Level Theory, Part 1

Axiomatizing the bare idea of a cumulative hierarchy of sets

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This document contains preprints of Level Theory, Parts 1–3. All three papers are forthcoming at *Bulletin of Symbolic Logic*.

Abstract. The following bare-bones story introduces the idea of a cumulative hierarchy of pure sets: 'Sets are arranged in stages. Every set is found at some stage. At any stage S: for any sets found before S, we find a set whose members are exactly those sets.We find nothing else at S'. Surprisingly, this story already guarantees that the sets are arranged in well-ordered levels, and suffices for quasi-categoricity. I show this by presenting Level Theory, a simplification of set theories due to Scott, Montague, Derrick, and Potter.

> What we shall try to do here is to axiomatize the types in as simple a way as possible so that everyone can agree that the idea is natural.
>
> Scott (1974: 208)

The following bare-bones story introduces the idea of a cumulative hierarchy of pure sets:[1]

> **The Basic Story.** Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find a set whose members are exactly those sets. We find nothing else at **s**.

This story says nothing at all about the height of any hierarchy, and apparently says almost nothing about the order-type of the stages. It lays down nothing more than the *bare idea* of a pure cumulative hierarchy. Surprisingly, though, this bare idea already guarantees that the sets are arranged in well-ordered levels. Indeed, this bare idea is quasi-categorical. Otherwise put: the Basic Story pins down any cumulative hierarchy completely, modulo that hierarchy's height, on which the Story takes no stance. The aim of this paper is to show all of this.

[1] See e.g. Shoenfield (1977: 323). I have modified Shoenfield's story in two ways. First: Shoenfield speaks of sets as 'formed' at stages; I avoid this way of speaking, to avoid begging the question against platonists. Second: Shoenfield speaks of forming 'collections consisting of sets' into sets; I simply speak plurally. Note that the Basic Story takes no stance on whether sets 'depend' upon their members in anything other than an heuristic sense (cf. Incurvati 2012, 2020: 51–69).

I begin by axiomatizing the Basic Story in the most obvious way possible, obtaining Stage Theory, ST. It is clear that any pure cumulative hierarchy satisfies ST. Unfortunately, ST has multiple primitives. To overcome this, I develop Level Theory, LT. Its only primitive is $\in$, but LT and ST say exactly the same things about sets (see §§1–4). As such, any cumulative hierarchy satisfies LT. Moreover, LT proves that the levels are well-ordered, and LT is quasi-categorical (see §§5–6).

My theory LT builds on work by Dana Scott, Richard Montague, John Derrick, and Michael Potter. I discuss their theories in §8, but I wish to be very clear at the outset: LT is significantly technically simpler than its predecessors, but it owes everything to them.

This paper is the first in a triptych. In Part 2, I explore potentialism, by considering a tensed variation of the Basic Story. In Part 3, I modify the Story again, to provide every set with a complement. Part 2 presuppose Part 1, but Parts 1 and 3 can be read in isolation.

0 Preliminaries

I use second-order logic throughout. Mostly, though, my use of second-order logic is just for convenience. Except when discussing quasi-categoricity (see §6), any second-order claim can be replaced with a first-order schema in the obvious way. In using second-order logic, I assume the Comprehension scheme, $\exists F \forall x(F(x) \leftrightarrow \phi)$, for any ϕ not containing 'F'.

For readability, I concatenate infix conjunctions, writing things like $a \subseteq r \in s \in t$ for $a \subseteq r \land r \in s \land s \in t$. I also use some simple abbreviations (where Ψ can be any predicate whose only free variable is x, and $\lhd$ can be any infix predicate):

$$(\forall x : \Psi)\phi := \forall x(\Psi(x) \to \phi) \qquad (\forall x \lhd y)\phi := \forall x(x \lhd y \to \phi)$$
$$(\exists x : \Psi)\phi := \exists x(\Psi(x) \land \phi) \qquad (\exists x \lhd y)\phi := \exists x(x \lhd y \land \phi)$$

When I announce a result or definition, I list in brackets the axioms I am assuming.

1 Stage Theory

The Basic Story, which introduces the bare idea of a cumulative hierarchy, mentions sets and stages. To begin, then, I will present a theory which quantifies distinctly over both sorts of entities. (It is a simple modification of Boolos's 1989 theory; see §§8.1–8.2.)

Stage Theory, ST, has two distinct sorts of variable, for *sets* (lower-case italic) and for **stages** (lower-case bold). It has three primitive predicates:

$\in$: a relation between sets; read '$a \in b$' as 'a is in b'
$<$: a relation between stages; read '$\mathbf{r} < \mathbf{s}$' as '$\mathbf{r}$ is before $\mathbf{s}$'
$\preceq$: a relation between a set and a stage; read '$a \preceq \mathbf{s}$' as 'a is found at $\mathbf{s}$'

For brevity, I write $a \prec \mathbf{s}$ for $\exists\mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})$, i.e. a is found before $\mathbf{s}$. Then ST has five axioms:[2]

Extensionality $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$

Order $\forall\mathbf{r}\forall\mathbf{s}\forall\mathbf{t}(\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t})$

Staging $\forall a \exists\mathbf{s}\ a \preceq \mathbf{s}$

Priority $\forall\mathbf{s}(\forall a \preceq \mathbf{s})(\forall x \in a)x \prec \mathbf{s}$

Specification $\forall F \forall\mathbf{s}((\forall x : F)x \prec \mathbf{s} \rightarrow (\exists a \preceq \mathbf{s})\forall x(F(x) \leftrightarrow x \in a))$

The first two axioms make implicit assumptions explicit: whilst I did not mention Extensionality in the Basic Story of a cumulative hierarchy, I take it as analytic that sets are extensional;[3] similarly, Order records the analytic fact that 'before' is a transitive relation. The remaining three axioms can then be read off the Basic Story directly: Staging says that every set is found at some stage; Priority says that a set's members are found before it; and Specification says that, if we find every F before $\mathbf{s}$, then we find the set of Fs at $\mathbf{s}$. So all of ST's axioms are obviously true of the Basic Story. Otherwise put: any cumulative hierarchy obviously satisfies ST.[4]

This is ST's chief virtue. Its chief drawback is that it contains multiple primitives. To see why this is a defect, suppose that we were forced to axiomatize the bare idea of a cumulative hierarchy using something like ST's two-sorted logic. In that case, our grasp of the (cumulative iterative) notion of *set* would unavoidably depend upon a concept which we had not rendered set-theoretically, namely, *stage of a hierarchy*. And that would somewhat undercut the commonplace ambition, that our notion of *set* might serve as a certain kind of autonomous foundation for mathematics.

2 Level Theory

To overcome this problem, I present Level Theory, LT. This theory's only primitive is $\in$, but it makes exactly the same claims about sets as ST does. I begin with a definition, due to Scott and Montague (see §8.3), which forms the linchpin of this paper:[5]

Definition 2.1: For any a, let a's *potentiation* be $\P a := \{x : \exists c(x \subseteq c \in a)\}$, if it exists.[6]

[2] Classical logic yields a 'cheap' proof of the existence of a stage and an empty set: by classical logic, there is some object, a; by Staging we have some $\mathbf{s}$ such that $a \preceq \mathbf{s}$; with $F(x)$ given by $x \neq x$, Specification yields a set, $\emptyset$, such that $\forall x\ x \notin \emptyset$. Those who find such proofs *too* cheap can adopt a free logic and then add explicit existence axioms; I will retain classical logic.

[3] For brevity, I am considering hierarchies of pure sets; I revisit this in §§A–B.

[4] Or, given footnote 2: any *non-null* hierarchy satisfies ST.

[5] Montague et al. (unpublished: Definition 22.4, p.161) and Scott (1974: 214). They used the '¶' symbol, but not the name 'potentiation'.

[6] By the notational conventions, $\P a = \{x : (\exists c \in a)x \subseteq c\} = \{x : \exists c(x \subseteq c \wedge c \in a)\}$. We do not initially assume that $\P a$ exists for every a; instead, we initially treat every expression of the form '$b = \P a$' as shorthand for '$\forall x(x \in b \leftrightarrow \exists c(x \subseteq c \in a))$', and must double-check whether $\P a$ exists. Ultimately, though, LT proves that $\P a$ exists for every a (Lemma 3.12.1).

The name *potentiation* emphasises the conceptual connection with powersets; note that $\P\{a\} = \wp a$.[7] The next two definitions employ this notion of potentiation (and thereby simplify definitions due to Derrick and Potter; see §8.4):[8]

Definition 2.2: Say that h is a *history*, written $Hist(h)$, iff $(\forall x \in h)x = \P(x \cap h)$. Say that s is a *level*, written $Lev(s)$, iff $(\exists h : Hist)s = \P h$.

The intuitive idea behind this definition is that a history is an initial sequence of levels, and that the levels go proxy for stages. It is not obvious that this will work as described; indeed, the next two sections are dedicated to establishing this fact. But, using the notion of a level, LT has just three axioms:[9]

Extensionality $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$
Separation $\forall F \forall a \exists b \forall x(x \in b \leftrightarrow (F(x) \wedge x \in a))$
Stratification $\forall a(\exists s : Lev)a \subseteq s$

3 The well-ordering of the levels

In §4, I will show that LT makes exactly the same claims about sets as ST does. First, I must develop the elements of set theory within LT. To do so, I need some more definitions:

Definition 3.1: Say that a is *transitive* iff $(\forall x \in a)x \subseteq a$. Say that a is *potent* iff $\forall x(\exists c(x \subseteq c \in a) \rightarrow x \in a)$.

Transitivity is completely familiar. *Potency* is discussed in a few places, albeit with no standard name.[10] As my choice of name suggests, though, there is a tight link between the operation of potentiation (see Definition 2.1) and the property of potency:

Lemma 3.2: If $\P a$ exists, then $\P a$ is potent.

Lemma 3.3 (Extensionality)**:** a is potent iff $a = \P a$.

Recall the conventions: Lemma 3.2 follows from the definitions alone, but Lemma 3.3 requires Extensionality. I leave the trivial proofs to the reader.

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[7] NB: by design, LT does not prove that every set has a powerset; for that, we have LT + Endless (see §7).

[8] Potter (2004: 41).

[9] For ultra-economy, we can replace Separation+Stratification with $\forall F(\exists a \forall x(F(x) \leftrightarrow x \in a) \leftrightarrow (\exists s : Lev)(\forall x : F)x \in s)$. We can read this as: *a property determines a set iff its instances are all in some level* (cf. Button and Walsh 2018: 183, Definition 8.9). As in footnote 2, above, the use of classical logic offers a 'cheap' proof of the existence of $\emptyset$. Moreover, LT has a model whose *only* denizen is $\emptyset$.

[10] Potter (1990: 19) uses 'hereditary'; Doets (1999: 78) and Button and Walsh (2018: 193) use 'supertransitive'. Mathias (2001: 487) and Burgess (2004: 208) use 'supertransitive' for sets which are both transitive and potent; Lévy and Vaught (1961: 1046) use 'supercomplete' for such sets.

My aim now is to prove several results about levels, in the sense of Definition 2.2.[11] These results do not need LT's Stratification axiom, since any sets which were not subsets of levels would be irrelevant.[12]

Lemma 3.4: Every level is potent and transitive.

Proof. Fix a level, s, so $s = ¶h$, for some history h. Potency follows by Lemma 3.2. For transitivity, fix $a \in s = ¶h$; so $a \subseteq c \in h$ for some c, and $c = ¶(c \cap h)$ as h is a history; so $a \subseteq ¶(c \cap h) \subseteq ¶h = s$. □

Lemma 3.5 (Extensionality, Separation)**:** If every F is potent and something is F, then there is an $\in$-minimal F. Formally: $\forall F((\exists x F(x) \land (\forall x : F)x \text{ is potent}) \rightarrow (\exists a : F)(\forall x : F)x \notin a)$.

Proof. Let F be as described and let u be an F. Using Separation twice, let:

$$c = \{x \in u : (\forall y : F)x \in y\} = \{x : (\forall y : F)x \in y\}$$
$$d = \{x \in c : x \notin x\}$$

Clearly $d \notin c$, since otherwise $d \in d \leftrightarrow d \notin d$; so there is some a which is F with $d \notin a$. Now if x is F, then $d \subseteq c \subseteq x$, but $d \notin a$ and a is potent, so $x \notin a$. □

Lemma 3.6 (Extensionality, Separation)**:** If some level is F, then there is an $\in$-minimal level which is F. Formally: $\forall F((\exists s : Lev)F(s) \rightarrow (\exists s : Lev)(F(s) \land (\forall r : Lev)(F(r) \rightarrow r \notin s)))$

Proof. All levels are potent, by Lemma 3.4; now use Lemma 3.5. □

Lemma 3.7 (Extensionality, Separation)**:** Every member of a history is a level.

Proof. For reductio, let h be a history with some non-level in it. Since $c = ¶(c \cap h)$ for all $c \in h$, every member of h is potent by Lemma 3.2. Using Lemma 3.5, let a be an $\in$-minimal non-level in h. Now $a = ¶(a \cap h)$; and $a \cap h = \{x \in a : x \in h\}$ exists by Separation. So, to obtain our desired contradiction, it suffices to show that $a \cap h$ is a history. Fix $b \in a \cap h$. So b is a level, by choice of a, and $b = ¶(b \cap h)$ as $b \in h$. If $x \in b$, then $x \subseteq b$, since b is transitive by Lemma 3.4; so $x \in a$, since a is potent as above; hence, $b \subseteq a$. So $b = ¶(b \cap h) = ¶(b \cap (a \cap h))$. Generalising, $a \cap h$ is a history. □

Lemma 3.8 (Extensionality, Separation)**:** $s = ¶\{r \in s : Lev(r)\}$, for any level s.

[11] The next few results simplify Potter (2004: 41–6). Lemma 3.5 is inspired by Potter's Proposition 3.6.4; Lemma 3.7 by Potter's Proposition 3.4.1; Lemma 3.8 by Potter's Proposition 3.6.8; and Lemma 3.9 by Potter's Proposition 3.6.11.

[12] Cf. Scott (1974: 211n.1).

Proof. Let s be a level. If $a \subseteq r \in s$, then $a \in s$, as s is potent by Lemma 3.4. If $a \in s$, then as $s = \P h$ for some history h, we have $a \subseteq r \in h \subseteq \P h = s$ for some r, and r is a level by Lemma 3.7. □

Lemma 3.9 (Extensionality, Separation)**:** All levels are comparable.[13] Formally: $(\forall s : Lev)(\forall t : Lev)(s \in t \lor s = t \lor t \in s)$

Proof. For reductio, suppose that some levels are incomparable. By Lemma 3.6, there is an $\in$-minimal level, s, which is incomparable with some level; and by Lemma 3.6 again, there is an $\in$-minimal level, t, which is incomparable with s. To complete the reductio, I will show that $s = t$.

To show that $s \subseteq t$, fix $a \in s$. So $a \subseteq r \in s$ for some level r, by Lemma 3.8. Now r is comparable with t, by choice of s. But if either $r = t$ or $t \in r$, then $t \in s$ as s is transitive, contradicting our assumption; so $r \in t$. Now $a \subseteq r \in t$, so that $a \in t$ as t is potent. Generalising, $s \subseteq t$.

Exactly similar reasoning, based on the choice of t, shows that $t \subseteq s$. So $t = s$. □

Rolling Lemmas 3.6 and 3.9 together, we obtain the fundamental theorem of level theory:

Theorem 3.10 (Extensionality, Separation)**:** The levels are well-ordered by membership.

Combining this result with Stratification, we obtain a powerful tool, which intuitively allows us to consider the level at which a set is first found:

Definition 3.11 (LT)**:** Let ℓa be the $\in$-least level with a as a subset; i.e., $a \subseteq \ell a$ and $\neg(\exists s : Lev) a \subseteq s \in \ell a$.

Lemma 3.12 (LT)**:** For all sets a, b, and all levels r, s:

(1) ℓa and $\P a$ both exist, and $\P a \subseteq \ell a$
(2) $a \notin \ell a$
(3) $r \subseteq s$ iff $s \notin r$
(4) $s = \ell s$
(5) if $b \subseteq a$, then $\ell b \subseteq \ell a$
(6) if $b \in a$, then $\ell b \in \ell a$
(7) $\ell a = \P\{\ell x : x \in a\}$
(8) if every member of a is a level, then $\P a = \ell a$

Proof. *(1)* ℓa exists by Stratification and Theorem 3.10. Now if $x \subseteq c \in a \subseteq \ell a$, then $x \in \ell a$ since ℓa is potent; so $\P a \subseteq \ell a$ exists by Separation.

(2) There is no level t with $a \subseteq t \in \ell a$, so $a \notin \ell a$ by Lemma 3.8.

(3) If $r \subseteq s$ then $s \notin r$ by the well-ordering of levels. Conversely, if $s \notin r$, then either $r \in s$ or $r = s$ by comparability; and $r \subseteq s$ either way, as s is transitive.

[13] Say that x is *comparable* with y iff $x \in y \lor x = y \lor y \in x$

(4) By (2), $s \notin \ell s$. By (3), $\ell s \notin s$. So $s = \ell s$, by comparability.

(5) If $b \subseteq a$ then $b \subseteq \ell a$. So $\ell a \notin \ell b$, by definition of ℓb, so $\ell b \subseteq \ell a$ by (3).

(6) If $b \in a$ then $b \in \ell a$. By (2), $b \notin \ell b$; so $\ell a \nsubseteq \ell b$, and hence $\ell b \in \ell a$ by (3).

(7) Let $k = \{\ell x : x \in a\}$. If $c \in \P k$ then $c \subseteq \ell x$ for some $x \in a$; now $\ell x \in \ell a$ by (6), so $c \in \ell a$. Conversely, if $c \in \ell a$ then $c \subseteq r \in \ell a$ for some level r by Lemma 3.8; since $a \nsubseteq r$ by definition of ℓa, there is some $x \in a \setminus r$; now $\ell x \notin r$ as r is potent, so that $r \subseteq \ell x$ by (3) and hence $c \subseteq \ell x$; so $c \in \P k$.

(8) In this case, $a = \{\ell x : x \in a\}$ by (4), so $\ell a = \P a$ by (7). □

4 The set-theoretic equivalence of ST and LT

Having explained how to work within LT, I will now make good on my earlier promise, and show that LT and ST make exactly the same claims about sets. More precisely, I will prove the following:

Theorem 4.1: ST $\vdash \phi$ iff LT $\vdash \phi$, for any LT-sentence ϕ.

To show that ST says no more about sets than LT does, I define a translation, $*$, from ST-formulas into LT-formulas. In effect, $*$ treats stages as levels, ordered by membership. Specifically, its non-trivial actions are as follows:[14]

$$(\mathbf{s} < \mathbf{t})^* := \mathbf{s} \in \mathbf{t} \qquad (x \preceq \mathbf{s})^* := x \subseteq \mathbf{s} \qquad (\forall \mathbf{s} \phi)^* := (\forall \mathbf{s} : Lev)(\phi^*)$$

After translation, we treat all first-order variables—whether bold or italic—as being of the same sort. Fairly trivially, for any LT-sentence ϕ, if ST $\vdash \phi$ then ST$^* \vdash \phi$. The left-to-right half of Theorem 4.1 now follows from this simple observation, together with the fact that $* : \text{ST} \longrightarrow \text{LT}$ is an interpretation:

Lemma 4.2 (LT)**:** ST* holds.

Proof. Extensionality* is Extensionality. Staging* is Stratification. Order* holds by Lemma 3.4. Note that Lemma 3.8 allows us to simplify $(x < \mathbf{s})^*$, i.e. $(\exists \mathbf{r}(x \preceq \mathbf{r} < \mathbf{s}))^*$, to $(x \in \mathbf{s})$. Now Priority* holds trivially. And Specification* holds as if $(\forall x : F)x \in \mathbf{s}$, then $\{x : F(x)\} \subseteq \mathbf{s}$ by Separation.[15] □

To obtain the right-to-left half of Theorem 4.1, I must first prove some quick results in ST:

Lemma 4.3 (ST)**:** Separation holds.

Proof. By Staging, $a \preceq \mathbf{s}$ for some $\mathbf{s}$. By Priority, $(\forall x \in a)x < \mathbf{s}$. Now use Specification. □

[14] So the other clauses are: $(\neg\phi)^* := \neg\phi^*$; $(\phi \wedge \psi)^* := (\phi^* \wedge \psi^*)$; $(\forall x \phi)^* := \forall x \phi^*$; $(\forall F \phi)^* := \forall F \phi^*$; and $\alpha^* := \alpha$ for all atomic formulas α which are not of the forms mentioned in the main text.

[15] Note that the $*$-translation of any ST-Comprehension instance is an LT-Comprehension instance.

Lemma 4.4 (ST)**:** $\forall \mathbf{s} \forall a(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$

Proof. Left-to-right is Priority. For right-to-left, suppose $(\forall x \in a)x \prec \mathbf{s}$; then $\{x : x \in a\} = a \preceq \mathbf{s}$ by Extensionality and Specification. □

I next introduce *slices*. These will turn out to be levels, in the sense of Definition 2.2. Here is the definition of a slice, and some elementary results concerning slices:

Definition 4.5: For each $\mathbf{s}$, let $\check{\mathbf{s}} = \{x : x \prec \mathbf{s}\}$, if it exists. Say that a is a *slice* iff $a = \check{\mathbf{s}}$ for some $\mathbf{s}$.

Lemma 4.6 (ST)**:** For any $\mathbf{s}$:

(1) $\check{\mathbf{s}}$ exists
(2) $\forall \mathbf{r} \forall a(a \preceq \mathbf{r} \leq \mathbf{s} \rightarrow a \preceq \mathbf{s})$
(3) $\forall a(a \subseteq \check{\mathbf{s}} \leftrightarrow a \preceq \mathbf{s})$
(4) $\check{\mathbf{s}}$ is transitive
(5) $\check{\mathbf{s}} = \P\{\check{\mathbf{r}} : \check{\mathbf{r}} \in \check{\mathbf{s}}\}$

Proof. (1) By Specification and Extensionality.

(2) Let $a \preceq \mathbf{r} \leq \mathbf{s}$. Now $(\forall x \in a)x \prec \mathbf{r}$ by Priority, so $(\forall x \in a)x \prec \mathbf{s}$ by Order, and $a \preceq \mathbf{s}$ by Lemma 4.4.

(3) $a \subseteq \check{\mathbf{s}}$ iff $(\forall x \in a)x \in \check{\mathbf{s}}$ iff $(\forall x \in a)x \prec \mathbf{s}$ iff $a \preceq \mathbf{s}$ by Lemma 4.4.

(4) If $a \in \check{\mathbf{s}}$, then $a \preceq \mathbf{r} < \mathbf{s}$ for some $\mathbf{r}$; hence $a \preceq \mathbf{s}$ and $a \subseteq \check{\mathbf{s}}$ by (2)–(3).

(5) If $a \in \check{\mathbf{s}}$, then $a \preceq \mathbf{r} < \mathbf{s}$ for some $\mathbf{r}$; hence $a \subseteq \check{\mathbf{r}} \preceq \mathbf{r} < \mathbf{s}$ by (3), so $a \subseteq \check{\mathbf{r}} \in \check{\mathbf{s}}$. If $a \subseteq \check{\mathbf{r}} \in \check{\mathbf{s}}$, then $a \subseteq \check{\mathbf{r}} \preceq \mathbf{t} < \mathbf{s}$ for some $\mathbf{t}$; now $a \subseteq \check{\mathbf{r}} \subseteq \check{\mathbf{t}}$ by (3), so $a \preceq \mathbf{t}$ by (3), i.e. $a \in \check{\mathbf{s}}$. □

It is now easy to show that $\in$ well-orders the slices: just transcribe the proofs of Lemmas 3.6 and 3.9 within ST, replacing 'levels' with 'slices', noting that ST proves Separation (see Lemma 4.3), and replacing appeal to Lemmas 3.4 and 3.8 with Lemma 4.6.4–5. We can then go on to prove that the levels are the slices.

Lemma 4.7 (ST)**:** s is a level iff s is a slice.

Proof. For induction on slices, suppose: $(\forall \check{\mathbf{q}} \in \check{\mathbf{s}})(\forall a \subseteq \check{\mathbf{q}})(a \text{ is a slice} \leftrightarrow Lev(a))$. I will show that $(\forall a \subseteq \check{\mathbf{s}})(a \text{ is a slice} \leftrightarrow Lev(a))$. The result will follow by Staging and Lemma 4.6.3.

First, fix a level $r \subseteq \check{\mathbf{s}}$. Let $h = \{q \in r : Lev(q)\}$; so $r = \P h$ by Lemma 3.8. (Note that ST proves all of Lemmas 3.2–3.9, verbatim, since ST proves Separation.) Fix $a \in r$; so $a \in \check{\mathbf{s}}$, so $a \subseteq \check{\mathbf{q}} \in \check{\mathbf{s}}$ for some $\check{\mathbf{q}}$ by Lemma 4.6.5; hence, by the induction hypothesis, a is a slice iff a is a level. So $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in r\}$. Noting that $h \subseteq \check{\mathbf{s}}$, let $\check{\mathbf{t}}$ be the $\in$-least slice such that $h \subseteq \check{\mathbf{t}}$. Since r is transitive and the slices are well-ordered, $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in \check{\mathbf{t}}\}$. So $r = \P h = \check{\mathbf{t}}$ by Lemma 4.6.5, i.e. r is a slice.

Next, fix $\check{\mathbf{r}} \subseteq \check{\mathbf{s}}$. Let $h = \{\check{\mathbf{q}} : \check{\mathbf{q}} \in \check{\mathbf{r}}\}$; so $\check{\mathbf{r}} = \P h$ by Lemma 4.6.5; and $h = \{q \in \check{\mathbf{r}} : Lev(q)\}$ by the induction hypothesis. Fix $q \in h$; since $\check{\mathbf{r}}$ is transitive, $q \cap h = \{p \in q : Lev(p)\}$, so that $q = \P(q \cap h)$ by Lemma 3.8. So h is a history and $\check{\mathbf{r}} = \P h$ is a level. □

This allows us to prove the last axiom of LT within ST:

Lemma 4.8 (ST)**:** Stratification holds.

Proof. Fix a; by Staging, $a \preceq \mathbf{s}$ for some $\mathbf{s}$, i.e. $a \subseteq \check{\mathbf{s}}$ by Lemma 4.6.3, and $\check{\mathbf{s}}$ is a level by Lemma 4.7. □

So ST ⊢ LT, completing the proof of Theorem 4.1.

5 The inevitability of well-ordering

A simple argument now establishes that LT axiomatizes the bare idea of a cumulative hierarchy of sets:

(a) Any cumulative hierarchy of sets satisfies ST (see §1).
(b) LT is set-theoretically equivalent to ST (see Theorem 4.1).
(c) So: any cumulative hierarchy of sets satisfies LT (from (a) and (b)).

Otherwise put: LT is true of the Basic Story I told at the start of this paper, and which I repeat here for ease of reference: *Sets are arranged in stages. Every set is found at some stage. At any stage* **s**: *for any sets found before* **s**, *we find a set whose members are exactly those sets. We find nothing else at* **s**.

In fact, (c) takes on an even deeper significance when we reflect on just *how* bare-bones this Basic Story is. The Story says that some stages are 'before' others, and we can safely assume that 'before' is a transitive relation on stages (hence ST's Order axiom).[16] But it is not obvious, for example, that it would be inconsistent to augment the Story by saying *for every stage there is an earlier stage*, or *between any two stages there is another stage*. This might prompt us to start entertaining cumulative hierarchies which are ordered like the integers, or the rationals, or more exotically still. A very simple argument, however, puts an abrupt end to such speculation:

(d) LT proves the well-ordering of the levels (see Theorem 3.10).
(e) So: any cumulative hierarchy of sets has well-ordered levels (from (c) and (d)).

Scott was the first to prove a well-ordering result from a similarly spartan starting point (see §8.3), and he put the point beautifully: 'This at first surprising result shows how little choice there is in setting up the type hierarchy.'[17] Scott's deep observation deserves to be much more widely known.

The connection between ST and LT also helps to demystify the definition of *level*. Working in ST, suppose that h is an initial sequence of slices; if $\check{\mathbf{s}} \in h$, then $\check{\mathbf{s}} \cap h$ is the set of all slices less than $\check{\mathbf{s}}$, so that $\check{\mathbf{s}} = \P(\check{\mathbf{s}} \cap h)$ by Lemma 4.6.5. These observations motivate Definition 2.2. We say that h is a history iff $(\forall x \in h)x = \P(x \cap h)$, in the hope that, so defined, a history will be an initial sequence of slices; if it is, then the

[16] In similar spirit, Shoenfield (1977: 323) says: 'We should certainly expect *before* to be a partial ordering of the stages; and this is the only fact about this relation which we need for our axioms.' But Shoenfield obtains well-ordering by arguing for Foundation using a proof due to Scott (see §8.1) and then using Replacement to define the V_αs; LT, of course, does not include Replacement (see §7).

[17] Scott (1974: 210).

next slice in the sequence is the potentiation of that history, by Lemma 4.6.5; and this is how we define levels.

6 The quasi-categoricity of LT

We just saw that every cumulative hierarchy of sets has well-ordered levels. In fact, we can push this point further. By design, LT says nothing about the height of any hierarchy. But, as I will show in this section, LT is quasi-categorical. Informally, we can spell out LT's quasi-categoricity as follows:

(f) Any two hierarchies satisfying LT are structurally identical for so far as they both run, but one may be taller than the other.

Since every cumulative hierarchy satisfies LT, we obtain:

(g) Any two cumulative hierarchies are structurally identical for so far as they both run, but one may be taller than the other (from (c) and (f)).

So, echoing Scott: when we set up a cumulative hierarchy, our only choice is how tall to make it.

It just remains to establish (f), i.e. to show that LT is quasi-categorical. In fact, there are at least two ways to explicate the informal idea of quasi-categoricity, and LT is quasi-categorical on both explications. (Note that both ways make essential use of second-order logic; this is the only section of the paper where my use of second-order logic is not merely for convenience.)

The first notion of quasi-categoricity is familiar from Zermelo. Working in some (set-theoretic) model theory, we define the V_αs as usual:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \wp V_\alpha; \qquad V_\alpha = \bigcup_{\beta\in\alpha} V_\beta \text{ when } \alpha \text{ is a limit}$$

Each V_α then naturally yields a set-theoretic structure, $\mathcal{V}_\alpha$, whose domain is V_α, and which interprets '$\in$' as membership-restricted-to-V_α, i.e. $\{\langle x, y\rangle \in V_\alpha \times V_\alpha : x \in y\}$. We then have the following result, using full second-order logic: $\mathcal{M} \vDash \text{ZF}$ iff $\mathcal{M} \cong \mathcal{V}_\alpha$ for some strongly inaccessible α.[18] There is an analogous quasi-categoricity result for LT:[19]

Theorem 6.1 (in full second-order logic)**:** $\mathcal{M} \vDash \text{LT}$ iff $\mathcal{M} \cong \mathcal{V}_\alpha$ for some $\alpha > 0$.

This shows that any two hierarchies satisfying LT (read that phrase as 'any models of LT') are structurally identical (read that phrase as 'are isomorphic') for so far as they both run (read that phrase in the light of the well-ordering of the V_αs, established in the model theory). In short, LT is quasi-categorical, on a model-theoretic ('external') way of understanding quasi-categoricity.

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[18] Zermelo (1930). For an accessible proof, see Button and Walsh (2018: §8.A).

[19] Button and Walsh (2018: §8.C) prove this for Potter's theory (see §8.4); the same proof works for LT. The same remark applies to the other results mentioned in this section. We could obtain external categoricity using only first-order logic, if we augmented LT with some axiom of the form 'there are exactly n levels'.

There is also, though, an object-language ('internal') way to understand quasi-categoricity.[20] Since this idea is less familiar, I will spend some time unpacking it.

In embracing Extensionality, LT assumes that everything is a pure set. There is a quick-and-dirty way to avoid this assumption. First, introduce a new predicate, *Pure*; intuitively, this should apply to the pure sets. Next, relativise LT to *Pure*, via the following formula:[21]

$$\begin{aligned}\mathrm{LT}(\mathit{Pure}, \varepsilon) \coloneqq\ & (\forall a : \mathit{Pure})(\forall b : \mathit{Pure})(\forall x(x \mathrel{\varepsilon} a \leftrightarrow x \mathrel{\varepsilon} b) \rightarrow a = b) \land \\ & \forall F(\forall a : \mathit{Pure})(\exists b : \mathit{Pure})\forall x(x \mathrel{\varepsilon} b \leftrightarrow (F(x) \land x \mathrel{\varepsilon} a)) \land \\ & (\forall a : \mathit{Pure})(\exists s : \mathit{Lev})a \subseteq s \land \\ & \forall x \forall y(x \mathrel{\varepsilon} y \rightarrow (\mathit{Pure}(x) \land \mathit{Pure}(y)))\end{aligned}$$

The first three conjuncts tell us that the pure sets satisfy LT;[22] the last says that, when we use 'ε', we restrict our attention to membership facts between pure sets. Using this formula, I can now state the internal quasi-categoricity result (I have labelled the lines to facilitate its explanation):

Theorem 6.2: This is a deductive theorem of impredicative second-order logic:

$$\begin{aligned}&(\mathrm{LT}(\mathit{Pure}_1, \varepsilon_1) \land \mathrm{LT}(\mathit{Pure}_2, \varepsilon_2)) \rightarrow \\ &\quad \exists R(\forall v \forall y(R(v,y) \rightarrow (\mathit{Pure}_1(v) \land \mathit{Pure}_2(y))) \land && (1)\\ &\qquad ((\forall v : \mathit{Pure}_1)\exists y R(v,y) \lor (\forall y : \mathit{Pure}_2)\exists v R(v,y)) \land && (2)\\ &\qquad \forall v \forall y \forall x \forall z((R(v,y) \land R(x,z)) \rightarrow (v \mathrel{\varepsilon_1} x \leftrightarrow y \mathrel{\varepsilon_2} z)) \land && (3)\\ &\qquad \forall v \forall y \forall z((R(v,y) \land R(v,z)) \rightarrow y = z) \land && (4)\\ &\qquad \forall y \forall v \forall x((R(v,y) \land R(x,y)) \rightarrow v = x) \land && (5)\\ &\qquad \forall v(\exists y R(v,y) \rightarrow (\forall x \subseteq_1 \ell_1 v)\exists z R(x,z)) \land && (6)\\ &\qquad \forall y(\exists v R(v,y) \rightarrow (\forall z \subseteq_2 \ell_2 y)\exists x R(x,z))) && (7)\end{aligned}$$

Intuitively, the point is this. Suppose two people are using their versions of LT, subscripted with '1' and '2' respectively. Then there is some second-order entity, a relation R, which takes us between their sets (1), exhausting the sets of one or the other person (2); which preserves membership (3); which is functional (4) and injective (5); and whose domain is an initial segment of one (6) or the other's (7) hierarchy. Otherwise put: LT is (internally) quasi-categorical.

As a bonus, this internal *quasi*-categoricity result can be lifted into an internal *total*-categoricity result. To explain how, consider this abbreviation (where 'P' is a second-order function-variable):

$$\underset{\infty}{\exists} x \Phi(x) \coloneqq \exists P(\forall x \Phi(P(x)) \land (\forall y : \Phi)\exists! x\ P(x) = y)$$

[20] This has been brought out by Parsons (1990, 2008), McGee (1997), and Väänänen and Wang (2015). The remainder of this section presents specific elements of Button and Walsh (2018: ch.11).

[21] Here, '$\subseteq$' and '*Lev*' should be defined in terms of 'ε' rather than '$\in$'; similarly for 'ℓ' in Theorem 6.2. For now, we can treat '*Pure*' as a primitive; but see Definition B.1.

[22] With one insignificant caveat (see footnotes 2 and 4): whereas classical logic guarantees that any model of LT contains an empty set, LT($\mathit{Pure}, \varepsilon$) allows that there may be no pure sets.

This formalizes the idea that there are as many Φs as there are objects *simpliciter*, i.e., that there is a bijection between the Φs and the universe. We can use this notation to state our internal total-categoricity result:

Theorem 6.3: This is a deductive theorem of impredicative second-order logic:

$$\begin{aligned}&(\text{LT}(Pure_1, \varepsilon_1) \wedge \underset{\infty}{\exists} x\, Pure_1(x) \wedge \text{LT}(Pure_2, \varepsilon_2) \wedge \underset{\infty}{\exists} x\, Pure_2(x)) \rightarrow \\ &\quad \exists R(\forall v \forall y(R(v,y) \rightarrow (Pure_1(v) \wedge Pure_2(y))) \wedge \\ &\qquad (\forall v : Pure_1)\exists! y R(v,y) \wedge (\forall y : Pure_2)\exists! v R(v,y) \wedge \\ &\qquad \forall v \forall y \forall x \forall z((R(v,y) \wedge R(x,z)) \rightarrow (v\ \varepsilon_1\ x \leftrightarrow y\ \varepsilon_2\ z)))\end{aligned}$$

Intuitively, if both LT-like hierarchies are as large as the universe, then there is a structure-preserving *bijection* between them. To see the significance of this result, note that it is common to claim that there are *absolutely infinitely many* pure sets. Whatever exactly this is meant to mean, it must surely entail that $\underset{\infty}{\exists} x\, Pure(x)$. So Theorem 6.3 tells us that absolutely infinite LT-like hierarchies are (internally) isomorphic.

7 LT as a subtheory of ZF

I have shown that any cumulative hierarchy satisfies LT, so that, in setting up a cumulative hierarchy, our only freedom of choice concerns its height. To make all of this more familiar, though, it is worth commenting on LT's relationship to ZF, the 'industry standard' set theory.

Unsurprisingly, ZF proves LT. In more detail: working in ZF, define the V_αs as usual; we can then show that the V_αs are the levels;[23] so Stratification holds as every set is a subset of some V_α.

Of course, ZF is much stronger than LT, since LT deliberately says nothing about the height of the cumulative hierarchy. If we want to set up a tall hierarchy, then three axioms naturally suggest themselves (where 'P' is a second-order function-variable in the statement of Unbounded):[24]

Endless $(\forall s : Lev)(\exists t : Lev) s \in t$

Infinity $(\exists s : Lev)((\exists q : Lev) q \in s \wedge (\forall q : Lev)(q \in s \rightarrow (\exists r : Lev) q \in r \in s))$

Unbounded $\forall P \forall a (\exists s : Lev)(\forall x \in a) P(x) \in s$

Endless says there is no last level. Infinity says that there is an infinite level, i.e. a level with no immediate predecessor. Unbounded states that the hierarchy of levels is so tall that no set can be mapped unboundedly into it. We now have some nice facts, whose proofs I leave to the reader:[25]

[23] *Proof sketch.* Working in ZF, fix α, and suppose for induction that $(\forall \beta < \alpha)(\forall x \subseteq V_\beta)(Lev(x) \leftrightarrow \exists \delta\ x = V_\delta)$. Fix $V_\gamma \subsetneq V_\alpha$; then $V_\gamma = ¶\{V_\delta : V_\delta \in V_\gamma\} = ¶\{s \in V_\gamma : Lev(s)\}$ by the induction hypothesis, which is a level by Lemma 3.8. Similarly, if $s \subsetneq V_\alpha$ is a level, then $\exists \delta\ s = V_\delta$.

[24] For Endless, cf. Montague (1965: 142), Scott (1974: 212), and Potter (1990: 20–1, 2004: 61–2). For Infinity, see Potter (2004: 68–70) and Boolos's (1989: 8) axiom Inf, which I discuss in §8.2.

[25] Cf. Scott (1974: 212) and Potter (1990: 20–4, 2004: 47–9, 61–2).

Proposition 7.1:

(1) LT proves Separation, Union, and Foundation.
(2) LT + Endless proves Pairing and Powersets.
(3) LT + Endless + Infinity proves Zermelo's axiom of infinity.[26]
(4) LT + Endless + ¬Infinity is equivalent to $\mathrm{ZF}_{\mathrm{fin}}$.[27]
(5) LT + Infinity + Unbounded proves Endless.
(6) LT + Infinity + Unbounded is equivalent to ZF.

Facts (1)–(3) show that LT + Endless + Infinity extends Zermelo's Z. This extension is strict, since Stratification is independent from Z.[28] Fact (6) then offers a neat way to conceive of ZF, as extending the theory which holds of any cumulative hierarchy, i.e. LT, with specific claims about the hierarchy's height.

8 Conclusion, and LT's predecessors

The theory LT holds of every cumulative hierarchy. Since LT is also quasi-categorical, the only choice we have, in setting up a cumulative hierarchy, is over the hierarchy's height.

I will close this paper by discussing LT's predecessors, in roughly chronological order.

8.1 Scott

At a talk in 1957, Scott presented what seems to have been the first theory of stages. This was an axiomatic theory of *ranks*, in the sense of the V_αs. Writing '$a < b$' for 'a has lesser rank than b', Scott's suggested axioms were Extensionality and:[29]

$$\forall a \forall b (a < b \leftrightarrow (\exists x < b) x \nless a)$$
$$\forall F (\forall a ((\forall x < a) F(x) \rightarrow F(a)) \rightarrow \forall a F(a))$$
$$\forall F \forall a \exists b \forall x (x \in b \leftrightarrow (F(x) \land x < a))$$

This 1957 theory is clearly satisfied in any $\mathcal{V}_\alpha$ with $\alpha > 0$, when $\in$ and $<$ are given the obvious interpretations. However, it has some unintended models.

Example 8.1: Let the domain have two sets: $\emptyset$ and a Quine atom $a = \{a\}$. Let $a < \emptyset$. This is a model of the 1957 theory, since $<$ is trivially a well-order, and since the only sets given by the third axiom are $\emptyset$ and $\{a\} = a$.

[26] i.e. $(\exists w \ni \emptyset)(\forall x \in w) x \cup \{x\} \in w$.
[27] The latter is the theory with all of ZF's axioms except that: (i) Zermelo's axiom of infinity is replaced with its negation; and (ii) it has a new axiom, $\forall a (\exists t \supseteq a)(t$ is transitive).
[28] Potter (2004: 293ff) makes a similar point. The independence is immediate from the fact that there are models of (even second-order) Z which fail to satisfy $\forall a (\exists c \supseteq a)(c$ is transitive); see Drake (1974: 111). For detailed discussions of Z's weaknesses, as either a first- or second-order theory, see Mathias (2001) and Uzquiano (1999). (As mentioned in the introduction, although I have formulated LT as a second-order theory, it has a natural first-orderization. Read uniformly as either first-order or second-order theories, and closing under provability, the point is: $\mathrm{Z} \subsetneq \mathrm{LT} + \mathrm{Endless} + \mathrm{Infinity} \subsetneq \mathrm{ZF}$.)
[29] Scott (1960); I have tweaked the presentation slightly.

Example 8.2: Let the domain have four sets: $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. Permute the usual rank relation, so that $\{\emptyset\} < \emptyset < \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$, with $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ incomparable.

At a talk in 1967, Scott provided a vastly improved theory of stages. I will present the 1967 theory in a slightly simplified form, starting with a definition given later by Potter (see §8.4):

Definition 8.3: For each set a, let $\text{Acc}a = \{x : (\exists c \in a)(x \in c \lor x \subseteq c)\}$, if it exists.

Scott's 1967 theory treats the notion of *level* as a primitive, which applies to certain sets. Temporarily using bold-face letters to range over these *levels*, the 1967 theory comprises just Extensionality, Separation, and two further axioms:[30]

Accumulation $\forall \mathbf{s}\ \mathbf{s} = \text{Acc}\{\mathbf{r} : \mathbf{r} \in \mathbf{s}\}$
Restriction $\forall a \exists \mathbf{s}\ a \subseteq \mathbf{s}$

Scott's 1967 theory (unlike his 1957 theory) does not explicitly state that the levels are well-ordered; instead, the 1967 theory *proves* the well-ordering of the levels (cf. §5).[31] We have Scott to thank for a truly remarkable bit of mathematics-cum-conceptual-analysis.

Scott's 1967 theory obviously inspires ST: compare his Restriction axiom with my Staging (and Stratification), and his Accumulation axiom with my Lemma 4.6.5 (and Lemma 3.8). Moreover, Scott's 1967 theory and ST make exactly the same claims about sets (cf. Theorem 4.1). But I used ST in §1, rather than Scott's 1967 theory, since ST is easier to motivate. In particular, Scott simply instructs us to write '$\mathbf{s} \in \mathbf{t}$' for '$\mathbf{s}$ is before $\mathbf{t}$', and his justification of Accumulation amounts to stipulating that 'a given level is *nothing more than* the accumulation of all the members and subsets of all the *earlier* levels'.[32] Both claims are very natural, and they are true in LT; but it is not immediately obvious that they are true of the Basic Story I told in the introduction. (In fairness to Scott, he does not start with that story, but with a related justificatory tale.)

8.2 Boolos and Shoenfield

The second source of inspiration for ST is Boolos. He first presented a theory of stages in 1971, which included explicit axioms stating that the stages are well-ordered;[33] this theory has several similarities with Shoenfield 1967.[34] Boolos then presented a better theory of stages in 1989, explicitly drawing from Scott's 1967 theory to prove (rather than assume) a principle of induction for stages.[35] My

30 Scott (1974: 208–9). Scott allowed urelements, which I am ignoring for ease of presentation (though see §A).
31 Scott's (1974: 211–2) proof uses the idea of a *grounded* set, introduced by Montague (1955).
32 Both quotes from Scott (1974: 209); his emphasis; variables adjusted to match surrounding text.
33 See Boolos's (1971: 223–4) I–IV and Induction Axioms.
34 Shoenfield (1967: 238–40).
35 Scott (1974: 211–2) and Boolos (1989: 11–12); Boolos cites Shoenfield's (1977: 327) presentation of Scott.

theory ST tweaks Boolos's 1989 theory in three ways.

First. Boolos has qualms about how to justify Extensionality;[36] I have no such qualms.

Second. Boolos aims to vindicate the traditional Zermelian axioms of Foundation, Union, Pairing, Separation, Powersets, and Infinity. To secure these last two axioms, his 1989 theory contains:

$$\textbf{Inf}\quad \exists \mathbf{t}(\exists \mathbf{r}\, \mathbf{r} < \mathbf{t} \wedge (\forall \mathbf{r} < \mathbf{t})\exists \mathbf{s}(\mathbf{r} < \mathbf{s} < \mathbf{t}))$$

$$\textbf{Net}\quad \forall \mathbf{r}\forall \mathbf{s}\exists \mathbf{t}(\mathbf{r} < \mathbf{t} \wedge \mathbf{s} < \mathbf{t})$$

Boolos's Inf guarantees there is a stage with infinitely many predecessors, and his Net guarantees that there is no last stage. Since ST is deliberately silent on the height of any cumulative hierarchy, it has no similar axioms. However, if I had wanted to augment ST with the claim that there is no last stage, I would have offered $\forall \mathbf{s}\exists \mathbf{t}\; \mathbf{s} < \mathbf{t}$ (cf. Endless, from §7). Boolos's Net says more than this; it guarantees that stages are directed. Boolos's proof of Pairing relies upon this directedness,[37] but I cannot see why Boolos felt independently entitled to adopt Net rather than the weaker principle.

Third. The remainder of Boolos's 1989 theory comprises Order, Staging, and these two axioms:[38]

$$\textbf{When}\quad \forall \mathbf{s}\forall a(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$$

$$\textbf{Spec}\quad \forall F\forall \mathbf{s}((\forall x : F)x \prec \mathbf{s} \rightarrow \exists a\forall x(F(x) \leftrightarrow x \in a))$$

In the presence of Extensionality, the axioms When+Spec are equivalent to ST's Priority+Specification; but we need Extensionality to prove the right-to-left direction of When from Priority+Specification (see Lemma 4.4). Moreover, given Boolos's qualms about Extensionality, he cannot provide an intuitive justification for the right-to-left direction of When. If $(\forall x \in a)x \prec \mathbf{s}$, then there should certainly be some $b \preceq \mathbf{s}$ such that $\forall x(x \in b \leftrightarrow x \in a)$; but only Extensionality can justify the assertion that $b = a$. Crucially for Boolos's aims, though, Powersets can fail if we replace When+Spec with Priority+Specification in Boolos's theory: without Extensionality or the right-to-left direction of When, we might keep finding new empty sets at every stage in the hierarchy; there will then be no stage by which every subset of a set has been found, and hence no stage at which any powerset can be found.

8.3 *Scott and Montague*

I now want to return to Scott's 1967 theory. As mentioned in §8.1, this theory initially takes the notion of *level* as primitive. However, Scott notes that the primitive can be eliminated, by proving within the 1967 theory that s is a level iff $¶s \subseteq s \wedge (\forall a \in$

[36] Boolos (1989: 10–11).

[37] Boolos's (1989: 19) proof is as follows. Fix a and b; by Staging, there are $\mathbf{r}$ and $\mathbf{s}$ with $a \preceq \mathbf{r}$ and $b \preceq \mathbf{s}$. By Net, there is some $\mathbf{t}$ after both $\mathbf{r}$ and $\mathbf{s}$. So by Spec there is a set whose members are exactly a and b.

[38] Boolos (1989: 8) formulates Spec as a first-order scheme, but considers the second-order axiom on the next page.

$s)(\exists h \in s)(\forall k \subseteq h)(\P k \in s \land (\P k \in h \lor a \subseteq \P k))$. Scott developed this ideologically-spartan theory in joint work with Montague; they described their theory as 'rank free', so I will call it RF.[39] It has just three axioms: Extensionality, Separation, and

Hierarchy $\forall a \exists h(\forall k \subseteq h)(\exists s = \P k)(s \in h \lor a \subseteq s)$

The point of calling it 'rank free' was to highlight that RF takes no stance on the number of ranks in the hierarchy. More precisely, we have the external quasi-categoricity result that $\mathcal{M} \vDash$ RF iff $\mathcal{M} \cong \mathcal{V}_\alpha$ for some $\alpha > 0$ (assuming full second-order logic; cf. Theorem 6.1). To establish this, Montague and Scott first say that h is a *hierarchy* iff $(\forall k \subseteq h)(h \subseteq \P k \lor \P k = \bigcap(h \setminus \P k))$. They then let $\mathrm{R}a \coloneqq \bigcap\{\P h : h \text{ is a hierarchy} \land a \subseteq \P h\}$ for each a, and show that $\mathrm{R}a$ serves the role of a's 'rank' (cf. LT's notion of ℓa, as laid down in Definition 3.11).

Unfortunately, as Scott himself put it, the deductions from these axioms and definitions 'are quite lengthy'.[40] This led Scott to dismiss the significance of RF, writing: 'there seems to be no technical or conceptual advantage in reducing the number of primitive notions to the minimum.'[41]

Still, these lengthy deductions were intended to form a section of a monograph on axiomatic set theory. A complete manuscript of this monograph exists,[42] containing very minor markups, handwritten notes to the printers, and an accompanying list of 'Things to be Done' which amounts to nothing more than writing an Introduction and dealing with the mundane logistics of publication. Everything, in short, was almost ready to print.

Sadly, it was never printed. This was a serious loss. As explained in §1, there are good philosophical reasons for 'reducing the number of primitive notions to the minimum.' Moreover, whilst Montague's and Scott's *deductions* were 'quite lengthy', the *axioms* of RF are quite elegant. The lengthiness of the deductions from RF is down to the awkwardness of the definitions of *hierarchy* and $\mathrm{R}a$. If Montague and Scott had been aware of the definition of *history* and *level*, as given in Definition 2.2, they could have given some much briefer deductions. Indeed, these definitions make it easy to prove that RF and LT are equivalent. One direction of this equivalence is easy:

Proposition 8.4 (LT)**:** RF holds.

Proof. It suffices to prove Hierarchy. Fix a, let $h = \{s \in \ell a : Lev(s)\}$ and fix $k \subseteq h$. Now $\P k = \ell k$ by Lemma 3.12.8; so if $\P k = \ell k \notin h$, then $\ell k \notin \ell a$, so $a \subseteq \ell a \subseteq \ell k = \P k$ by Lemma 3.12.3. □

For the other direction of the quivalence, I must first prove some quick facts in RF:

Lemma 8.5 (RF)**:** For all a:

39 Montague (1965: 139), Montague et al. (unpublished: 161–2), and Scott (1974: 214).

40 Scott (1974: 214). Indeed, it occupies 13 dense sides of Montague et al. (unpublished: 161–74). The two key definitions are 22.7 and 22.21.

41 Scott (1974: 214).

42 This is Montague et al. (unpublished).

(1) if $\P a$ exists, then $\P a \notin a$
(2) $\P a$ exists
(3) if every member of a is a level, then $\P a$ is a level

Proof. *(1)* If $\P a \in a$, then $(\forall c \subseteq \P a) c \in \P a$. But this is impossible: by Separation, let $d = \{x \in \P a : x \notin x\}$; then $d \notin \P a$

(2) Fix a, and let h witness Hierarchy. Let $k = h$, so that $\P h$ exists and $\P h \in h \vee a \subseteq \P h$, i.e. $a \subseteq \P h$ by (1). Since $\P h$ is potent by Lemma 3.2, $\P a \subseteq \P h$ exists by Separation on $\P h$. Now, clearly $\P a \subseteq \P\P a$.

(3) Using Separation and (2), let $h = \{s \in \P a : Lev(s)\}$. I will first prove that $\P h = \P a$, and then that h is a history, so that $\P h = \P a$ is a level.

To see that $\P a = \P h$: since $h \subseteq \P a$, we have $\P h \subseteq \P\P a = \P a$ by Lemmas 3.2–3.3; and if $x \in \P a$ then $x \subseteq r \in a$ for some level r, so $r \in h$, and hence $x \in \P h$.

To see that h is a history, fix $s \in h$; it suffices to show that $s = \P(s \cap h)$. Since s is a level, $\P(s \cap h) \subseteq \P s = s$ by Lemmas 3.3–3.4. To see that $s \subseteq \P(s \cap h)$, fix $x \in s$; now $x \subseteq r \in s$ for some level r by Lemma 3.8; and $r \subseteq s \in \P a$ by Lemma 3.4, so $r \in \P a$ by Lemma 3.2 and hence $r \in h$; so $x \subseteq r \in (s \cap h)$, i.e. $x \in \P(s \cap h)$. □

Proposition 8.6 (RF)**:** LT holds.

Proof. It suffices to prove Stratification. Fix a, and let h witness Hierarchy, i.e., $(\forall k \subseteq h)(\P k \in h \vee a \subseteq \P k)$. Let $k = \{s \in h : Lev(s)\}$. By Lemma 8.5.3, $\P k$ is a level. Now if $\P k \in h$, then $\P k \in k$, contradicting Lemma 8.5.1; so $a \subseteq \P k$. □

This last proof helps to explain the intuitive idea behind RF's axiom Hierarchy.[43] Roughly, the h guaranteed to exist by Hierarchy has this property: for any initial sequence of levels $k \subseteq h$, the next level after all of them is $\P k$; and if a is not a subset of $\P k$, then $\P k$ is in h; and *hence* (but here I invoke a transfinite induction) the members of h are all the levels up to the first level including a. In short, the fundamental idea behind RF is quite elegant.

8.4 Derrick and Potter

As mentioned in §2, my definition of *level* is inspired by Derrick and Potter,[44] but I have simplified it. Here is a little more detail about that simplification. In his 1990 book, Potter explicitly built on Scott's 1967 theory and also on Derrick's unpublished lecture notes.[45] Now, Scott's Accumulation axiom (see §8.1) formalizes the claim that 'a given level is *nothing more than* the accumulation of all the members and subsets of all the *earlier* levels'.[46] This suggests the use of the Acc-operator, and so Potter offers Definition 8.3.[47] Potter then supplies the definition of *history* and *level* given in Definition 2.2, but using Acc rather than ¶. So, Potter stipulates that h is a

[43] Cf. Montague et al. (unpublished: 162).
[44] See especially Potter (1990: 16–20, 2004: 41–7).
[45] Potter (1993: 183–4, 1990: 22, 2004: vii, 54).
[46] Scott (1974: 209).
[47] Potter (1990: 16, 2004: 41, 50).

history iff $(\forall x \in h)x = \text{Acc}(x \cap h)$, and that s is a level iff $s = \text{Acc}h$ for some history h. Potter then proves that, so defined, the levels are well-ordered. And his own theory of levels is, in effect, just LT, with this slightly different explicit definition of '*Lev*'.[48] But the use of ¶, rather than Acc, simplifies things significantly, as illustrated by the brevity of §3.

A Adding urelements

In this paper, I restricted my attention to pure sets.[49] This was only for ease of exposition; in this appendix and the next, I will remove this simplifying assumption.

To accommodate urelements, we must tweak the Basic Story. The easiest way to do this (which I revisit in §B) is to assume that the urelements are 'always' available to be collected into sets:

The Urelemental Story. Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any things, each of which is either a set found before **s** or an urelement, we find a set whose members are exactly those things. We find nothing else at **s**.

To formalize this Story, we need a new primitive predicate, enabling us to distinguish sets from urelements: we take *Set* as primitive, and define $Ur(x) :\!= \neg Set(x)$. Stage Theory with Urelements, STU, now has six axioms:[50]

Empty-U	$(\forall u : Ur)\forall x\ \ x \notin u$
Ext-U	$(\forall a : Set)(\forall b : Set)(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$
Order	$\forall \mathbf{r}\forall \mathbf{s}\forall \mathbf{t}(\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t})$
Staging-U	$(\forall a : Set)\exists \mathbf{s}\ \ a \preceq \mathbf{s}$
Priority-U	$\forall \mathbf{s}(\forall a : Set)(a \preceq \mathbf{s} \rightarrow (\forall x \in a)(Ur(x) \vee x \prec \mathbf{s}))$
Spec-U	$\forall F\forall \mathbf{s}((\forall x : F)(Ur(x) \vee x \prec \mathbf{s}) \rightarrow (\exists a : Set)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \in a)))$

Empty-U says that no urelement has any members; the other axioms relativise ST to sets. As in §1, any cumulative hierarchy obviously satisfies STU, on the assumption that the urelements are all 'always' available to be arbitrarily collected into sets.

We obtain Level Theory with Urelements, LTU, by tweaking LT's key definitions. Specifically, I offer the following re-definition:[51]

[48] There are three other small differences: (1) Potter allows urelements; (2) he provides a first-order theory; (3) he offers a slightly more restricted version of Separation, whose second-order formulation is $\forall F(\forall s : Lev)\exists b\forall x(x \in b \leftrightarrow (F(x) \wedge x \in s))$, but this trivially entails the unrestricted version of Separation given (Potter's version of) Stratification.

[49] Montague (1965: 139), Scott (1974: 214), and Potter (1990, 2004) accommodate urelements from the outset.

[50] As in footnote 2: STU gives us a stage **s** 'for free', so that $\{x : Ur(x)\}$ exists by Spec-U.

[51] The first level is therefore $\{x : Ur(x)\}$. This follows Montague (1965: 139) and Potter (1990: 16, 2004: 41). By contrast, Scott's (1974: 214) first level is $\emptyset$, and the urelements are members of every subsequent level.

Definition A.1 (for §A only)**:** Say that a is potent iff $\forall x((Ur(x) \vee (\exists c : Set)x \subseteq c \in a) \rightarrow x \in a)$. Let $¶a := \{x : Ur(x) \vee (\exists c : Set)x \subseteq c \in a\}$, if it exists. Say that $Hist(h)$ iff $(\forall x \in h)x = ¶(x \cap h)$. Say that $Lev(s)$ iff $(\exists h : Hist)s = ¶h$.

The axioms of LTU are then Empty-U, Ext-U, Stratification (with '*Lev*' as redefined) and:[52]

Sep-U $\forall F \forall a(\exists b : Set)\forall x(x \in b \leftrightarrow (F(x) \wedge x \in a))$

The proofs of §§3–4 go now through with trivial changes. Specifically, the (redefined) levels are well-ordered, and STU and LTU make exactly the same demands on sets and urelements.

The (quasi-)categoricity results of §6 also carry over to LTU. Let $\mathcal{A}$ and $\mathcal{B}$ be models of LTU in full second-order logic, and suppose there is a bijection between their respective collections of urelements, $Ur^{\mathcal{A}}$ and $Ur^{\mathcal{B}}$. This bijection can be lifted to a quasi-isomorphism: $\mathcal{A}$ and $\mathcal{B}$ are isomorphic 'so far as they go', but the levels of one may outrun the other. This external result can also be 'internalised', yielding results analogous to Theorems 6.2 and 6.3.

Note that LTU, like LT before it, takes no stance on the height of the hierarchy. In particular, it has no version of Replacement. In this regard, LTU differs sharply from ZF(C)U, which is something like the 'industry standard' for iterative set theory with urelements. It is particularly noteworthy that LTU allows that the set of urelements may be larger than any pure set.[53] (For a trivial example, suppose there are exactly 3 urelements and exactly 2 levels; for a less trivial example, suppose there are exactly $\beth_{\omega+1}$ urelements but only an $\omega + \omega$ sequence of levels.)

B Adding absolutely infinitely many urelements

The Urelemental Story accommodates urelements in a humdrum way. However, there has been recent interest in a less humdrum approach, according to which there are *absolutely infinitely many* urelements. Here is a brisk, three-premise argument in favour of that approach, inspired by Christopher Menzel:[54]

(a) There are absolutely infinitely many levels in the cumulative hierarchy.
(b) There are at least as many ordinals as there are levels in the cumulative hierarchy.
(c) Ordinals are not really sets; they are urelements.

[52] Together, Stratification and Sep-U deliver the existence of $\{x : Ur(x)\}$; see the previous two footnotes.

[53] LTU could therefore be used in place of e.g. Menzel's ZFCU′ (2014: 67–71), which is designed to accommodate the claim that the set of urelements is not equinumerous with any pure set.

[54] Menzel (1986: 41ff); cf. Rumfitt (2015: 271–5). Menzel (2014: 57) also offers a second (very different) argument to the same conclusion.

Each premise is not implausible,[55] and they jointly entail that there are absolutely infinitely many urelements. In this appendix, I will explore that idea (without endorsing it).

B.1 Preliminary motivations and observations

There is an immediate technical issue: in this kind of cumulative setting, no set has absolutely infinitely many members.[56] This follows from a simple version of Cantor's Theorem. For reductio, suppose that some set, a, has absolutely infinitely many members. As discussed in §6, this entails that $\underset{\infty}{\exists} x\, x \in a$, i.e. there is a map, P, such that $\forall x\, P(x) \in a$ and $(\forall y \in a)\exists! x\, P(x) = y$. By P's injectivity and Separation,[57] there is some $d = \{x \in a : x \notin P^{-1}(x)\}$. Since $P(d) \in a$, contradiction follows familiarly.

So: if there are absolutely infinitely many urelements, then there is no set of all urelements.[58] But the existence of such a set is a trivial consequence of Spec-U, as laid down in §A. So, those who think that there are absolutely infinitely many urelements must reject Spec-U. Furthermore, since Spec-U follows from the third sentence of the Urelemental Story of §A, they must change their story.

Many alternative stories are possible, but the simplest approach is simply to bolt a Limitation of Size principle onto the Urelemental Story, insisting that the Basic Story remains correct of the pure sets, whilst denying that any set is absolutely infinite. This leads to the following:[59]

The $\underset{\infty}{\text{U}}$relemental Story. Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any things—provided both that (i) there are not absolutely infinitely many of them, and that (ii) each of them is either a set found before **s** or an urelement—we find a set whose members are exactly those things. We find nothing else at **s**. (NB: since the Basic Story is correct of the pure sets, we do not find absolutely infinitely many pure sets before **s**.)

In the remainder of this appendix, I will briefly sketch (equivalent) stage-theoretic and level-theoretic formalizations of this Story. For readability, I leave all proofs to the reader, with hints in footnotes.

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[55] Claim (a) can be motivated by a principle of plenitude concerning sets. Claim (b) can be motivated by combining the fact that the levels of any (pure) cumulative hierarchy are well-ordered (see §5) with the idea that any system of well-ordered objects exemplifies an ordinal (provided that the objects are all members of some set). Claim (c) can be motivated by a kind of platonistic structuralism, according to which ordinals are indeed *objects*, but not *sets*, since sets have structure which is not purely order-theoretic. For the record, I do not subscribe to this kind of platonistic structuralism.

[56] *Pace* Menzel (1986: 44–51, 2014: 71–9). Note that my argument does not involve Powersets (which Menzel ultimately rejects). Menzel escapes formal inconsistency, whilst retaining (a first-order version of) Separation, only because his set-theoretic object language has no way to pick out a suitable map, P, which witnesses the absolute infinity of his set $\{x : Ur(x)\}$.

[57] I take it that rejecting Separation is not an option in this setting; though see Pt.3 for an approach which rejects Separation.

[58] Uzquiano (2015: 330–1) also suggests the use of a set theory with urelements but no set of urelements, though for somewhat different reasons.

[59] Cf. Uzquiano (2015: 331).

B.2 Stage-theoretic approach: $\mathrm{ST}\underset{\infty}{\mathrm{U}}$

To axiomatize the $\underset{\infty}{\mathrm{U}}$relemental Story, we need a predicate, '*Pure*', to pick out the pure sets (cf. §6). Since we have assumed that the Basic Story holds of the pure sets, we can define '*Pure*' explicitly:

Definition B.1: Say that a is pure, $Pure(a)$, iff both $Set(a)$ and there is some transitive $c \supseteq a$ whose members are all sets.

To axiomatize the $\underset{\infty}{\mathrm{U}}$relemental Story, we also need a way to formalize 'there are absolutely infinitely many Φs'. There are familiar concerns about the possibility of formalizing this idea.[60] Nonetheless, if there are absolutely infinitely many Φs, then certainly $\underset{\infty}{\exists}x\Phi(x)$ (cf. §6). Conversely, if $\underset{\infty}{\exists}x\Phi(x)$, then no property can have *more* instances than Φ. So, '$\underset{\infty}{\exists}x\Phi(x)$' will serve as our proxy for 'there are absolutely infinitely many Φs'.[61]

I can now lay down the theory $\mathrm{ST}\underset{\infty}{\mathrm{U}}$. Its axioms are Empty-U, Ext-U, Order, Staging-U, Priority-U, and the following:

Spec-$\underset{\infty}{\mathrm{U}}$ $\forall F \forall \mathbf{s}((\neg\underset{\infty}{\exists}xF(x) \wedge (\forall x : F)(Ur(x) \vee x \prec \mathbf{s})) \rightarrow (\exists a : Set)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \in a)))$

LoS-$\underset{\infty}{\mathrm{U}}$ $(\forall a : Set)\neg\underset{\infty}{\exists}x\, x \in a$

Pure-$\underset{\infty}{\mathrm{U}}$ $\forall F \forall \mathbf{s}((\forall x : F)(Pure(x) \wedge x \prec \mathbf{s}) \rightarrow \neg\underset{\infty}{\exists}xF(x))$

Many-$\underset{\infty}{\mathrm{U}}$ $\underset{\infty}{\exists}xUr(x)$

In brief: Spec-$\underset{\infty}{\mathrm{U}}$ restricts Spec-U to capture conditions (i)–(ii) of the $\underset{\infty}{\mathrm{U}}$relemental Story; LoS-$\underset{\infty}{\mathrm{U}}$ enshrines Limitation of Size, which follows from condition (i) plus the fact that 'we find nothing else' at any stage; Pure-$\underset{\infty}{\mathrm{U}}$ formalizes the parenthetical 'NB' of the Story; and Many-$\underset{\infty}{\mathrm{U}}$ formalizes the claim that there are absolutely infinitely many urelements.

B.3 Level-theoretic approach: $\mathrm{LT}\underset{\infty}{\mathrm{U}}$

$\mathrm{ST}\underset{\infty}{\mathrm{U}}$ is a multi-sorted, stage-theoretic, formalization of the $\underset{\infty}{\mathrm{U}}$relemental Story. That Story can instead be given a single-sorted formalization, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$. To do this, I start by tweaking LT's key definitions:

Definition B.2 (for §B only)**:** Say that a is potent iff $(\forall x : Set)(\exists c(x \subseteq c \in a) \rightarrow x \in a)$. Let $¶a = \{x : Set(x) \wedge \exists c(x \subseteq c \in a)\}$, if it exists. Say that $Hist(h)$ iff $(\forall x \in h)x = ¶(x \cap h)$. Say that $Lev(s)$ iff $(\exists h : Hist)s = ¶h$.

Using these redefinitions, we can prove analogues of Lemmas 3.4–3.9 from §3. Specifically, given Ext-U and Sep-U, we can prove that the levels (so defined) are potent, transitive, pure,[62] and well-ordered by $\in$.

[60] See e.g. McGee (1992: 279).

[61] Very little of what I say depends upon this particular choice of proxy. In particular, I rely upon its logical properties only when claiming that both $\mathrm{ST}\underset{\infty}{\mathrm{U}}$ and $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ prove Sep-U, and in my remarks on the quasi-categoricity of $\mathrm{LT}\underset{\infty}{\mathrm{U}}$.

[62] Since they are transitive, they witness their own purity.

I can now lay down $\mathrm{LT}\underset{\infty}{\mathrm{U}}$. It uses a primitive one-place function symbol, $\mathbf{L}$, where '$\mathbf{L}a$' should be read as *a's level-index*. (I discuss the use of this primitive in §B.4.) Then $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ has six axioms: Empty-U, Ext-U, LoS-$\underset{\infty}{\mathrm{U}}$, Many-$\underset{\infty}{\mathrm{U}}$, and two axioms governing $\mathbf{L}$:

Leveller $(\forall a : Set)((\exists s : Lev)\mathbf{L}a = s \wedge$
$(\forall x : Set)(x \in a \rightarrow \mathbf{L}x \in \mathbf{L}a) \wedge$
$(\forall s : Lev)(s \in \mathbf{L}a \rightarrow (\exists x : Set)(x \in a \wedge s \subseteq \mathbf{L}x)))$

Consolidation $\forall F((\neg \underset{\infty}{\exists} x F(x) \wedge \exists a(\forall x : F)(Ur(x) \vee \mathbf{L}x \in a)) \rightarrow$
$(\exists b : Set)\forall x(F(x) \leftrightarrow x \in b))$

To understand these axioms, note that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ guarantees that the (pure) levels are well-ordered by membership.[63] Now, Leveller ensures that the levels index the sets; intuitively, a's level-index is the least level greater than the level-index of every set in a. Consolidation then allows us to find all the impure sets we would want to find 'at' any given level. Finally, note that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ proves a pure-analogue of Stratification:[64]

Lemma B.3 ($\mathrm{LT}\underset{\infty}{\mathrm{U}}$)**:** If a is pure, then $a \subseteq \mathbf{L}a$

Consequently, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$'s pure sets can be thought of as satisfying LT. Indeed, if we define '$x \ \varepsilon \ y$ as '$Pure(x) \wedge Pure(y) \wedge x \in y$', then $\mathrm{LT}\underset{\infty}{\mathrm{U}} \vdash \mathrm{LT}(Pure, \varepsilon)$, as defined in §6. It follows that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ is externally and internally (quasi-)categorical: any two hierarchies satisfying $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ have quasi-categorical pure sets; moreover, if there is a bijection between the hierarchies' urelemental bases, their impure sets are quasi-categorical. (However, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$'s analogue of Theorem 6.1 is more restricted: if $\mathcal{M}$ is a standard, set-sized, model of $\mathrm{LT}\underset{\infty}{\mathrm{U}}$, then $|Ur^{\mathcal{M}}|$ is regular.)[65]

In fact, $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ and $\mathrm{ST}\underset{\infty}{\mathrm{U}}$ are provably equivalent, concerning sets and urelements. To prove that $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ interprets $\mathrm{ST}\underset{\infty}{\mathrm{U}}$, tweak the $*$-translation of §4, so that $(x \preceq \mathbf{s})^* :=$ $\mathbf{L}x \subseteq \mathbf{s}$.[66] It is then easy to show that $\mathrm{LT}\underset{\infty}{\mathrm{U}} \vdash \mathrm{ST}\underset{\infty}{\mathrm{U}}^*$ (cf. Lemma 4.2).

To show that $\mathrm{ST}\underset{\infty}{\mathrm{U}}$ interprets $\mathrm{LT}\underset{\infty}{\mathrm{U}}$, first note that $\mathrm{ST}\underset{\infty}{\mathrm{U}}$ proves Sep-U and the converse of Priority-U (cf. Lemmas 4.3–4.4). Then tweak Definition 4.5 (cf. Definition B.2):

Definition B.4 (for §B only)**:** Let $\check{\mathbf{s}} := \{x \prec \mathbf{s} : Pure(x)\}$. Say that a is a *slice* iff $a = \check{\mathbf{s}}$ for some $\mathbf{s}$.

---

63 To see this, note $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ proves Sep-U, and combine this with the remarks after Definition B.2.

64 Use induction on levels, together with the second conjunct of Leveller.

65 Assuming Choice. *Proof.* Let $\kappa = |Ur^{\mathcal{M}}|$. By Consolidation, every smaller-than-κ subset of $Ur^{\mathcal{M}}$ is in $Set^{\mathcal{M}}$. So κ is infinite, by Many-$\underset{\infty}{\mathrm{U}}$. For each $\lambda < \kappa$, there are κ^{λ} subsets of $Ur^{\mathcal{M}}$ with cardinality λ, so that $\kappa^{\lambda} \leq |Set^{\mathcal{M}}|$. So if $\mathrm{cf}(\kappa) < \kappa$, then by König's Theorem $\kappa < \kappa^{\mathrm{cf}(\kappa)} \leq |Set^{\mathcal{M}}|$, contradicting Many-$\underset{\infty}{\mathrm{U}}$; hence $\mathrm{cf}(\kappa) = \kappa$. (Thanks to Gabriel Uzquiano for suggesting I consider how $\mathrm{LT}\underset{\infty}{\mathrm{U}}$ interacts with regular cardinals.)

66 Stipulate that $(Set(x))^* := Set(x)$.

It follows that the slices are the levels (in the senses of Definitions B.2 and B.4; cf. Lemma 4.7).[67] We can then interpret LT$\underset{\infty}{\mathrm{U}}$'s unique primitive, **L**, via ρ, defined as follows:[68]

Definition B.5: For each set a, let $\rho a := \bigcap\{\check{\mathbf{s}} : a \preceq \mathbf{s} \land \neg\exists \mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})\}$.

Theorem B.6: ST$\underset{\infty}{\mathrm{U}}$ $\vdash \phi^\rho$ iff LT$\underset{\infty}{\mathrm{U}}$ $\vdash \phi$, for any LT$\underset{\infty}{\mathrm{U}}$-sentence ϕ, where ϕ^ρ is the formula which results from ϕ by replacing each instance of **L** with ρ.

The upshot is that no information about sets or urelements is lost or gained in moving from ST$\underset{\infty}{\mathrm{U}}$ to LT$\underset{\infty}{\mathrm{U}}$. Since any hierarchy which is described by the $\underset{\infty}{\mathrm{U}}$relemental Story satisfies ST$\underset{\infty}{\mathrm{U}}$, it also satisfies LT$\underset{\infty}{\mathrm{U}}$. And LT$\underset{\infty}{\mathrm{U}}$ is quasi-categorical. Our work on the $\underset{\infty}{\mathrm{U}}$relemental Story is complete.

B.4 Eliminating primitives and first-orderisation

Or rather: almost complete. Given the discussion of §1, we may want to eliminate LT$\underset{\infty}{\mathrm{U}}$'s primitive, **L**. This is easily done within second-order logic: just conjoin Leveller and Consolidation, and bind **L** with a (second-order) existential quantifier. But if we are willing to make some further assumptions, then we can eliminate **L** using certain *first-order* functions.[69]

Roughly, a *ranking-function*: (1) has a transitive domain (setting aside urelements); and (2) behaves like **L** where defined. More formally:

Definition B.7: Say that a function f is a *ranking-function* iff, for all $a \in \mathrm{dom}(f)$, both:

(1) $Set(a)$ and $(\forall x : Set)(x \in a \rightarrow x \in \mathrm{dom}(f))$; and
(2) $Lev(f(a))$ and $(\forall x : Set)(x \in a \rightarrow f(x) \in f(a))$ and $(\forall s : Lev)(s \in f(a) \rightarrow (\exists x \in a) s \subseteq f(x))$.

Say that $Ranks(f, a)$ iff f is a ranking-function with $a \in \mathrm{dom}(f)$.

It is easy to show that ranking-functions agree wherever they are defined, i.e.:

Lemma B.8 (Ext-U, Sep-U)**:** If $Ranks(f, a)$ and $Ranks(g, a)$, then $f(a) = g(a)$.

[67] For the analogue of Lemma 4.6: Ext-U, Pure-$\underset{\infty}{\mathrm{U}}$, and Spec-$\underset{\infty}{\mathrm{U}}$ guarantee that $\check{\mathbf{s}}$ exists for each stage $\mathbf{s}$; in clauses (2)–(3), the quantifier '$\forall a$' becomes '$(\forall a : Pure)$'; and note that each slice witnesses its own purity.

[68] Since ST$\underset{\infty}{\mathrm{U}}$ does not prove that all stages are comparable (cf. the discussion of Boolos's Net from §8.2), it takes several steps to vindicate Definition B.5. First: show that stages obey $<$-induction. Second: show that if $a \preceq \mathbf{s}$ and $\neg\exists \mathbf{r}(a \preceq r < \mathbf{s})$ and $a \preceq \mathbf{t}$ and $\neg\exists \mathbf{r}(a \preceq r < \mathbf{t})$ then $\check{\mathbf{s}} = \check{\mathbf{t}}$; it follows that ρa is a slice. Third: combine this with the fact that the slices are levels, to show that ρ behaves like **L**.

[69] Lévy and Vaught (1961: 1047) and Uzquiano (1999: 299) present a somewhat similar method for defining the rank of a set (via functions on ordinals). Here I treat first-order functions as sets of ordered pairs in the normal way, and $x \in \mathrm{dom}(f)$ abbreviates $\exists y\, \langle x, y\rangle \in f$. Of course, a fully first-order version of LT$\underset{\infty}{\mathrm{U}}$ would need to define '$\underset{\infty}{\exists} x F(x)$' differently (cf. footnote 61).

We can now replace Leveller, in $\mathrm{LT}\underset{\infty}{\mathrm{U}}$, with $(\forall a : \mathit{Set})\exists f\, \mathit{Ranks}(f, a)$. Note that this claim is *independent* of $\mathrm{LT}\underset{\infty}{\mathrm{U}}$: it guarantees that every set is a member of some set, and so guarantees that the hierarchy has no final stage (cf. Endless from §7). Still, this allows us to define $\mathbf{L}a \coloneqq \bigcap\{f(a) : \mathit{Ranks}(f, a)\}$. We can use this definition in Consolidation, and prove Leveller via Definition B.7.

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Level Theory, Part 2

Axiomatizing the bare idea of a potential hierarchy

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This document contains preprints of Level Theory, Parts 1–3. All three papers are forthcoming at *Bulletin of Symbolic Logic*.

Abstract. Potentialists think that the concept of set is importantly modal. Using tensed language as a heuristic, the following bare-bones story introduces the idea of a potential hierarchy of sets: 'Always: for any sets that existed, there is a set whose members are exactly those sets; there are no other sets'. Surprisingly, this story already guarantees well-foundedness and persistence. Moreover, if we assume that time is linear, the ensuing modal set theory is almost definitionally equivalent with non-modal set theories; specifically, with Level Theory, as developed in Part 1.

> What we need to do is to replace the language of time and activity by the more bloodless language of potentiality and actuality.
>
> Parsons (1977: 293)

Potentialists, such as Charles Parsons, Øystein Linnebo, and James Studd, think that the concept of *set* is importantly modal. Put thus, potentialism is a broad church; different potentialists will disagree on the precise details of the relevant modality.[1] My aim is shed light on potentialism, in general, using Level Theory, LT, as introduced in Part 1.

I start by formulating Potentialist Set Theory, PST. This uses a tensed logic to formalize the bare idea of a 'potential hierarchy of sets'.[2] Though PST is extremely minimal, it packs a surprising punch (see §§1–4).

In the vanilla version of PST, we need not assume that time is linear. However, if we make that assumption, then the resulting theory is almost definitionally equivalent to LT, its non-modal counterpart (see §§5–8). This equivalence allows me to clarify Hilary Putnam's famous claim, that modal and non-modal set theories express the same facts (see §9). Putting my cards on the table: I am not a potentialist, in part because I am so sympathetic with Putnam's claim.

This paper presupposes familiarity with Part 1. My notation conventions are as in Pt.1 §0, with the addition that I use $\vec{x}$ for an arbitrary sequence, writing things

[1] See e.g. Fine (2006), Linnebo (2013: 209, 2018a: 264–5, 2018b: 61–5), and Studd (2013: 706–7, 2019: 144–53).

[2] This is Linnebo's (2013) phrase.

like $F(\vec{x})$ rather than $F(x_1, \ldots, x_n)$. For readability, all proofs are relegated to the appendices.

1 Tense and possibility

Many potentialists hold that temporal language serves as a useful heuristic for their favoured mathematical modality. To illustrate the idea, consider what Studd calls the *Maximality Thesis*: 'any sets *can* form a set.'[3] This Thesis is given a modal formulation. But, as Studd notes, it can be glossed temporally: 'any sets *will* form a set'. Of course, no potentialist will take this temporal gloss literally. Nobody, after all, wants to countenance absurd questions like 'which pure sets existed at noon today?', or 'which pure sets will exist by teatime?'[4] The idea, to repeat, is just that temporal language is a useful *heuristic* for the potentialist's preferred modality.

To elaborate on this heuristic, consider the bare-bones story of (pure) sets, which I told and explored in Part 1, and which I will repeat here:

> **The Basic Story.** Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find a set whose members are exactly those sets. We find nothing else at **s**.

We can regard the stages of this Story as moments of time. Regarded thus, the Basic Story adopts the *tenseless* view of time, according to which moments are just a special kind of object. But this tenseless approach serves potentialists poorly. At no stage is there a set of all the sets which are found at any stage, so this tenseless Story falsifies the claim 'any sets *will* form a set'.

Familiarly, though, time can also be thought of in a *tensed* fashion. On the tensed approach, we do not quantify over moments or stages; rather, we use primitive temporal operators, like 'it *was* the case that. . . ' or 'previously: . . . '. And we can retell the Basic Story in tensed terms:

> **The Tensed Story.** Always: for any sets that existed, there is a set whose members are exactly those sets; there are no other sets.

Unlike the Basic Story, this Tensed Story is compatible with the claim '(always:) any sets *will* form a set'.

Note, though, that I say 'is compatible with', rather than 'entails'. If time abruptly ends, then some things will never form a set. And, by design, the Tensed Story is compatible both with the claim that time abruptly ends, and that time is endless. Otherwise put: it says nothing at all about the 'height' of any potential hierarchy. This silence is deliberate, for potentialists might disagree about questions of 'height'.

Still, once potentialists have agreed to use tense as an heuristic for their preferred modality, I do not see how they could doubt that the Tensed Story holds of every

[3] Studd (2013: 699). Linnebo (2013: 206–8) formulates a similar thesis.

[4] For further issues, see e.g. Parsons (1977: §II) and Studd (2013: 706, 2019: 49).

potential hierarchy of sets. In what follows, then, I take it for granted that the Tensed Story presents us with the *bare idea* of a potential hierarchy

2 Temporal logic for past-directedness

My first goal is to axiomatize the Tensed Story. For this, I will employ a temporal logic. In particular, I use a negative free second-order logic which assumes that time is past-directed. Here is a brief sketch of this *past-directed-logic*, with fuller explanations in footnotes. (Let me take this opportunity to flag that I am wholly indebted to Studd for the idea of investigating potentialism via temporal logic; see §10.2.)

We use '$\mathrm{E}(x)$' as an existence predicate; it abbreviates '$x = x$'. We prohibit consideration of never-existent entities, and we insist that quantification and atomic truth require existence.[5] We have three temporal operators (with their obvious duals):[6]

⬘: A past-tense operator; gloss '⬘ϕ' as 'previously: ϕ' or 'it *was* the case that ϕ' .

⬙: A future-tense operator; gloss '⬙ϕ' as 'eventually: ϕ or 'it *will be* the case that ϕ'.

$\Diamond$: An unlimited temporal operator; gloss '$\Diamond\phi$' as 'sometimes: ϕ'.

We have Necessitation rules: if ϕ is a theorem, then so are both ⬒ϕ and ⬓ϕ. We then lay down schemes as follows:[7]

⬒$(\phi \to \psi) \to ($⬒$\phi \to$ ⬒$\psi)$	⬓$(\phi \to \psi) \to ($⬓$\phi \to$ ⬓$\psi)$
⬘⬓$\phi \to \phi$	⬙⬒$\phi \to \phi$
⬒$\phi \to$ ⬒⬒ϕ	⬘$(\phi \land$ ⬒$\phi) \to$ ⬒$(\phi \lor$ ⬘$\phi)$

The first two schemes are familiar distribution principles. The second two schemes ensure appropriate past/future interaction. The fifth scheme ensures that *before* is transitive. The last scheme characterizes *past-directedness*.[8] Given past-directedness, 'sometimes: ϕ' amounts to 'it was, is, will be, or was going to be the case that ϕ'. So we adopt this scheme:

$$\Diamond\phi \leftrightarrow (\text{⬘}\phi \lor \phi \lor \text{⬙}\phi \lor \text{⬘⬙}\phi)$$

It follows that $\Diamond$ obeys S5. This completes my sketch of past-directed-logic.

In what follows, I will assume that potentialists are happy to use this temporal logic.[9] However, it is worth repeating that our potentialist only regards time as an

[5] So, we adopt the axiom scheme $\Diamond\mathrm{E}(x)$, and inference rules so that these schemes hold: (1) $\exists x\phi \to \exists x(\mathrm{E}(x) \land \phi)$ and $\exists F\phi \to \exists F(\mathrm{E}(F) \land \phi)$, for any formula ϕ; (2) $\alpha(\vec{x}) \to (\mathrm{E}(x_1) \land \ldots \land \mathrm{E}(x_n))$, for any atomic $\alpha(\vec{x})$ with all free variables displayed; (3) $F(\vec{x}) \to \mathrm{E}(F)$ for any 'F'. We also have second-order Comprehension, i.e. the scheme $\exists F\forall\vec{x}(F(\vec{x}) \leftrightarrow \phi)$, for any ϕ not containing 'F'.

[6] i.e. ⬒$\phi :=$ $\neg$⬘$\neg\phi$ and ⬓$\phi :=$ $\neg$⬙$\neg\phi$ and $\Box\phi := \neg\Diamond\neg\phi$.

[7] See e.g. Goldblatt (1992: 41) for all but the last scheme.

[8] i.e. this frame-condition: $(\forall \mathbf{v} \leq \mathbf{w})(\forall \mathbf{u} \leq \mathbf{w})(\exists \mathbf{t} \leq \mathbf{v})\mathbf{t} \leq \mathbf{u}$. Equivalently: if $\mathbf{v}$ and $\mathbf{u}$ are path-connected, then $(\exists \mathbf{t} \leq \mathbf{v})\mathbf{t} \leq \mathbf{u}$. We say that worlds are *path-connected* iff they are related by the reflexive, symmetric, transitive closure of the accessibility relation.

[9] Though note that not all potentialists have used temporal logics; see §10.1.

heuristic. Ultimately, they want $\Diamond$ to express their favoured mathematical modality. So they will need to explain how (and why) their favoured modality decomposes into other operators, ⬘ and ⬙, which obey past-directed-logic. This is a non-trivial demand; but, for the purposes of this paper, I assume it can be met.

3 Potentialist Stage Theory

Armed with past-directed-logic, the Tensed Story is easy to axiomatize. Let PST, for Potentialist Set Theory, be the result of adding these four axioms to past-directed-logic:

$$\begin{aligned}
&\textbf{Mem}_{\Diamond}\quad \forall a \Box \forall x(\Diamond x \in a \rightarrow \Box(\mathrm{E}(a) \rightarrow x \in a))\\
&\textbf{Ext}_{\Diamond}\quad \forall a \Box \forall b(\Box \forall x(\Diamond x \in a \leftrightarrow \Diamond x \in b) \rightarrow \Diamond a = b)\\
&\textbf{Priority}_{⬘}\quad \forall a(\forall x \in a)⬘\mathrm{E}(x)\\
&\textbf{Spec}_{⬘}\quad \forall F((\forall x : F)⬘\mathrm{E}(x) \rightarrow \exists a \forall x(F(x) \leftrightarrow x \in a))
\end{aligned}$$

The first two axioms are not explicit in the Tensed Story, but I take it they are supposed to be something like analytic: roughly, $\text{Mem}_{\Diamond}$ says that each set a has its members essentially, and $\text{Ext}_{\Diamond}$ says that if everything which could (ever) be in a could be in b, and vice versa, then $a = b$ (when they exist).[10] The next two axioms are explicit in the Story: $\text{Priority}_{⬘}$ says that a set's members existed before the set itself, and $\text{Spec}_{⬘}$ says that, if every F existed earlier, then the set of Fs exists. So all of PST's axioms are obviously true of the Tensed Story.

It is worth comparing PST with Stage Theory, ST (see Pt.1 §1). Indeed, we could equally think of PST as Potentialis*ed Stage* Theory, since it is little more than the most obvious reworking of ST using tensed operators.[11]

4 The inevitability of well-foundedness and persistence

I have just shown that PST is a good formalization of the Tensed Story. As explained in §1, though, this Story articulates the *bare idea* of a potential hierarchy of sets. It follows that any potential hierarchy satisfies PST. This is significant, since PST is surprisingly rich.

To gauge PST's depths, I will explain how it relates to Level Theory, LT, the non-modal theory which axiomatizes the (tenseless) Basic Story (see Pt.1 §§1–5). According to LT, the sets are arranged into well-ordered *levels*, where levels are sets which goes proxy for the stages of the Basic Story. Now, PST proves the following result (see §A):

Theorem 4.1 (PST)**:** Where $Max(s)$ abbreviates $(\mathrm{E}(s) \wedge \forall x\, x \subseteq s)$:

(1) LT holds

[10] See Parsons (1977: 286(3)), Studd (2013: 711–12, 2019), and Linnebo (2013: 215, 2018b: 211–2).

[11] But PST is indeed *more*: PST assumes past-directedness, and ST has no comparable assumption about stages. (Cf. the discussion of Boolos's (1989) Net in Pt.1 §8.2.) For the technical role of past-directedness, see the end of §A.

(2) $\forall x ⬓ \mathrm{E}(x)$
(3) $(\exists s : Lev)Max(s)$
(4) $(\forall s : Lev)\Diamond Max(s)$

If we consider a Kripke model of PST: (1) says that every possible world comprises a hierarchy of sets, arranged into well-ordered levels. Among other things, this yields *well-foundedness*, i.e. $\forall F(\exists x F(x) \rightarrow (\exists x : F)(\forall z : F)z \notin x)$.[12] Then (2) is a statement of *persistence*; it says that, once a set exists, it exists forever after. Last, (3) says that every world has a maximal level, and (4) says that every level is some world's maximal level. So, the worlds in a Kripke model of PST are, in effect, just arbitrary, persistent, initial segments of an (actualist) LT-hierarchy of pure sets.

I will develop the link between PST and LT over the next few sections. First, I want to highlight the significance of Theorem 4.1. The Tensed Story does not involve an explicit statement of well-foundedness or persistence. So one *might* try to entertain versions of the Tensed Story wherein well-foundedness or persistence fail: that is, one might try to entertain a potential hierachy wherein time had no beginning, or wherein sets fade in and out of existence. But the foregoing remarks show that all such speculation is incoherent: every potentialist hierarchy *must* obey well-foundedness and persistence, since every potentialist hierarchy obeys PST, and PST proves Theorem 4.1. Echoing Scott, then, we see 'how little choice there is in setting up' a potential hierarchy of sets.[13]

5 Linear Potentialist Stage Theory

So far, our potentialist has assumed that time is past-directed (to use the tensed-heuristic). If we also assume that time is *linear*, then we can obtain even deeper connections between PST and LT. I will spell out these connections in §§6–8; first, I must say a bit about linearity.

Formally, we can insist on linearity by adding these schemes to past-directed-logic:[14]

$$\Diamond\phi \leftrightarrow (⬘\phi \vee \phi \vee ⬙\phi) \qquad ⬘⬙\phi \rightarrow \Diamond\phi \qquad ⬙⬘\phi \rightarrow \Diamond\phi$$

As in §2, potentialists who want to use this linear-logic must explain why their favoured notion of mathematical possibility vindicates such linearity; this is a non-trivial challenge, but again I will not push it.[15] When using PST with this linear logic, I write LPST, for linear-PST.

By combining Theorem 4.1 with the assumption of linearity, we can simplify our ideology considerably. Intuitively, linearity allows us to gloss 'previously' as 'when there are fewer things', and to gloss 'eventually' as 'when there are more things'.

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[12] Indeed, PST proves a modal version of well-foundedness; see Lemma A.6.

[13] Scott (1974: 210). That quote is discussed in Pt.1 §5; this section 'modalizes' that discussion.

[14] See e.g. Goldblatt (1992: 78). These allow us to prove the schemes $\Diamond\phi \leftrightarrow (⬘\phi \vee \phi \vee ⬙\phi \vee ⬘⬙\phi)$ and $⬘(\phi \wedge ⬒\phi) \rightarrow ⬒(\phi \vee ⬘\phi)$ of §2.

[15] Though cf. the discussion of Boolos's 1971 theory in Pt.1 §8.2, and footnote 45.

More precisely, we recursively define a translation, •, whose only non-trivial clauses are as follows:[16]

$$(⬘\,\phi)^\bullet := \exists x \Diamond(\neg \mathrm{E}(x) \wedge \phi^\bullet)$$
$$(⬙\,\phi)^\bullet := (\exists x : Max)\Diamond(\exists v\; x \in v \wedge \phi^\bullet)$$

It is then easy to prove:

Proposition 5.1 (LPST)**:** $\phi \leftrightarrow \phi^\bullet$ for any LPST-formula ϕ

We can therefore rewrite LPST, without loss, as a modal theory which uses a *single* primitive modal operator, $\Diamond$, which obeys S5 (for more, see §B).

We can go even further, though, and eliminate *all* modal notions from LPST. The rough idea is straightforward. Theorem 4.1 says that *levels simulate possible worlds, and vice versa*. By assuming linearity, we can obtain results which say: *actual hierarchies simulate potential hierarchies, and vice versa*.

That way of putting things is, however, rather rough. The details of the simulation are in fact quite fiddly. I will therefore divide my discussion into three sections. In §6, I consider a *deductive* version of this simulation. This is suitable for first-order versions of LT and LPST, which I call LT_1 and LPST_1.[17] In §7, I consider a *semantic* version of this first-order simulation. Finally, in §8, I consider deductive and semantic versions of this simulation for (various) *second-order* versions of LT and LPST.

6 Near-synonymy: first order, deductive

To interpret LT_1 in LPST_1, we will simply replace what *happens* with what *could happen*. More precisely, we consider the following translation; following Studd, I call $\phi^\Diamond$ the *modalization* of ϕ:[18]

$$\alpha^\Diamond := \Diamond\alpha, \text{ for atomic } \alpha \qquad (\phi \wedge \psi)^\Diamond := (\phi^\Diamond \wedge \psi^\Diamond)$$
$$(\neg\phi)^\Diamond := \neg\phi^\Diamond \qquad (\exists x \phi)^\Diamond := \Diamond\exists x \phi^\Diamond$$

Conversely, to interpret LPST_1 in LT_1, we take the hint suggested by Theorem 4.1, and simply regard possible worlds as levels. More precisely, we consider the following translation; I call ϕ^s the *levelling* of ϕ:[19]

[16] So $(\exists x \phi)^\bullet := \exists x \phi^\bullet$, $(\exists X \phi)^\bullet := \exists X \phi^\bullet$, $(\Diamond\phi)^\bullet := \Diamond\phi^\bullet$, $(\neg\phi)^\bullet := \neg\phi^\bullet$, $(\phi \wedge \psi)^\bullet := (\phi^\bullet \wedge \psi^\bullet)$, and $\alpha^\bullet := \alpha$ for atomic α; we choose variables to avoid clashes.

[17] These arise just by replacing the single second-order axiom, Separation or $\text{Spec}_{⬘}$, with its obvious first-order schematisation, and abandoning Comprehension.

[18] Studd (2013: 708, 2019: 154); cf. also Linnebo (2010: 115–6, 2013: 213).

[19] Linnebo (2013: 224–5) and Studd (2013: 719, 2019: 173) consider similar maps. We choose new variables (to avoid clashes) in the clauses for $(⬘\,\phi)^s$, $(⬙\,\phi)^s$ and $(\Diamond\phi)^s$.

$$(x = y)^s := (x = y \subseteq s) \qquad (x \in y)^s := (x \in y \subseteq s)$$
$$(\phi \wedge \psi)^s := (\phi^s \wedge \psi^s) \qquad (\neg\phi)^s := \neg\phi^s$$
$$(\exists x\phi)^s := (\exists x \subseteq s)\phi^s \qquad (\Diamond\phi)^s := (\exists t : Lev)\phi^t$$
$$(\blacklozenge\phi)^s := (\exists t : Lev)(t \in s \wedge \phi^t) \qquad (\Diamond\phi)^s := (\exists t : Lev)(s \in t \wedge \phi^t)$$

Note that levelling is defined using variables; to illustrate: $(x \in y)^s$ is $(x \in y \subseteq s)$, but $(\Diamond x \in y)^s$ is $(\exists t : Lev)x \in y \subseteq t$. We now have a deep result about modalization and levelling (see §B.1):[20]

Theorem 6.1: For any LT_1-formula ϕ:

(1) If $\mathrm{LT}_1 \vdash \phi$, then $\mathrm{LPST}_1 \vdash \phi^\Diamond$
(2) $\mathrm{LT}_1 \vdash \phi \leftrightarrow (\phi^\Diamond)^s$

For any LPST_1-formula ϕ:

(3) If $\mathrm{LPST}_1 \vdash \phi$, then $\mathrm{LT}_1 \vdash Lev(s) \rightarrow \phi^s$
(4) $\mathrm{LPST}_1 \vdash Max(s) \rightarrow (\phi \leftrightarrow (\phi^s)^\Diamond)$

This result entails that modalization and levelling are faithful (see Corollary B.1). But Theorem 6.1 is much stronger than a statement of mutual faithful interpretability; it is almost a *definitional equivalence* between LT_1 and LPST_1. This claim, though, requires some explanation.[21]

Roughly speaking, to say that two theories are definitionally equivalent is to say that each interprets the other, and that combining the interpretations gets us back exactly where we began. To make this rough idea precise for the case of first-order theories, we say that S and T are definitionally equivalent iff there are interpretations I and J such that for any S-formula ϕ: (1) if $\mathrm{S} \vdash \phi$ then $\mathrm{T} \vdash \phi^I$; and (2) $\mathrm{S} \vdash \phi \leftrightarrow (\phi^I)^J$; and for any T-formula ϕ: (3) if $\mathrm{T} \vdash \phi$ then $\mathrm{S} \vdash \phi^J$; and (4) $\mathrm{T} \vdash \phi \leftrightarrow (\phi^J)^I$. Clauses (1) and (3) tell us we have interpretations; clauses (2) and (4) make precise the idea that 'combining the interpretations gets us back exactly where we began'.

The clauses of Theorem 6.1 are extremely similar to those of a paradigm definitional equivalence. So, Theorem 6.1 is almost a statement of definitional equivalence. Almost; but not quite. We must say something about s in clauses (3) and (4) of Theorem 6.1, thereby disrupting the similarity. So: we do not have a definitional equivalence; but we *almost* do.

Since 'almost-definitional-equivalence' is quite long-winded, and definitional equivalence is sometimes known as 'synonymy', I call this a (deductive) *near-synonymy* between LT_1 and LPST_1.

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[20] Studd proves similar results. Compare: (1) with Studd (2013: Theorem 23 p.719, 2019: Proposition 18 p.263); (2) with Studd (2013: Lemma 24 p.719, 2019: Lemma 20 p.263); (3) with Studd (2013: Lemma 25 p.719, 2019: Lemma 19 p.263); (4) with Studd (2013: 720, 2019: Proposition 22 p.263).

Clauses (1)–(3) do not require temporal-linearity. Clause (4) does. To see this, consider a model of PST with four worlds and accessibility relations exhaustively specified by: $\mathbf{w} < \mathbf{v} < \mathbf{u}$ and $\mathbf{w} < \mathbf{t}$ and $\mathbf{w} < \mathbf{u}$. Where $D(\mathbf{x})$ is $\mathbf{x}$'s first-order domain, let $D(\mathbf{w}) = \{\emptyset\}$; $D(\mathbf{v}) = D(\mathbf{t}) = \wp\{\emptyset\}$ and $D(\mathbf{u}) = \wp\wp\{\emptyset\}$.

[21] I know of no existing analogue of definitional equivalence between non-modal and modal theories (such as LT_1 and LPST_1); this is my best attempt to provide such an analogue. For a general overview to definitional equivalence in non-modal settings, see e.g. Button and Walsh (2018: ch.5).

7 Near-synonymy: first-order, semantic

Theorem 6.1 is deductive, but we can extract semantic content from it. (In what follows, my discussion of modal semantics should be understood in terms of *connected* Kripke structures, i.e. variable domain Kripke structures where all worlds are path-connected.)[22]

Modalization is defined syntactically, but it has obvious semantic import: as noted, it tells us to replace what *happens* with what *could happen*. This motivates a definition:[23]

Definition 7.1: Let $\mathcal{P}$ be any connected Kripe structure. Its *flattening*, $\flat\mathcal{P}$, is the following non-modal structure: $\flat\mathcal{P}$'s domain is $\mathcal{P}$'s global domain; and $\flat\mathcal{P} \vDash a \in b$ iff $\mathcal{P} \vDash \Diamond a \in b$.

Levelling has similar semantic import: it tells us to regard possible worlds as levels. So:

Definition 7.2: Let $\mathcal{A}$ be any non-modal structure. Its *potentialization*, $\sharp\mathcal{A}$, is the following connected Kripke structure: $\sharp\mathcal{A}$'s worlds are those s such that $\mathcal{A} \vDash \mathit{Lev}(s)$; accessibility is given by $r < s$ iff $\mathcal{A} \vDash r \in s$; $\sharp\mathcal{A}$'s global domain is just $\mathcal{A}$'s domain; $\sharp\mathcal{A} \vDash_s a \in b$ iff $\mathcal{A} \vDash a \in b \subseteq s$; and $\sharp\mathcal{A} \vDash_s a = b$ iff $\mathcal{A} \vDash a = b \subseteq s$.

By considering flattening and potentialization, we can move between models of LT_1 and connected Kripke models of LPST_1. To make this movement almost seamless (but only almost; see below), we need one last general construction; intuitively, this construction will allow us to take a Kripke structure, $\mathcal{P}$, and create a new structure, $\mathcal{P}_f$, by disrupting the 'identities' of $\mathcal{P}$'s worlds (and perhaps duplicating some worlds):

Definition 7.3: Let $\mathcal{P}$ be any connected Kripke structure. Let f be any surjection whose range is $\mathcal{P}$'s set of worlds. Then $\mathcal{P}_f$ is the following connected Kripke structure: $\mathcal{P}_f$'s set of worlds is $\mathrm{dom}(f)$; accessibility is given by $\mathbf{v} < \mathbf{w}$ in $\mathcal{P}_f$ iff $f(\mathbf{v}) < f(\mathbf{w})$ in $\mathcal{P}$; $\mathcal{P}_f$ has the same global domain as $\mathcal{P}$; and $\mathcal{P}_f \vDash_{\mathbf{w}} R(\vec{a})$ iff $\mathcal{P} \vDash_{f(\mathbf{w})} R(\vec{a})$ for all R (including identity).

We now have the following result (see §B.2):[24]

Theorem 7.4:

(1) If $\mathcal{P} \vDash \mathrm{LPST}_1$, then $\flat\mathcal{P} \vDash \mathrm{LT}_1$
(2) If $\mathcal{P} \vDash \mathrm{LPST}_1$, then there is a surjection f such that $\mathcal{P} = (\sharp\flat\mathcal{P})_f$
(3) If $\mathcal{A} \vDash \mathrm{LT}_1$, then $\sharp\mathcal{A} \vDash \mathrm{LPST}_1$
(4) If $\mathcal{A} \vDash \mathrm{LT}_1$, then $\mathcal{A} = \flat\sharp\mathcal{A}$

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[22] See footnote 8 for the definition of *path-connected*.
[23] See Studd (2019: 154–5).
[24] Clause (2) requires linearity, since $\flat\mathcal{P}$ has well-ordered levels.

This is a semantic reworking of Theorem 6.1. Consequently, it is *almost* a statement of (semantic) definitional equivalence. Recall that, roughly speaking, two theories are definitionally equivalent iff each interprets the other, and that combining the interpretations gets us back exactly where we began. In §6, I precisely defined this idea for (non-modal) first-order theories in deductive terms. The same idea can be defined in semantic terms. To say that S and T are definitionally equivalent is to say that they (respectively, and uniformly from interpretations) define operations, g and h, such that: if $\mathcal{B} \vDash \mathrm{T}$, then both (1) $g\mathcal{B} \vDash \mathrm{S}$ and (2) $\mathcal{B} = hg\mathcal{B}$; and if $\mathcal{A} \vDash \mathrm{S}$, then both (3) $h\mathcal{A} \vDash \mathrm{T}$ and (4) $\mathcal{A} = gh\mathcal{A}$. Clauses (1) and (3) tell us that we have interpretations; clauses (2) and (4) make precise the idea that 'combining the interpretations gets us back exactly where we began'.

Theorem 7.4 has a very similar shape. So it is almost a (semantic) statement of definitional equivalence between LT_1 and LPST_1. Again, though: almost, but not quite. Clause (2) of Theorem 7.4 does not tell us that $\mathcal{P} = \sharp\flat\mathcal{P}$, as a definitional equivalence would require, but introduces a slight wrinkle. So I will say that we have a semantic *near-synonymy*.

The wrinkle I just mentioned is unavoidable. Fix some $\mathcal{O} \vDash \mathrm{LPST}_1$ and f so that $\mathcal{O} \neq \mathcal{O}_f$. Clearly $\flat\mathcal{O} = \flat(\mathcal{O}_f)$, so that $\sharp\flat\mathcal{O} = \sharp\flat(\mathcal{O}_f)$; so we cannot in general have that $\mathcal{P} = \sharp\flat\mathcal{P}$. Moreover, this scarcely depends upon the specific definitions of flattening and potentialization; it is an inevitable consequence of the fact that modal semantics has an extra degree of freedom compared with non-modal semantics (the 'identities' of worlds, which f can disrupt).

8 Near-synonymy: second-order

I have outlined near-synonymies for the first-order theories LT_1 and LPST_1. I now want to consider near-synonymies for the second-order theories.

In what follows, I assume that LT's (non-modal) background logic treats second-order identity as co-extensionality, i.e. $\forall F \forall G(\forall \vec{x}(F(\vec{x}) \leftrightarrow G(\vec{x})) \rightarrow F = G)$. Similarly, I assume that all potentialists treat second-order identity as co-intensionality, i.e.:

Coint $\forall F \Box \forall G(\Box \forall x_1 \ldots \Box \forall x_n(\Diamond F(\vec{x}) \leftrightarrow \Diamond G(\vec{x})) \rightarrow \Diamond F = G)$

To take things further, though, I must separately consider two different approaches to second-order entities: *necessitism* and *contingentism*.[25]

8.1 *Second-order necessitism*

Second-order necessitism treats second-order entities as necessary existents. We can implement this formally via these axioms:

$\mathbf{Ex_n}$ $\mathrm{E}(F)$, for any second-order variable 'F'

$\mathbf{Comp_n}$ $\exists F \Box \forall x_1 \ldots \Box \forall x_n(\Diamond F(\vec{x}) \leftrightarrow \Diamond \phi)$, for any formula ϕ not containing 'F'

[25] I use 'necessitism' and 'contingentism' in roughly Williamson's (2013) sense, though note that the relevant modality here is *potentialist*.

$$\textbf{Inst}_{\text{n}}\ \ \forall F \Box \forall x_1 \ldots \Box \forall x_n (\Diamond F(\vec{x}) \to \Box((\mathrm{E}(x_1) \land \ldots \land \mathrm{E}(x_n)) \to F(\vec{x})))$$

The scheme Ex_n guarantees that every second-order entity is a necessary existent. Comp_n is a kind of potentialized Comprehension principle. Then Inst_n guarantees that second-order entities have their instances essentially (cf. $\text{Mem}_\Diamond$).

Let LPST_n, for necessitist-LPST, add these axioms and Coint to LPST.[26] Unsurprisingly, our earlier results are easily extended, to show that LT and LPST_n are deductively and semantically near-synonymous (see Theorems B.2 and B.6).

8.2 *Second-order contingentism*

In contrast with necessitism, second-order contingentism holds that a second-order entity exists iff all its (possible) instances do. Contingentists will therefore spurn Ex_n, Inst_n, and Comp_n, and instead adopt:

$$\textbf{Ex}_{\text{c}}\ \ \Diamond \mathrm{E}(F), \text{ for any second-order variable `}F\text{'}$$
$$\textbf{Inst}_{\text{c}}\ \ \forall F \Box \forall x_1 \ldots \Box \forall x_n (\Diamond F(\vec{x}) \to \Box(\mathrm{E}(F) \to F(\vec{x})))$$

retaining plain-vanilla Comprehension. Call the result LPST_c, for contingentist-LPST.

Potentialists who treat (monadic) second-order quantification as plural quantification are likely to be contingentists.[27] After all, necessitism proves $\exists F \neg \Diamond \exists a \Box \forall x(\Diamond F(a) \leftrightarrow \Diamond x \in a)$; read plurally, this contradicts the Maximality Thesis, that any sets can form a set (see §1). Moreover, the same example establishes that LT and LPST_c are *not* deductively near-synonymous. After all, LT proves $\exists F \neg \exists a \forall x(F(a) \leftrightarrow x \in a)$, whose modalization will contradict the Maximality Thesis.

Instead, LPST_c is deductively and semantically near-synonymous with a weakened version of LT. To obtain this weakening, note that contingentists, in effect, restrict second-order entities to the worlds at which their instances occur. Since worlds go proxy for levels, the non-modal equivalent should restrict second-order entities to those which are bounded by levels. Specifically, let $\vec{x} \subseteq s$ abbreviate $(x_1 \subseteq s \land \ldots \land x_n \subseteq s)$, and let $F \sqsubseteq s$ abbreviate $\forall \vec{x}(F(\vec{x}) \to \vec{x} \subseteq s)$. Then bounded Level Theory, LT_b, is the theory whose axioms are Extensionality, Separation, Stratification, and:

$$\textbf{Strat}_{\text{b}}\ \ \forall F (\exists s : Lev) F \sqsubseteq s$$
$$\textbf{Comp}_{\text{b}}\ \ (\forall s : Lev)(\exists F \sqsubseteq s)(\forall \vec{x} \subseteq s)(F(\vec{x}) \leftrightarrow \phi), \text{ for any } \phi \text{ not containing `}F\text{'}$$

with Comp_b *replacing* the usual Comprehension scheme. Our earlier results can then be extended, to show that LPST_c and LT_b are near-synonymous, both deductively and for a Henkin semantics (see Theorems B.3 and B.7).

So far, deductive and semantic results have gone hand-in-hand. However, they can be prised apart, by considering *full* semantics for second-order logic. For

[26] Note that we retain plain vanilla Comprehension; see footnote 5.

[27] This is Boolos's (1984) suggested interpretation of monadic second-order logic. For the link to contingentism, see Williamson (2013: 249) and Studd (2019: 157–62). The discussion in this paragraph closely follows Studd.

non-modal structures, full (actualist) semantics treats the (monadic) second-order domain as the powerset of the first-order domain. For connected Kripke structures, full contingentist semantics treats a world's (monadic) second-order domain as the powerset of that world's first-order domain. This full semantics is sufficiently rich, that LPST_c is not merely near-synonymous with LT_b, but with LT *itself* (see Theorem B.8).

9 The significance of the near-synonomies

The following table summarises the near-synonymies of §§6–8:

		deductive	semantic
LT_1	LPST_1	✓	✓
LT	LPST_n	✓	✓
LT_b	LPST_c	✓	✓
LT	LPST_c	×	full only

To appreciate the significance of these results, consider Paula, a potentialist who uses linear time as an heuristic for her favourite mathematical modality. Paula admires the mathematical work undertaken within ZF_1. However, she regards ZF_1 as lamentably *actualist*, since it lacks modal operators. Fortunately, there is an extension of LPST_1—call it LPZF_1— which is near-synonymous with ZF_1.[28] Leaning on this near-synonymy, Paula can regard (worryingly actualist) ZF_1 as a notational-variant of (reassuringly potentialist) LPZF_1. Indeed, by modalization and levelling, Paula can move fluidly between ZF_1 and LPZF_1.

The same idea cuts the other way. Actualist Alan may initially be somewhat perplexed by the boxes and diamonds which pepper Paula's work. But Alan need not remain confused for long: modalization and levelling allow him to make perfect sense of Paula, as using a notational-variant of ZF_1.

9.1 Outlining an Equivalence Thesis

The ease with which Paula and Alan can communicate with each other, despite their philosophical differences, suggests a further thought:

The Potentialist/Actualist Equivalence Thesis. Actualism and potentialism do not disagree; they are different but equivalent ways to express the same facts.

Putnam was the foremost proponent of such a Thesis.[29] I will say more about Putnam in §9.4; first, I want to assess the Equivalence Thesis directly. Specifically, I want to consider the following, concrete argument for the Equivalence Thesis:

(a) LT correctly axiomatizes the idea of an actual hierarchy of sets.

[28] Let $\text{LPZF}_1 = \text{LPST}_1 \cup \{\phi^\Diamond : \phi \in \text{ZF}_1\}$; the near-synomyny holds as $\text{LT}_1 \subset \text{ZF}_1$ (see Pt.1 §7).

[29] Putnam (1967: 8–9) specifically uses the phrase 'the same facts'.

(b) LPST correctly axiomatizes the idea of a (linear) potential hierarchy of sets.
(c) Theories like LT and LPST are near-synonymous.
So: the Equivalence Thesis obtains.

I am very sympathetic to this argument. However, I am not yet certain of its soundness. In the remainder of this section, I will explain how the argument is best resisted, but also suggest that the Equivalence Thesis remains plausible in the face of such resistance.

The first two premises of the argument are perfectly secure: I established (a) in Pt.1 §§1–5, and (b) in §§1–4 of this paper. But I should emphasise the caveat in (b). Whilst every potentialist should accept PST, embracing linearity requires a further step. So: this argument for the Equivalence Thesis can be resisted, straightforwardly, by denying that potentialists can/should assume linearity.

Premise (c), however, contains a sneaky weasel-clause, 'theories *like*. . . '. I will criticise this weaseling in §9.3. My more pressing concern, though, is whether we could even *hope* to infer the Equivalence Thesis from (a)–(c).[30]

9.2 *On drawing philosophical conclusions from formal equivalences*

Near-synonymy is an extremely tight, formal, equivalence between modal and non-modal theories. Still, theories can be a equivalent in some purely *formal* sense, whilst being non-equivalent in *other* important senses.

To illustrate, suppose Noddy systematically calls red things 'green' and green things 'red'. Defining interpretations by swapping colour-predicates, Noddy's theory of the empirical world may be definitionally equivalent with my own. Still, if we hold fixed the interpretation of colour-predicates, then we will say that Noddy's theory is simply mistaken; Noddy says 'grass is red', but grass is green.

This noddy example illustrates a simple moral: whether formally equivalent theories 'express the same facts' depends upon how firmly we have pinned down the interpretation of the theories' expressions. In the case of Noddy, the relevant expressions colour-predicates. In discussing the Potentialist/Actualist Equivalence Thesis, the relevant expressions are quantifiers and modal operators. And this indicates how discussions of the Equivalence Thesis are likely to play out.

Suppose you think that we have a firm grasp on the concepts used within the metaphysics of mathematics. In particular, suppose you are convinced that there is a clear difference in meaning between 'there is' and 'there could be' (as used by potentialists), which does not depend upon their use in any particular *formal* theories. The near-synonymies essentially ask you to move between what 'there is' and what 'there could be'. Given your prior conviction, you will regard this as a change in subject matter. So you will insist that actualism and potentialism make different claims, and reject the Equivalence Thesis.

Suppose instead, though, that you embrace a rather different attitude. You think that, in advance of any particular formal theorising, it is not entirely clear how one might go about distinguishing between the meanings of 'there is' and

[30] Button and Walsh (2018: §§5.6, 5.8, 14.7) offer some complementary thoughts, about the difficulties of drawing phliosophical conclusions from formal equivalences.

'there could be' (in mathematical contexts). Indeed, you think that any differences in their meaning would have to be revealed by differences in their use. In that case, you will likely find the argument of §9.1 extremely compelling. After all, the near-synonymies establish that there is no significant difference between '$\exists$' in LT_1 and '$\Diamond\exists$' in $LPST_1$.[31]

9.3 Equivalence and contingentist-potentialism

The case of LT_1 and $LPST_1$ is, though, the very simplest case. The situation concerning second-order theories is more complicated, and this merits scrutiny.

Consider Edna, a potentialist who (i) embraces contingentism and (ii) thinks that time is endless, who also (iii) uses second-order logic, whilst (iv) eschewing the full semantics. So Edna embraces an extension of $LPST_c$.[32] As we saw in §8.2, though, this theory is *not* near-synonymous (whether deductively or using Henkin semantics) with an extension of LT; we must retreat to LT_b. Edna therefore takes issue with the weasel-clause in premise (c) of the argument for the Equivalence Thesis. Indeed, she goes further, rebutting the argument as follows: actualists will insist that $\exists F \neg \exists a \forall x(F(x) \leftrightarrow x \in a)$; the modalization of this claim is $\Diamond\exists F \neg \Diamond\exists a \Box\forall x(\Diamond F(x) \leftrightarrow \Diamond x \in a)$; this is inconsistent with her favourite potentialist set theory; so potentialism and actualism genuinely *disagree*.[33]

This rebuttal of the Equivalence Thesis is exactly as strong as our grasp on the relevant ideology. *If* we have a firm grasp of Edna's intended potentialist modality (independently of the formalism), and how that modality contrasts with actuality, and of the sense of (higher-order) quantification, and why contingentism (but not the use of full second-order semantics) is suitable, *then* Edna's rebuttal will succeed. For, in that case, attempts to move between discussing what 'there is' and what 'there could be' will amount to a change in truth-value, and therefore also a change in subject matter. But if our grasp of the relevant ideology is insufficiently firm, then Edna's worry will melt away. Edna, then, presents us with an interesting way to resist the Equivalence Thesis, which dovetails with the line of resistance offered in §9.2.

The upshot is that the failure or success of the Equivalence Thesis turns on whether potentialists can supply us with a sufficiently firm grasp of their favoured metaphysical-mathematical-modal concepts. I am genuinely unsure whether they can, but I cheerfully present this as a challenge.

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[31] Soysal (2020: 588) makes a similar point against any potentialists who treat mathematical possibility as a primitive notion. However, Soysal states that 'the potential and [actual] iterative hierarchies are isomorphic, and modal and non-modal set theories are mutually interpretable'. Mutual interpretability is insufficient to support this point (see the *Second* point of §9.4); and it is imprecise to describe potentialist and actualist hierarchies as isomorphic. Soysal's point is better made by appealing to near-synonymy.

[32] See §C for details of Edna's theory. By the results of §8 and §C, if Edna drops any of (i)–(iv), then her favourite theory will be near-synonymous (in some salient sense) with LT itself, rather than LT_b.

[33] Thanks to Geoffrey Hellman and Øystein Linnebo for raising concerns along these lines.

9.4 Putnam on the equivalence of modal and non-modal theories

To conclude my discussion of the Equivalence Thesis, I want to revisit Putnam. As mentioned in §9.1, the Thesis is hugely indebted to Putnam, who claimed in 1967 that modal and non-modal theories are 'equivalent'. However, it is worth emphasizing a few of the differences between Putnam's 1967 claim and my Equivalence Thesis.

First. Putnam did not say much about the modality he had in mind, except to connect '$\Diamond$' with possible 'standard concrete models for Zermelo set theory'.[34] My discussion is restricted to a potentialist modality, though I have deliberately left room for various different versions of potentialism.[35]

Second. Putnam did not precisely define the formal notion of 'equivalence' he had in mind. He sometimes considers the *mutual interpretability* of modal and non-modal theories;[36] but mutual interpretability is far too weak to sustain anything like the Equivalence Thesis.[37] By contrast, my formal notion of 'equivalence' is *near-synonymy*.

Third. Putnam ultimately retracted his version of the Equivalence Thesis.[38] He claimed that mathematics is 'about proofs, ways of conceiving of mathematical problems, mathematical approaches, and much more', and worried that his interpretation would not preserve such things. Now, these considerations might tell against Putnam's 1967 claim; but they only highlight the plausibility of my Equivalence Thesis. My near-synonymies simply formalize the intuitive and obvious point that LT's levels simulate LPST's possible worlds, and vice versa (see §5); this simulation straightforwardly preserves proofs; and this is precisely why it is so plausible that LT and LPST do not really differ over 'ways of conceiving mathematical problems, mathematical approaches', or anything else that matters.

Fourth. Having decided that modal and non-modal formulations of set theory genuinely disagree, Putnam came to favour the former, on the grounds that non-modal set theories face 'a generalization of a problem first pointed out by Paul Benacerraf. . . e.g. are sets a kind of function or are functions a sort of set?'[39] Again, this might detract from Putnam's 1967 claim, but it has no force against my Equivalence Thesis. If LT and LPST are equally good in all other regards—as I think they might be—then choosing potentialism (with its distinctive modality) *over* actualism (with its distinctive ontology) is exactly as arbitrary as saying that functions are a kind of set (rather than vice versa).

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[34] Putnam (1967: 20–1).

[35] Linnebo (2018a: 262–6) offers good reasons to suggest that Putnam *should* have considered a potentialist modality.

[36] E.g. Putnam (1967: 8) 'the primitive terms of each admit of definition by means of the primitive terms of the other theory, and then each theory is a deductive consequence of the other.'

[37] Linnebo (2018a: 260–2) makes this point. To bring it out in another way, note that PA and PA + $\neg$Con(PA) are mutually interpretable, but are surely *not* equivalent ways to express the same facts.

[38] Putnam (2014: 11.Dec.2014).

[39] Putnam (2014: 13.Dec.2014).

10 Conclusion, and predecessors

The Tensed Story articulates the bare idea of a potential hierarchy of sets. PST axiomatizes that bare idea. Whilst it takes no stance on the height of any potential hierarchy, it ensures persistence and well-foundedness. Moreover, versions of PST are near-synonymous with versions of the non-modal theory LT. And these near-synonymies both sharpen and leave plausible the idea that there is no deep difference between actualism and potentialism.

I will close this paper by comparing PST with some alternative potentialist set theories.

10.1 Parsons and Linnebo

In formulating their modal set theories, Parsons and Linnebo do not use a temporal logic.[40] Instead, they use a single modal operator, $\Diamond$, whose background logic is S4.2, and which can be glossed as 'now and henceforth'.

The asymmetry of this operator generates a deep expressive problem.[41] Stated non-modally: there is a stage (the initial stage) at which nothing has any members. Potentialists should therefore want to be able to prove: *possibly, nothing has any members*, i.e. $\Diamond\forall x\forall y\ x \notin y$. But this cannot be a theorem for Parsons or Linnebo. To see why, suppose otherwise; then $\Box\Diamond\forall x\forall y\ x \notin y$ is also a theorem, by Necessitation; but this is catastrophic, for it catastrophically entails that there is always a later moment at which nothing has any members.

This problem does not arise in PST. There, $\Diamond$ obeys S5, and PST proves $\Diamond\forall x\forall y\, x \notin y$.

10.2 Studd

In using a tensed logic to formulate PST, I am entirely endebted to Studd. Moreover, Studd proves a result like Theorem 6.1 for his modal set theory, MST. So my PST is similar to Studd's MST, and owes a great deal to it. However, it is worth noting two differences.

The minor difference concerns our versions of Priority$_{\Diamond}$. Studd's MST has: all of a's members are found *together* before a is found.[42] My PST has: *each* of a's members is found before a is found. The slight difference emerges only at limit worlds:[43] in Studd's MST, a exists at a limit world iff a existed earlier; in my PST, a exists at a limit world iff all of a's members existed earlier.

The major difference concerns the richness of Studd's modal schemes. Studd's MST explicitly adopts modal axioms which guarantee *linearity*, *persistence*, *well-ordering*, and that time is *endless*.[44] My PST only assumes past-directedness, and

[40] Parsons (1977, 1983b) and Linnebo (2013, 2018b: ch.12).

[41] For related problems, see Studd (2013: 723–4, 2019: 169–71).

[42] Studd (2013: 712, 2019: 164–5).

[43] Where $\mathbf{w}$ is a limit world iff $(\forall \mathbf{u} < \mathbf{w})\exists \mathbf{v}(\mathbf{u} < \mathbf{v} < \mathbf{w})$.

[44] Studd (2013: 704, 2019: 152, 252) guarantees persistence via Barcan-formulas; see also Linnebo (2013: 210, 2018b: 207). Studd (2013: 702–4, 2019: 152, 251–2) guarantees well-ordering via a Löb-scheme; Parsons (1977: 296, 1983b: 318) and Linnebo (2013: 216, 2018b: 206) guarantee well-ordering

instead *proves* persistence and well-foundedness (see §4). Proof has three virtues over explicit assumption. First: my PST is considerably leaner than Studd's MST. Second: it will be strictly easier for potentialists to try to explain why they are entitled to assume past-directedness, than to try to justify Studd's richer assumptions.[45] Third: as in §4, the proofs of persistence and well-foundedness show 'how little choice there is in setting up' a potential hierarchy.

A Elementary results concerning PST

The time has come to prove the results stated in Part 2. I will start with some elementary results within PST, building up to Theorem 4.1 of §4. My proofs are semantic, relying on standard soundness and completeness results for (connected) Kripke frames. I use bold letters, $\mathbf{w}, \mathbf{v}, \mathbf{u}, \ldots$, for arbitrary worlds (note that this differs from my use of bold letters in Pt.1 and Pt.3).

In what follows, we must not assume that expressions like '$\{x : \phi(x)\}$' are rigid designators; we should read '$a = \{x : \phi(x)\}$' as abbreviating '$\forall x(x \in a \leftrightarrow \phi(x))$', which may be true in one world and false in another. Similarly, recall that '$a = \P b$' abbreviates '$\forall x(x \in a \leftrightarrow \exists c(x \subseteq c \in b))$'.

I start with two very elementary results:

Lemma A.1 (PST)**:** Extensionality holds.

Proof. Fix a and b at $\mathbf{w}$, and assume $\vDash_{\mathbf{w}} \forall x(x \in a \leftrightarrow x \in b)$. Fix x at world $\mathbf{u}$, now $\vDash_{\mathbf{u}} \Diamond x \in a$ iff $\vDash_{\mathbf{w}} x \in a$ (by Mem$_\Diamond$) iff $\vDash_{\mathbf{w}} x \in b$ iff $\vDash_{\mathbf{u}} \Diamond x \in b$; so $\vDash_{\mathbf{u}} \Diamond a = b$ by Ext$_\Diamond$. Hence $\vDash_{\mathbf{w}} a = b$. □

Lemma A.2 (PST)**:** Separation holds.

Proof. Using Comprehension, let G be given by $\forall x(G(x) \leftrightarrow (F(x) \land x \in a))$. If $G(x)$, then $\blacklozenge \mathrm{E}(x)$ by Priority$_\blacklozenge$; so some $b = \{x : G(x)\} = \{x \in a : F(x)\}$ exists by Spec$_\blacklozenge$. □

Since PST proves Extensionality and Separation, it proves the key results of Pt.1 §3, concerning the well-ordering of *levels*, in the sense of Pt.1 Definition 2.2. This next result establishes that all of the key notions of that Definition are (weakly) rigid:

Lemma A.3 (PST)**:**

(1) $\forall a(\forall b \subseteq a)\Box(\mathrm{E}(a) \rightarrow (\mathrm{E}(b) \land b \subseteq a))$
(2) $\forall a \exists b\, \P a = b$
(3) $\forall a(\forall b = \P a)\Box(\mathrm{E}(a) \rightarrow (\mathrm{E}(b) \land b = \P a))$

via non-modal means.

[45] To illustrate: Studd (2013: 144–53) glosses ⬒ as 'however the lexicon is interpreted by preceding interpretations' and ⬓ as 'however the lexicon is interpreted by succeeding interpretations'. I worry that Studd does not manage to show that, so glossed, these operators should obey the schemes for linearity, persistence, or well-ordering. However, past-directedness might well be justifiable; and from there we can prove persistence and well-foundedness, via Theorem 4.1.

(4) $(\forall h : \mathit{Hist})\Box(\mathrm{E}(h) \rightarrow \mathit{Hist}(h))$
(5) $(\forall s : \mathit{Lev})\Box(\mathrm{E}(s) \rightarrow \mathit{Lev}(s))$

Proof. *(1)* Fix a and b at $\mathbf{w}$ such that $\vDash_{\mathbf{w}} b \subseteq a$. Let a exist at $\mathbf{v}$; by Separation at $\mathbf{v}$ there is c at $\mathbf{v}$ such that $\vDash_{\mathbf{v}} c = \{x \in a : \Diamond x \in b\}$; I claim that $\vDash_{\mathbf{v}} c = b$. Fix x at $\mathbf{u}$: if $\vDash_{\mathbf{u}} \Diamond x \in c$, then $\vDash_{\mathbf{v}} x \in c$ by Mem$_\Diamond$, so $\vDash_{\mathbf{v}} \Diamond x \in b$ and $\vDash_{\mathbf{u}} \Diamond x \in b$; if $\vDash_{\mathbf{u}} \Diamond x \in b$, then $\vDash_{\mathbf{w}} x \in b \subseteq a$ by Mem$_\Diamond$, so that $\vDash_{\mathbf{v}} x \in a$ and $\vDash_{\mathbf{v}} \Diamond x \in b$, i.e. $\vDash_{\mathbf{v}} x \in c$, so that $\vDash_{\mathbf{u}} \Diamond x \in c$. Hence $\vDash_{\mathbf{v}} c = b$ by Ext$_\Diamond$.

(2) Fix a. If $\exists z(x \subseteq z \in a)$, then $⬘\mathrm{E}(x)$ by Priority$_{⬘}$ and (1). So using Spec$_{⬘}$ we have some b such that $b = \P a = \{x : \exists z(x \subseteq z \in a)\}$.

(3) Fix a and b at $\mathbf{w}$ with $\vDash_{\mathbf{w}} b = \P a$. Let a exist at $\mathbf{v}$, and using (2) fix c such that $\vDash_{\mathbf{v}} c = \P a$. Now $\vDash_{\mathbf{v}} b = c$, by Ext$_\Diamond$ and (1).

(4)–(5) By (1) and (3). □

We can now show that levels persist, and also that every world has a maximal level:

Lemma A.4 (PST)**:** $(\forall s : \mathit{Lev})⬓(\mathrm{E}(s) \wedge \mathit{Lev}(s))$

Proof. Let s be a level in $\mathbf{w}$. For induction on levels (i.e. Pt.1 Theorem 3.10), suppose that $\vDash_{\mathbf{w}} (\forall r : \mathit{Lev})(r \in s \rightarrow ⬓(\mathrm{E}(r) \wedge \mathit{Lev}(r)))$. Fix $\mathbf{v} > \mathbf{w}$; using Spec$_{⬘}$ fix t such that $\vDash_{\mathbf{v}} t = \P\{x : (\exists r : \mathit{Lev})(x \subseteq r \wedge \Diamond r \in s)\}$. I claim that $\vDash_{\mathbf{v}} s = t$; the result will then follow by induction on levels in $\mathbf{w}$ and Lemma A.3.5.

If $\vDash_{\mathbf{u}} \Diamond x \in s$, then $\vDash_{\mathbf{w}} x \in s$; so by Pt.1 Lemma 3.8 there is some r such that $\vDash_{\mathbf{w}} x \subseteq r \in s \wedge \mathit{Lev}(r)$; now $\vDash_{\mathbf{v}} \mathrm{E}(r) \wedge \mathit{Lev}(r)$ by the induction hypothesis and Lemma A.3.5, and $\vDash_{\mathbf{v}} x \subseteq r$ by Lemma A.3.1; so $\vDash_{\mathbf{v}} x \in t$ and hence $\vDash_{\mathbf{u}} \Diamond x \in t$. The converse is similar. So $\vDash_{\mathbf{u}} \Diamond s = t$, and $\vDash_{\mathbf{v}} s = t$ by Ext$_\Diamond$. □

Lemma A.5 (PST)**:** $(\exists s : \mathit{Lev})(\forall r : \mathit{Lev})(r \subseteq s \wedge (r \neq s \leftrightarrow ⬘\mathrm{E}(r)))$

Proof. Using Spec$_{⬘}$, let $h = \{r : \mathit{Lev}(r) \wedge ⬘\mathrm{E}(r)\}$. I claim that h is a history. Fix $r \in h$. Clearly $\P(r \cap h) \subseteq \P r = r$ as levels are potent. Conversely, if $a \in r$ then there is some level q such that $a \subseteq q \in r$ by Pt.1 Lemma 3.8, and since $⬘\mathrm{E}(r)$ we have $⬘\mathrm{E}(q)$; so $q \in r \cap h$ and hence $a \in \P(r \cap h)$. Generalising, $r \subseteq \P(r \cap h)$. So h is a history. Using Lemma A.3.2, let $s = \P h$. By construction, s is a level. I claim that s has the required properties.

For reductio, suppose that $⬘\mathrm{E}(s)$; then $s \in h \subseteq s$, contradicting the well-ordering of levels; so $\neg⬘\mathrm{E}(s)$.

Suppose $r \neq s$. Then either $r \in s$ or $s \in r$ by the well-ordering of levels; but if $s \in r$ then $⬘\mathrm{E}(s)$ by Priority$_{⬘}$, a reductio. So $r \in s$. Hence $⬘\mathrm{E}(r)$ by Priority$_{⬘}$, and also $r \subseteq s$ as s is transitive. □

From here, we can prove a Löb-like scheme for PST:

Lemma A.6 (PST)**:** $\Box(⬒\phi \rightarrow \phi) \rightarrow \Box\phi$, for all ϕ

Proof. For reductio, suppose this is false at $\mathbf{w}$, i.e. $\vDash_{\mathbf{w}} \Box(⬒\phi \rightarrow \phi)$ but $\vDash_{\mathbf{w}} \Diamond\neg\phi$. So $\vDash_{\mathbf{v}} \neg\phi$ for some $\mathbf{v}$. Since $\vDash_{\mathbf{v}} ⬒\phi \rightarrow \phi$, there is $\mathbf{u} < \mathbf{v}$ such that $\vDash_{\mathbf{u}} \neg\phi$. For brevity, let:

$$\psi(x) \text{ abbreviate } (\neg\phi \wedge Lev(x) \wedge \neg⬘\mathrm{E}(x) \wedge (\forall q : Lev)q \subseteq x)$$

Now $\vDash_{\mathbf{v}} (\exists s : Lev)⬘\psi(s)$, by Lemmas A.4–A.5. Using induction on levels, let s be the $\in$-minimal level in $\mathbf{v}$ such that $\vDash_{\mathbf{v}} ⬘\psi(s)$. So there is $\mathbf{t} < \mathbf{v}$ with $\vDash_{\mathbf{t}} \psi(s)$. Since $\vDash_{\mathbf{t}} \neg\phi$ and $\vDash_{\mathbf{t}} ⬒\phi \rightarrow \phi$ by assumption, there is $\mathbf{t}_0 < \mathbf{t}$ with $\vDash_{\mathbf{t}_0} \neg\phi$. Using Lemma A.5, fix r such that $\vDash_{\mathbf{t}_0} \psi(r)$. Now $\vDash_{\mathbf{t}} Lev(r) \wedge ⬘\mathrm{E}(r)$ by Lemma A.4, so $\vDash_{\mathbf{t}} r \in s$ by Lemma A.5 and choice of s. So $\vDash_{\mathbf{v}} Lev(r) \wedge r \in s \wedge ⬘\psi(r)$ by Lemma A.4, contradicting the choice of s. □

This effectively licenses schematic induction on worlds, enabling us to prove the main result of §4:

Theorem 4.1 (PST)**:** Where $Max(s)$ abbreviates $(\mathrm{E}(s) \wedge \forall x\, x \subseteq s)$:

(1) LT holds
(2) $\forall x ⬓\mathrm{E}(x)$
(3) $(\exists s : Lev)Max(s)$
(4) $(\forall s : Lev)\Diamond Max(s)$

Proof. *(1)* It suffices to prove Stratification, i.e. that $\forall a(\exists s : Lev)a \subseteq s$. Fix $\mathbf{w}$, and suppose for induction on worlds that $\vDash_{\mathbf{v}} \forall a(\exists s : Lev)a \subseteq s$ for all $\mathbf{v} < \mathbf{w}$. Using Lemma A.5, fix s such that $\vDash_{\mathbf{w}} Lev(s) \wedge \neg⬘\mathrm{E}(s) \wedge (\forall r : Lev)r \subseteq s$. Suppose $\vDash_{\mathbf{w}} x \in a$; by Priority$_{⬘}$ there is some $\mathbf{u} < \mathbf{w}$ such that $\vDash_{\mathbf{u}} \mathrm{E}(x)$; by assumption there is r such that $\vDash_{\mathbf{u}} Lev(r) \wedge x \subseteq r$; now $\vDash_{\mathbf{w}} x \subseteq r \in s$ by Lemmas A.3–A.4, so that $x \in s$ as s is potent. Hence $\vDash_{\mathbf{w}} a \subseteq s$. The result follows by Lemma A.6.

(2)–(3) Combine Stratification with Lemmas A.3–A.6.

(4) Fix $\mathbf{w}$, and suppose for induction on worlds that $\vDash_{\mathbf{v}} (\forall s : Lev)\Diamond Max(s)$ for all $\mathbf{v} < \mathbf{w}$. Let s be such that $\vDash_{\mathbf{w}} Lev(s)$. If $\vDash_{\mathbf{w}} ⬘\mathrm{E}(s)$ then $\vDash_{\mathbf{w}} \Diamond Max(s)$ by our supposition and Lemma A.3. Otherwise, $\vDash_{\mathbf{w}} (\forall r : Lev)r \subseteq s$ by the well-ordering of levels and Lemma A.5, so that $\vDash_{\mathbf{w}} Max(s)$ by Stratification. The result follows by Lemma A.6. □

To round things off, note that LT's key notions are robust under modalization:

Lemma A.7 (PST)**:**

(1) $\phi^{\Diamond}(\vec{x})$ iff $\Box\phi^{\Diamond}(\vec{x})$, for any LT-formula $\phi(\vec{x})$
(2) if $\mathrm{E}(b) \wedge b \subseteq a$, then $(b \subseteq a)^{\Diamond}$
(3) if $(b \subseteq a)^{\Diamond}$, then $\Box(\mathrm{E}(a) \rightarrow b \subseteq a)$
(4) if $\mathrm{E}(b)$ and $(b = \P a)^{\Diamond}$, then $\mathrm{E}(a)$ and $b = \P a$
(5) if $\mathrm{E}(b)$ and $b = \P a$, then $\mathrm{E}(a)$ and $(b = \P a)^{\Diamond}$
(6) if $\mathrm{E}(h)$, then $Hist(h) \leftrightarrow Hist^{\Diamond}(h)$
(7) if $\mathrm{E}(s)$, then $Lev(s) \leftrightarrow Lev^{\Diamond}(s)$

Proof. *(1)* A routine induction on complexity, using the fact that $\Diamond$ obeys S5.

(2)–(3) Straightforward.

(4) Suppose $\vDash_{\mathbf{w}} \mathrm{E}(b)$ and $\vDash_{\mathbf{w}} (b = \P a)^{\diamond}$, i.e. $\vDash_{\mathbf{w}} \Box\forall x(\diamond x \in b \leftrightarrow (\exists z(x \subseteq z \in a))^{\diamond})$.

I first show that $\vDash_{\mathbf{w}} \mathrm{E}(a)$. By Separation there is c at $\mathbf{w}$ such that $\vDash_{\mathbf{w}} c = \{x \in b : \diamond x \in a\}$; I claim $a = c$ using $\mathrm{Ext}_{\diamond}$. Fix x at $\mathbf{u}$. If $\vDash_{\mathbf{u}} \diamond x \in c$ then clearly $\vDash_{\mathbf{u}} \diamond x \in a$. Conversely, if $\vDash_{\mathbf{u}} \diamond x \in a$, then letting $x = z$ we have $\vDash_{\mathbf{u}} (\exists z(x \subseteq z \in a))^{\diamond}$ by (2), hence $\vDash_{\mathbf{w}} \diamond x \in b$ so that $\vDash_{\mathbf{w}} x \in b$ and hence $\vDash_{\mathbf{w}} x \in c$ i.e. $\vDash_{\mathbf{u}} \diamond x \in c$.

I now show that $\vDash_{\mathbf{w}} b = \P a$. If $\vDash_{\mathbf{w}} x \in b$, then $\vDash_{\mathbf{w}} (\exists z(x \subseteq z \in a))^{\diamond}$, i.e. there is $\mathbf{u}$ and z such that $\vDash_{\mathbf{u}} (x \subseteq z \in a)^{\diamond}$; now $\vDash_{\mathbf{w}} x \subseteq z \in a$ by (3) and as $\vDash_{\mathbf{w}} \mathrm{E}(a)$. Conversely, if $\vDash_{\mathbf{w}} x \subseteq z \in a$ for some z, then $\vDash_{\mathbf{w}} (\exists z(x \subseteq z \in a))^{\diamond}$ by (2), so $\vDash_{\mathbf{w}} \diamond x \in b$ and so $\vDash_{\mathbf{w}} x \in b$.

(5) Similar to (4).

(6)–(7) By (1) and (4)–(5). □

All the results of this appendix can be first-orderized straightforwardly. Keen readers will also notice that the proofs of this appendix have made no apparent use of the assumption of past-directedness. Indeed: the only role for past-directedness is to supply us with a possibility operator, $\diamond$, which is unrestricted and obeys S5.

B Results concerning LPST

I will now turn from PST to LPST. As mentioned in §5, linearity allows us to define away $\blacklozenge$ and $\lozenge$ via the map $\phi \mapsto \phi^{\bullet}$. To guarantee that this is so, we use the results of §A to prove Lemma 5.1 by a simple induction on complexity; I leave this to the reader.

Evidently, LPST$^{\bullet}$ is a unimodal S5 theory. However, it may be worth noting that it can be given a simpler presentation. Let MLT be a unimodal S5 theory whose set-theoretic axioms are $\mathrm{Mem}_{\diamond}$, $\mathrm{Ext}_{\diamond}$, Separation, and clauses (3)–(4) of Theorem 4.1. The proofs of Lemmas A.1–A.3 go through in MLT with only tiny adjustments; and it is easy to show that MLT $\vdash \diamond\phi^{\bullet} \leftrightarrow (\blacklozenge\phi \vee \phi \vee \lozenge\phi)^{\bullet}$ for each LPST-formula ϕ. It follows that LPST$^{\bullet}$ $\dashv\vdash$ MLT. By Lemma 5.1, then, LPST and MLT are (strictly) definitionally equivalent.

B.1 Deductive near-synonymy

The key results concerning LPST, though, are the near-synonymies. I will start with the first-order deductive near-synonymy:

Theorem 6.1: For any LT_1-formula ϕ not containing s:

(1) If $\mathrm{LT}_1 \vdash \phi$, then $\mathrm{LPST}_1 \vdash \phi^{\diamond}$

(2) $\mathrm{LT}_1 \vdash \phi \leftrightarrow (\phi^{\diamond})^{s}$

For any LPST_1-formula ϕ not containing s:

(3) If $\mathrm{LPST}_1 \vdash \phi$, then $\mathrm{LT}_1 \vdash \mathit{Lev}(s) \rightarrow \phi^{s}$

(4) $\mathrm{LPST}_1 \vdash \mathit{Max}(s) \rightarrow (\phi \leftrightarrow (\phi^{s})^{\diamond})$

Proof. (1) Extensionality$^{\diamond}$ is $\mathrm{Ext}_{\diamond}$. For Stratification$^{\diamond}$, use Theorem 4.1.3 and Lemma A.7. For the Separation$_1^{\diamond}$-instances, fix suitable ϕ; fix a at $\mathbf{w}$; by Separation we have some b in $\mathbf{w}$ such that $\vDash_{\mathbf{w}} b = \{x \in a : \phi^{\Diamond}\}$. Fix x at $\mathbf{u}$; now $\vDash_{\mathbf{u}} \Diamond x \in b$ iff $\vDash_{\mathbf{w}} x \in b$ iff $\vDash_{\mathbf{w}} \phi^{\Diamond} \wedge x \in a$ iff $\vDash_{\mathbf{u}} \phi^{\Diamond} \wedge \Diamond x \in a$ by Lemma A.7.

(2) A routine induction on complexity.

(3) The well-ordering and potency of levels yields the levelling of each underlying logical principle. It is then straightforward to obtain the levelling of each LPST_1 axiom is then straightforward.

(4) An induction on complexity. The cases of atomic formulas, conjunctions and quantifiers are easy, relying on $\text{Mem}_{\Diamond}$ and Lemma A.7.2–3.

For quantifiers: using the induction hypothesis, LPST proves that, if $Max(s)$ then: $(\exists x \phi)$ iff $(\exists x \subseteq s)$ iff $(\exists x \subseteq s)(\phi^s)$ iff $\Diamond(\exists x \subseteq s)(\phi^s)^{\Diamond}$ iff $((\exists x \phi)^s)^{\Diamond}$.

For modal operators, I will prove the case for $\blacklozenge$ (the others are similar). Fix $\mathbf{w}$ and, using Theorem 4.1.3, let $\vDash_{\mathbf{w}} Max(s)$; I claim that $\vDash_{\mathbf{w}} \blacklozenge\phi \leftrightarrow ((\blacklozenge\phi)^s)^{\Diamond}$.

Suppose $\vDash_{\mathbf{w}} \blacklozenge\phi$, i.e. there is $\mathbf{v} < \mathbf{w}$ such that $\vDash_{\mathbf{v}} \phi$. Using Theorem 4.1.3, let $\vDash_{\mathbf{v}} Max(r)$. By the induction hypothesis, $\vDash_{\mathbf{v}} \phi \leftrightarrow (\phi^r)^{\Diamond}$; so $\vDash_{\mathbf{w}} (\phi^r)^{\Diamond}$. Hence $\vDash_{\mathbf{w}} Lev(r) \wedge \Diamond r \in s \wedge (\phi^r)^{\Diamond}$; now by Lemma A.7 we have $\vDash_{\mathbf{w}} \Diamond\exists r(Lev^{\Diamond}(r) \wedge \Diamond r \in s \wedge (\phi^r)^{\Diamond})$, i.e. $\vDash_{\mathbf{w}} ((\blacklozenge\phi)^r)^{\Diamond}$

Suppose $\vDash_{\mathbf{w}} ((\blacklozenge\phi)^r)^{\Diamond}$, i.e. for some $\mathbf{v}$ and some r at $\mathbf{v}$ we have $\vDash_{\mathbf{v}} Lev^{\Diamond}(r) \wedge \Diamond r \in s \wedge (\phi^r)^{\Diamond}$. Using Theorem 4.1.4 and Lemma A.7, fix $\mathbf{u}$ such that $\vDash_{\mathbf{u}} Max(r)$; note that $\vDash_{\mathbf{u}} (\phi^r)^{\Diamond}$, so that $\vDash_{\mathbf{u}} \phi$ by the induction hypothesis. Moreover, $\mathbf{u} < \mathbf{w}$, as $\vDash_{\mathbf{v}} \Diamond r \in s$ and we have assumed linearity. So $\vDash_{\mathbf{w}} \blacklozenge\phi$. ◻

Theorem 6.1 straightforwardly entails that modalization and levelling are *faithful*:

Corollary B.1:

(1) $\text{LPST}_1 \vdash \phi^{\Diamond}$ iff $\text{LT}_1 \vdash \phi$, for any LT_1-formula ϕ

(2) $\text{LT}_1 \vdash Lev(s) \rightarrow \phi^s$ iff $\text{LPST}_1 \vdash \phi$, for any LPST_1-formula ϕ not containing s

I leave the proof to the reader. The reader can also prove these two second-order versions of Theorem 6.1, mentioned in §8:

Theorem B.2: Theorem 6.1 holds for LT and LPST_{n}, where we enrich modalization and levelling with these clauses:

$$\alpha^{\Diamond} := \Diamond\alpha, \text{ for atomic } \alpha \qquad (\exists F\phi)^{\Diamond} := \Diamond\exists F\phi^{\Diamond}$$

$$(F = G)^s := F = G \qquad F(\vec{x})^s := (F(\vec{x}) \wedge \vec{x} \subseteq s) \qquad (\exists F\phi)^s := \exists F\phi^s$$

Theorem B.3: Theorem 6.1 holds for LT_{b} and LPST_{c}, where we enrich modalization as above, but instead enrich levelling as follows:

$$(F = G)^s := (F = G \sqsubseteq s) \qquad F(\vec{x})^s := (F(\vec{x}) \wedge F \sqsubseteq s) \qquad (\exists F\phi)^s := (\exists F \sqsubseteq s)\phi^s$$

B.2 Semantic near-synonymy

I now consider the semantic near-synonymies. The first-order result follows from two lemmas, which are proved by a routine induction on complexity:

Lemma B.4: If $\mathcal{P} \vDash \text{LPST}_1$, then $\mathcal{P} \vDash \phi^{\Diamond}(\vec{a})$ iff $\flat\mathcal{P} \vDash \phi(\vec{a})$, for any $\vec{a}$ in $\flat\mathcal{P}$'s domain and any LT_1-formula $\phi(\vec{x})$ with free variables displayed.

Lemma B.5: If $\mathcal{A} \vDash \text{LT}_1$, then $\mathcal{A} \vDash \phi^r(\vec{a})$ iff $\sharp\mathcal{A} \vDash_r \phi(\vec{a})$, for any $\vec{a}$ from $\mathcal{A}$'s domain, any r such that $\mathcal{A} \vDash \mathit{Lev}(r)$, and any LPST_1-formula $\phi(\vec{x})$ with free variables displayed.

Theorem 7.4:

(1) If $\mathcal{P} \vDash \text{LPST}_1$, then $\flat\mathcal{P} \vDash \text{LT}_1$
(2) If $\mathcal{P} \vDash \text{LPST}_1$, then there is a surjection f such that $\mathcal{P} = (\sharp\flat\mathcal{P})_f$
(3) If $\mathcal{A} \vDash \text{LT}_1$, then $\sharp\mathcal{A} \vDash \text{LPST}_1$
(4) If $\mathcal{A} \vDash \text{LT}_1$, then $\mathcal{A} = \flat\sharp\mathcal{A}$

Proof. *(1)* By Theorem 6.1.1 and Lemma B.4.

(2) Let W be the set of $\mathcal{P}$'s worlds; let $L = \{s : \flat\mathcal{P} \vDash \mathit{Lev}(s)\}$ be the set of $\sharp\flat\mathcal{P}$'s worlds. Using Theorem 4.1.3, for each $\mathbf{w} \in W$, let $f(\mathbf{w})$ be the maximal level in $\mathbf{w}$.

I claim that $f : W \longrightarrow L$ is a surjection. To show that $L \subseteq \text{ran}(f)$, fix $s \in L$, i.e. $\flat\mathcal{P} \vDash \mathit{Lev}(s)$. Let $\mathbf{w}$ be such that $\mathcal{P} \vDash_{\mathbf{w}} \text{E}(s)$; now $\mathcal{P} \vDash_{\mathbf{w}} \mathit{Lev}(s)$ by Lemmas B.4 and A.7, and there is $\mathbf{v}$ such that $\mathcal{P} \vDash_{\mathbf{v}} \mathit{Max}(s)$ by Theorem 4.1.4; so $f(\mathbf{v}) = s$. The proof that $\text{ran}(f) \subseteq L$ is similar but simpler.

Now $\mathcal{P}$ and $\sharp\flat\mathcal{P}$ share a global domain, since $\Diamond\text{E}(x)$ is a schema of our logic (see footnote 5). They agree on membership and identity by construction. So $\mathcal{P} = (\sharp\flat\mathcal{P})_f$.

(3) By Theorem 6.1.3 and Lemma B.5.

(4) By Stratification, $\mathcal{A}$ and $\flat\sharp\mathcal{A}$ have the same domain, and they agree on membership by construction. ◻

As discussed in §8, we also have two second-order versions of Theorem 7.4 which hold for full or Henkin semantics.

Theorem B.6: Theorem 7.4 holds for LT and LPST_n, where we extend flattening and potentialization with these clauses:

Flattening: $\flat\mathcal{P}$'s second-order domain is $\mathcal{P}$'s global second-order domain; and $\flat\mathcal{P} \vDash F(\vec{a})$ iff $\mathcal{P} \vDash \Diamond F(\vec{a})$.

Potentialization: $\sharp\mathcal{A}$'s global second-order domain is $\mathcal{A}$'s second-order domain; and $\sharp\mathcal{A} \vDash_s F(\vec{a})$ iff $\mathcal{A} \vDash F(\vec{a}) \land \vec{a} \subseteq s$.

Theorem B.7: Theorem 7.4 holds for LT_b and LPST_c, where we extend flattening and potentialization as above, *and* add a further clause for potentialization, to allow variable second-order domains: $\sharp\mathcal{A} \vDash_s \text{E}(F)$ iff $\mathcal{A} \vDash F \sqsubseteq s$.

As mentioned in §8.2, if we invoke *full* semantics, we can obtain a final semantic result. Recall that, with full semantics, first-order domains determine second-order domains. (In the modal setting: full contingentist semantics specifies that a world's monadic second-order domain is the powerset of that world's first-order domain.) So, when we are using full semantics, we can forget about second-order entities, allowing them to 'take care of themselves', and simply use the definitions of flattening and potentialization that were given for *first-order* theories. We then have a near-synonymy as follows:

Theorem B.8: Using full semantics, Theorem 7.4 holds for LT and LPST_c, with flattening and potentialization exactly as defined in §7.

Proof. Clauses (2)–(4) are left to the reader. To establish (1), suppose $\mathcal{P} \vDash \text{LPST}_\text{c}$. So $\mathcal{P} \vDash_{\mathbf{w}}$ LT for each world **w**, by Theorem 4.1. Now LT is externally quasi-categorical by Pt.1 Theorem 6.1, and membership is modally robust by $\text{Mem}_\Diamond$ and $\text{Ext}_\Diamond$. So, given any two worlds of $\mathcal{P}$, one is an initial segment of the other. Hence $\flat\mathcal{P} \vDash$ LT. □

C Equivalences concerning LT_b

In §9.3, I considered Edna, a contingentist who holds that time is endless. To formalise the claim 'time is endless', we have the modal scheme $\Diamond\top$. Let $\text{LPST}_{\text{c}+}$ be the result of adding this scheme to LPST_c. So, Edna's theory is $\text{LPST}_{\text{c}+}$.

By contrast, consider the principle $\Box\bot \vee \Diamond\Box\bot$. Over LPST_c, this amounts to the statement 'time has an end'. Call this theory $\text{LPST}_{\text{c}-}$.

Actualists can mirror such talk about the 'end of time'. The sentence Endless, from Pt.1 §7, states that the (actualist) hierarchy has no last level. For brevity, let $\text{LT}_{\text{b}+}$ be LT_b + Endless, and let $\text{LT}_{\text{b}-}$ be LT_b + ¬Endless. It is easy to confirm that $\text{LPST}_{\text{c}+}$ is near-synonymous with $\text{LT}_{\text{b}+}$, and that $\text{LPST}_{\text{c}-}$ is near-synonymous with $\text{LT}_{\text{b}-}$.

However, $\text{LT}_{\text{b}+}$ and $\text{LT}_{\text{b}-}$ merit discussion in their own right. Fairly trivially, $\text{LT}_{\text{b}-}$ is identical to LT + ¬Endless. More interestingly, $\text{LT}_{\text{b}+}$ can be regarded as a notational variant of the *first-order* theory LT_1 + Endless, i.e. LT_{1+}. Specifically: there is an interpretation which is identity over the first-order entities and bi-interpretability over the second-order entities.[46] Here is the point in detail. We interpret $\text{LT}_{\text{b}+}$ in LT_{1+} using a translation, $\Downarrow$, which tells us to regard n-place second-order variables as an odd way to talk about sets of n-tuples. Formally, its only non-trivial clauses are:

$$(Y^n(x_1, \ldots, x_n))^{\Downarrow} := \langle x_1, \ldots, x_n \rangle \in Y^n$$
$$(\forall Y^n \phi)^{\Downarrow} := \forall Y^n((\forall z \in Y^n)(z \text{ is an } n\text{-tuple}) \rightarrow \phi^{\Downarrow})$$

where we treat n-tuples via Wiener–Kuratowski,[47] and regard each capital, superscripted, variable as just a new first-order variable. This yields a very tight connection between LT_{1+} and $\text{LT}_{\text{b}+}$:

Theorem C.1:

(1) $\text{LT}_{1+} \vdash \phi$ iff $\text{LT}_{\text{b}+} \vdash \phi$, for first-order ϕ

46 Thanks to James Studd, Albert Visser, and Sean Walsh for discussion of this case.

47 So e.g. $(Y^2(x_1, x_2))^{\Downarrow}$ is $(\exists z \in Y^2)\forall y(y \in z \leftrightarrow (\forall w(w \in y \leftrightarrow w = x_1) \vee \forall w(w \in y \leftrightarrow (w = x_1 \vee w = x_2))))$.

(2) $\mathrm{LT_{b+}} \vdash \phi$ iff $\mathrm{LT_{1+}} \vdash \phi^{\Downarrow}$, for second-order ϕ

Moreover, $\mathrm{LT_{b+}}$ proves that $\Downarrow$ is identity over the first-order entities and an isomorphism over the second-order entities.

Proof. (1) It suffices to show that $\mathrm{LT_{b+}}$ proves the Separation scheme. Fix a formula ϕ. Fix a. By Stratification, there is some level $s \supseteq a$. Using $\mathrm{Comp_b}$, there is $F \sqsubseteq s$ such that $(\forall x \subseteq s)(F(x) \leftrightarrow \phi)$. By Extensionality and the Separation axiom, we have $b = \{x \in a : F(x)\}$; now $b = \{x \in a : \phi\}$, as required, since levels are transitive.

(2) To establish $\mathrm{Comp_b}^{\Downarrow}$: fix a level s; using Endless, let t be the $2n$+1^{th} level after s; then use the Separation scheme to obtain $F^n = \{\langle x_1, \ldots, x_n\rangle \in t : \phi^{\Downarrow}\}$, noting that $\langle x_1, \ldots, x_n\rangle \in t$ iff $x_i \subseteq s$ for all $1 \leq i \leq n$. $\mathrm{Strat_b}^{\Downarrow}$ follows from Stratification.

Moreover. In $\mathrm{LT_{b+}}$, stipulate that $\tau(F^n) = \{\langle x_1, \ldots, x_n\rangle : F^n(\vec{x})\}$. □

Note that Theorem C.1 is *not* a definitional equivalence: definitional equivalence is unavailable, since $\mathrm{LT_{1+}}$ and $\mathrm{LT_{b+}}$ have different grammars. This difference aside, $\mathrm{LT_{1+}}$ and $\mathrm{LT_{b+}}$ are as tightly linked as we could want. Moreover, since $\mathrm{LPST_{c+}}$ and $\mathrm{LT_{b+}}$ are near-synonymous, Theorem C.1 allows us to regard $\mathrm{LPST_{c+}}$, which is a modal second-order theory, as a notational variant of $\mathrm{LT_{1+}}$, which is a non-modal first-order theory.[48]

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[48] Cf. Studd (2019: 179–80).

Level Theory, Part 3

A boolean algebra of sets arranged in well-ordered levels

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This document contains preprints of Level Theory, Parts 1–3. All three papers are forthcoming at *Bulletin of Symbolic Logic*.

Abstract. On a very natural conception of sets, every set has an absolute complement. The ordinary cumulative hierarchy dismisses this idea outright. But we can rectify this, whilst retaining classical logic. Indeed, we can develop a boolean algebra of sets arranged in well-ordered levels. I show this by presenting Boolean Level Theory, which fuses ordinary Level Theory (from Part 1) with ideas due to Thomas Forster, Alonzo Church, and Urs Oswald. BLT neatly implement Conway's games and surreal numbers; and a natural extension of BLT is definitionally equivalent with ZF.

> Like all walls it was ambiguous, two-faced. What was inside it and what was outside it depended upon which side you were on.
>
> Le Guin (1974: 1)

Building on work by Alonzo Church and Urs Oswald, Thomas Forster has provided a pleasingly different way to think about sets. As in the ordinary cumulative hierarchy, the sets are stratified into well-ordered levels. But, unlike the ordinary cumulative picture, the sets form a boolean algebra. In particular, every set has an absolute complement, in the sense that $\forall a \exists c \forall x(x \in a \leftrightarrow x \notin c)$. In this paper, I develop an axiomatic theory for this conception of set: Boolean Level Theory, or BLT.

I start by outlining the bare-bones idea of a complemented hierarchy of sets, according to which sets are arranged in stages, but where each set is found alongside its complement. I axiomatize this bare-bones story in the most obvious way possible, obtaining Boolean Stage Theory, BST. It is clear that any complemented hierarchy satisfies BST (see §§1–2). Unfortunately, BST has multiple primitives. To overcome this, I develop Boolean Level Theory, BLT. The only primitive of BLT is $\in$, but BLT and BST say exactly the same things about sets. As such, any complemented hierarchy satisfies BLT. Moreover, BLT is quasi-categorical (see §§3–5). I then provide two interpretations using BLT_{ZF} (an obvious extension of BLT): we can regard ZF as a proper part of BLT_{ZF}; but ZF is definitionally equivalent to BLT_{ZF}

(see §§6–7). I close by explaining how to implement Conway's games and surreal numbers in BLT (see §8).

This paper is the third in a triptych. It closely mirrors Part 1, but can be read in isolation. Let me repeat, though, that Part 1 is hugely indebted to the work of Dana Scott, Richard Montague, George Boolos, John Derrick, and Michael Potter; this paper inherits those debts.[1]

Some remarks on notation (which is exactly as in Pt.1 §0). I use second-order logic throughout. Mostly, though, this is just for convenience. Except when discussing quasi-categoricity (see §5), any second-order claim can be replaced with a first-order schema in the obvious way. I use some simple abbreviations (where Ψ can be any predicate whose only free variable is x, and $\lhd$ can be any infix predicate):

$$(\forall x : \Psi)\phi := \forall x(\Psi(x) \rightarrow \phi) \qquad (\forall x \lhd y)\phi := \forall x(x \lhd y \rightarrow \phi)$$
$$(\exists x : \Psi)\phi := \exists x(\Psi(x) \land \phi) \qquad (\exists x \lhd y)\phi := \exists x(x \lhd y \land \phi)$$

I also concatenate infix conjunctions, writing things like $a \subseteq r \in s \in t$ for $a \subseteq r \land r \in s \land s \in t$. And I run these devices together; so $(\forall x \notin x \in a)x \subseteq a$ abbreviates $\forall x((x \notin x \land x \in a) \rightarrow x \subseteq a)$. When I announce a result or definition, I list in brackets the axioms I am assuming. For readability, all proofs are relegated to the appendices.

1 The Complemented Story

Here is a very natural image of sets: *sets are not just collections of objects; sets partition the universe, and both sides of the partition yield a set.* There is the set of sheep; and there is the set of non-sheep. There is the set of natural numbers; and there is the set of everything else. There is the empty set; and there is the universal set.

Many will reject this image out of hand. Supposedly, the paradoxes of naïve set theory have taught us that there is no universal set; for if there were a universal set $V = \{x : x = x\}$, then Separation would entail the existence of the Russell set $\{x : x \notin x\}$, which is a contradiction.

That reasoning, though, is too quick. Separation is incompatible with the existence of V.[2] More generally, Separation is incompatible with the principle of Complementation (i.e. with the principle that every set has an absolute complement). But it does not immediately follow that Complementation is false; only that we must choose between Separation and Complementation.

Both principles are very natural. Separation, however, has the weight of history behind it; and this might not merely be a historical accident. There is a serious argument in favour of Separation and against Complementation, which runs as follows. The paradoxes of naïve set theory forced us to develop a less naïve conception of *set*. The best such conception (according to this argument) is the cumulative iterative conception, as articulated by this bare-bones story (recycled from Pt.1):

[1] See in particular Montague (1965: 139), Montague et al. (unpublished: §22), Scott (1960, 1974), Boolos (1971: 8–11, 1989), and Potter (1990: 16–22, 2004: ch.3).

[2] NB: I assume classical logic throughout.

The Basic Story. Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find a set whose members are exactly those sets. We find nothing else at **s**.

It is easy to see that this conception of set yields Separation rather than Complementation: any subset of a set a occurs at (or before) any stage at which a itself occurs. So (the argument concludes) we should embrace Separation and reject Complementation.

I take this argument very seriously. However, its success hinges on whether the ordinary cumulative iterative conception really is the 'best' conception of *set*. Whatever exactly 'best' is supposed to mean, the argument lays down a challenge: produce an equally good or better conception of *set*, which accepts Complementation and rejects Separation.

This paper considers a very specific reply to this challenge, due to Forster's development of work by Church and Oswald.[3] Forster's idea is to make a small tweak to the story of the ordinary hierarchy, so that 'each time we [find] a new set. . . we also [find] a companion to it which is to be its complement'.[4] In slightly more detail, we offer the following bare-bones story:

The Complemented Story. Sets are arranged in stages. Every set is found at some stage. At any stage **s**: for any sets found before **s**, we find both

(*Lo*) a set whose *members* are exactly those sets, and

(*Hi*) a set whose *non-members* are exactly those sets.

We find nothing else at **s**.

According to our new story, we find each set using either clause (*Lo*) or clause (*Hi*). Moreover, if we find a set using clause (*Lo*), then we find its absolute complement using clause (*Hi*), and vice versa. This is the absolute complement since, in clause (*Hi*), we quantify over all sets that will ever be discovered, not just those discovered before stage **s**. This story therefore secures Complementation; it describes the bare

[3] Church (1974) and Oswald (1976); see also Mitchell (1976) and Sheridan (2016). Forster (2001) includes a nice summary of the technicalities behind the original Church–Oswald idea.

[4] Forster (2008: 100). Note that I speak of 'finding' sets, whereas Forster speaks of 'creating' them. Talk of 'creation' leads Forster to say that the members of V change, stage-by-stage, as more sets are created, so that V is 'intensional', in a way that $\emptyset$ is not (2008: 100). I think that Forster should regard $\emptyset$ as equally 'intensional', since what $\emptyset$ omits changes, stage-by-stage. However, if sets are discovered (rather than created) stage-by-stage, then all issues concerning intensionality can be side-stepped: all that changes, stage-by-stage, is our knowledge about V's members and $\emptyset$'s non-members.

If we admit contingently-existing urelements, then the discussion of intensionality becomes much more complicated. In the actual world, Boudica $\in \{x : x = x\}$; but in a possible world where she never existed, Boudica $\notin \{x : x = x\}$; by contrast, in all possible worlds, Boudica $\notin \{x : x \neq x\}$. From this, one might infer that V is intensional whereas $\emptyset$ is not. But this inference is not immediate; it requires two substantial, further, assumptions: (1) that the descriptions '$\{x : x \neq x\}$' and '$\{x : x = x\}$' *rigidly* designate $\emptyset$ and V respectively, and (2) that intensionality concerns trans-world variation of *members* rather than trans-world variation of *non-members*. I hope to explore both assumptions elsewhere. (Thanks to James Studd, Timothy Williamson, Stephen Yablo, and an anonymous referee for this journal, for pushing me on this point.)

idea of a *complemented* hierarchy of sets. But it only describes the *bare* idea, since, for example, it says nothing about the height of the hierarchy.

In what follows, I will develop an axiomatic theory of this story, and explore that theory's behaviour. To be clear: I am not claiming that we should reject the ordinary hierarchy in favour of the complemented. My aim is only to provide a coherent (and surprisingly elegant) conception of *set* which allows for Complementation rather than Separation.

In what follows, I will speak of *low* sets and *high* sets.[5] A set is low iff we find it using clause (*Lo*); we characterize low sets by saying 'exactly these things, which we found earlier, are this set's members'. The limiting case of a low set is the empty set, $\emptyset$. A set is high iff we find it using clause (*Hi*); we characterize high sets by saying 'exactly these things, which we found earlier, are omitted from this set'. The limiting case of a high set is the universe, V. (Note that low sets can have high sets as members, e.g. $\{V\}$ would be a low set with a high member.)

2 Boolean Stage Theory

Given a model of ZF, there are simple methods for constructing models of the complemented hierarchy.[6] However, if the idea of a complemented hierarchy is genuinely to rival that of the ordinary hierarchy, it cannot remain parasitic upon ZF; it needs a fully autonomous theory. I will provide such a theory over the next two sections.[7]

The Complemented Story, which introduces the bare-bones idea of a complemented hierarchy, speaks of both stages and sets. To begin, then, I will present a theory which quantifies distinctly over both sorts of entities. Boolean Stage Theory, or BST, has two distinct sorts of first-order variable, for *sets* (lower-case italic) and for **stages** (lower-case bold). It has five primitive predicates:

$\in$: a relation between sets; read '$a \in b$' as 'a is in b'
$<$: a relation between stages; read '$\mathbf{r} < \mathbf{s}$' as '$\mathbf{r}$ is before $\mathbf{s}$'
$\preceq$: a relation between a set and a stage; read '$a \preceq \mathbf{s}$' as 'a is found at $\mathbf{s}$'
Lo: a property of sets; read '$Lo(a)$' as 'a is low', i.e. we find a using clause (*Lo*)
Hi: a property of sets; read '$Hi(a)$' as 'a is high', i.e. we find a using clause (*Hi*)

For brevity, I write $a \prec \mathbf{s}$ for $\exists \mathbf{r}(a \preceq \mathbf{r} < \mathbf{s})$, i.e. a is found before $\mathbf{s}$. Then BST has eight axioms:[8]

Extensionality $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$

[5] Note that every set will be low or high. This terminology departs somewhat from Church's. Church (1974: 298) defined 'a *low* set as a set which has a one-to-one relation with a well-founded set' and 'a *high* set as a set which is the complement of a low set'. This leaves logical space for sets which are neither low nor high (in Church's terms), and Church (1974: 305) used such sets to provide a Frege–Russell definition of cardinal numbers.

[6] See Forster (2001: §§1–2, 2008: 106–8); and my interpretation I in §D.1.

[7] The approach in this section follows Scott and Boolos, but in the setting of complemented hierarchies rather than the ordinary hierarchies; see Pt.1 §§1 and 8.

[8] Using classical logic yields 'cheap' proofs of the existence of a stage, an empty set, and a universal set, via Staging, Specification$_{Lo}$ and Specification$_{Hi}$. Those who find such proofs *too* cheap might wish to add some explicit existence axioms. (Cf. Pt.1 footnote 2.)

Order $\forall \mathbf{r}\forall \mathbf{s}\forall \mathbf{t}(\mathbf{r} < \mathbf{s} < \mathbf{t} \rightarrow \mathbf{r} < \mathbf{t})$

Staging $\forall a \exists \mathbf{s}\ a \preceq \mathbf{s}$

Cases $\forall a(Lo(a) \vee Hi(a))$

Priority$_{Lo}$ $\forall \mathbf{s}(\forall a : Lo)(a \preceq \mathbf{s} \rightarrow (\forall x \in a)x \prec \mathbf{s})$

Priority$_{Hi}$ $\forall \mathbf{s}(\forall a : Hi)(a \preceq \mathbf{s} \rightarrow (\forall x \notin a)x \prec \mathbf{s})$

Specification$_{Lo}$ $\forall F \forall \mathbf{s}((\forall x : F)x \prec \mathbf{s} \rightarrow (\exists a : Lo)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \in a)))$

Specification$_{Hi}$ $\forall F \forall \mathbf{s}((\forall x : F)x \prec \mathbf{s} \rightarrow (\exists a : Hi)(a \preceq \mathbf{s} \wedge \forall x(F(x) \leftrightarrow x \notin a)))$

I will now explain how to justify each axiom.

The first two axioms make implicit assumptions explicit. Whilst I did not mention Extensionality when I told the story of the complemented hierarchy, I take it as analytic that sets are extensional.[9] Similarly, Order records the analytic fact that 'before' is transitive. Note, though, that I do not explicitly assume that the stages are well-ordered,[10] as it is unclear at this point what would justify that assumption. (After all, if we are willing to countenance entities as ill-founded as V, then it is not immediately obvious that we should refuse to countenance a hierarchy with infinite descending chains of stages. And the Complemented Story does not explicitly require that the stages be well-ordered.)

Informally, Staging says that every set is discovered at some stage; this claim appears verbatim in the Complemented Story. Likewise, Cases says that every set is either low or high, and this is immediate from the fact that every set is discovered using either clause (*Lo*) or clause (*Hi*). (Note, though, that I do not assume at the outset that this is an exclusive disjunction; initially, we should be open to the thought that one set could be discovered using both clauses.)[11]

Next, Priority$_{Lo}$ and Priority$_{Hi}$ say that if we find a low set at a stage, then we find all its members earlier, and if we find a high set at a stage, then we find all its non-members earlier; both claims follow from clauses (*Lo*) and (*Hi*). Finally, Specification$_{Lo}$ and Specification$_{Hi}$ say that if every F was found before a certain stage, then at that stage we find both the low set of all Fs, and the high set of all non-Fs; again, both claims follow from (*Lo*) and (*Hi*).

Since all eight axioms hold of the Complemented Story, any complemented hierarchy satisfies BST.

3 Boolean Level Theory

Unfortunately, BST contains rather a lot of primitives. Fortunately, most of them can be eliminated. In this section, I present Boolean Level Theory, or BLT. This theory's only primitive is $\in$, but it makes exactly the same claims about sets as BST does.[12] I start with a key definition:[13]

9 For brevity of exposition, I am considering hierarchies of pure sets.

10 Here I part company with Forster (2008: 100), who explicitly stipulates that the stages are well-ordered. Ultimately, BST proves a well-ordering result (Theorem 4.1).

11 Ultimately, BST proves that no set is discovered using both clauses (Lemma B.7).

12 The approach in this section mirrors Pt.1 §§2 and 4, which builds on work by Montague, Scott, Derrick and Potter; see also Pt.1 §8.

13 Compare Montague's and Scott's ¶-operation, presented in Pt.1 Definition 2.1.

Definition 3.1: For any set a, let a's *absolute complement* be $\overline{a} = \{x : x \notin a\}$, if it exists. Let $\mathbf{P}a = \{x : (\exists c \notin c \in a)(x \subseteq c \lor \overline{x} \subseteq c)\}$, if it exists.[14]

The definition of $\overline{a}$ needs no comment, but the definition of $\mathbf{P}a$ merits explanation. It turns out that BST proves that a is low iff $a \notin a$, and a is high iff $a \in a$ (see Lemma B.7). Seen in this light, $\mathbf{P}a$ collects together all the subsets of low members of a, and all the complements of such subsets. As a specific example, if b is low, then $\mathbf{P}\{b\} = \{x : x \subseteq b \lor \overline{x} \subseteq b\}$, i.e. it is the result of closing b's powerset under complements. We use this operation in this next definition (where 'bistory' is short for 'boolean-history', and 'bevel' is short for 'boolean-level'):[15]

Definition 3.2: Say that h is a *bistory*, written $Bist(h)$, iff $h \notin h \land (\forall x \in h)x = \mathbf{P}(x \cap h)$. Say that s is a *bevel*, written $Bev(s)$, iff $(\exists h : Bist)s = \mathbf{P}h$.

The intuitive idea behind Definition 3.2 is that the bevels go proxy for the stages of the Complemented Story, and each bistory is an initial sequence of bevels. (It is far from obvious that these definitions work as described, but we will soon see that they do.) Using these definitions, BLT has just four axioms:[16]

Extensionality $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$
Complements $\forall a(\exists c = \overline{a})(a \notin a \leftrightarrow c \in c)$
Separation$_{\notin}$ $\forall F(\forall a \notin a)(\exists b \notin b)\forall x(x \in b \leftrightarrow (F(x) \land x \in a))$
Stratification$_{\notin}$ $(\forall a \notin a)(\exists s : Bev)a \subseteq s$

Intuitively, Complements tells us that every set has a complement, and a set is low iff its complement is high; Separation$_{\notin}$ tells us that arbitrary subsets of low sets exist (and are low); and Stratification$_{\notin}$ tells us that every low set is a subset of some bevel (which corresponds to the thought that it is found at some stage). These axioms and definitions are vindicated by this next result, which shows that BLT has exactly the same set-theoretic content as BST (see §B for the proof):

Theorem 3.3: BST $\vdash \phi$ iff BLT $\vdash \phi$, for any BLT-sentence ϕ.

Otherwise put: no information about sets is gained or lost by moving between BST and BLT. Moreover, since every complemented hierarchy satisfies BST, every complemented hierarchy satisfies BLT. In what follows, then, I will treat BLT as the canonical theory of complemented hierarchies.

---

[14] By the notational conventions, $\mathbf{P}a = \{x : \exists c(c \in a \land c \notin c \land (x \subseteq c \lor \overline{x} \subseteq c))\}$. BLT's axiom Complements guarantees that $\overline{a}$ exists for every a. However, we do not initially assume that $\mathbf{P}a$ exists for every a; instead, we initially treat every expression of the form '$b = \mathbf{P}a$' as shorthand for '$\forall x(x \in b \leftrightarrow (\exists c \notin c \in a)(x \subseteq c \lor (\exists z \subseteq c)\forall y(y \in z \leftrightarrow y \notin x)))$', and must double-check whether $\mathbf{P}a$ exists. Ultimately, though, BLT proves that $\mathbf{P}a$ exists for every a: if $a \notin a$ then $\mathbf{P}a \subseteq \mathbf{B}a$ (see Definition 4.3); if $a \in a$ then $\mathbf{P}a = V$.

[15] Compare Pt. 1 Definition 2.2, which simplifies the Derrick–Potter definition of 'level'. Here, 'bistory' is short for 'boolean-history'; 'bevel' is short for 'boolean level'.

[16] As in footnote 8, classical logic yields a 'cheap' proof of the existence of $\emptyset$ and V.

4 Characteristics and extensions of BLT

To give a sense of how BLT behaves, I will state some of its 'characteristic' results (the proofs are in §A). The first two results allow us to characterize BLT with a simple slogan: *a boolean algebra of sets arranged in well-ordered levels.*

Theorem 4.1 (BLT)**:** The bevels are well-ordered by $\in$.

Theorem 4.2 (BLT)**:** The sets form a boolean algebra under complementation, $\cap$ and $\cup$.

This first result is quite surprising:[17] the Complemented Story does not *explicitly* specify that the stages must be well-ordered (see §2); but, since every complemented hierarchy satisfies BLT (see §3), every complemented hierarchy has well-ordered levels.

The well-ordering of the bevels yields a powerful tool, which intuitively allows us to consider the bevel at which a set is first found:

Definition 4.3 (BLT)**:** If $a \notin a$, let $\mathbf{B}a$ be the $\in$-least bevel with a as a subset; i.e., $a \subseteq \mathbf{B}a$ and $\neg(\exists s : Bev)a \subseteq s \in \mathbf{B}a$. If $a \in a$, let $\mathbf{B}a = \mathbf{B}\overline{a}$.

Note that $\mathbf{B}a$ exists for any a, by Stratification$_{\notin}$, Complements and Theorem 4.1.

A third characteristic result is that there is a *contra-automorphism* on the universe.[18] Roughly put: replacing membership with non-membership (and vice versa) yields an isomorphic universe. Formally:

Definition 4.4: We recursively define a's *negative*, written $-a$, as follows:

$$-a := \overline{\{-x : x \in a\}}, \text{ if } a \notin a \qquad\qquad -a := \{-x : x \notin a\}, \text{ if } a \in a$$

Theorem 4.5 (BLT)**:** $\forall a \forall b(a \in b \leftrightarrow -a \notin -b)$

This immediately yields a nice duality:

Corollary 4.6 (BLT)**:** $\phi \leftrightarrow \phi^{\mho}$, for any BLT-sentence ϕ, where $\phi^{\mho}$ is the sentence which results from ϕ by replacing every '$\in$' with '$\notin$' and vice versa.

These results highlight some of BLT's deductive strengths. Now let me comment on its (deliberate) weakness. By design, BLT axiomatizes only the *bare* idea of a complemented hierarchy, and so makes no comment on the hierarchy's height.[19] If we

[17] It will be much less surprising for those who have read Pt.1 §5.

[18] See Forster (2001: Definition 16 and subsequent comments). This result inspires my epigraph, from Le Guin. I owe the point to Brian King: in 2006, he arrived at an idea like the Complemented Story (independently of Forster) and explained it using Le Guin's image.

[19] Beyond the fact that classical logic guarantees the existence of at least one stage; see footnotes 8 and 16.

want to ensure that our hierarchy is reasonably tall, three axioms suggest themselves (where 'P' is a second-order function-variable in the statement of Unbounded$_{\notin}$):

$$\begin{array}{rl} \textbf{Endless}_{\notin} & (\forall s : Bev)(\exists t : Bev)s \in t \\ \textbf{Infinity}_{\notin} & (\exists s : Bev)((\exists q : Bev)q \in s \land (\forall q : Bev)(q \in s \to (\exists r : Bev)q \in r \in s)) \\ \textbf{Unbounded}_{\notin} & \forall P(\forall a \notin a)(\exists s : Bev)(\forall x \in a)P(x) \in s \end{array}$$

Endless$_{\notin}$ says there is no last bevel. Infinity$_{\notin}$ says that there is an infinite bevel, i.e. a bevel with no immediate predecessor. Unbounded$_{\notin}$ states that the hierarchy of bevels is so tall that no low set can be mapped unboundedly into it (recall that the low sets are precisely the non-self-membered sets).

To make all of this more familiar, here are some simple facts relating BLT to ZF. Let BLT$_{+}$ stand for BLT+Endless$_{\notin}$, and BLT$_{ZF}$ stand for BLT+Infinity$_{\notin}$+Unbounded$_{\notin}$; then:[20]

Proposition 4.7:

(1) BLT proves the Axiom of Empty Set, i.e. $\exists a \forall x \; x \notin a$.
(2) BLT proves Union, i.e. $\forall a(\bigcup a$ exists).
(3) BLT$_{+}$ proves Pairing, i.e. $\forall a \forall b(\{a, b\}$ exists), but BLT does not.
(4) BLT$_{+}$ proves Powersets-restricted-to-low-sets, i.e. $(\forall a \notin a)(\wp a$ exists), but BLT does not.
(5) BLT contradicts Powersets, i.e. it proves $\exists a \neg \exists b \forall x(x \in b \leftrightarrow x \subseteq a)$.
(6) BLT proves Foundation-restricted-to-high-sets, i.e. $(\forall a \in a)(\exists x \in a)a \cap x = \emptyset$.
(7) BLT$_{+}$ contradicts Foundation, i.e. it proves $(\exists a \neq \emptyset)(\forall x \in a)a \cap x \neq \emptyset$.
(8) BLT$_{ZF}$ proves Endless$_{\notin}$.

If we want to state this result with maximum shock value: of the standard axioms of ZF, BLT validates only Extensionality, Empty Set, and Union (though BLT is also consistent with Pairing and standard formulations of Infinity).

5 The quasi-categoricity of BLT

We have seen that every complemented hierarchy satisfies BLT, so that every complemented hierarchy has well-ordered bevels. In fact, we can push this point further, by noting that BLT is quasi-categorical.[21]

Informally, we can spell out BLT's quasi-categoricity as follows: *Any two complemented hierarchies are structurally identical for so far as they both run, but one may be taller than the other*. So, when we set up a complemented hierarchy, our only choice is how tall to make it.

[20] Since BLT$_{+}$ proves Pairing, BLT$_{+}$ extends NF$_{2}$, the sub-theory of Quine's NF whose axioms are Extensionality, Pairing, and Theorem 4.2. However, BLT$_{+}$ does not extend NF$_{O}$, the theory which adds to NF$_{2}$ the axiom that $\{x : a \in x\}$ exists for every a; in particular, $\{x : \emptyset \in x\}$ does not exist; see the proof of Proposition 5.1.5 in §A. For discussion of NF$_{2}$ and NF$_{O}$, see Forster (2001: §2).

[21] This mirrors the discussion of LT's quasi-categoricity; see Pt.1 §6.

In fact, there are at least two ways to explicate the informal idea of quasi-categoricity, and BLT is quasi-categorical on both explications.[22] The first notion of quasi-categoricity should be familiar from Zermelo's results for ZF, and uses the full semantics for second-order logic:

Theorem 5.1: Given full second-order logic:

(1) The bevels of any model of BLT are well-ordered.[23]

(2) For any ordinal $\alpha > 0$, there is a model of BLT whose bevels form an α-sequence.[24]

(3) Given any two models of BLT, one is isomorphic to an initial segment of the other.[25]

Since this result involves semantic ascent, it is an *external* quasi-categoricity result. There is also an *internal* quasi-categoricity result for BLT, which is a theorem of the (second-order) object language, but this point requires a little more explanation.

In embracing Extensionality, BLT assumes that everything is a pure set. Here is an easy way to avoid making that assumption. Consider the following formula, which relativises BLT to a new primitive predicate, *Pure*:[26]

$$\begin{aligned}\mathrm{BLT}(\mathit{Pure}, \varepsilon) \coloneqq\ & (\forall a : \mathit{Pure})(\forall b : \mathit{Pure})(\forall x(x \mathrel{\varepsilon} a \leftrightarrow x \mathrel{\varepsilon} b) \rightarrow a = b) \land \\ & (\forall a : \mathit{Pure})(\exists c : \mathit{Pure})((\forall x : \mathit{Pure})(x \mathrel{\varepsilon} c \leftrightarrow x \notin a) \land (a \notin a \leftrightarrow c \mathrel{\varepsilon} c)) \land \\ & \forall F(\forall a : \mathit{Pure})(a \notin a \rightarrow \\ & \qquad (\exists b : \mathit{Pure})(b \notin b \land \forall x(x \mathrel{\varepsilon} b \leftrightarrow (F(x) \land x \mathrel{\varepsilon} a)))) \land \\ & (\forall a : \mathit{Pure})(a \notin a \rightarrow (\exists s : \mathit{Bev})a \subseteq s) \land \\ & \forall x \forall y(y \mathrel{\varepsilon} x \rightarrow (\mathit{Pure}(x) \land \mathit{Pure}(y)))\end{aligned}$$

The first four conjuncts say that the pure sets satisfy BLT;[27] the last says that, when we use 'ε', we restrict our attention to membership facts between pure sets. This avoids the assumption that everything is a pure set. Moreover, I can use this formula to state our internal quasi-categoricity result (I have labelled the lines to facilitate its explanation):[28]

Theorem 5.2: This is a deductive theorem of impredicative second-order logic:

---

[22] Both ways make essential use of second-order logic, albeit in different ways.

[23] i.e. if $\mathcal{M} \vDash \mathrm{BLT}$ then $\{s \in M : \mathcal{M} \vDash \mathit{Bev}(s)\}$ is well-ordered by $\in^{\mathcal{M}}$.

[24] i.e. there is some $\mathcal{M} \vDash \mathrm{BLT}$ such that $\{s \in M : \mathcal{M} \vDash \mathit{Bev}(s)\}$ is isomorphic to α.

[25] When $\mathcal{A}$ and $\mathcal{M}$ are models of BLT, say that $\mathcal{A}$ is an *initial segment* of $\mathcal{M}$ iff either $\mathcal{A} = \mathcal{M}$ or there is some s such that $\mathcal{M} \vDash \mathit{Bev}(s)$ and $\mathcal{A}$ is isomorphic to the substructure of $\mathcal{M}$ whose domain is $\{x \in M : \mathcal{M} \vDash \mathbf{B}x \in s\}$.

[26] Here, '$\subseteq$' and '*Bev*' should be defined in terms of ε rather than $\in$; similarly for '$\mathbf{B}$' in the statement of Theorem 5.2.

[27] With one insignificant caveat (see footnote 16): whereas classical logic guarantees that any model of BLT contains an empty set and a universal set, $\mathrm{LT}(\mathit{Pure}, \varepsilon)$ allows that there may be no pure sets.

[28] Button and Walsh's (2018: ch.11) proofs carry over straightforwardly to BLT.

$$
\begin{aligned}
&(\mathrm{BLT}(Pure_1, \varepsilon_1) \land \mathrm{BLT}(Pure_2, \varepsilon_2)) \to \\
&\quad \exists R(\forall v \forall y(R(v, y) \to (Pure_1(v) \land Pure_2(y))) \land && (1)\\
&\quad\quad ((\forall v : Pure_1)\exists y R(v, y) \lor (\forall y : Pure_2)\exists v R(v, y)) \land && (2)\\
&\quad\quad \forall v \forall y \forall x \forall z((R(v, y) \land R(x, z)) \to (v\ \varepsilon_1\ x \leftrightarrow y\ \varepsilon_2\ z)) \land && (3)\\
&\quad\quad \forall v \forall y \forall z((R(v, y) \land R(v, z)) \to y = z) \land && (4)\\
&\quad\quad \forall v \forall x \forall y((R(v, y) \land R(x, y)) \to v = x) \land && (5)\\
&\quad\quad \forall v \forall x \forall y((\mathbf{B}_1 x \subseteq_1 \mathbf{B}_1 v \land R(v, y)) \to \exists z R(x, z)) \land && (6)\\
&\quad\quad \forall v \forall y \forall z((\mathbf{B}_2 z \subseteq_2 \mathbf{B}_2 y \land R(v, y)) \to \exists x R(x, z))) && (7)
\end{aligned}
$$

Intuitively, the point is this. Suppose two people are using their versions of BLT, subscripted with '1' and '2' respectively. Then there is some second-order entity, a relation R, which takes us between their sets (1), exhausting the sets of one or the other person (2); which preserves membership (3); which is functional (4) and injective (5); and whose domain is an initial segment of one (6) or the other's (7) hierarchy. Otherwise put: BLT is (internally) quasi-categorical.

As a bonus, this internal quasi-categoricity result can be lifted into an internal *total*-categoricity result. To explain how, consider this abbreviation (where 'P' is a second-order function-variable):

$$\underset{\infty}{\exists} x \Phi(x) := \exists P(\forall x \Phi(P(x)) \land (\forall y : \Phi)\exists! x\ P(x) = y)$$

This formalizes the idea that there is a bijection between the Φs and the universe (see Pt.1 §6). Using this notation, we can state our internal total-categoricity result:

Theorem 5.3: This is a deductive theorem of impredicative second-order logic:

$$
\begin{aligned}
&(\mathrm{BLT}(Pure_1, \varepsilon_1) \land \underset{\infty}{\exists} x\, Pure_1(x) \land \mathrm{BLT}(Pure_2, \varepsilon_2) \land \underset{\infty}{\exists} x\, Pure_2(x)) \to \\
&\quad \exists R(\forall v \forall y\, (R(v, y) \to\ (Pure_1(v) \land Pure_2(y))) \land \\
&\quad\quad (\forall v : Pure_1)\exists! y R(v, y) \land (\forall y : Pure_2)\exists! v R(v, y) \land \\
&\quad\quad \forall v \forall y \forall x \forall z\, ((R(v, y) \land R(x, z)) \to (v\ \varepsilon_1\ x \leftrightarrow y\ \varepsilon_2\ z)))
\end{aligned}
$$

Intuitively, if both BLT-like hierarchies are as large as the universe, then there is a structure-preserving *bijection* between them.

6 Ordinary set theory as a proper part of BLT

The Complemented Story provides two clauses for finding sets. Clause (*Lo*) tells us that, at each stage **s** and for any sets found before **s**, we find a set whose members are exactly those sets. But this is exactly what we would find according to the Basic Story (see §1), which deals with ordinary, uncomplemented hierarchies. Intuitively, then, we should be able to recover an ordinary hierarchy by considering a complemented hierarchy whilst ignoring any use of clause (*Hi*). This intuitive idea is exactly right; the aim of this section is to explain it carefully.

First, I must formalize the notion of a set which we find without ever using clause (*Hi*). I call such sets *hereditarily low*, or *helow* for short. So: helow sets are low, their members are low, the members of their members are low, etc. Here is the precise definition:

Definition 6.1: Say that a is *helow*, or $Helo(a)$, iff there is some transitive $c \supseteq a$ such that $(\forall x \in c)x \notin x$.

To restrict our attention to the ordinary (uncomplemented) hierarchy, we then just restrict our attention to the helow sets. To implement this formally, for any formula ϕ, let $\phi^{\triangledown}$ be the formula which results by restricting all of ϕ's quantifiers to helow sets. Using this notation, we can then prove results of this shape: *If some theory of uncomplemented hierarchies proves ϕ, then some suitable theory of complemented hierarchies proves $\phi^{\triangledown}$.*

To state these results precisely, we need a suitable theory of uncomplemented hierarchies. That theory is LT, discussed in Pt.1. In a nutshell: LT stands to uncomplemented hierarchies exactly as BLT stands to complemented hierarchies. I will now briefly recap LT's key elements. To formalize the Basic Story, we define a predicate, *Lev*, to capture the notion of a *level* of an uncomplemented hierarchy (Pt.1 Definition 2.2); then LT is the theory whose axioms are Extensionality, Separation, and Stratification, which states that $\forall a(\exists s : Lev)a \subseteq s$ (see Pt.1 §2). It transpires that LT is quasi-categorical, and that every uncomplemented hierarchy satisfies LT, no matter how tall or short it is (see Pt.1 §§5–6). If we want to secure a tall uncomplemented hierarchy, we can consider the axioms Endless, Infinity and Unbounded (see Pt.1 §7); these are exactly like Endless$_{\notin}$, Infinity$_{\notin}$ and Unbounded$_{\notin}$ (see §3 of this part), except that they replace '*Bev*' with '*Lev*'. Let LT$_+$ stand for LT + Endless; it turns out that ZF is deductively equivalent to LT + Infinity + Unbounded; so LT, LT$_+$, and ZF are three theories which axiomatize uncomplemented hierarchies, making successively stronger demands on the hierarchy's height. With this background in place, here is the result which intuitively states that the helow part of any complement hierarchy is an ordinary (uncomplemented) hierarchy (see §C for the proof):

Theorem 6.2: For any LT-sentence ϕ:

(1) If LT $\vdash \phi$, then BLT $\vdash \phi^{\triangledown}$
(2) If LT$_+ \vdash \phi$, then BLT$_+ \vdash \phi^{\triangledown}$
(3) If ZF $\vdash \phi$, then BLT$_{ZF} \vdash \phi^{\triangledown}$

7 Definitional equivalence

Theorem 6.2.3 allows us to regard ZF as the result of restricting attention to the helow-fragment of BLT$_{ZF}$'s universe of sets. But we also have a much deeper interpretative result, as follows (see §D):[29]

[29] Forster conjectured that a result of this shape should hold.

Theorem 7.1: ZF and BLT_{ZF} are definitionally equivalent, as are LT_+ and BLT_+.

As an immediate consequence, ZF and BLT_{ZF} are *equiconsistent*, as are LT_+ and BLT_+. However, definitional equivalence is much stronger than mere equiconsistency.

Roughly, to say that two theories are definitionally equivalent is to say that each theory can define all the primitive expressions of the other, such that each theory can simulate the other perfectly, and where combining the two simulations gets you back exactly where you began.[30] So, in some purely formal sense, ZF and BLT_{ZF} can be regarded as notational variants; as wrapping the same deductive content in different notational packaging.

One might be tempted to go further, and suggest that Theorem 7.1 shows that there is *no* relevant difference between ZF and BLT_{ZF}. That, however, would require further argument.[31] Precisely because definitional equivalence is a purely formal property, it ignores all non-formal matters, and these may be philosophically significant. There is more philosophical discussion to be had about the significance of Theorem 7.1, but that must wait for another time.

8 Conway games and surreal numbers in BLT

Since ZF and BLT_{ZF} are definitionally equivalent, there is a sense in which each can do anything that the other can. Still, BLT_{ZF} can do some things more easily than ZF. This is neatly illustrated by considering John Conway's theory of games and surreal numbers.[32]

Consider two-player games in which players move alternately, with no element of chance, where the game must end in a win or loss. (Think of chess, but without the possibility of stalemate.) Abstractly, such games can be thought of as specifications of permissible positions: to make a move in such a game is just to select a new position which is permissible given the current game state; and you lose when it is your turn to move but there is no permissible position. (Think of being checkmated: you must move to a position where your King is not in check, but no such move is available.) Crucially, any position in any such game can be considered as a game in its own right. (Imagine the version of chess which always starts with the pieces arranged as after the Queen's Gambit in regular chess.) So every game can be regarded, abstractly, as nothing other than a specification of which games each player can move to. Otherwise put, if we call the two game-players Low and High, then a game is just a specification of *low options*, i.e. games that Low can move to, and *high options*, i.e. games that High can move to.

The idea is very natural. However, as Conway remarked, formalizing it 'in ZF destroys a lot of its symmetry.' He therefore suggested that 'the proper set theory in which to perform such a formalisation would be one with two kinds of

[30] For a precise statement of what definitional equivalence requires, see Button and Walsh (2018: ch.5).

[31] Compare Pt.2 §9.

[32] Joel David Hamkins suggested this application of BLT to me; many thanks to him, both for the initial suggestion, and for much subsequent correspondence.

membership': a game would just be a set with 'low-members' (low options) and 'high-members' (high options).[33] However, we can easily implement this idea in BLT, using only *one* kind of membership. We start by saying that the games are the sets, and then stipulate:

Definition 8.1 (BLT)**:** If a is low, the set of a's *low options* is $\mathrm{L}a := \{x \in a : x \notin x\}$; the set of a's *high options* is $\mathrm{H}a := \{x \in a : x \in x\}$. If a is high, $\mathrm{L}a := \mathrm{L}\bar{a}$ and $\mathrm{H}a := \mathrm{H}\bar{a}$.

Intuitively, then, a and $\bar{a}$ represent the same game. Moreover, there is a natural algebra on the games, given as follows (I explain the definitions below):[34]

Definition 8.2 (BLT)**:** With $-$ as in Definition 4.4, define $+$ and $\leq$ recursively:

$$a + c := \{x + c : x \in \mathrm{L}a\} \cup \{a + x : x \in \mathrm{L}c\} \cup \{\overline{y + c} : y \in \mathrm{H}a\} \cup \{\overline{a + y} : y \in \mathrm{H}c\}$$
$$a \leq c \text{ iff } (\forall y \in \mathrm{H}c) y \nleq a \land (\forall x \in \mathrm{L}a) c \nleq x$$

We stipulate that $a \equiv c$ iff $a \leq c \leq a$, and define $a - c := a + (-c)$.

We can make these algebraic operations intuitive as follows. To take the *negative* of a game is to reverse the players' roles (cf. Theorem 4.5). To *add* two games is to place them side-by-side, allowing a player to move in one game without affecting the other. But the *partial-order* requires slightly more explanation. Suppose High plays first on the game a; then Low has a winning strategy *iff* whatever move High makes, i.e. for all $y \in \mathrm{H}a$, if Low plays first on y then High has no winning strategy. Similarly, suppose Low plays first on a; then High has a winning strategy *iff* for all $x \in \mathrm{L}a$, if High plays first on x then Low has no winning strategy. So, if we gloss '$\emptyset \leq z$' as 'Low has a winning strategy as second player on z' and gloss '$z \leq \emptyset$' as 'High has a winning strategy as second player on z', this motivates two important special cases of the partial order:

$$\emptyset \leq a \text{ iff } (\forall y \in \mathrm{H}a) y \nleq \emptyset \qquad\qquad a \leq \emptyset \text{ iff } (\forall x \in \mathrm{L}a) \emptyset \nleq x$$

The remainder of the definition is then set up so that $a - b \leq \emptyset$ iff $a \leq b$. More generally, we have the following foundational result:

Theorem 8.3 (BLT)**:** The sets form a partially-ordered abelian Group, with $\emptyset = 0$ and $+, -, \leq$ as in Definition 8.2, all modulo $\equiv$.[35]

We can obtain a totally-ordered Field by restricting our attention to surreals:

[33] Conway (1976: 66). Cox and Kaye (2012) take up this suggestion and offer an axiomatic theory with two kinds of membership; they prove it is definitionally equivalent with ZF. By Theorem 7.1, it is definitionally equivalent with $\mathrm{BLT}_{\mathrm{ZF}}$.

[34] The well-ordering of bevels guarantees determinacy, and licenses induction and recursive definitions (see footnote 37, below). Definition 8.2 and 8.4 are BLT-implementations of Conway's (1976: chs.0–1) definitions. (As defined, the sum of two low sets is always low; an arbitrary choice was required.) For Theorem 8.3, see Conway's (1976: 78); for Theorem 8.5, see Conway (1976: ch.1). For an accessible presentation, see also Schleicher and Stoll (2006: §§2–4).

[35] To quotient by $\equiv$, define $[a] := \{b \equiv a : (\forall x \equiv a)\mathbf{B}b \subseteq \mathbf{B}x\}$; cf. Scott (1955) and Conway (1976: 65).

Definition 8.4 (BLT)**:** We specify that a is *surreal* iff: for all $x \in \mathrm{L}a$ and all $y \in \mathrm{H}a$, both x and y are surreal and $x \ngeq y$. We define multiplication on surreals thus:

$$\begin{aligned} a \cdot c \coloneqq \{&x \cdot c + a \cdot y - x \cdot y : (x \in \mathrm{L}a \land y \in \mathrm{L}c) \lor (x \in \mathrm{H}a \land y \in \mathrm{H}c)\} \cup \\ \{&\overline{x \cdot c + a \cdot y - x \cdot y} : (x \in \mathrm{L}a \land y \in \mathrm{H}c) \lor (x \in \mathrm{H}a \land y \in \mathrm{L}c)\} \end{aligned}$$

We say that a is a *surreal-ordinal* iff a is both helow and surreal.

Theorem 8.5 (BLT)**:** The surreals form a totally-ordered Field, modulo $\equiv$.

Summing up: Conway's beautifully rich, nonstandard, theory of surreal numbers is available, essentially off-the-shelf, within BLT.

9 Conclusion

The Complemented Story lays down a conception of *set* which rivals the (ordinary) cumulative notion, but which accepts Complementation and rejects Separation (see §1).

I have shown that any complemented hierarchy satisfies BLT (see §§2–3). So, given the characteristic results of BLT, the sets of any complemented hierarchy are arranged into well-ordered bevels, and constitute a boolean algebra (see §4). Moreover, BLT is quasi-categorical (see §5); so our only choice, in setting up a complemented hierarchy, is how tall to make it.

The theory BLT_{ZF} arises from BLT just by adding axioms which state that the complemented hierarchy *is* quite tall (see §4). And we can regard ZF as either a proper part of BLT_{ZF} (see §6), or as a notational variant (in a purely formal sense) of BLT_{ZF} (see §7). But both interpretations suggest that there is no obvious *a priori* reason to favour Separation over Complementation. And in some settings, such as the discussion of Conway games, using Complementation is extremely natural (see §8)

A Characteristics of BLT

The remainder of this paper consists of proofs of the results discussed in the main text. Many of the simpler proofs are similar to results for Pt.1; in such cases, I omit the proof and refer interested readers to the appropriate result from Pt.1.

This first appendix deals with the results from §4. Initially, I will work in ECS, the subtheory of BLT whose only axioms are Extensionality, Complements and $\text{Separation}_{\notin}$ (see §3). I start with some simple results and definitions:

Lemma A.1 (ECS)**:** If $c \subseteq a \notin a$, then $c \notin c$; if $a \in a \subseteq c$, then $c \in c$.

Proof. If $c \subseteq a \notin a$, then $c \notin c = \{x \in a : x \in c\}$ by $\text{Separation}_{\notin}$ and Extensionality. If $a \in a \subseteq c$, then $\overline{c} \subseteq \overline{a} \notin \overline{a}$ by Complements, so that $\overline{c} \notin \overline{c}$ as before, and $c \in c$ by Complements. □

Definition A.2: Say that a is *potent*$_{\notin}$ iff $\forall x(\exists c(x \subseteq c \notin c \in a) \rightarrow x \in a)$. Say that a is *transitive*$_{\notin}$ iff $(\forall x \notin x \in a)x \subseteq a$. Say that a is *complement-closed* iff $\forall x(x \in a \leftrightarrow \overline{x} \in a)$.

Lemma A.3 (ECS)**:** If $\mathbf{P}a$ exists (see Definition 3.1), then:

(1) $(\forall x \notin x \in \mathbf{P}a)\exists c(x \subseteq c \notin c \in a)$.
(2) $\mathbf{P}a$ is potent$_{\notin}$.
(3) $\mathbf{P}a$ is complement-closed.

Proof. (1) Fix $x \notin x \in \mathbf{P}a$; so for some $c \notin c \in a$, either $x \subseteq c$ or $\overline{x} \subseteq c$. But $\overline{x} \in \overline{x}$ by Complements, so $\overline{x} \nsubseteq c$ by Lemma A.1.

(2) Fix $x \subseteq c \notin c \in \mathbf{P}a$; so $x \subseteq c \subseteq b \notin b \in a$ for some b by (1); hence $x \in \mathbf{P}a$.

(3) Fix $x \in \mathbf{P}a$. If $x \subseteq c$ for some $c \notin c \in a$, then $x = \overline{\overline{x}} \subseteq c$ so that $\overline{x} \in \mathbf{P}a$; if $\overline{x} \subseteq c$ for some $c \notin c \in a$, then $\overline{x} \in \mathbf{P}a$ straightforwardly. □

It follows that bevels (see Definition 3.2) have several important closure properties:

Lemma A.4 (ECS)**:** Every bevel is transitive$_{\notin}$, potent$_{\notin}$, complement-closed, and non-self-membered.

Proof. Let s be a bevel, i.e. $s = \mathbf{P}h$ for some bistory h. So s is potent$_{\notin}$ and complement-closed by Lemma A.3. For transitivity$_{\notin}$, fix $a \notin a \in s = \mathbf{P}h$; so $a \subseteq c \notin c \in h$ for some c by Lemma A.3.1; and $c = \mathbf{P}(c \cap h)$ as h is a bistory; so $a \subseteq \mathbf{P}(c \cap h) \subseteq \mathbf{P}h = s$. To see $s \notin s$, suppose $s \in s$ for reductio. Then $\overline{s} \notin \overline{s} \in s$ by Complements, so $\overline{s} \subseteq s$ by transitivity$_{\notin}$, so $s = V$. Since $h \notin h$ by definition, and $h \in V = s = \mathbf{P}h$, by Lemma A.3.1 there is some c such that $h \subseteq c \notin c \in h$. Since h is a bistory, $c = \mathbf{P}(h \cap c) = \mathbf{P}h = V$, contradicting the fact that $c \notin c$. □

From here, we can prove the well-ordering of the bevels, by proving a sequence of results like those from Pt.1 §3; I leave this to the reader:[36]

Lemma A.5 (ECS)**:** If there is an F, and all Fs are non-self-membered and potent$_{\notin}$, then there is an $\in$-minimal F. Formally: $\forall F((\exists x F(x) \wedge (\forall x : F)(x \notin x \wedge x \text{ is potent}_{\notin})) \rightarrow (\exists a : F)(\forall x : F)x \notin a)$

Lemma A.6 (ECS)**:** If some bevel is F, then there is an $\in$-minimal bevel which is F. Formally: $\forall F((\exists s : Bev)F(s) \rightarrow (\exists s : Bev)(F(s) \wedge (\forall r : Bev)(F(r) \rightarrow r \notin s)))$

Lemma A.7 (ECS)**:** Every member of a bistory is a bevel.

Lemma A.8 (ECS)**:** $s = \mathbf{P}\{r \in s : Bev(r)\}$, for any bevel s.

Lemma A.9 (ECS)**:** All bevels are comparable, i.e. $(\forall s : Bev)(\forall t : Bev)(s \in t \vee s = t \vee t \in s)$

[36] For Lemma A.7, first note that if h is a history and $c \in h$, then $c = \mathbf{P}(c \cap h) \subseteq \mathbf{P}h \notin \mathbf{P}h$ by Lemma A.4, so $c \notin c$ by Lemma A.1. For Lemmas A.8–A.9, reason about non-self-membered sets in the first instance, then deal with self-membered sets using Complements and complement-closure.

Combining Lemmas A.6 and A.9, ECS proves that the bevels are well-ordered by $\in$; this is Theorem 4.1. This licenses our use of the $\mathbf{B}$-operator (see Definition 4.3). Here are some simple results about that operator, which can be proved by tweaking the proof of Pt.1 Lemma 3.12:

Lemma A.10 (BLT)**:** For any sets a, c, and any bevels r, s:

(1) $\mathbf{B}a$ exists
(2) $a \notin \mathbf{B}a$
(3) $r \subseteq s$ iff $s \notin r$
(4) $s = \mathbf{B}s$
(5) if $c \subseteq a \notin a$ or $a \in a \subseteq c$, then $\mathbf{B}c \subseteq \mathbf{B}a$
(6) if $c \in a \notin a$ or $c \notin a \in a$, then $\mathbf{B}c \in \mathbf{B}a$

Moreover, we can now show that sets are closed under arbitrary pairwise intersection:

Lemma A.11 (BLT)**:** For any sets a and c, the set $a \cap c = \{x : x \in a \wedge x \in c\}$ exists.

Proof. First suppose that either $a \notin a$ or $c \notin c$ (or both); without loss of generality, suppose $a \notin a$; now $a \cap c = \{x \in a : x \in c\}$ exists by Separation$_{\notin}$. Next suppose that both $a \in a$ and $c \in c$. So both $\overline{a} \notin \overline{a}$ and $\overline{c} \notin \overline{c}$ by Complements. Let s be the maximum of $\mathbf{B}\overline{a}$ and $\mathbf{B}\overline{c}$. Since s is potent$_{\notin}$, both $\overline{a} \subseteq s$ and $\overline{c} \subseteq s$, so $\overline{a} \cup \overline{c} = \{x \in s : x \in \overline{a} \vee x \in \overline{c}\}$ exists by Separation$_{\notin}$. Now $a \cap c = \overline{\overline{a} \cup \overline{c}}$ exists by Complements. ◻

This immediately entails that the sets form a boolean algebra, which is Theorem 4.2. Our next result shows that the universe is contra-automorphic:[37]

Theorem 4.5 (BLT): $\forall a \forall b(a \in b \leftrightarrow -a \notin -b)$

Proof. Recall that negative is given as in Definition 4.4 by

$$-a := \overline{\{-x : x \in a\}}, \text{ if } a \notin a \qquad -a := \{-x : x \notin a\}, \text{ if } a \in a$$

Fix a bevel s and for induction suppose that, for any $x, y \in s$:

(1) $-x$ is well-defined and $\mathbf{B}x = \mathbf{B}(-x)$; and
(2) $x = y$ iff $-x = -y$.

It suffices to show that both properties hold of a, b when $\mathbf{B}a = \mathbf{B}b = s$.

Concerning (1). Suppose $a \notin a$. If $x \in a$, then $\mathbf{B}(-x) = \mathbf{B}x \in \mathbf{B}a$ by induction assumption (1) and Lemma A.10.6. Using Separation$_{\notin}$, let $c \notin c = \{v \in \mathbf{B}a : (\exists x \in a) v = -x\} = \{-x : x \in a\}$. Moreover, $\mathbf{B}c = \mathbf{B}a$, by the well-ordering of bevels

[37] Theorem 4.1 licenses recursive definitions. We can regard as defining second-order entities. If we are using second-order logic, such definitions yield a second-order entity. If we are using first-order logic, then (as usual) we define a term by considering a strictly increasing sequence of first-order 'bounded approximations' (specifying the behavior of the term over the last few bevels manually, if there is a last bevel).

and since $\mathbf{B}(-x) = \mathbf{B}x \in \mathbf{B}a$ for all $x \in a$. Now $\bar{c} \in \bar{c} = -a$ by Complements; so $\mathbf{B}a = \mathbf{B}c = \mathbf{B}\bar{c} = \mathbf{B}(-a)$. The case when $a \in a$ is similar, defining $c \notin c = \{v \in \mathbf{B}a : (\exists x \notin a)v = -x\} = \{-x : x \notin a\} = -a$.

Concerning (2). If $a \in a \leftrightarrow b \in b$, then $a = b$ iff $-a = -b$ by induction assumption (2). Without loss of generality, suppose that $a \in a$ and $b \notin b$; in establishing (1), we found that $-a \notin -a$ and $-b \in -b$; so $a \neq b$ and $-a \neq -b$. □

I ended §4 by stating some simple facts about extensions of BLT. I will prove the distinctively boolean facts, leaving the remainder to the reader:

Proposition 5.1, fragment:

(2) BLT proves Union, i.e. $\forall a(\bigcup a$ exists)

(5) BLT contradicts Powersets, i.e. it proves $\exists a \neg \exists b \forall x(x \in b \leftrightarrow x \subseteq a)$

(6) BLT proves Foundation-restricted-to-high-sets, i.e. $(\forall a \in a)(\exists x \in a)a \cap x = \emptyset$.

(7) BLT_+ contradicts unrestricted Foundation, i.e. it proves $(\exists a \neq \emptyset)(\forall x \in a)a \cap x \neq \emptyset$.

Proof. (2) If $a \in a$, then $\bigcup a = \overline{\{x \in \bar{a} : (\forall y \in a)x \notin y\}}$, which exists by $\text{Separation}_{\notin}$ and Complements. If $a \notin a$, then using $\text{Separation}_{\notin}$ let $a_0 = \{x \in a : x \notin x\}$ and let $a_1 = \{x \in a : x \in x\}$. I will show that $\bigcup a_0$ and $\bigcup a_1$ exist, so that, using Complements and Lemma A.11:

$$\bigcup a = \bigcup a_0 \cup \bigcup a_1 = \overline{\overline{\bigcup a_0} \cap \overline{\bigcup a_1}}$$

Clearly $\bigcup a_0$ exists by $\text{Separation}_{\notin}$ on $\mathbf{B}a$. If $a_1 = \emptyset$ then $\bigcup a_1 = \emptyset$; otherwise, $\overline{\bigcup a_1} = \bigcap\{\bar{x} : x \in a_1\}$, which exists by Complements and $\text{Separation}_{\notin}$ on $\mathbf{B}a$.

(5) If there is only one bevel, then the only sets are $\emptyset$ and $V = \{\emptyset, V\}$, so that $\wp\emptyset = \{\emptyset\}$ does not exist. Otherwise, we find $\overline{\{\emptyset\}}$ at the second bevel, and if $\wp\overline{\{\emptyset\}}$ existed it would be $\{x : \emptyset \notin x\}$. So suppose for reductio that $a = \{x : \emptyset \notin x\}$. Then $\emptyset \notin \emptyset$, so $\emptyset \in a$, so $a \notin a$. Now $\bar{a} \in \bar{a} = \{x : \emptyset \in x\}$ by Complements, so that $\emptyset \in \bar{a}$, contradicting that $\emptyset \in a$.

(6) If $a \in a$ then $\bar{a} \in a$ by Complements, and $a \cap \bar{a} = \emptyset$.

(7) We find $\{V\}$ at the second bevel, and $\{V\} \cap V \neq \emptyset$. □

B The set-theoretic equivalence of BST and BLT

I now want to prove Theorem 3.3, which states that BLT and BST say exactly the same things about sets. (This mirrors Pt.1 §4.)

To show that BST says no more about sets than BLT does, I define a translation $* : \text{BST} \longrightarrow \text{BLT}$, whose non-trivial actions are as follows:[38]

$$Lo(x) := x \notin x \qquad Hi(x) := x \in x$$

$$(\mathbf{s} < \mathbf{t})^* := \mathbf{s} \in \mathbf{t} \qquad (x \preceq \mathbf{s})^* := (x \subseteq \mathbf{s} \vee \bar{x} \subseteq \mathbf{s}) \qquad (\forall \mathbf{s}\phi)^* := (\forall \mathbf{s} : Bev)(\phi^*)$$

[38] So the other clauses are: $(\neg\phi)^* := \neg\phi^*$; $(\phi \wedge \psi)^* := (\phi^* \wedge \psi^*)$; $(\forall x\phi)^* := \forall x\phi^*$; $(\forall F\phi)^* := \forall F\phi^*$; and $\alpha^* := \alpha$ for all atomic formulas α which are not of the forms mentioned in the main text.

After translation, I treat all first-order variables as being of the same sort. Fairly trivially, for any BLT-sentence ϕ, if BST $\vdash \phi$ then BST$^* \vdash \phi$. The left-to-right half of Theorem 3.3 now follows as $*$ is an interpretation:

Lemma B.1 (BLT)**:** BST* holds.

Proof. Extensionality* is Extensionality. Order* holds by Lemma A.4; Staging* holds by Stratification$_{\notin}$ and Complements; and Cases* is trivial. Next, by Lemmas A.4 and A.8, we can simplify $(x \prec \mathbf{s})^*$ to $x \in \mathbf{s}$. So, using Lemmas A.1 and A.4, we can simplify Priority$_{Lo}$* thus:

$$(\forall s \in Bev)(\forall a \notin a)((a \subseteq s \lor \bar{a} \subseteq s) \rightarrow (\forall x \in a)x \in s)$$
$$\text{i.e. } (\forall s \in Bev)(\forall a \subseteq s)(\forall x \in a)x \in s$$

which is trivial; then Priority$_{Hi}$* holds similarly, by Complements. A similar simplification allows us to obtain Specification$_{Lo}$* via Separation$_{\notin}$; then Specification$_{Hi}$* holds similarly, by Complements.[39] □

To obtain the right-to-left half of Theorem 3.3, I will work in BST. I start by defining *slices*, which will go proxy for stages, and will turn out to be bevels, and then stating a few elementary results (for proofs, tweak those of Pt.1 §4):

Definition B.2 (BST)**:** For each $\mathbf{s}$, let $\check{\mathbf{s}} = \{x : x \prec \mathbf{s}\}$. Say that a is a slice iff $a = \check{\mathbf{s}}$ for some stage $\mathbf{s}$.

Lemma B.3 (BST)**:** $\forall F(\forall a : Lo)(\exists b : Lo)\forall x(x \in b \leftrightarrow (F(x) \land x \in a))$

Lemma B.4 (BST)**:** $\forall \mathbf{s}(\forall a : Lo)(a \preceq \mathbf{s} \leftrightarrow (\forall x \in a)x \prec \mathbf{s})$

Lemma B.5 (BST)**:** For any $\mathbf{s}$:

(1) $\check{\mathbf{s}}$ exists and is low
(2) $\forall \mathbf{r}(\forall a : Lo)(a \preceq \mathbf{r} \preceq \mathbf{s} \rightarrow a \preceq \mathbf{s})$
(3) $(\forall a : Lo)(a \subseteq \check{\mathbf{s}} \leftrightarrow a \preceq \mathbf{s})$

We must now part company slightly with the strategy of Pt.1 §4, to handle low and high sets, and their relation to (non-)self-membership:

Lemma B.6 (BST)**:** If some slice is F, then there is an $\in$-minimal slice which is F.

Proof. Every slice is low, by Lemma B.5.1. Subsets of low sets are low, by a result like Lemma A.1. From this, and Lemma B.5, it follows that $\forall \check{\mathbf{s}} \forall x((\exists c : Lo)x \subseteq c \in \check{\mathbf{s}} \rightarrow x \in \check{\mathbf{s}})$. The result now follows, reasoning as in Pt.1 Lemma 3.5. □

Lemma B.7 (BST)**:** a is low iff $a \notin a$; and a is high iff $a \in a$.

[39] Note that the $*$-translation of any BST-Comprehension instance is a BLT-Comprehension instance.

Proof. Suppose for reductio that $a \in a$ is low. Using Staging and Lemma B.6, let $\check{\mathbf{s}}$ be an $\in$-minimal slice such that $\exists \mathbf{t}(a \preceq \mathbf{t} \wedge \check{\mathbf{t}} = \check{\mathbf{s}})$; let $\mathbf{t}$ witness this. Since $a \in a \preceq \mathbf{t}$ and a is low, $a \preceq \mathbf{r} < \mathbf{t}$ for some $\mathbf{r}$ by Priority$_{Lo}$; so $\check{\mathbf{r}} \in \check{\mathbf{t}} = \check{\mathbf{s}}$ by Lemma B.5, contradicting $\check{\mathbf{s}}$'s minimality. Discharging the reductio: if a is low, then $a \notin a$. Similarly: if a is high, then $a \in a$. The biconditionals follow by Cases. □

Lemma B.8 (BST)**:** $\bar{a}$ exists; and $a \notin a \leftrightarrow \bar{a} \in \bar{a}$; and $\forall \mathbf{s}(a \preceq \mathbf{s} \leftrightarrow \bar{a} \preceq \mathbf{s})$.

Proof. Using Staging, let $a \preceq \mathbf{s}$. If $a \notin a$, then a is low by Lemma B.7, so $(\forall x \in a) x \prec \mathbf{s}$ by Priority$_{Lo}$, so that by Specification$_{Hi}$ and Extensionality $\{x : x \notin a\} = \bar{a} \preceq \mathbf{s}$ exists and is high, i.e. $\bar{a} \in \bar{a}$ by Lemma B.7. If $a \in a$, reason similarly using Priority$_{Hi}$ and Specification$_{Lo}$. □

Note that BST $\vdash$ ECS by Lemmas B.3, B.7, and B.8. So Lemmas A.1–A.9 hold verbatim within BST. We can now complete our reasoning about slices, by resuming the proof-strategy of Pt.1 §4; at this point, I leave the remaining details to the reader:

Lemma B.9 (BST)**:** $\check{\mathbf{s}} \notin \check{\mathbf{s}}$; and $\check{\mathbf{s}}$ is transitive$_{\notin}$; and $\check{\mathbf{s}} = \mathbf{P}\{\check{\mathbf{r}} : \check{\mathbf{r}} \in \check{\mathbf{s}}\}$.

Lemma B.10 (BST)**:** All slices are comparable, i.e. $\forall \check{\mathbf{s}} \forall \check{\mathbf{t}}(\check{\mathbf{s}} \in \check{\mathbf{t}} \vee \check{\mathbf{s}} = \check{\mathbf{t}} \vee \check{\mathbf{t}} \in \check{\mathbf{s}})$.

Lemma B.11 (BST)**:** s is a bevel iff s is a slice.

It follows that BST proves Stratification$_{\notin}$, delivering Theorem 3.3.

C Helow sets

In this appendix I prove Theorem 6.2, which shows how to recover ordinary, uncomplemented hierarchies via helow sets (see Definition 6.1). For readability, I refer to non-self-membered sets as *low*, and self-membered sets as *high* (cf. Lemma B.7). Note that every helow set is low, since all its members are low (i.e. non-self-membered). Now:

Definition C.1 (BLT)**:** If a is low, let $a_{\triangledown} \coloneqq \{x \in a : x \text{ is helow}\}$; by Separation$_{\notin}$, $a_{\triangledown}$ exists and is low.

Lemma C.2 (BLT)**:** a is helow iff every member of a is helow.

Proof. Left-to-right. Where c witnesses that a is helow, if $x \in a$, then $x \in c$ and hence $x \subseteq c$, so c also witnesses that x is helow. *Right-to-left*. Let every member of a be helow. Every member of a is low, so a itself is low; hence $a \subseteq (\mathbf{B}a)_{\triangledown}$. Now $(\mathbf{B}a)_{\triangledown}$ witnesses that a is helow: if $x \in c \in (\mathbf{B}a)_{\triangledown}$ then c is helow so x is helow (by left-to-right), so $x \in (\mathbf{B}a)_{\triangledown}$ as $\mathbf{B}a$ is transitive$_{\notin}$. □

I can now begin to show that $\triangledown : \text{LT} \longrightarrow \text{BLT}$, which simply restricts all quantifiers to helow sets (see §6), is an interpretation of LT:

Lemma C.3 (BLT)**:** Both Extensionality$^\triangledown$ and Separation$^\triangledown$ hold.

Proof. For Extensionality$^\triangledown$, fix helow a and b and suppose that $(\forall x : Helo)(x \in a \leftrightarrow x \in b)$; then $\forall x(x \in a \leftrightarrow x \in b)$ by Lemma C.2, so $a = b$ by Extensionality. Similarly, repeated use of Lemma C.2 shows that Separation$^\triangledown$ follows from Separation$_\notin$. □

The next task is to connect bevels with levels$^\triangledown$. (See Pt.1 Definitions 2.1–3.1 for the definitions of *potent*, ¶, *Hist* and *Lev*.)

Lemma C.4 (BLT)**:** For any bevels r, s:

(1) $s_\triangledown$ is helow, potent and transitive
(2) $r \in s$ iff $r_\triangledown \in s_\triangledown$
(3) $s = \mathbf{B}(s_\triangledown)$
(4) $s_\triangledown = \P h = \P^\triangledown(h)$, where $h = \{r_\triangledown \in s_\triangledown : Bev(r)\}$.
(5) $s_\triangledown$ is a level$^\triangledown$

Proof. *(1)* By Lemma C.2, $s_\triangledown$ is helow; then $s_\triangledown$ is potent and transitive as s is potent$_\notin$ and transitive$_\notin$.

(2) *Left-to-right*. By (1). *Right-to-left*. Let $r_\triangledown \in s_\triangledown$. So $r \neq s$, since $r_\triangledown \notin r_\triangledown$. Similarly, $s_\triangledown \notin r_\triangledown$, since $s_\triangledown$ is transitive; so $s \notin r$ by *left-to-right*. So $r \in s$, by Lemma A.9.

(3) Induction on bevels, using (2).

(4) By (1) and Lemma C.2, h is helow. If $a \in \P h$, then $a \in s_\triangledown$ as $s_\triangledown$ is potent by (1). Conversely, if $a \in s_\triangledown$, then $a \subseteq r \in s$ for some bevel r by Lemma A.8, and $a \subseteq r_\triangledown \in s_\triangledown$ by (2) and Lemma C.2, so $a \in \P h$. So $s_\triangledown = \P h$. Repeated use of Lemma C.2, as in Lemma C.3, now yields that $\P h = \P^\triangledown(h)$.

(5) With h as in (4), since $s = \P^\triangledown(h)$ it suffices to show that $Hist^\triangledown(h)$. If $r_\triangledown \in h$, then $r_\triangledown \cap h = \{q_\triangledown \in r_\triangledown : Bev(q)\}$, by (1); so $r_\triangledown = \P^\triangledown(r_\triangledown \cap h)$ by (4). □

Lemma C.5 (BLT)**:** The levels$^\triangledown$ are the bevels$_\triangledown$, i.e.: $Lev^\triangledown(a)$ iff $(\exists s : Bev) a = s_\triangledown$.

Proof. By Lemma C.4, if s is a bevel then both $Lev^\triangledown(s_\triangledown)$ and $\mathbf{B}(s_\triangledown) = s$. To complete the proof, it suffices to note that if p and q are distinct levels$^\triangledown$, then $\mathbf{B}p \neq \mathbf{B}q$; this follows from Lemma A.10.6 and the fact that the levels$^\triangledown$ are well-ordered by $\in$. (The well-ordering of levels$^\triangledown$ is Pt.1 Theorem 3.10$^\triangledown$, which holds via Lemma C.3.) □

Corollary C.6 (BLT)**:** Stratification$^\triangledown$ holds; Endless$_\notin$ proves Endless$^\triangledown$; Infinity$_\notin$ proves Infinity$^\triangledown$; and Unbounded$_\notin$ proves Unbounded$^\triangledown$.

Recalling that LT + Infinity + Unbounded is equivalent to ZF (see §6), Lemmas C.3 and C.6 yield Theorem 6.2.

D Definitional equivalence

In this appendix, I prove the definitional equivalence discussed in §7.[40]

D.1 *Interpreting* BLT_{ZF} *in* ZF

I first define an interpretation, I, to simulate (extensions of) BLT within (extensions of) LT. The key idea is to use $\emptyset$ as a flag to indicate whether to treat a set as low or high. To allow $\emptyset$ to play this role, I define a bijection $\sigma : V \longrightarrow V \setminus \{\emptyset\}$:[41]

$$\sigma(a) := \begin{cases} \{a\} & \text{if } a \text{ is a Zermelo number} \\ a & \text{otherwise} \end{cases}$$

where the Zermelo numbers are $0 = \emptyset$ and $n+1 = \{n\}$. I then interpret membership thus:

$$x \in^I a \text{ iff } (\sigma(x) \in a \leftrightarrow \emptyset \notin a)$$

Since $\sigma(a) \notin a$ for all a, it follows that $a \notin^I a$ iff $\emptyset \notin a$ (i.e. a is treated as low), and $a \in^I a$ iff $\emptyset \in a$ (i.e. a is treated as high). I will now prove a sequence of results which establish that I is an interpretation of BLT. The first few are straightforward:

Lemma D.1 (LT$_+$)**:** Where $a \subseteq^I b$ abbreviates $(\forall x \in^I a)x \in^I b$:

(1) If $\emptyset \notin a$ and $\emptyset \notin b$, then: $a \subseteq b$ iff $a \subseteq^I b$
(2) If $\emptyset \in a$ and $\emptyset \in b$, then: $a \supseteq b$ iff $a \subseteq^I b$.

Proof. *(1)* Since σ is a bijection $V \longrightarrow V \setminus \{\emptyset\}$, $a \subseteq b$ iff $\forall x(\sigma(x) \in a \rightarrow \sigma(x) \in b)$ iff $a \subseteq^I b$.

(2) Similarly, $a \supseteq b$ iff $\forall x(\sigma(x) \notin a \rightarrow \sigma(x) \notin b)$ iff $a \subseteq^I b$. □

Lemma D.2 (LT$_+$)**:** Extensionality^I holds.

Proof. Suppose $\forall x(x \in^I a \leftrightarrow x \in^I b)$. If $a \notin^I a$ but $b \in^I b$, then $\forall x(\sigma(x) \in a \leftrightarrow \sigma(x) \notin b)$, so that $a \cup b = V$, which is impossible. Generalising, $a \in^I a$ iff $b \in^I b$. Now apply Lemma D.1. □

Lemma D.3 (LT$_+$)**:** $\text{Separation}_{\notin}{}^I$ holds.

Proof. Fix F and $a \notin^I a$, i.e. $\emptyset \notin a$. Using Separation, let $b = \{\sigma(x) \in a : F(x)\}$. Since $\emptyset \notin b$ we have $\forall x(x \in^I b \leftrightarrow (F(x) \wedge x \in^I a))$. □

[40] Recall: both LT and BLT (and their extensions) are formulated as second-order theories. I continue to frame my discussion in second-order terms in this appendix. However, the theories can easily be reformulated as first-order formulations, and the definitional equivalences hold for these first-orderisations (only the quasi-categoricity results of §5 require second-order resources).

[41] Many thanks to Randall Holmes for discussion of this construction (and other constructions); the proof in this section is much more self-contained than it would have been, had it not been for his input. Thanks also to Thomas Forster, for encouraging me to consider the question of definitional equivalence. The proof-strategy is similar to Löwe (2006).

The interpretation of complementation is obvious: $\overline{a}^I = a \cup \{\emptyset\}$ if $a \notin^I a$, and $\overline{a}^I = a \setminus \{\emptyset\}$ if $a \in^I a$. The next result follows trivially:

Lemma D.4 (LT$_+$)**:** $\forall a \forall x(x \in^I a \leftrightarrow x \notin^I \overline{a}^I)$, and Complements^I holds.

The only intricate part of this interpretation concerns the treatment of bevels. Within LT$_+$, we can define the von Neumann ordinals, and recursively define the following:

$$W_\gamma = \{\sigma(x) : (\exists \beta < \gamma) x \subseteq W_\beta \cup \{\emptyset\}\}$$

Now LT$_+$ proves that W_γ exists for each γ, and that these are the bevels^I:

Lemma D.5 (LT$_+$)**:** $Bev^I(s)$ iff $s = W_\gamma$ for some γ.

Proof. Lemmas D.2–D.4 show that LT$_+$ proves ECS^I. Hence LT$_+$ proves Theorem 4.1^I, i.e. that the bevels^I are well-ordered by $\in^I$. For induction on γ, suppose that if $\beta < \gamma$ then W_β is the β^{th} bevel^I. Let s be the γ^{th} bevel^I. By Lemma D.1:

$$\begin{aligned} W_\gamma &= \{\sigma(x) : (\exists \beta < \gamma) x \subseteq W_\beta \cup \{\emptyset\}\} \\ &= \{\sigma(x) : (\exists \beta < \gamma)(x \subseteq^I W_\beta \vee \overline{x}^I \subseteq^I W_\beta)\} \\ &= \{\sigma(x) : (\exists W_\beta \notin^I W_\beta \in^I s)(x \subseteq^I W_\beta \vee \overline{x}^I \subseteq^I W_\beta)\} \\ &= (\mathbf{P}\{w \in s : Bev(w)\})^I \end{aligned}$$

So $W_\gamma = s$ by Lemma A.8^I. By induction, the bevels^I are the W_γs. □

I can now prove the crucial proposition:

Lemma D.6 (LT$_+$)**:** Stratification$_{\notin}{}^I$ holds.

Proof. By Lemma D.5, it suffices to show that $(\forall a \notin^I a)\exists \gamma\ a \subseteq^I W_\gamma$. Since the levels are well-ordered by $\in$ (Pt.1 Theorem 3.10), we can write V_γ for the γ^{th} level. I claim: if $a \notin^I a \subseteq V_\gamma$, then $a \subseteq W_\gamma$. For induction, suppose this holds for all ordinals $\beta < \gamma$. Fix $a \notin^I a \subseteq V_\gamma$. If $\gamma = 0$, then $a = \emptyset \subseteq^I W_0 = \emptyset$. Otherwise, fix $x \in^I a$, i.e. $\sigma(x) \in a \subseteq V_\gamma$; now $x \subseteq V_\beta$ for some $\beta < \gamma$, by Pt.1 Lemma 3.12, so that $x \subseteq W_\beta \cup \{\emptyset\}$ by the induction hypothesis; so $\sigma(x) \in W_\gamma$, i.e. $x \in^I W_\gamma$. Generalising, $a \subseteq^I W_\gamma$. □

Lemma D.7: LT$_+ \vdash$ BLT$^I_+$ and ZF $\vdash$ BLT$^I_{\text{ZF}}$.

Proof. Lemmas D.2–D.6 establish that LT$_+ \vdash$ BLT^I. And LT$_+ \vdash$ Endless$_{\notin}{}^I$, using Endless and our explicitly defined bevels^I, the W_γs. Evidently, Infinity yields Infinity$_{\notin}{}^I$. For Unbounded$_{\notin}{}^I$, fix P and $a \notin^I a$; by Unbounded, the set $c = \{\sigma(P(x)) : \sigma(x) \in a\}$ exists; by construction, $\emptyset \notin c$ and $(\forall x \in^I a)P(x) \in^I c$. The result follows, since ZF is equivalent to LT + Infinity + Unbounded (see §6). □

D.2 *Interpreting* ZF *in* $\mathrm{BLT}_{\mathrm{ZF}}$

I now switch to working in BLT_+. Using σ—i.e. using verbatim the same definitions of 'Zermelo number' and of σ in BLT_+ as we used in LT_+—consider this function:

$$\eta(a) = \begin{cases} \{\sigma(\eta(x)) : x \in a\} & \text{if } a \notin a \\ \{\sigma(\eta(x)) : x \notin a\} \cup \{\emptyset\} & \text{if } a \in a \end{cases}$$

I will prove that η is a bijection $V \longrightarrow \mathit{Helo}$. I then define a translation, J, by stipulating:

$$x \in^J a \text{ iff } \eta(x) \in \eta(a)$$

It will follow that J is an interpretation of LT_+ in BLT_+.

Lemma D.8 (BLT_+)**:** If $\eta(a) = \eta(b)$, then $a = b$.

Proof. Let $\eta(a) = \eta(b)$, so that $a \notin a \leftrightarrow b \notin b$. For induction, suppose that $\eta(x) = \eta(y) \rightarrow x = y$ for all x, y with $\mathbf{B}x, \mathbf{B}y \in \mathbf{B}a \cup \mathbf{B}b$. If $a \notin a$ and $b \notin b$, then $\{\sigma(\eta(x)) : x \in a\} = \{\sigma(\eta(x)) : x \in b\}$, so that $a = b$ by the induction hypothesis and the injectivity of σ. The case when $a \in a$ is similar. □

Lemma D.9 (BLT_+)**:** $\eta(a)$ is helow, for any a.

Proof. For induction, suppose that $\eta(x)$ is helow for all x with $\mathbf{B}x \in \mathbf{B}a$. Suppose $a \notin a$; since $\sigma(\eta(x))$ is helow iff $\eta(x)$ is helow, every member of $\eta(a)$ is helow; so $\eta(a)$ is helow by Lemma C.2. The case when $a \in a$ is similar. □

Lemma D.10 (BLT_+)**:** If a is helow, then $a = \eta(c)$ for some c.

Proof. By Lemma D.8, η^{-1} is functional. For induction, suppose that for all helow $z \in \mathbf{B}a$, we have that $\eta^{-1}(z)$ is defined and $\mathbf{B}(\eta^{-1}(z)) \subseteq \mathbf{B}z$.

If $\emptyset \notin a$, let $c \notin c = \{\eta^{-1}(\sigma^{-1}(x)) \in \mathbf{B}a : x \in a\}$ using $\text{Separation}_{\notin}$. Fix $x \in a$; then $\sigma^{-1}(x) \in \mathbf{B}a$ and $\sigma^{-1}(x)$ is helow, recalling that a is helow and using Lemma C.2). Now $\mathbf{B}(\eta^{-1}(\sigma^{-1}(x))) \subseteq \mathbf{B}(\sigma^{-1}(x)) \in \mathbf{B}a$ by the induction hypothesis, i.e. $\eta^{-1}(\sigma^{-1}(x)) \in \mathbf{B}a$. So $c = \{\eta^{-1}(\sigma^{-1}(x)) : x \in a\}$, so that $a = \eta(c)$ and $\mathbf{B}c \subseteq \mathbf{B}a$.

If $\emptyset \in a$, then instead let $c = \{\eta^{-1}(\sigma^{-1}(x)) : \emptyset \neq x \in a\}$; now $a = \eta(\overline{c})$. □

Lemma D.11: $\mathrm{BLT}_+ \vdash \mathrm{LT}_+^J$ and $\mathrm{BLT}_{\mathrm{ZF}} \vdash \mathrm{ZF}^J$.

Proof. By Lemmas D.8–D.10, $\eta : V \longrightarrow \mathit{Helo}$ is a bijection; now use Theorem 6.2. □

D.3 *The interpretations are inverse*

It remains to show that I and J are mutually inverse, in the sense required for definitional equivalence.[42] The key lies in their treatments of the Zermelo numbers.

[42] Via Friedman and Visser (2014: Corollary 5.5), to establish Theorem 7.1 we could instead verify that I and $\triangledown$ (from §C) are bi-interpretations.

Working informally, let z_n be the n^{th} Zermelo number, and let v_n be defined similarly, but starting from V rather than $\emptyset$, i.e.:

$$\mathrm{z}_n = \overbrace{\{\dots\{}^{n \text{ times}} \emptyset \} \dots \} \qquad \mathrm{v}_n = \overbrace{\{\dots\{}^{n \text{ times}} V \} \dots \}$$

We can now consider two sequences:

$$\begin{array}{llllllll} \mathrm{z}_0, & \mathrm{z}_1, & \mathrm{z}_2, & \mathrm{z}_3, & \dots, & \mathrm{z}_{2n}, & \mathrm{z}_{2n+1}, & \dots \\ \mathrm{z}_0, & \mathrm{v}_0, & \mathrm{z}_1, & \mathrm{v}_1, & \dots, & \mathrm{z}_n, & \mathrm{v}_n, & \dots \end{array}$$

Inutitively, I treats the former sequence as the latter, and J treats the latter as the former. The proof that I and J are mutually inverse simply builds on this intuitive thought.

Here are two facts which make the intuitive thought precise:

Lemma D.12 (LT$_+$)**:** $\forall x \; x \notin^I \emptyset$, and $\forall x \; x \in^I \{\emptyset\}$, and $\forall x (x \in^I \mathrm{z}_{n+2} \leftrightarrow x = \mathrm{z}_n)$ for all n.

Lemma D.13 (BLT$_+$)**:** $\eta(\mathrm{z}_n) = \mathrm{z}_{2n}$ and $\eta(\mathrm{v}_n) = \mathrm{z}_{2n+1}$, for all n.

The proofs of both facts are trivial. Using the second fact, though, I can build up to the proof in BLT$_+$ that $x \in a$ iff $(x \in^I a)^J$:

Lemma D.14 (BLT$_+$)**:** The function σ^J, i.e. the J-interpretation of LT's definition of σ, maps $\mathrm{z}_n \mapsto \mathrm{v}_n \mapsto \mathrm{z}_{n+1}$, and $x \mapsto x$ otherwise.

Proof. Note that $\mathrm{z}_{2n} \in \mathrm{z}_{2n+1} \in \mathrm{z}_{2n+2}$, with these membership facts unique. So $\eta(\mathrm{z}_n) \in \eta(\mathrm{v}_n) \in \eta(\mathrm{z}_{n+1})$, by Lemma D.13, i.e. $\mathrm{z}_n \in^J \mathrm{v}_n \in^J \mathrm{z}_{n+1}$. □

Lemma D.15 (BLT$_+$)**:** $\eta(\sigma^J(a)) = \sigma(\eta(a))$, for all a.

Proof. By Lemmas D.13–D.14, we have $\eta(\sigma^J(\mathrm{z}_n)) = \eta(\mathrm{v}_n) = \mathrm{z}_{2n+1} = \sigma(\mathrm{z}_{2n}) = \sigma(\eta(\mathrm{z}_n))$ and $\eta(\sigma^J(\mathrm{v}_n)) = \eta(\mathrm{z}_{n+1}) = \mathrm{z}_{2n+2} = \sigma(\mathrm{z}_{2n+1}) = \sigma(\eta(\mathrm{v}_n))$. Now suppose $a \neq \mathrm{z}_n$ and $a \neq \mathrm{v}_n$ for any n, so that $\sigma^J(a) = a$ and hence $\eta(\sigma^J(a)) = \eta(a)$; moreover, $\eta(a) \neq \mathrm{z}_n$ for any n by Lemma D.13; so $\eta(\sigma^J(a)) = \eta(a) = \sigma(\eta(a))$. □

Lemma D.16 (BLT$_+$)**:** $\eta(\sigma^J(x)) \in \eta(a) \leftrightarrow a \notin a$ iff $x \in a$

Proof. If $a \notin a$ then $\eta(a) = \{\eta(\sigma^J(x)) : x \in a\}$ by Lemma D.15. If $a \in a$ then $\eta(a) = \{\eta(\sigma^J(x)) : x \notin a\} \cup \{\emptyset\}$, and note that $\emptyset \neq \eta(\sigma^J(x)) = \sigma(\eta(x))$ for all x. □

Lemma D.17 (BLT$_+$)**:** $x \in a$ iff $(x \in^I a)^J$

Proof. Using Lemma D.16 and the fact that $a \notin a$ iff $\eta(\emptyset) = \emptyset \notin \eta(a)$, note the following chain of equivalent formulas:

(1) $x \in a$

(2) $\eta(\sigma^J(x)) \in \eta(a) \leftrightarrow \eta(\emptyset) \notin \eta(a)$
(3) $(\sigma(x) \in a \leftrightarrow \emptyset \notin a)^J$
(4) $(x \in^I a)^J$ □

It remains to show in LT_+ that $x \in a$ iff $(x \in^J a)^I$. Working in BLT_+, define μ as a map sending $z_{n+1} \mapsto v_n \mapsto z_n$ and $x \mapsto x$ otherwise; by Lemma D.14, if $x \neq \emptyset$ then $\mu^{-1}(x) = \sigma^J(x)$. We then have two quick results:

Lemma D.18 (BLT_+)**:** $\eta(x) \in \eta(a)$ iff $(x = \emptyset \wedge a \in a) \vee (x \neq \emptyset \wedge (\mu(x) \in a \leftrightarrow a \notin a))$

Proof. If $x = \emptyset$, then $\eta(\emptyset) = \emptyset \in \eta(a)$ iff $a \in a$. If $x \neq \emptyset$; use Lemma D.16. □

Lemma D.19 (LT_+)**:** If $x \neq \emptyset$, then $\sigma(\mu^I(x)) = x$.

Proof. By Lemma D.12, μ^I maps $z_{n+2} \mapsto z_{n+1} \mapsto z_n$, and $x \mapsto x$ otherwise. □

Lemma D.20 (LT_+)**:** $x \in a$ iff $(x \in^J a)^I$

Proof. Using Lemmas D.19 and D.18^I, note the following chain of equivalent formulas:

(1) $x \in a$
(2) $(\emptyset = x \wedge x \in a) \vee (\emptyset \neq x \wedge x \in a)$
(3) $(\emptyset = x \wedge x \in a) \vee (\emptyset \neq x \wedge \sigma(\mu^I(x)) \in a)$
(4) $(\emptyset = x \wedge a \in^I a) \vee (\emptyset \neq x \wedge (\mu^I(x) \in^I a \leftrightarrow \emptyset \notin a))$
(5) $(\emptyset = x \wedge a \in^I a) \vee (\emptyset \neq x \wedge (\mu^I(x) \in^I a \leftrightarrow a \notin^I a))$
(6) $((\emptyset = x \wedge a \in a) \vee (\emptyset \neq x \wedge (\mu(x) \in a \leftrightarrow a \notin a)))^I$
(7) $(\eta(x) \in \eta(a))^I$
(8) $(x \in^J a)^I$ □

Theorem 7.1 now follows from Lemmas D.7, D.11, D.17, and D.20.

D.4 Finitary cases of definitional equivalences

The base theories, LT and BLT, are not definitionally equivalent. To see this, consider:

$$\mathrm{lt}(1) := 1 \qquad \mathrm{blt}(1) := 2$$
$$\mathrm{lt}(n+1) := 2^{\mathrm{lt}(n)} \qquad \mathrm{blt}(n+1) := 2^{\mathrm{blt}(n)+1}$$

Any model of LT with n levels has $\mathrm{lt}(n)$ sets, and any model of BLT with n bevels has $\mathrm{blt}(n)$ sets. In particular, there is a model of LT with four sets, but no model of BLT has four sets. So LT and BLT are not definitionally equivalent.

There is, though, a nice definitional equivalence when we insist that there are infinitely many sets but that every set is finite. Concretely: let LT_{fin} be $LT_+ + \neg$Infinity, and let BLT_{fin} be $BLT_+ + \neg\text{Infinity}_{\notin}$. Our earlier results immediately entail that LT_{fin}

and BLT_{fin} are definitionally equivalent. Moreover, as noted in Pt.1 §7, LT_{fin} is equivalent to ZF_{fin}. Finally, ZF_{fin} and PA are definitionally equivalent.[43] So:

Lemma D.21: PA, ZF_{fin}, LT_{fin}, and BLT_{fin} are definitionally equivalent.

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[43] Kaye and Wong (2007: Theorems 3.3, 6.5, 6.6). ZF_{fin} is the theory with all of ZF's axioms except that: (i) Zermelo's axiom of infinity is replaced with its negation; and (ii) it has a new axiom, $\forall a(\exists t \supseteq a)(t \text{ is transitive})$.

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