (underlying Theorem 4.4) is actually larger than the class based on solutions $y(t)$ of parameter free polynomial systems $y' = f(t, y)$.

In [9, section 7.2] a number of recursively unsolvable propositions associated with polynomial systems are given. Our Theorem 4.4 may be considered as an addendum to this list.

We would also like to mention a series of papers on the recursive analysis of differential equations by Weihrauch and Zhong which, while not directly related to our subject, are nevertheless in the same direction; see e.g. [30] and the references therein. We also point out [2], where various open problems related to the computability of real numbers are discussed. In the proof of Theorem 4.4, Hilbert's 10th problem is involved via Theorem 1 in [3]; for further undecidable propositions in analysis based on it, see [15, chapter 9]. We also point out [1], where ODEs are used in a different context that deals with the simulation of Turing machines; there the time variable, t, ranges over the infinite interval $[0, \infty)$ in contrast to the finite interval $[0, T]$ used here.

The paper is organized as follows. Section 2 deals with computable functions. Section 3 proves Theorem 3.2 which is our main result concerning computable solutions of ODEs with a parameter. Section 4 contains the proof of Theorem 4.4 which is our main result concerning recursively enumerable sets. In the Appendix we conclude with a lemma about the existence of Lipschitz approximations needed in the proof of Theorem 3.2.

2. Preliminaries

In this section we review a number of known properties of computable real numbers and functions. Since gathering proofs from the literature for the statements in exactly the form needed later on is somewhat laborious, we outline them for the convenience of the reader.

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, respectively, the sets of all natural, integer, rational, real and complex numbers. In order to deal with finite sequences we select (e.g. using [8, chapter 3]), for any integer $s \geq 2$, a one-to-one recursive mapping $\langle\ ,\dots,\ \rangle_s$ from $\mathbb{N}^s$ onto $\mathbb{N}$ and recursive mappings $k_1^s, \dots, k_s^s$ from $\mathbb{N}$ onto $\mathbb{N}$ in such a way that $\langle k_1^s(z), \dots, k_s^s(z)\rangle_s = z$ for all $z \in \mathbb{N}$. We complete this notation by setting $\langle z\rangle_1 = k_1^1(z) = z$. If there is no ambiguity we drop the index s, writing e.g. $\langle\langle a, b, c\rangle, \langle d, e\rangle\rangle$ instead of $\langle\langle a, b, c\rangle_3, \langle d, e\rangle_2\rangle_2$, etc.

Throughout the paper we use the following recursive (but not one-to-one) enumeration of $\mathbb{Q}$:

$$\zeta_q \stackrel{\text{def}}{=} (k_1^3(q) - k_2^3(q))/(1 + k_3^3(q)), \quad q \in \mathbb{N} \tag{2.1}$$

([11, p. 236]). We extend (2.1) to vectors $\zeta \in \mathbb{Q}^s$ by stipulating

$$\zeta_q^s \stackrel{\text{def}}{=} (\zeta_{a_1}, \dots, \zeta_{a_s}), \quad \text{where } a_j = k_j^s(q). \tag{2.2}$$

For $a \in \mathbb{N}$ we use Kleene's symbol $(a)_j$ to denote the exponent of the j-th prime in the prime factorization of a (i.e. $(24)_0 = 3$, etc.). Finally, for $x = (x_1, \dots, x_s) \in \mathbb{R}^s$ we set $|x| = |x_1| + \cdots + |x_s|$.

Definition 2.1. $\eta \in \mathbb{R}^s$ is *computable* if there exists a recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ and a constant $c \in \mathbb{N}$ such that

$$|\eta - \zeta_{\sigma(l)}^s| \leq \frac{c}{l}, \quad l \geq 1. \tag{2.3}$$

Remarks. (i) On taking $\sigma'(l) = \sigma(c \cdot l)$, $l \in \mathbb{N}$, as a new recursive function it follows that in (2.3) we may take $c = 1$. (ii) It is easily seen that $\eta = (\eta_1, \dots, \eta_s)$ is computable iff all components η_j, $j \leq s$, are computable.

In order to define the computability of functions in a similar way we need a recursive enumeration of all polynomials $P(x_1, \dots, x_s)$ with rational coefficients. Among the various possibilities for doing so the following is convenient. For given s, any $r \in \mathbb{N}$ has the unique representation $r = \langle\langle a, b, c\rangle, \langle n_1, \dots, n_s\rangle\rangle$, giving rise to the monomial

$$m_r(x_1, \dots, x_s) = (a - b)(1 + c)^{-1} x_1^{n_1} \cdots x_s^{n_s}$$

(the fact that this enumeration is not one-to-one does not matter). Now let $q \in \mathbb{N}$, $q \geq 2$. Then q has the unique prime factorization $q = p_0^{a_0} \cdots p_N^{a_N}$, with $a_N > 0$ and $p_0, \dots, p_N$ the prime numbers from $p_0 = 2$ to p_N, listed in increasing order. This allows us to define the polynomial

$$\psi_q^s(x_1, \dots, x_s) \stackrel{\text{def}}{=} \sum_{j=0}^{N} m_{r_j}(x_1, \dots, x_s), \quad \text{where } r_j = (q)_j. \tag{2.4}$$

For completeness we also set $\psi_0^s = \psi_1^s = 0$. An enumeration of the polynomial functions with rational coefficients and values in $\mathbb{R}^n$ is then given by

$$\psi_q^{s,n} \stackrel{\text{def}}{=} (\psi_{q_1}^s, \dots, \psi_{q_n}^s), \quad \text{where; } q_j = k_j^n(q), j = 1, \dots, n. \tag{2.5}$$

Note that by our convention, $\psi_q^{s,1} = \psi_q^s$.

Definition 2.2. Let $U \subset \mathbb{R}^s$ be a bounded subset. A function $f : U \to \mathbb{R}^n$ is *computable* on U if there exists $c \in \mathbb{N}$ and a recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sup_{x \in U} |f(x) - \psi_{\sigma(l)}^{s,n}(x)| \leq \frac{c}{l}, \quad \text{for all } l \geq 1. \tag{2.6}$$

Remarks. (i) Occasionally, we use the notation $|g|_D = \sup_{z \in D} |g(z)|$, for functions $g : D \to \mathbb{R}^n$, so (2.6) assumes the form

$$|f - \psi_{\sigma(l)}^{s,n}|_U \leq \frac{c}{l}, \quad \text{for all } l \geq 1.$$

(ii) The remarks subsequent to Definition 2.1 apply also to Definition 2.2. (iii) By our conventions, Definitions 2.1 and 2.2 include the scalar cases (2.1) and (2.4). (iv) Definitions 2.1 and 2.2 differ from the corresponding ones in [19] but are easily seen to be equivalent. (v) It would be straightforward to extend our considerations to complex valued functions. However, a splitting into real and imaginary parts reduces the complex case to the real case. Thus, without loss of generality, we restrict ourselves to the real domain (see footnote (5), p. 5 in [19]).

The next two lemmas allow us to pass from Theorem 3.2 to its corollaries.

Lemma 2.3. *Let $a < b$ be computable real numbers, $K \in \mathbb{N}$, and set $\mathscr{K} = [-K, K]^n$. Then there exist recursive functions $\Pi : \mathbb{N}^2 \to \mathbb{N}$ and $\Gamma : \mathbb{N} \to \mathbb{N}$ such that the following hold for $y, z \in \mathscr{K}$, $t \in [0, 1]$ and integers $p \geq 1$, $q \geq 0$:*

$$\left|\psi_q^{n+1}(y, (b-a)t + a) - \psi_{\Pi(q,p)}^{n+1}(y, t)\right| \leq \tfrac{\Gamma(q)}{p}, \tag{a}$$

$$\left|\psi_{\Pi(q,p)}^{n+1}(y, t) - \psi_{\Pi(q,p)}^{n+1}(z, t)\right| \leq \left\{\tfrac{\Gamma(q)}{p} + \sup\left|\psi_q^{n+1}(\zeta, \lambda) - \psi_q^{n+1}(\eta, \lambda)\right| / |\zeta - \eta|\right\} |y - z|, \tag{b}$$

where the 'sup' ranges over all $\zeta, \eta \in \mathscr{K}$, $\zeta \neq \eta$, and all $\lambda \in [a, b]$.

Proof. We restrict ourselves to giving a sketch. With $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. The polynomial ψ_q^{n+1} may then be written in the following form:

$$\psi_q^{n+1}(y, \lambda) = \sum_{\alpha,k} b_{\alpha,k} y^\alpha \lambda^k, \quad \text{with } b_{\alpha,k} = b_{\alpha_1,\ldots,\alpha_n,k} \in \mathbb{Q}. \tag{2.7}$$

By our assumption and Definition 2.1, there are recursive functions $\sigma_a, \sigma_b : \mathbb{N} \to \mathbb{N}$ such that

$$|a - a_l| \leq \tfrac{1}{l}, \qquad |b - b_l| \leq \tfrac{1}{l}, \quad \text{for } a_l := \zeta_{\sigma_a(l)}, b_l := \zeta_{\sigma_b(l)}. \tag{2.8}$$

On the basis of (2.4), (2.7), (2.8), it is then easily seen that there exists a recursive function $\Pi : \mathbb{N}^2 \to \mathbb{N}$ such that

$$\sum_{\alpha,k} b_{\alpha,k} y^\alpha ((b_p - a_p)t + a_p)^k = \psi_{\Pi(q,p)}^{n+1}(y, t) \tag{2.9}$$

(with notation as in (2.8)). From (2.8) and (2.9) one now infers that there exists a recursive function Γ such that (a) holds. Clause (b) is obtained via a similar analysis with the help of the following identity in which we use the abbreviations $\gamma_p = b_p - a_p$, $\gamma = b - a$:

$$\begin{aligned}&\sum_{\alpha,k} b_{\alpha,k} y^\alpha (\gamma_p t + a_p)^k - \sum_{\alpha,k} b_{\alpha,k} z^\alpha (\gamma_p t + a_p)^k \\ &\quad = \sum_{\alpha,k} b_{\alpha,k}(y^\alpha - z^\alpha)\left((\gamma_p t + a_p)^k - (\gamma t + a)^k\right) + \sum_{\alpha,k} b_{\alpha,k}(y^\alpha - z^\alpha)(\gamma t + a)^k.\end{aligned}$$

(By taking it large enough we may use the same $\Gamma(q)$ in (a) and (b).) □

By passing to components we immediately get a version for vector valued polynomials.

Corollary 2.4. *Lemma 2.3 also holds, for any m, with $\psi_q^{n+1,m}$ in place of ψ_q^{n+1}.*

The second auxiliary lemma is:

Lemma 2.5. *Let $f = f(x_1, \ldots, x_s)$ be computable on $\prod_1^s [a_j, b_j]$, let $\gamma_j, \alpha_j, j \leq s$, be computable and assume that $\gamma_j t + \alpha_j \in [a_j, b_j]$, for $t \in [A_j, B_j]$, $j \leq s$. Then $f(\gamma_1 t_1 + \alpha_1, \ldots, \gamma_s t_s + \alpha_s)$ is computable on $\prod_1^s [A_j, B_j]$.*

Lemma 2.5 asserts that the computability of a function is preserved if its arguments are subject to linear transformations with computable coefficients. The proof proceeds by standard approximation arguments and is omitted. By a passage to components one obtains a vector version of the lemma:

Corollary 2.6. *Lemma 2.5 also holds for $f : \prod_1^s [a_j, b_j] \to \mathbb{R}^n$.*

Our next question is whether a continuous function that is computable on adjacent intervals $[a, b]$ and $[b, c]$ is also computable on $[a, c]$. Lemma 2.7 will provide an affirmative answer for $a, b, c \in \mathbb{Q}$. For the proof we shall use the first-order theory $\mathscr{R}_c$ of real closed fields based on the first-order predicate calculus, the predicate symbols $<, =$, the function symbols $+, \cdot, -$, and (e.g.) the rationals, $\mathbb{Q}$, as the set of constants. For a closed formula G of $\mathscr{R}_c$ we write $\mathbb{R} \models G$ if G is true under its standard interpretation on $\mathbb{R}$. A classical result (e.g. [22, theorem 4.7, p. 212]) states

$$\text{the predicate } \mathbb{R} \models G \text{ is decidable.} \tag{2.10}$$

Therefore, if $[G]$ denotes the Gödel number of G (in any standard setting) then we have

$$\text{the set } E = \{[G] \mid \mathbb{R} \models G\} \text{ is recursive.} \tag{2.11}$$

The next lemma is familiar, but in view of the fact that some of the arguments will be needed in a decisive place in the Appendix, we shall provide a more detailed proof.

Lemma 2.7. *Let $f(x, \lambda)$ be continuous on $[a, c] \times [A, B]$ and computable on each of the parts $[a, b] \times [A, B]$, $[b, c] \times [A, B]$, for some $b \in (a, c)$. If $a, b, c, A, B \in \mathbb{Q}$, then $f(x, \lambda)$ is computable on $[a, c] \times [A, B]$.*

Proof. We set $J_1 = [a, b], J_2 = [b, c], J = [a, c], I = [A, B]$. By Definition 2.2 and our assumptions, there exist recursive functions σ, μ such that

$$|f - \psi^2_{\sigma(l)}|_{J_1\times I} \le \tfrac{1}{l}, \qquad |f - \psi^2_{\mu(l)}|_{J_2\times I} \le \tfrac{1}{l}, \tag{2.12}$$

where we use the shorthand $|g|_D = \sup_{z\in D} |g(z)|$ (Remark (i) subsequent to Definition 2.2). For convenience we set

$$\varphi_l = \psi^2_{\sigma(l)}, \qquad \phi_l = \psi^2_{\mu(l)}$$

and define $\Gamma_l, \varphi'_l, \phi'_l$ via

$$\Gamma_l(\lambda) = \tfrac{1}{2}(\phi_l(b, \lambda) - \varphi_l(b, \lambda)),$$
$$\varphi'_l(x, \lambda) = \varphi_l(x, \lambda) + \Gamma_l(\lambda), \qquad \phi'_l(x, \lambda) = \phi_l(x, \lambda) - \Gamma_l(\lambda).$$

(Γ_l is not related to the earlier function $\Gamma(p)$.) We then have

$$\varphi'_l(b, \lambda) = \phi'_l(b, \lambda)) = \tfrac{1}{2}(\varphi_l(b, \lambda) + \phi_l(b, \lambda)). \tag{2.13}$$

On $J \times I$ we now define a piecewise polynomial function $\Pi_l(x, \lambda)$ by stipulating

$$\begin{aligned} \Pi_l(x, \lambda) &= \varphi'_l(x, \lambda), \quad \text{for } x \in J_1, \lambda \in I;\\ \Pi_l(x, \lambda) &= \phi'_l(x, \lambda), \quad \text{for } x \in J_2, \lambda \in I. \end{aligned} \tag{2.14}$$

By (2.13), Π_l is continuous on $J \times I$. For $\Gamma_l(\lambda)$ we obtain the estimate

$$\begin{aligned} |\Gamma_l(\lambda)| &\le \tfrac{1}{2}|\varphi_l(b, \lambda) - f(b, \lambda)| + \tfrac{1}{2}|\phi_l(b, \lambda) - f(b, \lambda)|\\ &\le \tfrac{1}{2}|\varphi_l - f|_{J_1\times I} + \tfrac{1}{2}|\phi_l - f|_{J_2\times I} \le \tfrac{1}{l}\,. \end{aligned}$$

This entails

$$|f - \varphi'_l|_{J_1\times I} \le |f - \varphi_l|_{J_1\times I} + |\Gamma_l|_I \le \tfrac{2}{l}$$

and likewise

$$|f - \phi'_l|_{J_2\times I} \le |f - \phi_l|_{J_2\times I} + |\Gamma_l|_I \le \tfrac{2}{l}.$$

Hence,

$$\begin{aligned} |f - \Pi_l|_{J\times I} &= \max(|f - \Pi_l|_{J_1\times I}, |f - \Pi_l|_{J_2\times I})\\ &= \max(|f - \varphi'_l|_{J_1\times I}, |f - \phi'_l|_{J_2\times I}) \le \tfrac{2}{l}. \end{aligned} \tag{2.15}$$

We now define a predicate $P \subset \mathbb{N}^3$ via

$$P(l, q, p) \overset{\text{def}}{\Longleftrightarrow} \{|\Pi_l - \psi^2_q|_{J\times I} \le \tfrac{1}{p} \text{ and } l \ge 1\} \text{ or } l = 0. \tag{2.16}$$

Since $a, b, c, A, B \in \mathbb{Q}$, the right hand side of (2.16) may be expressed as a closed formula $G(l, q, p)$ for the language $\mathscr{R}_c$, and its Gödel number $[G(l, q, p)]$ is a recursive function of l, q, p. Recalling (2.11) we infer that $P(l, q, p)$ holds iff $[G(l, q, p)] \in E$ (see (2.11)). Since E is recursive this implies

$$\text{the predicate } P \text{ is recursive.} \tag{2.17}$$

Since Π_l is continuous on $J \times I$ it follows from the Weierstrass approximation theorem that for given l, p there is q such that $P(l, q, p)$ holds, i.e.,

$$(\forall\, l, p)(\exists q)P(l, q, p). \tag{2.18}$$

Thus, there is a recursive function $\nu(l, p)$ such that

$$(\forall p, l)P(l, \nu(l, p), p). \tag{2.19}$$

Setting $p = l$ and recalling (2.16) we get

$$|\Pi_l - \psi^2_{\nu(l,l)}|_{J\times I} \le \tfrac{1}{l}, \quad l \ge 1. \tag{2.20}$$

Since, by (2.15) and (2.20),

$$|f - \psi^2_{\nu(l,l)}|_{J\times I} \le |f - \Pi_l|_{J\times I} + |\Pi_l - \psi^2_{\nu(l,l)}|_{J\times I} \le \tfrac{3}{l}, \quad l \ge 1,$$

f is computable on $J \times I$. □

Remark. The resorting to (2.10), (2.11) may look somewhat surprising. In fact, an explicit construction of approximating polynomials can be carried out using classical approximation theory (see [18] for a reference). However, this turns out to be rather involved and the use of (2.10), (2.11) is much more practical.

The following generalization of Lemma 2.7 is clear.

Corollary 2.8. *Let $a_0, \ldots, a_N, A, B \in \mathbb{Q}$, $a_0 < a_1 < \cdots < a_N$, and let $f = (f_1, \ldots f_n) : [a_0, a_N] \times [A, B] \to \mathbb{R}^n$ be a continuous function. If f is computable on $[a_j, a_{j+1}] \times [A, B]$, for $j = 0, \ldots, N - 1$, then f is computable on $[a_0, a_N] \times [A, B]$.*

3. Computable solutions of ODEs

We first specify the setting for Theorem 3.2. Let $N \in \mathbb{N}$ and $a \leq b$; set $\mathscr{K} = [-N, N]^n$ and $\mathscr{D} = \mathscr{K} \times [a, b]$. On $\mathscr{D}$ we are given a vector function

$$f(y, \lambda) = (u_1(y, \lambda), \dots, u_n(y, \lambda), \quad y = (y_1, \dots, y_n),$$

subject to the following two assumptions: (**A**): there is a recursive function $\sigma : \mathbb{N} \to \mathbb{N}$ such that for the polynomials $f_l(y, \lambda) := \psi^{n+1,n}_{\sigma(l)}(y, \lambda)$ we have

$$\sup_{(y,\lambda)\in\mathscr{D}} |f(y, \lambda) - f_l(y, \lambda)| \leq \tfrac{1}{l}, \quad l \geq 1; \tag{3.1}$$

(**B**): there exists a constant $C > 0$ such that

$$|f_l(y_2, \lambda) - f_l(y_1, \lambda)| \leq C|y_2 - y_1|, \quad (y_j, \lambda) \in \mathscr{D},\ l \geq 1. \tag{3.2}$$

By (3.1), (3.2) we have (letting $l \to \infty$)

$$|f(y_2, \lambda) - f(y_1, \lambda)| \leq C|y_2 - y_1|, \quad (y_j, \lambda) \in \mathscr{D}. \tag{3.3}$$

Moreover, there exists a constant M such that

$$|f(y, \lambda)|, |f_l(y, \lambda)| \leq M, \quad (y, \lambda) \in \mathscr{D},\ l \geq 1. \tag{3.4}$$

We are also given an $n \times n$-matrix $D(y) = \big(d_{jk}(y)\big)$ whose entries $d_{jk}(y)$, $j, k \leq n$, are polynomials in $y_1, \dots, y_n$ with computable coefficients. The system of ODEs to be investigated is

$$D(y)y_t = f(y, \lambda), \quad \lambda \in [a, b], \tag{3.5}$$

where y is now a function of a real variable t with parameter $\lambda \in [a, b]$ and y_t is the derivative of y with respect to t.

Definition 3.1. A continuous mapping $y : [0, T] \times [a, b] \to \mathbb{R}^n$ is an *admissible family* of solutions of (3.5) if:

(a) there exists $d > 0$ such that $|y(t, \lambda)| \leq N - d$ for $t \in [0, T]$, $\lambda \in [a, b]$;
(b) $y(\cdot, \lambda) \in C^1([0, T])$, and $y(\cdot, \lambda)$ satisfies Eq. (3.5) pointwise on the interval $[0, T]$ for $\lambda \in [a, b]$;
(c) $\det D(y(t, \lambda)) \neq 0$ for $t \in [0, T]$, $\lambda \in [a, b]$;
(d) the function $y(0, \lambda)$, $\lambda \in [a, b]$, is computable.

Theorem 3.2. *Assume $a, b, T \in \mathbb{Q}$; let $y(t, \lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$, be an admissible family of solutions of* (3.5)*. Then y is computable on $[0, T] \times [a, b]$.*

The proof goes through several preparatory steps with the final approach in Step 5. The strategy is to approximate $y(t, \lambda)$ with solutions $y_l(t, \lambda)$ of differential equations that are based on f_l, where the computability will be visible via an iteration process.

Step 1. We abbreviate $P(y) = \det(D(y))$. By our assumptions, P is a polynomial in $y \in \mathbb{R}^n$ with computable coefficients, and by Definition 3.1(c), there is $\mu > 0$ such that

$$|P(y(t, \lambda))| \geq \mu, \quad t \in [0, T],\ \lambda \in [a, b]. \tag{3.6}$$

On the set $\mathscr{M} \subset \mathbb{R}^n$ of all y with $P(y) \neq 0$, the matrix $D(y)$ has an inverse which, by Cramer's rule, has the form

$$D(y)^{-1} = P(y)^{-1}G(y), \quad G(y) = \big(g_{ik}(y)\big),\ i, k \leq n, \tag{3.7}$$

where the g_{ik} are polynomials in y with computable coefficients. Thus, on $\mathscr{M}$ the system (3.5) is equivalent to

$$y_t = P(y)^{-1}F(y, \lambda), \quad \text{with } F(y, \lambda) := G(y)f(y, \lambda) \text{ for } (y, \lambda) \in \mathscr{D}. \tag{3.8}$$

On the basis of (3.1), (3.2) and the structure of $G(y)$ one easily checks that F in (3.8) has properties analogous to those of f, i.e. there is a recursive function $\alpha : \mathbb{N} \to \mathbb{N}$ and a constant $C_F > 0$ such that for the polynomials $F_l(y, \lambda) := \psi^{n+1,n}_{\alpha(l)}(y, \lambda)$ we have

$$\sup_{(y,\lambda)\in\mathscr{D}} |F(y, \lambda) - F_l(y, \lambda)| \leq \tfrac{1}{l}, \quad l \geq 1, \tag{3.9}$$

$$|F_l(y_2, \lambda) - F_l(y_1, \lambda)| \leq C_F|y_2 - y_1|, \quad (y_j, \lambda) \in \mathscr{D},\ l \geq 1. \tag{3.10}$$

Furthermore, letting $l \to \infty$, we also see that

$$|F(y_2, \lambda) - F(y_1, \lambda)| \leq C_F|y_2 - y_1|, \quad (y_j, \lambda) \in \mathscr{D}. \tag{3.11}$$

Step 2. For $P(y)^{-1}F(y, \lambda)$, only local Lipschitz constants are available, i.e. only for neighborhoods of points in $\mathscr{M}$. We will give such constants along the trajectory of $y(t, \lambda)$. Since P is a polynomial, there is a constant $c = c(N)$ such that

$$|P(\zeta) - P(\eta)| \leq \frac{1}{2}c|\zeta - \eta|, \quad \zeta, \eta \in [-(N + 1), N + 1]^n. \tag{3.12}$$

We take c so large that, in addition,

$$\mu \le c, \qquad \frac{\mu}{c} \le d \tag{3.13}$$

(see (3.6), Definition 3.1(a)). Next, we pick $\sigma \in [0, T]$, set $y(\sigma, \lambda) = \eta(\lambda)$, $\lambda \in [a, b]$, and consider the following neighborhoods:

$$\mathscr{U}_\lambda = \left\{ \xi \in \mathbb{R}^n \mid |\xi - \eta(\lambda)| \le \frac{\mu}{2c} \right\}, \quad \lambda \in [a, b] \tag{3.14}$$

(where the dependence on σ has been suppressed). In this setting we have:

Proposition 3.3. *Let $\xi, \zeta \in \mathscr{U}_\lambda$. Then*

(a) $|\xi| \le N - \frac{d}{2}$,
(b) $|P(\xi)| \ge \frac{\mu}{2}$,
(c) $|P(\xi)^{-1} - P(\zeta)^{-1}| \le \frac{4c}{\mu^2}|\xi - \zeta|$.

Proof. The first inequality comes from (3.13), (3.14) and Definition 3.1(a): $|\xi| \le |\eta(\lambda) - \xi| + |\eta(\lambda)| \le \frac{\mu}{2c} + N - d \le N - \frac{d}{2}$. For the second inequality we use (3.6):

$$|P(\xi)| \ge |P(\eta(\lambda))| - |P(\xi) - P(\eta(\lambda))| \ge \mu - c|\eta(\lambda) - \xi| \ge \frac{\mu}{2}.$$

Finally, by (3.12),

$$|P(\xi)^{-1} - P(\zeta)^{-1}| \le |P(\xi)P(\zeta)|^{-1}|P(\zeta) - P(\xi)| \le \frac{4c}{\mu^2}|\xi - \zeta|. \quad \square$$

Remark. The constants in (a)–(c) do not depend on σ and λ.

For the following we use the abbreviation

$$H(y, \lambda) = \frac{1}{P(y)} F(y, \lambda), \quad (y, \lambda) \in \mathscr{D},\ P(y) \ne 0, \tag{3.15}$$

with $F(y, \lambda)$ as in (3.8).

Proposition 3.4. *There are constants C_1, M_1 such that for all $\xi, \zeta \in \mathscr{U}_\lambda$ and $\lambda \in [a, b]$ we have:*

(a) $|H(\xi, \lambda)| \le M_1$,
(b) $|H(\xi, \lambda) - H(\zeta, \lambda)| \le C_1|\xi - \zeta|$.

Proof. By (3.4), there is M_0 such that $|F(\xi, \lambda)| \le M_0$ for $(\xi, \lambda) \in \mathscr{D}$. Combined with Proposition 3.3(b), this yields (a). By Proposition 3.3, (3.11) and (a) we have

$$\begin{aligned} |H(\xi, \lambda) - H(\zeta, \lambda)| &\le \left| \left(\frac{1}{P(\xi)} - \frac{1}{P(\zeta)} \right) F(\xi, \lambda) \right| + \frac{1}{|P(\zeta)|} |F(\xi, \lambda) - F(\zeta, \lambda)| \\ &\le \frac{4}{\mu^2} c|\xi - \zeta| M_0 + \frac{2}{\mu} C_F |\xi - \zeta|, \end{aligned}$$

which yields (b). $\square$

Corollary 3.5. *Let $\tau \ge 0$ be such that $M_1\tau \le \frac{\mu}{2c}$ and $\tau C_1 \le \frac{1}{2}$. Then the solution $z(t, \lambda)$ of*

$$z_t = H(z, \lambda), \qquad z(\sigma, \lambda) = \eta(\lambda), \quad \lambda \in [a, b], \tag{3.16}$$

exists on $[\sigma, \sigma + \tau] \times [a, b]$ and satisfies $z(t, \lambda) \in \mathscr{U}_\lambda$ for $(t, \lambda) \in [\sigma, \sigma + \tau] \times [a, b]$.

Remarks. The proof is omitted here since it is based on a straightforward analysis of the integral equation

$$z(t, \lambda) = \eta(\lambda) + \int_\sigma^t H(z(s, \lambda), \lambda) ds, \quad t \in [\sigma, \sigma + \tau], \tag{3.17}$$

in terms of well known iteration arguments [7,10]. Such arguments will be met again below in similar situations. By the uniqueness of the solution in (3.17) and since $z(\sigma, \lambda) = \eta(\lambda) = y(\sigma, \lambda)$, the above solution $z(t, \lambda)$ coincides with our given solution family, i.e. $z(t, \lambda) = y(t, \lambda)$ for $(t, \lambda) \in [\sigma, \sigma + \tau] \times [a, b]$.

Step 3. In this step we prove a local version of Theorem 3.2. Hence, we now assume that our point $\sigma \in [0, T]$ satisfies the following two conditions: (**C1**): $\sigma \in \mathbb{Q}$, (**C2**): the vector function $\eta(\lambda) = y(\sigma, \lambda)$ as a function of $\lambda \in [a, b]$ is computable, i.e. there is a recursive function β such that the function $\eta_l(\lambda) := \psi^{1,n}_{\beta(l)}(\lambda)$ satisfies

$$|\eta(\lambda) - \eta_l(\lambda)| \le \tfrac{1}{l}, \quad \lambda \in [a, b],\ l \ge 1. \tag{3.18}$$

We may choose β such that, in addition,

$$|\eta(\lambda) - \eta_l(\lambda)| \le \tfrac{\mu}{4c}, \quad \lambda \in [a, b],\ l \ge 1. \tag{3.19}$$

The proof of the next lemma is rather technical and will be given in the Appendix.

Lemma 3.6. *There exists a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ and constants C_2, C_2', M_2 depending only on $P(\ )$, μ, c, such that the polynomials $P_l(y, \lambda) := \psi^{n+1}_{\gamma(l)}(y, \lambda)$ satisfy the following for $y, \xi, \zeta \in \mathscr{U}_\lambda$, $\lambda \in [a, b]$, $l \ge 1$:*

(a) $|P(y)^{-1} - P_l(y, \lambda)| \le \frac{C_2}{l}$,
(b) $|P_l(y, \lambda)| \le M_2$,
(c) $|P_l(\xi, \lambda) - P_l(\zeta, \lambda)| \le C_2'|\xi - \zeta|$.

A local polynomial approximation to $H(y, \lambda)$ in (3.15) is now provided by

$$H_l(y, \lambda) = P_l(y, \lambda)F_l(y, \lambda). \tag{3.20}$$

Lemma 3.7. *There exist constants C_3, C_3', M_3, such that for all $y, \xi, \zeta \in \mathscr{U}_\lambda$ and $\lambda \in [a, b]$,*

(a) $|H_l(y, \lambda)| \le M_3$,
(b) $|H_l(\xi, \lambda) - H_l(\zeta, \lambda)| \le C_3'|\xi - \zeta|$,
(c) $|H(y, \lambda) - H_l(y, \lambda)| \le \frac{C_3}{l}$.

Proof. By (3.9), (3.10), there is m_0 such that $|F(y, \lambda)|, |F_l(y, \lambda)| \le m_0$, for $(y, \lambda) \in \mathscr{D}$, $l \ge 1$. Combined with Lemma 3.6(b), this proves (a). For (b) we use (3.10) and Lemma 3.6(b), (c):

$$|H_l(\xi, \lambda) - H_l(\zeta, \lambda)| \le |P_l(\xi, \lambda) - P_l(\zeta, \lambda)||F_l(\xi, \lambda)| + |P_l(\zeta, \lambda)||F_l(\xi, \lambda) - F_l(\zeta, \lambda)| \le (C_2'm_0 + M_2C_F)|\xi - \zeta|.$$

For (c) we use (3.9) and Lemma 3.6(a), (b):

$$|H(y, \lambda) - H_l(y, \lambda)| \le |P(y)^{-1} - P_l(y, \lambda)||F(y, \lambda)| + |P_l(y, \lambda)||F(y, \lambda) - F_l(y, \lambda)| \le \frac{1}{l}(C_2m_0 + M_2). \quad \Box$$

Step 4. Next we establish a connection between our considerations and computability as discussed in Section 2. The strategy is as follows. With the notation of (3.8) and (3.15), the family $y(t, \lambda)$ in Theorem 3.2 is a solution of the equation $y_t = H(y, \lambda)$. Now we first look at H_l instead of H and prove the computability of the solutions of $(y_l)_t = H_l(y_l, \lambda)$, postponing the original question to Step 5.

To this end we invoke a standard iteration process in which iterates y_{lm}, $m = 0, 1, 2, \ldots$, are defined via

$$y_{l,m+1}(t, \lambda) = \eta_l(\lambda) + \int_\sigma^t H_l(y_{lm}(s, \lambda), \lambda)\, ds, \quad y_{l0}(t, \lambda) = \eta_l(\lambda), \tag{3.21}$$

where H_l, η_l are given by (3.20), (3.18). We associate with (3.21) the mapping $z \mapsto \tilde{z}$ given by

$$\tilde{z}(t, \lambda) = \eta_l(\lambda) + \int_\sigma^t H_l(z(s, \lambda), \lambda)\, ds, \tag{3.22}$$

where $z(t, \lambda)$ ranges over the set $\{\psi^{2,n}_q \mid q \in \mathbb{N}\}$ given by (2.5). We now re-expand the abbreviations involved in the presentation of (3.22), i.e. recalling the definitions in (3.9), (3.18), (3.20) and Lemma 3.6 we rewrite (3.22) more explicitly:

$$\tilde{z}(t, \lambda) = \psi^{1,n}_{\beta(l)}(\lambda) + \int_\sigma^t \psi^{n+1}_{\gamma(l)}(z(s, \lambda), \lambda)\, \psi^{n+1,n}_{\alpha(l)}(z(s, \lambda), \lambda)\, ds,$$
$$\text{with } z(t, \lambda) = \psi^{2,n}_q(t, \lambda) \text{ for some } q \in \mathbb{N}. \tag{3.23}$$

Since $\sigma \in \mathbb{Q}$ (condition (**C1**) at the beginning of Step 3), $\tilde{z}(t, \lambda)$ is again a polynomial vector function in t and λ with coefficients in $\mathbb{Q}$ and therefore of the form $\psi^{2,n}_p$, for some p. More precisely we have:

Lemma 3.8. *Given $\sigma \in \mathbb{Q}$, there exists a recursive function $\Pi = \Pi_\sigma : \mathbb{N}^2 \to \mathbb{N}$, with the following property. If $z = \psi^{2,n}_q$ for some $q \in \mathbb{N}$, then the image $\tilde{z}$ of z, defined via (3.23), is given by $\tilde{z} = \psi^{2,n}_{\Pi(l,q)}$.*

Proof. We restrict ourselves to an intuitive argument. Our stipulations in (2.4), (2.5) are such that the expressions for $\psi^{n+1}_{\gamma(l)}(y,\lambda)$, $\psi^{2,n}_{q}(y,\lambda)$, $\psi^{n+1,n}_{\alpha(l)}(y,\lambda)$, $\psi^{1,n}_{\beta(l)}(y,\lambda)$ and, hence, the expression for the integrand in (3.23) may be written out by a Turing machine as a function of l and q. As integration of polynomials is carried out by algebraic operations and the initial point σ is rational, there is also a Turing machine that writes down the expression for the polynomial $\tilde{z}$ in (3.23) as a function of l and q. It is then possible to scan through $\psi^{2,n}_0, \psi^{2,n}_1, \ldots$, until one gets $\tilde{z} = \psi^{2,n}_p$, and hence p as a recursive function of l and q. □

Lemma 3.9. *There is a recursive function $\Sigma : \mathbb{N}^2 \to \mathbb{N}$ such that the iterates y_{lm} in (3.21) are given by*

$$y_{lm}(t,\lambda) = \psi^{2,n}_{\Sigma(l,m)}(t,\lambda). \tag{3.24}$$

Proof. We recall that a polynomial vector function $\mathscr{P}(\lambda)$ with values in $\mathbb{R}^n$ and rational coefficients has the representation $\mathscr{P}(\lambda) = \psi^{1,n}_q(\lambda)$ for suitable $q \in \mathbb{N}$. It is also represented in the form $\mathscr{P}(\lambda) = \psi^{2,n}_p$, for a certain $p \in \mathbb{N}$ with t not occurring in $\psi^{2,n}_p$. On the basis of the encoding leading to (2.4), (2.5) one easily shows that there is a recursive function $\delta : \mathbb{N} \to \mathbb{N}$ such that

$$\psi^{1,n}_q(\lambda) = \psi^{2,n}_{\delta(q)}(\lambda). \tag{3.25}$$

We now define the function Σ recursively by stipulating

$$\Sigma(l,0) = \delta(\beta(l)), \quad \Sigma(l,m+1) = \Pi(l,\Sigma(l,m)),\ m \geq 0, \tag{3.26}$$

and show by induction that (**D**): $\psi^{2,n}_{\Sigma(l,m)} = y_{l,m}$. Indeed, if $m = 0$, then

$$\psi^{2,n}_{\Sigma(l,0)}(t,\lambda) = \psi^{2,n}_{\delta(\beta(l))} = \psi^{1,n}_{\beta(l)}(\lambda) = y_{m0}(t,\lambda).$$

For the step from m to $m+1$ we observe that

$$\psi^{2,n}_{\Sigma(l,m+1)}(t,\lambda) = \psi^{2,n}_{\Pi(l,\Sigma(l,m))}(t,\lambda),$$

by (3.26). By the induction hypothesis we have

$$\psi^{2,n}_{\Sigma(l,m)}(t,\lambda) = y_{lm}(t,\lambda).$$

Since $y_{l,m+1}$ is the image of y_{lm} via (3.23) we conclude, by combining this with Lemma 3.8 and (3.26),

$$y_{l,m+1}(t,\lambda) = \psi^{2,n}_{\Pi(l,\Sigma(l,m))}(t,\lambda) = \psi^{2,n}_{\Sigma(l,m+1)}(t,\lambda).$$

This proves (**D**) and, hence, the lemma. □

Step 5: Proof of Theorem 3.2. For simplicity, we replace the constants C_1, M_1 in Proposition 3.4 and C_3, C_3', M_3 in Lemma 3.7 by

$$M_4 = \max(M_1, M_3), \qquad C_4 = \max(C_1, C_3, C_3'). \tag{3.27}$$

In order to track the function $y(t,\lambda)$ we subdivide the interval $[0,T]$ using division points:

$$\tau_k = \frac{k}{K}T, \quad k = 0,\ldots,K, \quad \tau = \tau_1 = \frac{1}{K}T,$$

where K is taken large enough that

$$\tau M_4 \leq \frac{\mu}{4c}, \qquad \tau C_4 \leq \frac{1}{2}, \quad \tau \leq 1, \tag{3.28}$$

with μ and c as in (3.13). Point $\sigma \in [0,T]$ introduced in Step 2 is now set to be $\sigma = \tau_k, k < K$. The functions $\eta(\lambda) = y(\sigma,\lambda)$, $\eta_l(\lambda)$, $\lambda \in [a,b]$, and the neighborhoods $\mathscr{U}_\lambda$, etc. are then the same as before, on the basis of this choice of σ.

Our first goal is to find the solutions of the equation

$$y_l(t,\lambda) = \eta_l(\lambda) + \int_\sigma^t H_l(y_l(s,\lambda),\lambda)\,\mathrm{d}s, \quad t \in [\sigma, \sigma+\tau]. \tag{3.29}$$

To this end we have introduced the iterates y_{lm} in (3.21) and shown in Lemma 3.9 that they are effectively describable polynomials. We now turn to their analytic properties. We first claim that

$$y_{lm}(t,\lambda) \in \mathscr{U}_\lambda, \quad \text{for } t \in [\sigma,\sigma+\tau], \lambda \in [a,b], m \geq 0. \tag{3.30}$$

We proceed by induction. For $m = 0$, i.e. for $y_{l0} = \eta_l$, the claim follows from (3.14) and (3.19). For the step from m to $m+1$ we infer the following from (3.21), using (3.19), Lemma 3.7 and the induction hypothesis:

$$|y_{l,m+1}(t,\lambda) - \eta(\lambda)| \leq |\eta(\lambda) - \eta_l(\lambda)| + \int_\sigma^{\sigma+\tau} |H_l(y_{lm}(s,\lambda),\lambda)|\,\mathrm{d}s \leq \frac{\mu}{2c}.$$

This concludes the proof of (3.30). Let us now define

$$\mathscr{B}_\sigma = [\sigma, \sigma + \tau] \times [a, b].$$

From (3.21) we infer

$$|y_{l,m+1}(t,\lambda) - y_{lm}(t,\lambda)| \le \int_\sigma^{\sigma+\tau} |H_l(y_{lm}(s,\lambda),\lambda) - H_l(y_{l,m-1}(s,\lambda),\lambda)|\,\mathrm{d}s.$$

In view of (3.30) we may apply Lemma 3.7(b) and (3.28) to the right hand side of this inequality so as to get

$$\sup_{(t,\lambda)\in B_\sigma} |y_{l,m+1}(t,\lambda) - y_{lm}(t,\lambda)| \le \frac{1}{2} \sup_{(t,\lambda)\in B_\sigma} |y_{lm}(t,\lambda) - y_{l,m-1}(t,\lambda)|.$$

From Lemma 3.7(a) and (3.28) we then get by iteration and using (3.21)

$$\sup_{(t,\lambda)\in B_\sigma} |y_{l,m+1}(t,\lambda) - y_{lm}(t,\lambda)| \le \frac{1}{2^m} \int_\sigma^{\sigma+\tau} |H_l(\eta_l(\lambda),\lambda)|\,\mathrm{d}s \le \frac{1}{2^m}\,\tau M_4 \le \frac{1}{2^m}\,\frac{\mu}{4c}\,. \tag{3.31}$$

By (3.31), the sequence y_{lm}, $m = 0, 1, \ldots,$ is Cauchy on $\mathscr{B}_\sigma$ with respect to the sup-norm and converges uniformly toward a limit function $y_l(t,\lambda)$, $(t,\lambda) \in \mathscr{B}_\sigma$. Moreover, y_l is a solution of (3.29) and satisfies

$$\sup_{(t,\lambda)\in B_\sigma} |y_l(t,\lambda) - y_{lm}(t,\lambda)| \le \frac{1}{2^m}\,\frac{\mu}{2c}, \qquad y_l(t,\lambda) \in \mathscr{U}_\lambda, \text{ for } (t,\lambda) \in \mathscr{B}_\sigma. \tag{3.32}$$

In order to relate the iterates y_{lm} to the given solution family $y(t,\lambda)$, $t \in [0,T]$, $\lambda \in [a,b]$, we recall that the latter satisfies the ODEs

$$y_t(t,\lambda) = H(y(t,\lambda),\lambda), \quad t \in [0,T],\ \lambda \in [a,b],$$

and hence the integral equation

$$y(t,\lambda) = \eta(\lambda) + \int_\sigma^t H(y(s,\lambda),\lambda)\,\mathrm{d}s, \quad t \in [\sigma,T],\ \lambda \in [a,b]. \tag{3.33}$$

By our choice of τ, Corollary 3.5 applies, whence

$$y(t,\lambda) \in \mathscr{U}_\lambda, \quad (t,\lambda) \in \mathscr{B}_\sigma. \tag{3.34}$$

We now combine (3.29), (3.33) so as to get

$$\begin{aligned}|y(t,\lambda) - y_l(t,\lambda)| &\le |\eta(\lambda) - \eta_l(\lambda)| + \int_\sigma^{\sigma+\tau} |H(y(s,\lambda),\lambda) - H_l(y(s,\lambda),\lambda)|\,\mathrm{d}s \\ &\quad + \int_\sigma^{\sigma+\tau} |H_l(y(s,\lambda),\lambda) - H_l(y_l(s,\lambda),\lambda)|\,\mathrm{d}s, \quad (t,\lambda) \in \mathscr{B}_\sigma.\end{aligned}$$

In view of (3.32), (3.34), Lemma 3.7 is applicable to $y(s,\lambda)$ and $y_l(s,\lambda)$. On the basis of (3.27), (3.28) and (3.18), we thus infer from the last inequality

$$\sup_{(t,\lambda)\in\mathscr{B}_\sigma} |y(t,\lambda) - y_l(t,\lambda)| \le \frac{1}{l} + \frac{\tau C_4}{l} + \frac{1}{2} \sup_{(t,\lambda)\in\mathscr{B}_\sigma} |y(t,\lambda) - y_l(t,\lambda)|.$$

In view of (3.28) therefore,

$$\sup_{(t,\lambda)\in\mathscr{B}_\sigma} |y(t,\lambda) - y_l(t,\lambda)| \le \frac{3}{l}. \tag{3.35}$$

We next combine (3.32), (3.35) by means of the triangle inequality and invoke Lemma 3.9; setting $m = l$ we get

$$\sup_{(t,\lambda)\in\mathscr{B}_\sigma} |y(t,\lambda) - \psi^{2,n}_{\Sigma(l,l)}(t,\lambda)| \le \left(3 + \frac{\mu}{2c}\right)\frac{1}{l}, \tag{3.36}$$

for $l \ge 1$. By Definition 2.2, this means that $y(t,\lambda)$, $t \in [0,T]$, $\lambda \in [a,b]$, is computable on $\mathscr{B}_\sigma = [\sigma, \sigma+\tau] \times [a,b]$.

Now, (3.36) has been proved under the assumptions that $\sigma = \tau_k$, for some $k < K$, and that the function $\lambda \mapsto y(\sigma,\lambda)$, $\lambda \in [a,b]$, is computable (cf. (3.18)). For $\sigma = 0$, these assumptions are satisfied by our solution family $y(t,\lambda)$, $t \in [0,T]$, $\lambda \in [a,b]$, which is subject to Definition 3.1. Thus, (3.36) holds on $[0,\tau_1] \times [a,b]$ ($\tau_1 = \tau$), i.e. y is computable on $[0,\tau_1] \times [a,b]$. This implies in particular that the function $\lambda \mapsto y(\tau_1,\lambda)$, $\lambda \in [a,b]$ is computable. Setting $\sigma = \tau_1$ we may thus apply (3.36) to the rectangle $[\tau_1,\tau_2] \times [a,b]$ and conclude that y is computable on it. Proceeding in this way we obtain

$$y(t,\lambda),\ (t,\lambda) \in [\tau_k,\tau_{k+1}] \times [a,b], \text{ is computable for } k = 0,\ldots,K-1. \tag{3.37}$$

On the basis of (3.37) we may apply Corollary 2.8 so as to get the computability of y on $[0,T] \times [a,b]$. This concludes the proof of Theorem 3.2. □

Corollary 3.10. *Assume that* $a, b, T \in \mathbb{R}$ *are computable,* $a \le b$, $T > 0$. *If* $y(t, \lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$, *is an admissible family of solutions of* (3.5), *then* y *is computable on* $[0, T] \times [a, b]$.

Proof. We define $\tilde{y}$ via

$$\tilde{y}(s, \tilde{\lambda}) = y(sT, (b-a)\tilde{\lambda} + a), \quad s, \tilde{\lambda} \in [0, 1]. \tag{3.38}$$

Since y is an admissible solution family of (3.5), $\tilde{y}$ satisfies the ODEs

$$\begin{aligned} &T^{-1}D(\tilde{y}(s, \tilde{\lambda}))\tilde{y}_s(s, \tilde{\lambda}) = f(\tilde{y}(s, \tilde{\lambda}), (b-a)\tilde{\lambda} + a), \quad s, \tilde{\lambda} \in [0, 1], \\ &\tilde{y}(0, \tilde{\lambda}) = y(0, (b-a)\tilde{\lambda} + a), \quad \tilde{\lambda} \in [0, 1]. \end{aligned} \tag{3.39}$$

Since $y(0, \lambda)$, $\lambda \in [a, b]$, is computable, by Definition 3.1, the function $\tilde{y}(0, \tilde{\lambda})$, $\tilde{\lambda} \in [0, 1]$, is computable by Lemma 2.5. It follows that $\tilde{y}(s, \tilde{\lambda})$, $s, \tilde{\lambda} \in [0, 1]$, is an admissible solution family of (3.39). In order to apply Theorem 3.2 to (3.39) and $\tilde{y}$ we seek an approximating family corresponding to the family $\psi^{n+1,n}_{\sigma(\lambda)}$ related to $f(\ ,\ )$ via (3.1), (3.2). To this end we recall the recursive functions Γ, Π in Lemma 2.3 and Corollary 2.4, which depend on a, b and their approximants via Definition 2.1. We set

$$\nu(l) = \Pi(\sigma(l), l\Gamma(\sigma(l))), \quad l \in \mathbb{N} \tag{3.40}$$

(so $|\psi^{n+1,n}_{\sigma(l)}(y, (b-a)\tilde{\lambda} + a) - \psi^{n+1,n}_{\nu(l)}(y, \tilde{\lambda})| \le \frac{1}{l}$). An elaborate but straightforward argument, based on (3.40), (3.1), (3.2) and Lemma 2.3, then shows that the polynomial vector functions

$$\tilde{f}_l(y, \tilde{\lambda}) = \psi^{n+1,n}_{\nu(l)}(y, \tilde{\lambda}), \quad l \ge 1,$$

are related to $f(y, (b-a)\tilde{\lambda} + a)$, $y \in \mathscr{K}$, $\tilde{\lambda} \in [0, 1]$, via (3.1), (3.2) with ν in place of σ. Thus, Theorem 3.2 is applicable to (3.39) and $\tilde{y}$, implying that $\tilde{y}(s, \tilde{\lambda})$, $s, \tilde{\lambda} \in [0, 1]$, is computable. This fact together with Lemma 2.5 and (3.38) implies the computability of $y(t, \lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$. □

Remarks. While the system (3.5) is autonomous, there is a non-autonomous case that is subsumed under Theorem 3.2. This case arises if we consider the system

$$D(t, y)y_t = f(t, y, \lambda), \quad (t, y) \in [-N, N]^{n+1}, \ \lambda \in [a, b]. \tag{3.41}$$

Here the matrix $D(t, y) = \big(d_{jk}(t, y)\big)$, $j, k \le n$, is polynomial in t and $y = (y_1, \dots, y_n)$, with computable coefficients, while f is subject to the condition

$$|f(t, \xi, \lambda) - f(s, \zeta, \lambda)| \le C(|t-s| + |\xi - \zeta|), \tag{3.42}$$

for some C, where $(t, \xi), (s, \zeta) \in [-N, N]^{n+1}$, $\lambda \in [a, b]$. The approximants f_l, $l \ge 1$, of f are then also required to satisfy (3.42) with a constant C independent of $l \ge 1$. The reduction to Theorem 3.2 is then achieved by putting (3.41) into the autonomous form

$$D(z, y)y_t = f(z, y, \lambda), \quad z_t = 1. \tag{3.43}$$

These assumptions are e.g. satisfied if $f(t, y, \lambda)$ is itself polynomial in t, y, λ, with computable coefficients. The systems considered in [19] are of this type. If we relax (3.42) by dropping the term $|t-s|$, then (3.41) is not directly subsumed under Theorem 3.2. However, an inspection shows that only minor modifications of the proof are necessary in order to adapt it to this situation.

4. Recursively enumerable sets

In this section we show that the sets E_f defined via (1.1) are recursively enumerable.

The definitions of $\mathscr{A}$ and $\mathscr{A}_F$ will be given in (4.11). In a slight digression from [3] we work with real valued functions adding the necessary modifications for the complex setting at the end. The term "computable" always means computable in the sense of Definitions 2.1 and 2.2.

Let $\mathscr{M}$ be an arbitrary set of real valued functions $f = f(x_1, \dots, x_s)$, defined and continuous on some domain $\mathscr{D} = \prod_{j=1}^{s}[a_j, b_j]$, with a_j, b_j computable; $\mathscr{D}$ and s may vary from one function in $\mathscr{M}$ to another. The space $\mathscr{H}_0(\mathscr{M})$ is then defined as the smallest set $\mathscr{E}$ of functions such that $\mathscr{M} \subset \mathscr{E}$ and such that $\mathscr{E}$ is closed under the following operations: (**E**): if f, g are defined on $\mathscr{D}$ and $f, g \in \mathscr{E}$, then $f+g$, $f-g$ and fg are in $\mathscr{E}$; furthermore, if $f \ne 0$ on $\mathscr{D}$, then $1/f \in \mathscr{E}$; (**F**): if $f \in \mathscr{E}$ is defined on $\mathscr{D}$, and if $A_j, B_j, C_j, a'_j, b'_j, a''_j, b''_j$ are computable and satisfy

$$A_j + B_j y_j + C_j z_j \in [a_j, b_j], \quad \text{for } y_j \in [a'_j, b'_j], \ z_j \in [a''_j, b''_j], j \le s,$$

then the function

$$f(A_1 + B_1 y_1 + C_1 z_1, \dots, A_s + B_s y_s + C_s z_s), \quad y_j \in [a'_j, b'_j], \ z_j \in [a''_j, b''_j],$$

is in $\mathscr{E}$; (**G**): if $f(x_1, \dots, x_s, t)$, defined on $\mathscr{D} \times [a, b]$, is in $\mathscr{E}$, then the function $\int_a^b f(x_1, \dots, x_s, t)\,dt$, defined on $\mathscr{D}$, is in $\mathscr{E}$. The proof of the following proposition is by straightforward induction and will be omitted.

Proposition 4.1. *If all members of $\mathscr{M}$ are computable, then each $f \in \mathscr{H}_0(\mathscr{M})$ is computable.*

In the following we look at a set $\mathscr{M}^1$ of functions generated by a certain subclass of ODEs of type (3.5). In order to make this precise, set $y = (y_1, \dots, y_n)$, $w = (w_1, \dots, w_n)$, $z = (z_1, \dots, z_m)$. Let $D(y)$, $L(w)$ be $n\times n$-matrices and $H(z)$ an $m\times m$-matrix whose entries are polynomials in the indicated variables with computable coefficients. Let $S(y, z)$, $R(w)$ be n-vectors, $Q(z)$ an m-vector, whose components are polynomials in the indicated variables with computable coefficients. We then consider the coupled system

$$D(y)y_t = S(y, z), \qquad H(z)z_\lambda = Q(z), \qquad L(w)w_\lambda = R(w), \tag{4.1}$$

and seek solutions $y(t, \lambda)$, $z(\lambda)$, $w(\lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$, such that

$$\begin{aligned} &\det(D(y(t, \lambda))) \det(H(z(\lambda))) \det(L(w(\lambda))) \neq 0, \quad t \in [0, T], \lambda \in [a, b], \\ &y(0, \lambda) = w(\lambda), \quad \lambda \in [a, b]; \\ &z(0), w(0), a, b, T \text{ are computable.} \end{aligned} \tag{4.2}$$

With this specified we define $\mathscr{M}^1$ via: (**H**): $f \in \mathscr{M}^1$ iff there is a system of type (4.1) and a solution $y(t, \lambda) = (y_1, \dots, y_n)$, $z(\lambda)$, $w(\lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$, via (4.2) such that $f(t, \lambda) = y_j(t, \lambda)$, $t \in [0, T]$, $\lambda \in [a, b]$, for some j.

Proposition 4.2. *If $f \in \mathscr{M}^1$, then f is computable.*

Proof. Let y, z, w be solutions of some system (4.1) subject to (4.2). Fix $N \in \mathbb{N}$ sufficiently large such that

$$y(t, \lambda), w(\lambda) \in [-N, N]^n, \qquad z(\lambda) \in [-N, N]^m, \quad t \in [0, T], \lambda \in [a, b], \tag{4.3}$$

Now we remark that if $P(v)$ is any k-vector function, polynomial in $v = (v_1, \dots, v_s)$, with computable coefficients, then one easily shows that there is a recursive function $\alpha = \alpha_N : \mathbb{N} \to \mathbb{N}$ and a constant $C = C_N$ such that

$$|P(\xi) - \psi^{s,k}_{\alpha(l)}(\xi)| \le \frac{1}{l}, \qquad |\psi^{s,k}_{\alpha(l)}(\xi) - \psi^{s,k}_{\alpha(l)}(\zeta)| \le C|\xi - \zeta|,$$

for $\xi, \zeta \in [-N, N]^s$, $l \ge 1$. This holds in particular for $Q(z)$, $R(w)$ in (4.1), i.e. $Q(z)$ satisfies (3.1), (3.2) for suitable $\sigma = \sigma_N$ and $C = C_N$, and likewise with $R(w)$. By virtue of (4.2), the assumptions of Theorem 3.2 (resp. Corollary 3.10) are thus satisfied by the systems $H(z)z_\lambda = Q(z)$ and $L(w)w_\lambda = R(w)$ and by their respective solutions $z(\lambda)$, $w(\lambda)$, $\lambda \in [a, b]$. It thus follows that $z(\lambda)$, $w(\lambda)$, $\lambda \in [a, b]$, are computable. In particular, there is a recursive function $\vartheta : \mathbb{N} \to \mathbb{N}$ such that

$$|z(\lambda) - \eta_l(\lambda)| \le \frac{1}{l}, \quad \lambda \in [a, b], \text{ where } \eta_l(\lambda) := \psi^{1,m}_{\vartheta(l)}(\lambda). \tag{4.4}$$

We now insert $z(\lambda)$ for z in the first system of (4.1) so as to get a system of type (3.5), i.e.

$$D(y)y_t = S(y, z(\lambda)), \quad \lambda \in [a, b]. \tag{4.5}$$

It remains to show that $S(y, z(\lambda))$, $y \in [-N, N]^n$, $\lambda \in [a, b]$, satisfies (3.1), (3.2) for a suitable recursive function σ and a suitable constant C.

In order to see this we fix $M \ge N+1$ such that $\eta_l(\lambda) \in [-M, M]^m$, $\lambda \in [a, b]$. We also note that, since $S(y, z)$ is polynomial in the variables y_j, z_k and has computable coefficients, there is a constant $C = C_M$ and a recursive function $\alpha = \alpha_M : \mathbb{N} \to \mathbb{N}$ such that S_l given by

$$S_l(y, z) = \psi^{n+m,n}_{\alpha(l)}(y, z), \quad l \ge 1, \tag{4.6}$$

has the properties

$$\begin{aligned} &|S(y, z) - S_l(y, z)| \le \frac{1}{l}, \quad l \ge 1, \\ &|S_l(y', z') - S_l(y, z)| \le C(|y' - y| + |z' - z|), \quad l \ge 1, \end{aligned} \tag{4.7}$$

where $y, y' \in [-M, M]^n$, $z, z' \in [-M, M]^m$. By combining (4.4) with (4.7) by means of the triangle inequality we find

$$\begin{aligned} &|S(y, z(\lambda)) - S_l(y, \eta_l(\lambda))| \le \frac{1}{l}(1 + C) \\ &|S_l(y', \eta_l(\lambda)) - S_l(y, \eta_l(\lambda))| \le C|y' - y|, \end{aligned} \tag{4.8}$$

where $y, y' \in [-M, M]^n$, $\lambda \in [a, b]$, $l \ge 1$. It follows from (4.8) that the vector functions $f(y, \lambda) = S(y, z(\lambda))$ and $f_l(y, \lambda) = S_l(y, \eta_l(\lambda))$, $l \ge 1$, satisfy conditions (3.1), (3.2), up to a constant factor.

One still has to show that $f_l(y, \lambda)$ admits a representation in the form $f_l = \psi^{n+1,n}_{\pi(l)}$. Now (4.4), (4.6) entail

$$f_l(y, \lambda) = S_l(y, \eta_l(\lambda)) = \psi^{m+n,n}_{\alpha(l)}(y, \psi^{1,m}_{\vartheta(l)}(\lambda)). \tag{4.9}$$

From this it is clear that there is an effective procedure that, upon input of l, writes out the expression for f_l, and so one can effectively determine a $p \in \mathbb{N}$ for which $f_l(y,\lambda) = \psi_p^{n+1,n}(y,\lambda)$, and hence, a recursive function $\pi : \mathbb{N} \to \mathbb{N}$ with $p = \pi(l)$ and $f_l(y,\lambda) = \psi_{\pi(l)}^{n+1,n}(y,\lambda)$, $l \geq 1$.

So far we have shown that the system (4.5) satisfies all conditions that guarantee the applicability of Theorem 3.2 and Corollary 3.5. Now by assumption, $y(t,\lambda)$, $t \in [0,T]$, $\lambda \in [a,b]$, is a solution of (4.5) subject to (4.2) and such that $y(0,\lambda) = w(\lambda)$, $\lambda \in [a,b]$. Since, as noted above, $w(\lambda)$, $\lambda \in [a,b]$, is computable, Theorem 3.2 now says that $y(t,\lambda)$, $t \in [0,T]$, $\lambda \in [a,b]$, and all its components are computable. □

Now we make the following general remark.

Proposition 4.3. *Let $g(x)$, $h(x)$, $x \in [0,2\pi]$, be computable via Definition 2.2 and set $f(x) = g(x) + \mathrm{i}\,h(x)$. Then the set E defined by*

$$E = \left\{ n \in \mathbb{N} \,\middle|\, \int_0^{2\pi} f(x)\,\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \neq 0 \right\} \tag{4.10}$$

is recursively enumerable.

The proof, which is by straightforward approximation arguments based on Definitions 2.1 and 2.2 and applied to $g(x)$, $h(x)$ and $\mathrm{e}^{\mathrm{i}nx}$, is omitted.

Finally, we define (with $\mathscr{A}_{\mathrm{F}}$ slightly more general than in the introduction)

$$\begin{aligned} \mathscr{A} &= \{f = g + \mathrm{i}\,h \mid g, h \in \mathscr{H}_0(\mathscr{M}^1)\} \\ \mathscr{A}_{\mathrm{F}} &= \{f \in \mathscr{A} \mid f \text{ is defined on } [0,2\pi]\}. \end{aligned} \tag{4.11}$$

The above propositions together with Theorem 1 from [3] then yield our second main result:

Theorem 4.4. *A set $E \subset \mathbb{N}$ is recursively enumerable iff it admits the representation (4.10) with $f \in \mathscr{A}_{\mathrm{F}}$.*

Remarks. (1) Theorem 4.4 gives a characterization of recursively enumerable sets in terms of concepts of analysis. For other possibilities, depending on Hilbert's tenth problem we refer the reader to [15, chapter 9].

(2) In the present paper we have restricted our considerations to real valued functions while in [3] the functions are complex. But, since all non-linearities in [3] are polynomial, a passage to real and imaginary parts reduces the setting in [3] to the present form. That is, with $\mathscr{H}_0(\mathscr{M}_1)$ as in Theorem 1 of [3], and with $\mathscr{H}_0(\mathscr{M}^1)$ defined here by (**E**), (**F**), (**G**), (**H**), one shows by induction that $g + \mathrm{i}\,h \in \mathscr{H}_0(\mathscr{M}_1)$ iff $g, h \in \mathscr{H}_0(\mathscr{M}^1)$.

(3) As pointed out in [3, section 5], we are not able to prove the "only if" part of Theorem 4.4 without recourse to systems of type (4.1). This forced us to study parameter dependent systems (3.5). Whether one can represent recursively enumerable sets by functions that are based on ODEs without parameters is still an open problem.

Appendix. Proof of Lemma 3.6

In this section we prove Lemma 3.6 concerning the polynomial approximation of $P(y)^{-1}$ for y in the neighborhoods

$$\mathscr{U}_\lambda = \left\{ \xi \in \mathbb{R}^n \mid |\xi - \eta(\lambda)| \leq \frac{\mu}{2c} \right\}, \quad \lambda \in [a,b],$$

of the points $y(\sigma,\lambda) = \eta(\lambda)$ (see (3.14)). What makes the proof lengthy is that we require uniform Lipschitz constants for the approximating polynomials P_l.

We first rewrite (3.18) as follows:

$$|\eta(\lambda) - \eta_l(\lambda)| \leq \frac{1}{kl}, \quad k, l \geq 1,\ \lambda \in [a,b], \tag{A.1}$$

where we have set $\eta_l(\lambda) := \psi_{\beta(kl)}^{1,n}(\lambda)$ as in (3.18), but with k suppressed in the index. We fix k such that

$$\frac{1+c}{k} \leq \frac{1}{100}\,\frac{\mu}{(1+c)}, \tag{A.2}$$

where c is the Lipschitz constant for the polynomial P on the domain $[-(N+1), N+1]^n$ as in (3.12), taken large enough that also (3.13) holds. We also recall from Proposition 3.3(b) that

$$|P(y)| \geq \frac{\mu}{2}, \quad \text{for all } y \in \mathscr{U}_\lambda, \lambda \in [a,b], \tag{A.3}$$

and from Definition 3.1(a) that

$$|\eta(\lambda)| < N, \quad \lambda \in [a,b]. \tag{A.4}$$

Since P is a polynomial with computable coefficients, there exists a recursive function $\rho : \mathbb{N} \to \mathbb{N}$ such that the functions $R_l(y) := \psi^n_{\rho(l)}(y)$ satisfy

$$|P(y) - R_l(y)| \le \frac{1}{l}, \qquad |R_l(y) - R_l(z)| \le c|y - z|, \quad l \ge 1,\ y, z \in [-(N+1), N+1]^n. \tag{A.5}$$

While $R_l(y)$ approximates $P(y)$ on $[-(N+1), N+1]^n$ via (A.5), this does not necessarily hold for $R_l(y)^{-1}$ and $P(y)^{-1}$, in view of possible zeros of P. What can be asserted, though, are uniform approximation properties on the neighborhoods $\mathscr{U}_\lambda$, $\lambda \in [a, b]$. To see this, we first note that, by (A.4), (A.1), and (A.5), (A.2),

$$|\eta_l(\lambda)| \le N + 1, \qquad |R_l(\eta_l(\lambda)) - P(\eta(\lambda))| \le 10^{-2}\mu, \quad \lambda \in [a, b].$$

This, together with (3.6), (A.3), (A.5), (A.2), implies the following statements for $\lambda \in [a, b]$ and $l \ge 1$:

$$|R_l(\eta_l(\lambda))| \ge \tfrac{99}{100}\mu, \tag{A.6}$$

$$\text{if } y \in \mathscr{U}_\lambda \text{ then } |R_l(y)| \ge \tfrac{2}{5}\mu. \tag{A.7}$$

Arguing as in the proof of Proposition 3.3 using (A.3), (A.7), (A.5), we get

$$|P(y)^{-1} - R_l(y)^{-1}| \le \frac{5}{\mu^2 l}, \quad y \in \mathscr{U}_\lambda,\ \lambda \in [a, b],\ l \ge 1. \tag{A.8}$$

We now define

$$Q_l(y, \lambda) := \frac{R_l(\eta_l(\lambda)) - R_l(y)}{R_l(\eta_l(\lambda))}, \qquad \mathscr{P}^{\,l}_m(y, \lambda) := \frac{1}{R_l(\eta_l(\lambda))} \sum_{p=0}^{m} Q_l(y, \lambda)^p, \tag{A.9}$$

for $y \in \mathscr{U}_\lambda$, $\lambda \in [a, b]$, and note that

$$|Q_l(y, \lambda)| \le \frac{2}{3}, \quad y \in \mathscr{U}_\lambda, \lambda \in [a, b]. \tag{A.10}$$

This is proved using (A.6), (A.5), (A.1), (A.2):

$$\begin{aligned} |Q_l(y, \lambda)| &\le \left|\frac{R_l(\eta_l(\lambda)) - R_l(\eta(\lambda))}{R_l(\eta_l(\lambda))}\right| + \left|\frac{R_l(\eta(\lambda)) - R_l(y)}{R_l(\eta_l(\lambda))}\right| \\ &\le \frac{100}{99}\,\frac{1}{\mu}\,c(|y - \eta(\lambda)| + |\eta(\lambda) - \eta_l(\lambda)|) \le \frac{100}{99}\,\frac{1}{\mu}\,c\left(\frac{1}{2}\,\frac{\mu}{c} + \frac{1}{kl}\right). \end{aligned}$$

As a consequence of (A.10) we have the representation

$$\frac{1}{R_l(y)} = \frac{1}{R_l(\eta_l(\lambda))} \sum_{p=0}^{\infty} Q_l(y, \lambda)^p, \quad y \in \mathscr{U}_\lambda, \lambda \in [a, b], \tag{A.11}$$

where the convergence is absolute and uniform for $\lambda \in [a, b]$ and $y \in \mathscr{U}_\lambda$.

In the following we shall always assume that $\lambda \in [a, b]$ and $y \in \mathscr{U}_\lambda$. From (A.9), (A.11) and (A.6), (A.10), we infer

$$\left|\frac{1}{R_l(y)} - \mathscr{P}^{\,l}_l(y, \lambda)\right| \le \frac{100}{33}\,\frac{c_1}{\mu\, l}, \tag{A.12}$$

where c_1 is a constant such that $(\frac{2}{3})^{l+1} \le c_1 \frac{1}{l}$, $l \ge 1$. In order to construct the approximating polynomials $P_l(y, \lambda)$ as in Lemma 3.6, we observe that

$$R_l(\eta_l(\lambda)) = \psi^n_{\rho(l)}(\psi^{1,n}_{\beta(kl)}(\lambda))$$

(see the definitions before (A.5) and after (A.1)), with k fixed as above. It is then routine to show that there exists a recursive function $\omega : \mathbb{N} \to \mathbb{N}$ such that

$$R_l(\eta_l(\lambda)) = \psi^1_{\omega(l)}(\lambda). \tag{A.13}$$

The construction of $P_l(y, \lambda)$ is based on the following lemma whose proof is postponed to the end of the section.

Lemma A.1. *There exists a recursive function $r : \mathbb{N}^2 \to \mathbb{N}$ such that for the function $\varphi_{lp}(\lambda) := \psi^1_{r(l,p)}(\lambda)$ we have*

$$\left|1 - (R_l(\eta_l(\lambda)))^p \varphi_{lp}(\lambda)\right| \le \frac{1}{l}, \quad \lambda \in [a, b],\ l \ge 1,\ p \ge 1.$$

We now introduce the functions $P_l(y, \lambda)$.

$$P_l(y,\lambda) \stackrel{\text{def}}{=} \varphi_{l1}(\lambda) \sum_{p=0}^{l} Q_l(y,\lambda)^p (R_l(\eta_l(\lambda)))^p \varphi_{lp}(\lambda). \tag{A.14}$$

By (A.9), the functions P_l are polynomials in $y = (y_1, \dots, y_n)$ and λ, with rational coefficients. In order to show that they have the properties asserted by Lemma 3.6, we set $\eta_l = \eta_l(\lambda)$, $\varphi_{lp} = \varphi_{lp}(\lambda)$, and note that, by (A.6) and Lemma A.1,

$$|\varphi_{l1}| \le \frac{1}{|R_l(\eta_l)|} |R_l(\eta_l)\varphi_{l1} - 1| + \frac{1}{|R_l(\eta_l)|} \le \frac{100}{99}\left(\frac{1}{\mu\, l} + \frac{1}{\mu}\right) \le \frac{3}{\mu}. \tag{A.15}$$

We now consider the identity

$$\mathscr{P}_l^l(y,\lambda) - P_l(y,\lambda) = \frac{1}{R_l(\eta_l)}\left(1 - \varphi_{l1} R_l(\eta_l)\right) \sum_{p=0}^{l} Q_l(y,\lambda)^p + \varphi_{l1} \sum_{p=0}^{l} Q_l(y,\lambda)^p \left(1 - \varphi_{lp}(R_l(\eta_l))^p\right).$$

Using (A.6), the inequality of Lemma A.1 and (A.10), (A.15) we get

$$\left|\mathscr{P}_l^l(y,\lambda) - P_l(y,\lambda)\right| \le \frac{100}{99}\frac{3}{\mu\, l} + \frac{9}{\mu\, l} \le \frac{13}{\mu\, l}.$$

Combined with (A.8), (A.12) and (A.2), this implies

$$\left|\frac{1}{P(y)} - P_l(y,\lambda)\right| \le \frac{C_2}{l},$$

for some constant C_2. Clause (a) of Lemma 3.6 is now proved. In view of (A.3) we also have clause (b). For clause (c) we consider the identity

$$P_l(y,\lambda) - P_l(z,\lambda) = \varphi_{l1} \sum_{p=1}^{l} \left(\sum_{j=0}^{p-1} Q_l(y,\lambda)^{p-1-j} Q_l(z,\lambda)^j \right) (R_l(\eta_l))^p \varphi_{lp} \left(Q_l(y,\lambda) - Q_l(z,\lambda)\right),$$

which follows from (A.14). By the inequality in Lemma A.1, we have

$$|(R_l(\eta_l))^p \varphi_{lp}| \le 1 + \frac{1}{l} \le 2.$$

Combining this with (A.15), (A.10), (A.6) and (A.5) we obtain

$$|P_l(y,\lambda) - P_l(z,\lambda)| \le \frac{3}{\mu}\left(\sum_{p=1}^{\infty} p \left(\frac{2}{3}\right)^{p-1}\right) 2\, \frac{100}{99}\frac{1}{\mu}\, c\, |y-z| \le C_2' |y-z|,$$

where C_2' collects all the constants. Hence clause (c). The existence of a recursive function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$P_l(y,\lambda) = \psi_{\gamma(l)}^{n+1}(y,\lambda), \tag{A.16}$$

may be seen by analyzing the defining Eq. (A.14). Without details, the idea is as follows: each term (from (A.5), (A.13) and Lemma A.1), and hence the entire expression for P_l, in (A.14) is computable by a Turing machine as a function of l; one has therefore also a Turing machine that computes $\gamma(l)$ in (A.16).

To conclude the proof of Lemma 3.6 we prove Lemma A.1. To this end we consider the statement

$$l = 0 \text{ or } \left\{ \sup_{\lambda\in[a,b]} \left|1 - \left(\psi_{\omega(l)}^1(\lambda)\right)^p \psi_q^1(\lambda)\right| \le \frac{1}{l} \text{ and } l > 0 \right\}. \tag{A.17}$$

Since $a, b \in \mathbb{Q}$, the statements in (A.17) may be formalized in the first-order theory $\mathscr{R}_c$ of real closed fields (Section 2) giving rise to a closed formula $G(l, p, q)$ satisfying

$$\mathbb{R} \models G(l,p,q) \quad \text{iff} \quad \text{(A.17) holds.} \tag{A.18}$$

By (2.10), the relation $\mathbb{R} \models G(l, p, q)$ is recursive, i.e. given l, p, q it is decidable whether $\mathbb{R} \models G(l, p, q)$ holds. From (A.6) and the Weierstrass approximation theorem it follows that, given l, p, there exists q such that (A.17) holds, i.e. $(\forall l, p)(\exists q)(\mathbb{R} \models G(l, p, q))$. This implies that there is a recursive function $r : \mathbb{N}^2 \to \mathbb{N}$ satisfying the requirements of Lemma A.1. □

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