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Natural Dualities Through Product Representations: Bilattices and Beyond

Abstract. This paper focuses on natural dualities for varieties of bilattice-based algebras. Such varieties have been widely studied as semantic models in situations where information is incomplete or inconsistent. The most popular tool for studying bilattices-based algebras is product representation. The authors recently set up a widely applicable algebraic framework which enabled product representations over a base variety to be derived in a uniform and categorical manner. By combining this methodology with that of natural duality theory, we demonstrate how to build a natural duality for any bilattice-based variety which has a suitable product representation over a dualisable base variety. This procedure allows us systematically to present economical natural dualities for many bilattice-based varieties, for most of which no dual representation has previously been given. Among our results we highlight that for bilattices with a generalised conflation operation (not assumed to be an involution or commute with negation). Here both the associated product representation and the duality are new. Finally we outline analogous procedures for pre-bilattice-based algebras (so negation is absent).

Keywords: Product representation, Natural duality, Bilattice, Conflation, Double Ockham algebra.

Mathematics Subject Classification: Primary: 08C20, Secondary: 03G10, 03G25, 06B10, 06D50.

1. Introduction

Bilattices, with and without additional operations, have been identified by researchers in artificial intelligence and in philosophical logic as of value for analysing scenarios in which information may be incomplete or inconsistent. Over twenty years, a bewildering array of different mathematical models has been developed which employ bilattice-based algebras in such situations; [15,19,23,26] give just a sample of the literature. Within a logical context, bilattices have been used to interpret truth values of formal systems. The range of possibilities is illustrated by [1,2,4,16–18,25,27].

To date, the structure theory of bilattices has had two main strands: product representations (see in particular [5,9,11] and references therein)

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and topological duality theory [8, 22, 24]. In this paper we entwine these two strands, demonstrating how a dual representation and a product representation can be expected to fit together and to operate in a symbiotic way. Our work on distributive bilattices in [8] provides a prototype. Crucially, as in [8], we exploit the theory of natural dualities; see Sect. 3.

In [9] we set up a uniform framework for product representation. We introduced a formal definition of *duplication* of a base variety of algebras which gives rise to a new variety with additional operations built by combining suitable algebraic terms in the base language and coordinate manipulation (details are recalled in Sect. 2). This construction led to a very general categorical theorem on product representation [9, Theorem 3.2] which makes overt the intrinsic structure of such representations. Our Duality Transfer Theorem (Theorem 3.1) demonstrates how a natural duality for a given base class immediately yields a natural duality for any duplicate of that class. Moreover, the dualities for duplicated varieties mirror those for the base varieties, as regards both advantageous properties and complexity (note the concluding remarks in Sect. 4). By combining the Duality Transfer Theorem with product representation we can set up dualities for assorted bilattice-based varieties (see Sect. 4, Table 1). In almost all cases the dualities are new. The varieties in question arise as duplicates of $\mathcal{B}$ (Boolean algebras), $\mathcal{D}$ (bounded distributive lattices) $\mathcal{K}$ (Kleene algebras), $\mathcal{DM}$ (De Morgan algebras), and $\mathcal{DB}$ (bounded distributive bilattices), all of which have amenable natural dualities (see [10] and also [8]). Variants are available when lattice bounds are omitted.

We contrast key features of our natural duality approach with earlier work on dualities for bilattice-based algebras. We stress that our methods lead directly to dual representations which are categorical: morphisms do not have to be treated case-by-case as an overlay to an object representation (as is done in [22, 24]). Others' work on dualities in the context of distributive bilattices has sought instead, for a chosen class of algebras, a dual category which is an enrichment of a subcategory of Priestley spaces, that is, they start from Priestley duality, applied to the distributive lattice reducts of their algebras, and then superimpose extra structure to capture the suppressed operations. This strategy has been successfully applied to very many classes of distributive-lattice-based algebras, but it has drawbacks. Although the underlying Priestley duality is natural, the enriched Priestley space representation rarely is. Accordingly one cannot expect the rewards a natural duality offers. These rewards include instant access to free algebras, a simple treatment of coproducts, and a good description of duals of homomorphisms. Furthermore, if a natural duality has the added virtue

of being strong (see Sect. 3), then one can easily translate into dual form algebraic problems expressible in terms of injective or surjective morphisms.

Section 5 focuses on the variety $\mathcal{DB}_-$ of (bounded) distributive bilattices with a conflation operation—which is not assumed to be an involution or to commute with the negation. This variety has not been investigated before and would not have been susceptible to earlier methods. We realise $\mathcal{DB}_-$ as a duplicate of the variety dO of double Ockham algebras and set up a natural duality for dO, whence we obtain a duality for $\mathcal{DB}_-$. Both results are new. This example is also a novelty within bilattice theory since it takes us outside the realm of finitely generated varieties without losing the benefits of having a natural duality.

In Sect. 6 we consider the negation-free setting of pre-bilattice-based algebras, and link the ideas of [9, Sect. 9] with dual representations. Again, a very general theorem enables us to transfer a known duality from a base variety to a suitably constructed duplicate. Here multisorted duality theory is needed. Nonetheless the ideas and the categorical arguments are simple, and the proof of Theorem 3.1 is easily adapted.

We should comment on the scope of the applications we present in this paper. Our companion paper [9] focused on bilattice-based varieties and its product representation theorem was derived with applications to such varieties in mind. To align with [9] we shall illustrate our results by calling only on bilattice-based varieties. The range of such varieties is sufficiently diverse for us to demonstrate the applicability of the various duality techniques. However we emphasise that Theorems 3.1 and 6.3 are available more widely, in fact whenever the base variety is dualisable. However, to have explored applications to non-lattice-based varieties would have involved delving deeper into duality theory than space allowed.

2.   The General Product Representation Theorem Recalled

We shall assume that readers are familiar with the basic notions concerning bilattices. A summary can be found, for example, in [5] and a bare minimum in [9, Sect. 2]. Here we simply draw attention to some salient points concerning notation and terminology since usage in the literature varies. Except in Sect. 6 we assume that a negation operator is present.

A (unbounded) bilattice is an algebra $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \neg)$, where the reducts $\mathbf{A}_t := (A; \vee_t, \wedge_t)$ and $\mathbf{A}_k := (A; \vee_k, \wedge_k)$ are lattices (respectively the *truth lattice* and *knowledge lattice*). The operation $\neg$, capturing negation, is an endomorphism of $\mathbf{A}_k$ and a dual endomorphism of $\mathbf{A}_t$.

Bilattice models come in two flavours: with and without bounds. Which flavour is preferred (or appropriate) may depend on an intended application, or on mathematical considerations. A subscript $_u$ on the symbol denoting a category will indicate that we are working in the unbounded setting. So, for example, $\mathbf{D}$ denotes the category of bounded distributive lattices and $\mathbf{D}_u$ the category of all distributive lattices.

All the bilattices considered in this paper are distributive, meaning that each of the four lattice operations distributes over each of the other three. The weaker condition of interlacing is necessary and sufficient for a bilattice to have a product representation. However varieties of interlaced bilattice-based algebras seldom come within the scope of natural duality theory.

Our investigations involve classes of algebras, viewed both algebraically and categorically. We draw, lightly, on some of the basic formalism and theory of universal algebra, specifically regarding varieties (*alias* equational classes) and prevarieties; a standard reference for this material is [6]. A class of algebras over a common language will be regarded as a category in the usual way: the morphisms are all the homomorphisms. The variety generated by a family $\mathcal{M}$ of algebras of common type is denoted $\mathbb{V}(\mathcal{M})$. Equivalently $\mathbb{V}(\mathcal{M})$ is the class $\mathbb{HSP}(\mathcal{M})$ of homomorphic images of subalgebras of products of algebras in $\mathcal{M}$. The *prevariety* generated by $\mathcal{M}$ is the class $\mathbb{ISP}(\mathcal{M})$ whose members are isomorphic images of subalgebras of products of members of $\mathcal{M}$. Usually the algebras in $\mathcal{M}$ will be finite.

We now recall our general product representation framework [9, Sect. 3]. We fix an arbitrary algebraic language Σ and let $\mathcal{N}$ be a family of Σ-algebras. Let Γ be a set of pairs of Σ-terms such that, for $(t_1, t_2) \in \Gamma$, the terms t_1 and t_2 have common even arity, denoted $2n_{(t_1,t_2)}$. We view Γ as an algebraic language for a family of algebras $\mathsf{P}_\Gamma(\mathbf{N})$ ($\mathbf{N} \in \mathcal{N}$), where the arity of $(t_1, t_2) \in \Gamma$ is $n_{(t_1,t_2)}$. We write $[t_1, t_2]$ when the pair (t_1, t_2) is regarded as belonging to Γ, *qua* language. For $\mathbf{A} \in \mathbb{V}(\mathcal{N})$ we define a Γ-algebra $\mathsf{P}_\Gamma(\mathbf{A}) = (A \times A; \{[t_1, t_2]^{\mathsf{P}_\Gamma(\mathbf{A})} \mid (t_1, t_2) \in \Gamma\})$, in which the operation $[t_1, t_2]^{\mathsf{P}_\Gamma(\mathbf{A})} \colon (A \times A)^n \to A \times A$ is given by

$$
[t_1, t_2]^{\mathsf{P}_\Gamma(\mathbf{A})}((a_1, b_1), \ldots, (a_n, b_n))
$$
$$
= (t_1^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n), t_2^{\mathbf{A}}(a_1, b_1, \ldots, a_n, b_n)),
$$

where $n = n_{(t_1,t_2)}$ and $(a_1, b_1), \ldots, (a_n, b_n) \in A \times A$. It is easy to check that the assignment $\mathbf{A} \mapsto \mathsf{P}_\Gamma(\mathbf{A})$ (on objects) and $h \mapsto h \times h$ (on morphisms) defines a functor $\mathsf{P}_\Gamma \colon \mathbb{V}(\mathcal{N}) \to \mathbb{V}(\mathsf{P}_\Gamma(\mathcal{N}))$. We shall also need the following notation. Given a set X the map $\delta^X \colon X \to X \times X$ is given by $\delta^X(x) = (x, x)$ and $\pi_1^X, \pi_2^X \colon X \times X \to X$ denote the projection maps.

We are ready to recall a key definition from [9, Sect. 3], where further details can be found. We say that Γ *duplicates* $\mathcal{N}$ and that $\mathcal{A} = \mathbb{V}(\mathsf{P}_\Gamma(\mathcal{N}))$ is *a duplicate of* $\mathcal{B} = \mathbb{V}(\mathcal{N})$ if the following conditions on $\mathcal{N}$ and Γ are satisfied:

(L) for each n-ary operation symbol $f \in \Sigma$ and each $i \in \{1,2\}$ there exists an n-ary Γ-term t (depending on f and i) such that $\pi_i^N \circ t^{\mathsf{P}_\Gamma(\mathbf{N})} \circ (\delta^N)^n = f^{\mathbf{N}}$ for each $\mathbf{N} \in \mathcal{N}$;

(M) there exists a binary Γ-term v such that $v^{\mathsf{P}_\Gamma(\mathbf{N})}((a,b),(c,d)) = (a,d)$ for $\mathbf{N} \in \mathcal{N}$ and $a,b,c,d \in N$;

(P) there exists a unary Γ-term s such that $s^{\mathsf{P}_\Gamma(\mathbf{N})}(a,b) = (b,a)$ for $\mathbf{N} \in \mathcal{N}$ and $a,b \in N$.

We now present the Product Representation Theorem [9, Theorem 3.2].

THEOREM 2.1. *Assume that Γ duplicates a class of algebras $\mathcal{N}$. Then the functor $\mathsf{P}_\Gamma \colon \mathcal{B} \to \mathcal{A}$ sets up a categorical equivalence between $\mathcal{B} = \mathbb{V}(\mathcal{N})$ and its duplicate $\mathcal{A} = \mathbb{V}(\mathsf{P}_\Gamma(\mathcal{N}))$.*

The classes of algebras arising in this section have principally been varieties. In the next section we concentrate on singly-generated prevarieties. The following corollary tells us how the class operators $\mathbb{HSP}$ and $\mathbb{ISP}$ behave with respect to duplication. It is an almost immediate consequence of the fact that P_Γ is a categorical equivalence; assertion (c) follows directly from (a) and (b).

COROLLARY 2.2. *Assume that Γ duplicates a class of algebras $\mathcal{N}$. The following statements hold for each $\mathbf{A} \in \mathbb{V}(\mathcal{N})$:*

(a) $\mathbb{HSP}(\mathsf{P}_\Gamma(\mathbf{A}))$ *is categorically equivalent to* $\mathbb{HSP}(\mathbf{A})$.

(b) $\mathbb{ISP}(\mathsf{P}_\Gamma(\mathbf{A}))$ *is categorically equivalent to* $\mathbb{ISP}(\mathbf{A})$.

(c) *If* $\mathbb{V}(\mathbf{A}) = \mathbb{ISP}(\mathbf{A})$ *then* $\mathbb{V}(\mathsf{P}_\Gamma(\mathbf{A})) = \mathbb{ISP}(\mathsf{P}_\Gamma(\mathbf{A}))$.

3. Natural Duality and Product Representation

It is appropriate to recall only in brief the theory of natural dualities as we shall employ it. A textbook treatment is given in [10] and a summary geared to applications to distributive bilattices in [8, Sects. 3 and 5].

Our object of study in this section will be a prevariety $\mathcal{A}$ generated by an algebra $\mathbf{M}$, so that $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$. (Only in Sect. 6 will we replace the single algebra $\mathbf{M}$ by a family of algebras $\mathcal{M}$. We shall then need to bring multisorted duality theory into play.)

Traditionally (and in [10] in particular) $\mathbf{M}$ is assumed to be finite. This suffices for our applications in Sect. 4. However our application to bilattices with generalised conflation will depend on the more general theory presented in [13]. Therefore we shall assume that $\mathbf{M}$ can be equipped with a compact Hausdorff topology $\mathcal{T}$ with respect to which it becomes a topological algebra. When $\mathbf{M}$ is finite $\mathcal{T}$ is necessarily discrete.

Our aim is to find a second category $\mathcal{X}$ whose objects are topological structures of common type and which is dually equivalent to $\mathcal{A}$ via functors $\mathsf{D} \colon \mathcal{A} \to \mathcal{X}$ and $\mathsf{E} \colon \mathcal{X} \to \mathcal{A}$. Moreover—and this is a key feature of a natural duality—we want each algebra $\mathbf{A}$ in $\mathcal{A}$ to be concretely representable as an algebra of continuous structure-preserving maps from $\mathsf{D}(\mathbf{A})$ (the *dual space* of $\mathbf{A}$) into $\underset{\sim}{\mathbf{M}}$, where $\underset{\sim}{\mathbf{M}} \in \mathcal{X}$ has the same underlying set M as does $\mathbf{M}$. For this to succeed, some compatibility between the structures $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ will be necessary. We consider a topological structure $\underset{\sim}{\mathbf{M}} = (M; G, R, \mathcal{T})$ where

- $\mathcal{T}$ is a topology on M (as demanded above);
- G is a set of operations on M, meaning that, for $g \in G$ of arity $n \geqslant 1$, the map $g \colon \mathbf{M}^n \to \mathbf{M}$ is a continuous homomorphism (any nullary operation in G will be identified with a constant in the type of $\mathbf{M}$);
- R is a set of relations on M such that if $r \in R$ is n-ary ($n \geqslant 1$) then r is the universe of a topologically closed subalgebra $\mathbf{r}$ of $\mathbf{M}^n$.

We refer to such a topological structure $\underset{\sim}{\mathbf{M}}$ as an *alter ego* for $\mathbf{M}$ and say that $\underset{\sim}{\mathbf{M}}$ and $\mathbf{M}$ are *compatible*. Of course. the topological conditions imposed on G and R are trivially satisfied if M is finite. (The general theory in [10] allows an alter ego also to include partial operations, but they do not arise in our intended applications.) We use $\underset{\sim}{\mathbf{M}}$ to build a new category $\mathcal{X}$. We first consider structures of the same type as $\underset{\sim}{\mathbf{M}}$. These have the form $\mathbf{X} = (X; G^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}})$ where $\mathcal{T}^{\mathbf{X}}$ is a compact Hausdorff topology and $G^{\mathbf{X}}$ and $R^{\mathbf{X}}$ are sets of operations and relations on X in bijective correspondence with those in G and R, with matching arities. Isomorphisms between such structures are defined in the obvious way. For any non-empty set S we give M^S the product topology and lift the elements of G and R pointwise to M^S. We then form $\mathcal{X} := \mathbb{IS}_\mathrm{c}\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, the class of isomorphic copies of closed substructures of non-empty powers of $\underset{\sim}{\mathbf{M}}$ (with $^+$ indicating that the empty index set is not included). We make $\mathcal{X}$ into a category by taking all continuous structure-preserving maps as the morphisms.

As a consequence of the compatibility of $\underset{\sim}{\mathbf{M}}$ and $\mathbf{M}$, and the topological conditions imposed, the following assertions are true. Let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. Then $\mathcal{A}(\mathbf{A}, \mathbf{M})$ may be seen as a closed substructure of $\underset{\sim}{\mathbf{M}}^A$ and $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ as a subalgebra of $\mathbf{M}^X$. We can set up well-defined contravariant hom-functors $\mathsf{D} \colon \mathcal{A} \to \mathcal{X}$ and $\mathsf{E} \colon \mathcal{X} \to \mathcal{A}$;

on objects: $\mathsf{D} \colon \mathbf{A} \mapsto \mathcal{A}(\mathbf{A}, \mathbf{M})$,

on morphisms: $\mathsf{D} \colon x \mapsto - \circ x$,

and

on objects: $\mathsf{E} \colon \mathbf{X} \mapsto \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$,

on morphisms: $\mathsf{E} \colon \phi \mapsto - \circ \phi$,

The following assertions are part of the standard framework of natural duality theory. Details can be found in [10, Chap. 2]; see also [13, Sect. 2]. Given $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$, we have natural evaluation maps $e_{\mathbf{A}} \colon a \mapsto - \circ a$ and $\varepsilon_{\mathbf{X}} \colon x \mapsto - \circ x$, with $e_{\mathbf{A}} \colon \mathbf{A} \to \mathsf{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to \mathsf{DE}(\mathbf{X})$. Moreover $(\mathsf{D}, \mathsf{E}, e, \varepsilon)$ is a dual adjunction. Each of the maps $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ is an embedding. We say that $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}$, or simply that $\underset{\sim}{\mathbf{M}}$ *dualises* $\mathbf{M}$, if each $e_{\mathbf{A}}$ is surjective, so that it is an isomorphism $e_{\mathbf{A}} \colon \mathbf{A} \cong \mathsf{ED}(\mathbf{A})$. A dualising alter ego $\underset{\sim}{\mathbf{M}}$ plays a special role in the duality it sets up: it is the dual space of the free algebra on one generator in $\mathcal{A}$. This fact is a consequence of compatibility. More generally, the free algebra generated by a non-empty set S has dual space $\underset{\sim}{\mathbf{M}}^S$.

Assume that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and in addition that each $\varepsilon_{\mathbf{X}}$ is surjective and so an isomorphism. Then we say $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\mathbf{M}$ or that the duality yielded by $\underset{\sim}{\mathbf{M}}$ is *full*. In this case $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent. Full dualities are particularly amenable if they are *strong*; this is the requirement that the alter ego be injective in the topological prevariety it generates. We do not need here to go deeply into the topic of strong dualities (see [10, Chap. 3] for a full discussion) but we do note in passing that each of the functors D and E in a strong duality interchanges embeddings and surjections—a major virtue if a duality is to be used to transfer algebraic problems into a dual setting.

We are ready to present our duality theorem for duplicated (pre)varieties. Our notation is chosen to match that in Theorem 2.1.

THEOREM 3.1. (Duality Transfer Theorem) *Let* $\mathbf{N}$ *be an algebra and assume that* Γ *duplicates* $\mathbf{N}$. *If the topological structure* $\underset{\sim}{\mathbf{N}} = (N; G, R, \mathfrak{T})$ *yields a duality on* $\mathcal{B} = \mathbb{ISP}(\mathbf{N})$ *with dual category* $\mathcal{Y} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$, *then* $\mathbf{N}^2$ *yields a*

duality on $\mathbf{\mathcal{A}} = \mathbb{ISP}(\mathsf{P}_\Gamma(\mathbf{N}))$, again with $\mathbf{\mathcal{Y}}$ as the dual category. If the former duality is full, respectively strong, then the same is true of the latter.

PROOF. For the purposes of the proof we shall assume that N, and hence also M, is finite. It is routine to check that the topological conditions which come into play when N is infinite lift to the duplicated set-up.

We claim that $\underset{\sim}{\mathbf{N}}^2$ acts as a legitimate alter ego for $\mathbf{M} := \mathsf{P}_\Gamma(\mathbf{N})$. Certainly these structures have the same universe, namely $N \times N$. It follows from the definition of the operations of $\mathsf{P}_\Gamma(\mathbf{N})$ that $\mathsf{P}_\Gamma(\mathbf{r})$, whose universe is $r \times r$, is a subalgebra of $(\mathsf{P}_\Gamma(\mathbf{N}))^n$ whenever $r \in R$ is the universe of a subalgebra $\mathbf{r}$ of $\mathbf{N}^n$. But $R^{\underset{\sim}{\mathbf{N}}^2}$ consists of the relations $r \times r$, for $r \in R$. Likewise, an n-ary operation g in G gives rise to the same operation, *viz.* $g \times g$, of $\mathsf{P}_\Gamma(\mathbf{N})$ and in the structure $\underset{\sim}{\mathbf{N}}^2$. Hence $g \times g$ is compatible with $\mathsf{P}_\Gamma(\mathbf{N})$.

We now set up the functors for the existing duality for $\mathbb{ISP}(\mathbf{N})$ and for the duality sought for $\mathbb{ISP}(\mathbf{M})$. Let $\mathbf{\mathcal{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}}^2)$. Then $\mathbf{\mathcal{Y}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}}^2) = \mathbf{\mathcal{X}}$ too. Let $\mathsf{D}_{\mathbf{\mathcal{B}}} \colon \mathbf{\mathcal{B}} \to \mathbf{\mathcal{Y}}$ and $\mathsf{E}_{\mathbf{\mathcal{B}}} \colon \mathbf{\mathcal{Y}} \to \mathbf{\mathcal{B}}$ be the functors determined by $\underset{\sim}{\mathbf{N}}$ and $\mathsf{D}_{\mathbf{\mathcal{A}}} \colon \mathbf{\mathcal{A}} \to \mathbf{\mathcal{X}}$ and $\mathsf{E}_{\mathbf{\mathcal{A}}} \colon \mathbf{\mathcal{X}} \to \mathbf{\mathcal{A}}$ those determined by $\underset{\sim}{\mathbf{N}}^2$. Since $\mathbf{\mathcal{Y}} = \mathbf{\mathcal{X}}$, the functors $\mathsf{D}_{\mathbf{\mathcal{B}}}$ and $\mathsf{D}_{\mathbf{\mathcal{A}}}$ have a common codomain.

Let $\mathbf{A} \in \mathbf{\mathcal{A}}$. By Corollary 2.2, we may assume that $\mathbf{A} = \mathsf{P}_\Gamma(\mathbf{B})$, for some $\mathbf{B} \in \mathbf{\mathcal{B}}$. By Theorem 2.1 and the definition of P_Γ on morphisms,

$$\mathbf{\mathcal{A}}(\mathbf{A}, \mathsf{P}_\Gamma(\mathbf{N})) = \mathsf{P}_\Gamma(\mathbf{\mathcal{B}}(\mathbf{B}, \mathbf{N})) = \{\, y \times y \mid y \in \mathbf{\mathcal{B}}(\mathbf{B}, \mathbf{N}) \,\}.$$

Let $\alpha \in \mathsf{E}_{\mathbf{\mathcal{A}}} \mathsf{D}_{\mathbf{\mathcal{A}}}(\mathbf{A}) = \mathbf{\mathcal{X}}(\mathsf{D}_{\mathbf{\mathcal{A}}}(\mathbf{A}), \underset{\sim}{\mathbf{N}}^2)$. For $i = 1, 2$, define $\alpha_i \colon \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B}) \to \underset{\sim}{\mathbf{N}}$ by $\alpha_i(y) = \pi_i^N(\alpha(y \times y))$ for $y \in \mathbf{\mathcal{B}}(\mathbf{B}, \mathbf{N})$. It is straightforward to see that $\alpha_i \in \mathsf{E}_{\mathbf{\mathcal{B}}} \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B})$. Therefore, for $i = 1, 2$, there exists $b_i \in B$ such that $\alpha_i(y) = y(b_i)$ for $y \in \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B})$. We claim that $\alpha(x) = x(b_1, b_2)$ for all $x \in \mathbf{\mathcal{A}}(\mathbf{A}, \mathsf{P}_\Gamma(\mathbf{N}))$. We can write $x = y \times y$ where $y \in \mathbf{\mathcal{B}}(\mathbf{B}, \mathbf{N})$. Then

$$\alpha(x) = \alpha(y \times y) = (\pi_1^N(\alpha(y \times y)), \pi_2^N(\alpha(y \times y)))$$
$$= (\alpha_1(y), \alpha_2(y)) = (y(b_1), y(b_2)) = (y \times y)(b_1, b_2) = x(b_1, b_2).$$

This proves that $e_{\mathbf{A}} \colon \mathbf{A} \to \mathsf{E}_{\mathbf{\mathcal{A}}} \mathsf{D}_{\mathbf{\mathcal{A}}}(\mathbf{A})$ is surjective for each $\mathbf{A} \in \mathbf{\mathcal{A}}$, so that we do indeed have a duality for $\mathbf{\mathcal{A}}$ based on the alter ego $\underset{\sim}{\mathbf{M}} = \underset{\sim}{\mathbf{N}}^2$.

We now claim that if $\underset{\sim}{\mathbf{N}}$ fully dualises $\mathbf{N}$ then $\underset{\sim}{\mathbf{M}}$ fully dualises $\mathbf{M}$. To do this we shall show that the bijection $\eta \colon \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B}) \to \mathsf{D}_{\mathbf{\mathcal{A}}}(\mathbf{A})$, defined by $\eta(y) = y \times y$ for each $y \in \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B})$, is an isomorphism (of topological structures) from $\mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B})$ onto $\mathsf{D}_{\mathbf{\mathcal{A}}}(\mathbf{A})$, where, as before, $\mathbf{A} = \mathsf{P}_\Gamma(\mathbf{B})$, see [10, Lemma 3.1.1]. Let r be an n-ary relation in $\underset{\sim}{\mathbf{N}}$. For $y_1, \dots, y_n \in \mathsf{D}_{\mathbf{\mathcal{B}}}(\mathbf{B})$,

$(y_1, \ldots, y_n) \in r^{\mathsf{D}_{\mathcal{B}}(\mathbf{B})}$

$$\iff \forall a \in N \left((y_1(a), \ldots, y_n(a)) \in r\right)$$
$$\iff \forall (a_1, a_2) \in M \left(((y_1(a_1), y_1(a_2)), \ldots, (y_n(a_1), y_n(a_2))) \in r \times r\right)$$
$$\iff (y_1 \times y_1, \ldots, y_n \times y_n) \in (r \times r)^{\mathsf{D}_{\mathcal{A}}(\mathbf{A})}.$$

A similar argument applies to operations.

The map η has compact codomain and Hausdorff domain and hence is a homeomorphism provided η^{-1} is continuous. To prove this it will suffice to show that each map $\pi_b \circ \eta^{-1}$ is continuous, where π_b denotes the projection from $\mathsf{D}_{\mathcal{B}}(\mathbf{B})$, regarded as a subspace of $\underset{\sim}{\mathbf{N}}^B$, onto the b-coordinate, for $b \in B$. The map $\pi_{(b,b)}$ is defined likewise. Let U be open in N. For $y \times y \in \mathsf{D}_{\mathcal{A}}(\mathbf{A})$,

$$y \times y \in (\pi_b \circ \eta^{-1})^{-1}(U) \iff \pi_b(y) \in U \iff y(b) \in U$$
$$\iff (y \times y)(b, b) \in U \times U \iff \pi_{(b,b)}(y \times y) \in U \times U$$
$$\iff (y \times y) \in (\pi_{(b,b)})^{-1}(U \times U).$$

This proves the continuity assertion.

Finally, since $\underset{\sim}{\mathbf{N}}$ is injective in $\mathcal{X} = \mathcal{Y}$ if and only if $\underset{\sim}{\mathbf{N}}^2$ is, $\underset{\sim}{\mathbf{N}}$ yields a strong duality on $\mathcal{B}$ if and only if $\underset{\sim}{\mathbf{N}}^2$ yields a strong duality on $\mathcal{A}$, by [10, Theorem 3.2.4]. ∎

Theorem 3.1 should not be disparaged because it is simple to prove. It needs to be remembered that the derivation of workable natural dualities can be arduous. The theorem shows how to build a wide class of such dualities with ease, so giving access at a stroke and in a systematic way to a multitude of potential applications.

Of course, though, Theorem 3.1 is only useful when we have a (strong) duality to hand for the base class $\mathbb{ISP}(\mathbf{N})$ we wish to employ. Nothing we have said about natural dualities so far tells us how to find an alter ego $\underset{\sim}{\mathbf{N}}$ for $\mathbf{N}$, or even whether a duality exists. Fortunately, simple and well-understood strong dualities exist for the base varieties $\mathbb{ISP}(\mathbf{N})$ which support the miscellany of logic-oriented examples presented in Sect. 4. In all cases considered there, $\mathbf{N}$ is a small finite algebra with a lattice reduct. Existence of such a reduct guarantees dualisability [10, Sect. 3.4]: a brute-force alter ego $\underset{\sim}{\mathbf{N}} = (N; \mathbb{S}(\mathbf{N}^2), \mathcal{T})$ is available. However this default choice is likely to yield a tractable duality only when $\mathbf{N}$ is very small. Otherwise the subalgebra lattice $\mathbb{S}(\mathbf{N}^2)$ is generally unwieldy. Methodology exists for slimming down a given dualising alter ego to yield a potentially more workable duality (see [10, Chap. 8]), but it is preferable to obtain an economical duality from the outset. This is often possible when $\mathbf{N}$ is a distributive lattice, not

necessarily finite: in many such cases one can apply the piggyback method which originated with Davey and Werner (see [10, Chap. 7] and [13]). We shall demonstrate its use in Sect. 5, where we develop a duality for double Ockham algebras, our base variety for studying generalised conflation.

Against this background we can appreciate the merits of Theorem 3.1. Suppose we have a class $\mathbb{ISP}(\mathbf{M})$ (with $\mathbf{M}$ finite) which is expressible as a duplicate of a dualisable base variety $\mathbb{ISP}(\mathbf{N})$. Then $|M| = |N|^2$ and, on cardinality grounds alone, finding an amenable duality directly for $\mathbb{ISP}(\mathbf{M})$ could be challenging, whereas the chances are much higher that we have available, or are able to set up, a simple dualising alter ego $\underset{\sim}{\mathbf{N}}$ for $\mathbf{N}$. And then, given $\underset{\sim}{\mathbf{N}}$ we can immediately obtain an alter ego $\underset{\sim}{\mathbf{M}}$ for $\mathbf{M}$, with the same number of relations and operations in $\underset{\sim}{\mathbf{M}}$ as in $\underset{\sim}{\mathbf{N}}$.

4.   Examples of Natural Dualities via Duplication

We now present a miscellany of examples. All involve bilattices but, as noted earlier, the scope of our methods is potentially wider. We derive (strong) dualities for certain (finitely generated) duplicated varieties given in [9] by calling on well-known (strong) dualities for their base varieties. A catalogue of base varieties and duplicates is assembled in [9, Appendix, Table 1], with references to where in the paper these examples are presented. Table 1 lists alter egos for dualities for base varieties. These dualities are discussed in [10], with their sources attributed. Natural dualities for the indicated duplicated varieties, also strong, can be read off from the table, using the Duality Transfer Theorem. When specifying a generator for each base variety, we adopt abbreviations for standard sets of operations:

$$\mathcal{F}_{\mathbf{L}} = \{\vee, \wedge, 0, 1\}, \qquad \mathcal{F}_{\mathbf{B}} = \mathcal{F}_{\mathbf{DM}} = \mathcal{F}_{\mathbf{K}} = \mathcal{F}_{\mathbf{L}} \cup \{\sim\};$$

we have elected to denote negation in Boolean algebras ($\mathbf{B}$), De Morgan ($\mathbf{DM}$) algebras and Kleene algebras ($\mathbf{K}$) by $\sim$, to distinguish it from bilattice negation, $\neg$.

The top row of Table 1 should be treated as a prototype, both algebraically and dually. There the base variety is $\mathbf{D}$, the variety of bounded distributive lattices. The duplicated variety in this case is the variety $\mathbf{DB}$ of bounded distributive bilattices. It is generated (as a prevariety) by the four-element algebra in $\mathbf{DB}$. Full details of the natural duality for $\mathbf{DB}$ and its relationship to Priestley duality for the base variety $\mathbf{D}$ appear in [8]. All the other examples in the table work in essentially the same way. The examples we list may be grouped into two types. In one type, the duplicator Γ

Table 1. Examples of natural dualities (bounded case)

Base variety and its natural duality			Duplicate variety
Variety	Generator	Alter ego	Non-bilattice operation added
Bounded DL's	$(\{0,1\}; \mathcal{F}_{\mathcal{L}})$	$(\{0,1\}; \leqslant, \mathcal{T})$ [10, §4.3.1]	N/A
Boolean Algebras	$(\{0,1\}; \mathcal{F}_{\mathcal{B}})$	$(\{0,1\}; \mathcal{T})$ [10, §4.1.2]	Implication, $\supset$ [1], [5, §2] Moore's epistemic operator, L [20] Negation-by-failure, / [26, §3]
De Morgan Algebras	$(\{0,1\}^2; \mathcal{F}_{\mathcal{DM}})$	$(\{0,1\}^2; \preccurlyeq, g, \mathcal{T})$ [10, §4.3.15]	Conflation, $-$ (with bounds) [22]
Kleene Algebras	$(\{0,a,1\}; \mathcal{F}_{\mathcal{K}})$	See [10, §4.3.9]	Negation-by-failure, / [26, §4]

includes the set of terms used to duplicate the variety of bounded lattices to create bounded bilattices, augmented with additional terms to capture other operations from terms in the base language; this applies to $\mathbf{DB}$ itself, to implicative bilattices, to distributive bilattices with conflation, to the varieties carrying Moore's operator. In examples of the second type the base-level generator $\mathbf{N}$ is already equipped with a (distributive) bilattice structure and Γ includes all the terms used to create $\mathbf{DB}$ plus terms to create any extra operation present in $\mathbf{N}$. This is the situation with negation-by-failure.

For the natural dualities recorded in Table 1, we note that, apart from $\mathbf{D}$, the base variety in each case is De Morgan algebras or a subvariety thereof. The alter ego includes a partial order $\preccurlyeq$ known as the *alternating order* in [10, Theorem 4.3.16]; in the case of $\mathbf{DM}$, the relation $\preccurlyeq$ on universe $\{0,1\}^2$ of the four-element generator $\mathbf{4}_{\mathcal{DM}}$ is the knowledge order. The map g is the involution swapping the coordinates.

Only simple modifications are needed to handle the case when the language of a lattice-based variety does not include lattice bounds as nullary operations. It is an old result that Priestley duality for the variety $\mathbf{D}_u$ can be set up in much the same way as that for $\mathbf{D}$, with the dual category being bounded (*alias* pointed) Priestley spaces. The alter ego for

$\mathbf{2}_u = (\{0,1\}; \vee, \wedge)$ is $(\{0,1\}; \leqslant, 0, 1)$; note that when we move to the unbounded setting $\{0\}$ and $\{1\}$ become universes of subalgebras (alternatively, 0 and 1 are nullary operations). The constant maps $\mathbf{0}$ and $\mathbf{1}$ from a distributive lattice $\mathbf{L}$ into $(\{0,1\}; \vee, \wedge)$ are $\mathbf{D}_u$ homomorphisms and are the bounds of the dual space of $\mathbf{L}$. Details are given in [10, Sects. 1.2 and 4.3.1].

Natural dualities for duplicates of $\mathbf{D}_u$ are derived from those for corresponding duplicates of $\mathbf{D}$ simply by adding to the alter ego nullary operations $(0,0)$ and $(1,1)$. Compare with [8, Sect. 4], which provides a direct treatment of duality for $\mathbf{DB}_u$; here, even more than in the bounded case, we see the merit of the automatic process that Theorem 3.1 supplies. A duality for $\mathbf{DM}_u$ (De Morgan lattices) is obtained by adding the top and bottom elements for the partial order $\preccurlyeq$ to the alter ego for $\mathbf{DM}$. Our transfer theorem then applies to unbounded distributive bilattices with conflation.

5.   Bilattices with Generalised Conflation

In this section we break new ground, both in relation to product representation and in relation to natural duality.

The bilattice-based variety $\mathbf{DB}_-$ that we study—(bounded) distributive bilattices with generalised conflation—has not been considered before. Previous authors who have studied product representation when conflation is present have assumed that this operation is an involution that commutes with negation (see [14, Theorem 8.3], [5] and our treatment in [9, Sect. 5]). We shall demonstrate that neither assumption is necessary for the existence of a product representation.

Our focus in this paper is on developing theoretical tools. Nevertheless we should supply application-oriented reasons to justify investigating generalised conflation. We first note that it is often, but not always, natural to assume that conflation be an involution. On the other hand, the justification for the commutation condition is less clear cut. Indeed, both the original definition in [14] and that in [25] omit commutation, and this is brought in only later. In [25, Sect. 3] the emphasis is on truth values. The authors' desired interpretation then leads them to consider a special algebra SIXTEEN$_3$, in which the conflation operation does commute with negation. In [18, Sect. 2] conflation is used to study (knowledge) consistent and exact elements of a lattice. The investigations in both [25] and [18] are intrinsically connected to the product representation for bilattices with conflation. Our product representation would permit similar interpretations when commutation fails and/or conflation is not an involution. In a different setting,

conflation has been used in [15] to present an algebraic model of the logic system of revisions in databases, knowledge bases, and belief sets introduced in [23]. In this model the coordinates of a pair in a product representation of a bilattice are interpreted as the degrees of confidence for including in a database an item of information and for excluding it. Conflation then models the transformation of information that reinterprets as evidence for inclusion whatever did not previously count as evidence against, and vice versa. That is, conflation comprises two processes: given the information against (for) a certain argument, these capture information for (against) the same argument. In [15] these two transformations coincide, and are mutually inverse. Our work on generalised conflation would allow these assumptions to be weakened so facilitating a wider range of models.

The class $\mathcal{DB}_-$ consists of algebras of the form

$$\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \neg, -, 0, 1),$$

where the reduct of $\mathbf{A}$ obtained by suppressing – belongs to $\mathcal{DB}$ and – is an endomorphism of $\mathbf{A}_t$ and a dual endomorphism of $\mathbf{A}_k$. Here we elect to include bounds. The variety $\mathcal{DBC}$ of (bounded) distributive bilattices with conflation (where by convention conflation and negation do commute) is a subvariety of $\mathcal{DB}_-$. However $\mathcal{DB}_-$ and $\mathcal{DBC}$ behave quite differently: even though $\neg$ is an involution, $-$ is not. As a consequence the monoid these operations generate is not finite, as is the case in $\mathcal{DBC}$. (We note that the unbounded case of generalised conflation could also be treated by making appropriate modifications to the above definition and throughout what follows.)

Our product representation for $\mathcal{DB}_-$ uses as its base variety the class $\mathsf{d\mathcal{O}}$ of double Ockham algebras. This is a new departure as regards representations of bilattice expansions. A *double Ockham algebra* is a $\mathcal{D}$-based algebras equipped with two dual endomorphisms f and g of the $\mathcal{D}$-reducts. An Ockham algebra carries just one such operation. The variety $\mathcal{O}$ of Ockham algebras, which includes Boolean algebras, De Morgan algebras and Kleene algebras among its subvarieties, has been exhaustively studied, both algebraically and via duality methods, as indicated by the texts [3,10] and many articles. The variety $\mathsf{d\mathcal{O}}$ is much less well explored. The remainder of the section is accordingly organised as follows. Proposition 5.1 presents the product representation for $\mathcal{DB}_-$ over the base variety $\mathsf{d\mathcal{O}}$. We then set $\mathcal{DB}_-$ aside while we develop the theory of $\mathsf{d\mathcal{O}}$ which we need if we are to apply our Duality Transfer Theorem to $\mathcal{DB}_-$. This requires us first to identify an algebra $\mathbf{M}$ such that $\mathsf{d\mathcal{O}} = \mathbb{ISP}(\mathbf{M})$ (Proposition 5.2). We then set up an alter ego $\underset{\sim}{\mathbf{M}}$ for $\mathbf{M}$ and call on [13, Theorem 4.4] to obtain a natural

duality for dO (Theorem 5.6). This is then combined with Theorem 3.1 to arrive at a natural duality for $\mathcal{DB}_-$ (Theorem 5.7).

To motivate how we can realise $\mathcal{DB}_-$ as a duplicate of dO we briefly recall from [9, Sect. 5] how $\mathcal{DBC}$ arises as a duplicate of $\mathcal{DM}$. We adopt the notation introduced in [9, Sect. 4]. Let Σ be a language and t be an n-ary Σ-term. For $m \geqslant n$ and $i_1, \ldots, i_n \in \{1, \ldots, m\}$ we denote by $t^m_{i_1 \ldots i_n}$ the m-ary term $t^m_{i_1 \ldots i_n}(x_1, \ldots, x_m) = t(x_{i_1}, \ldots, x_{i_n})$. Using this notation, we can capture the extra operation $-$ on the generator $\mathbf{16}_{\mathcal{DBe}}$ of $\mathcal{DBC}$ using the De Morgan negation $\sim$, combined with coordinate-flipping: the family of terms $\Gamma_{\mathcal{DBe}} = \Gamma_{\mathcal{DB}} \cup \{(\sim^2_2, \sim^2_1)\}$ acts as a duplicator for $\mathcal{DM}$ with $\mathcal{DBC}$ as the duplicated variety; here $\Gamma_{\mathcal{DB}}$ duplicates bounded lattices. (See [9, Sect. 5] for an explanation as to why the form of the operations in $\mathcal{DBC}$ dictates that $\mathcal{DM}$ should be used as the base variety.)

We now present our duplication result linking dO and $\mathcal{DB}_-$.

PROPOSITION 5.1. *Let $\{\vee, \wedge, f, g, 0, 1\}$ be the language of double Ockham algebras dO. The set $\Gamma_{\mathcal{DB}_-} = \Gamma_{\mathcal{DB}} \cup \{(f^2_2, g^2_1)\}$ duplicates dO. Moreover, $\mathcal{DB}_- = \mathbb{V}\big(\mathsf{P}_{\Gamma_{\mathcal{DB}_-}}(\mathsf{dO})\big)$, where $\Sigma_{\Gamma_{\mathcal{DB}_-}}$ is identified with the language of $\mathcal{DB}_-$.*

PROOF. Certainly $\Gamma_{\mathcal{DB}_-}$ duplicates dO because $(f^2_2, g^2_1) \in \Gamma_{\mathcal{DB}_-}$ and $\Gamma_{\mathcal{DB}}$ is a duplicate for $\Sigma_{\mathcal{D}}$ on $\mathcal{D}$.

Now let $\mathbf{A} \in \mathcal{DB}_-$. By the product representation of $\mathcal{DB}$ over $\mathcal{D}$, the bilattice reduct $\mathbf{A}_{\mathcal{DB}} \cong \mathsf{P}_{\Gamma_{\mathcal{DB}}}(\mathbf{L})$, for some $\mathbf{L} \in \mathcal{D}$. We identify A and $L \times L$ and define $f, g \colon L \to L$ by $f(a) = \pi_1(-(0, a))$ and $g(a) = \pi_2(-(a, 0))$, for $a \in L$. For $a, b \in L$,

$$g(a \vee b) = \pi_2(-(a \vee b, 0)) = \pi_2(-((a, 0) \vee_k (b, 0))) = \pi_2(-(a, 0) \wedge_k -(b, 0))$$
$$= \pi_2(-(a, 0) \vee_k -(b, 0)) = \pi_2(-(a, 0)) \wedge \pi_2(-(b, 0)) = g(a) \wedge g(b),$$
$$g(a \wedge b) = \pi_2(-(a \wedge b, 0)) = \pi_2(-((a, 0) \wedge_k (b, 0)))$$
$$= \pi_2(-(a, 0)) \vee \pi_2(-(b, 0)) = g(a) \vee g(b),$$
$$g(1) = \pi_2(-(1, 0)) = \pi_2(1, 0) = 0, \quad g(0) = \pi_2(-(0, 0)) = \pi_2(1, 1) = 1,$$

and similarly for f. Hence $\mathbf{B} = (L; \vee, \wedge, f, g, 0, 1) \in \mathsf{dO}$. Observe that

$$\pi_1(-(a, 0)) = \pi_1(-((a, 0) \vee_t (0, 0))) = \pi_1(-(a, 0) \vee_t (1, 1)) = 1;$$
$$\pi_2(-(0, b)) = \pi_2(-((0, b) \wedge_k (0, 1))) = \pi_2(-(0, b) \vee_k (0, 1)) = 1.$$

Hence

$$-(a, b) = -((a, 0) \vee_k (0, b)) = -(a, 0) \wedge_k -(0, b)$$
$$= (\pi_1(-(a, 0)), \pi_2(-(a, 0))) \wedge_k (\pi_1(-(0, b)), \pi_2(-(0, b)))$$

$$= (1, \pi_2(-(a,0))) \wedge_k (\pi_1(-(0,b)), 1)$$
$$= (1, g(a)) \wedge_k (f(b), 1) = (f(b), g(a)) = [f_2^2, g_1^2](a,b).$$

Therefore $\mathbf{A} \cong \mathsf{P}_{\Gamma_{\mathcal{DB}_-}}(\mathbf{B})$. ∎

This theorem gives insight into the effect of reinstating the assumptions customarily imposed on conflation and which we removed in passing from $\mathcal{DBC}$ to $\mathcal{DB}_-$. From the product representation for $\mathcal{DB}_-$, it follows that $-$ is involutive if and only if f and g are. The resulting subvariety of $\mathcal{DB}_-$ is a duplicate of double De Morgan algebras (that is, algebras in $\mathsf{d\mathcal{O}}$ such that both unary operations are involutions). Similarly, $-$ commutes with $\neg$ if and only if $f = g$. This time we obtain a subvariety of $\mathcal{DB}_-$ which duplicates $\mathcal{O}$.

We now want to identify an (infinite) algebra which generates our base variety $\mathsf{d\mathcal{O}}$ as a prevariety. We take our cue from the variety $\mathcal{O}$ of Ockham algebras: $\mathcal{O}$ is generated as a prevariety by an algebra $\mathbf{M}$ whose universe is $\{0,1\}^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0,1,2\ldots\}$; lattice operations and constants are obtained pointwise from the two-element bounded lattice and, identifying the elements as infinite binary strings, negation is given by a left shift followed by pointwise Boolean complementation on $\{0,1\}$. See for example [13, Sect. 4] for details. We may view the exponent $\mathbb{N}_0$ as the free monoid on one generator e, with 0 as identity and n acting as the n-fold composite of e.

For $\mathsf{d\mathcal{O}}$, analogously, we first consider the free monoid $E = \{e_1, e_2\}^*$ on two generators e_1 and e_2 and identify it with the set of all finite words in the language with e_1 and e_2 as function symbols, with the empty word corresponding to the identity element 1; the monoid operation $\cdot$ is given by concatenation. For $s \in E$, we denote the length of s by $|s|$.

For us, $\mathsf{d\mathcal{O}}$ will serve as a base variety. Accordingly we align our notation with that in Theorem 3.1. We now consider the algebra $\mathbf{N}$ with universe $\{0,1\}^E$ with lattice operations and constants given pointwise. The lattice $\{0,1\}^E$ is in fact a Boolean lattice, whose complementation operation we denote by c. The dual endomorphisms f and g are given as follows. For $a \in \{0,1\}^E$ we have $f(a)(s) = c(a(s \cdot e_1))$ and $g(a) = c(a(s \cdot e_2))$ for every $s \in E$. This gives us an algebra $\mathbf{N} := (\{0,1\}^E; \vee, \wedge, f, g, 0, 1) \in \mathsf{d\mathcal{O}}$.

For future use we show how to assign to each word $s \in E$ a unary term t_s in the language of $\mathsf{d\mathcal{O}}$, as follows. If $s = 1$ (the empty word) then t_s is the identity map; if $s = e_1 \cdot s'$ then $t_s = f \circ t_{s'}$; and if $s = e_2 \cdot s'$ then $t_s = g \circ t_{s'}$. Structural induction shows that the term function $t_s^{\mathbf{N}}$ is given by

$$(t_s^{\mathbf{N}}(a))(e) = \begin{cases} a(s \cdot e) & \text{if } |s| \text{ is even,} \\ 1 - a(s \cdot e) & \text{if } |s| \text{ is odd,} \end{cases}$$

for every $a \in N$ and $s \in E$.

PROPOSITION 5.2. *Let* $\mathbf{N}$ *be defined as above. Then* $\mathsf{d\mathcal{O}} = \mathbb{ISP}(\mathbf{N})$.

PROOF. It will suffice to show that given any $\mathbf{A} \in \mathsf{d\mathcal{O}}$ and any $a \neq b$ in $\mathbf{A}$, there exists a $\mathsf{d\mathcal{O}}$-morphism h from $\mathbf{A}$ into $\mathbf{N}$ such that $h(a) \neq h(b)$; see [10, Theorem 1.3.1]. By the Prime Ideal Theorem there exists a $\mathcal{D}$-morphism x from (the $\mathcal{D}$-reduct of) $\mathbf{A}$ into $\mathbf{2}$ with $x(a) \neq x(b)$. Define $\varphi \colon \mathbf{A} \to \mathbf{N}$ by

$$\varphi(c)(s) = \begin{cases} x(t_s(c)) & \text{if } |s| \text{ is even,} \\ 1 - x(t_s(c)) & \text{if } |s| \text{ is odd,} \end{cases}$$

for $c \in \mathbf{A}$ and $s \in E$. It is routine to check that φ is a $\mathcal{D}$-morphism which preserves f and g. Finally, $\varphi(c)(1) = x(c)$, whence $\varphi(a) \neq \varphi(b)$. ∎

We now seek a natural duality for $\mathsf{d\mathcal{O}}$ which parallels that which is already known for the category $\mathcal{O}$ of Ockham algebras. Our treatment follows the same lines as that given for $\mathcal{O}$ in [13, Sect. 4], whereby a powerful version of the piggyback method is deployed. (The duality for $\mathcal{O}$ was originally developed by Goldberg [21] and re-derived as an early example of a piggyback duality by Davey and Werner [12].) A general description of the piggybacking method and the ideas underlying it can be found in [13, Sect. 3]. We wish to apply to $\mathsf{d\mathcal{O}}$ a special case of [13, Theorem 4.4]. We first make some comments and establish notation. We piggyback over Priestley duality between $\mathcal{D} = \mathbb{ISP}(\mathbf{2})$ and $\mathcal{P} = \mathbb{IS_c P^+}(\underset{\sim}{\mathbf{2}})$ (where $\mathbf{2}$ and $\underset{\sim}{\mathbf{2}}$ are the two-element objects in $\mathcal{D}$ and $\mathcal{P}$ with universe $\{0,1\}$, defined in the usual way). We denote the hom-functors setting up the dual equivalence between $\mathcal{D}$ and $\mathcal{P}$ by H and K. The aim is to find an element $\omega \in \mathcal{D}(\mathbf{N}^\flat, \mathbf{2})$ which, together with endomorphisms of $\mathbf{N}$, captures enough information to build an alter ego $\underset{\sim}{\mathbf{N}}$ of $\mathbf{N}$ which yields a full duality, in fact, a strong duality.

We now work towards showing that we can apply [13, Theorem 4.4] to $\mathsf{d\mathcal{O}} = \mathbb{ISP}(\mathbf{N})$, where $\mathbf{N}$ is as defined above. We shall take $\omega \colon \mathbf{N} \to \mathbf{2}$ to be the projection map given by $\omega(a) = a(1)$. We want to set up an alter ego $\underset{\sim}{\mathbf{N}} = (\{0,1\}^E; G, R, \mathcal{T})$ so that in particular $\underset{\sim}{\mathbf{N}}$ has a Priestley space reduct $\underset{\sim}{\mathbf{N}}^\flat$ such that $\omega \in \mathcal{P}(\underset{\sim}{\mathbf{N}}^\flat, \underset{\sim}{\mathbf{2}})$. Moreover we need the structure $\underset{\sim}{\mathbf{N}}$ to be chosen in such a way that the conditions (1)–(3) in [13, Theorem 4.4] are satisfied. We define $\mathcal{T}$ to be the product topology on $N = \{0,1\}^E$ derived from the discrete topology on $\{0,1\}$; this is compact and Hausdorff and makes $\mathbf{N}$ into a topological algebra. We now need to specify G and R. We would expect R

to contain an order relation $\preccurlyeq$ such that $(\{0,1\}^E; \preccurlyeq, \mathcal{T}) \in \boldsymbol{\mathcal{P}}$. For Ockham algebras—where one uses the free monoid on one generator as the exponent rather than E—the corresponding order relation is the *alternating order* in which alternate coordinates are order-flipped; see [10, Sect. 7.5] (and recall the comment about De Morgan algebras, a subvariety of $\boldsymbol{\mathcal{O}}$, in Sect. 4). The key point is that a composition of an even (respectively odd) number of order-reversing self-maps on an ordered set is order-preserving (respectively order-reversing). Hence the definition of $\preccurlyeq$ in Lemma 5.3 is entirely natural.

LEMMA 5.3. *Let* **N** *be as above. Then* $\preccurlyeq$*, given by*

$$a \preccurlyeq b \Longleftrightarrow \forall z \in E \begin{cases} a(s) \leqslant b(s) & \text{if } |s| \text{ is even,} \\ a(s) \geqslant b(s) & \text{if } |s| \text{ is odd,} \end{cases}$$

is an order relation making $(\{0,1\}^E; \preccurlyeq, \mathcal{T})$ *a Priestley space. Moreover* $\preccurlyeq$ *is the universe of a subalgebra of* $\mathbf{N}^2$ *and this subalgebra is the unique maximal subalgebra of* $(\omega, \omega)^{-1}(\leqslant) = \{ (a,b) \in N^2 \mid \omega(a) \leqslant \omega(b) \}$.

PROOF. Each of $\underset{\sim}{\mathbf{2}}$ and the structure $\underset{\sim}{\mathbf{2}}^{\partial}$ (that is, $\underset{\sim}{\mathbf{2}}$ with the order reversed) is a Priestley space. It follows that the topological structure $(\{0,1\}^E; \preccurlyeq, \mathcal{T})$ is a product of Priestley spaces and so itself a Priestley space.

Take a, b, c, d in N such that $a \preccurlyeq b$ and $c \preccurlyeq d$ and let $s \in E$. Then

$$(a \wedge c)(s) = a(s) \wedge c(s) \leqslant b(s) \wedge d(s) = (b \wedge d)(s) \qquad \text{if } |s| \text{ is even,}$$
$$(a \wedge c)(s) = a(s) \wedge c(s) \geqslant b(s) \wedge d(s) = (b \wedge d)(s) \qquad \text{if } |s| \text{ is odd.}$$

Hence $a \wedge c \preccurlyeq b \wedge d$. Similarly $a \vee c \preccurlyeq b \vee d$. Also $0 \preccurlyeq 0$ and $1 \preccurlyeq 1$. If $|s|$ is even, $f(a)(s) = (c \circ a \circ e_1)(s) = 1 - (a(e_1 \cdot s)) \leqslant 1 - (b(e_1 \cdot s)) = f(b)(s)$, since $a \preccurlyeq b$ and $|e_1 \cdot s|$ is odd. Similarly, if $|s|$ is odd then $f(a)(s) \geqslant f(b)(s)$. Therefore $f(a) \preccurlyeq f(b)$. Likewise $g(a) \preccurlyeq g(b)$. Thus $\preccurlyeq$ is indeed the universe of a subalgebra of $\mathbf{N}^2$.

Now let r be the universe of a subalgebra of $\mathbf{N}^2$ maximal with respect to inclusion in $(\omega, \omega)^{-1}(\leqslant)$. Then, with t_s as defined earlier for $s \in E$, we have

$$(a,b) \in r \implies (\forall s \in E)\big((t_s(a), t_s(b)) \in r\big) \implies (\forall s \in E)\big(t_s(a) \leqslant t_s(b)\big)$$
$$\implies (\forall s \in E)(\forall e \in E)\big(t_s(a)(e) = 1 \implies t_s(b)(e) = 1\big).$$

But

$$(t_s(a))(e) = \begin{cases} a(s \cdot e) & \text{if } |s| \text{ is even,} \\ 1 - a(s \cdot e) & \text{if } |s| \text{ is odd.} \end{cases}$$

We deduce that r is a subset of $\preccurlyeq$. In addition $a \preccurlyeq b$ implies $\omega(a) \leqslant \omega(b)$: consider $s = 1$. Maximality of r implies that r equals $\preccurlyeq$. Consequently $\preccurlyeq$ is the unique maximal subalgebra contained in $(\omega, \omega)^{-1}(\leqslant)$. ∎

We now introduce the operations we shall include in our alter ego $\underset{\sim}{\mathbf{N}}$. Let the map $\gamma_i \colon E \to E$ be given by $\gamma_i(s) = s \cdot e_i$. Then we can define an endomorphism u_i of $\mathbf{N}$ by $u_i(a) = a \circ \gamma_i$, for $i = 1, 2$. These maps are continuous with respect to the topology $\mathcal{T}$ we have put on N. We define

$$\underset{\sim}{\mathbf{N}} := (\{0,1\}^E; u_1, u_2, \preccurlyeq, \mathcal{T}).$$

Then $\underset{\sim}{\mathbf{N}}$ is compatible with $\mathbf{N}$. We let $\mathcal{Y} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$ be the topological prevariety generated by $\underset{\sim}{\mathbf{N}}$ and by $^\flat$ the forgetful functor from $\mathcal{Y}$ into $\mathcal{P}$ which suppresses the operations u_1 and u_2. We note that now ω, as defined earlier, may be seen to belong to $\mathcal{D}(\mathbf{N}^\flat, \mathbf{2}) \cap \mathcal{P}(\underset{\sim}{\mathbf{N}}^\flat, \underset{\sim}{\mathbf{2}})$. The following two lemmas concern the interaction of $\mathbf{N}$, $\underset{\sim}{\mathbf{N}}$ and ω as regards separation properties.

LEMMA 5.4. *Assume that $\mathbf{N}$, $\underset{\sim}{\mathbf{N}}$ and ω are defined as above. Then, given $a \neq b$ in N, there exists a unary term u in the language of $(N; u_1, u_2)$ such that $\omega(u(a)) \neq \omega(u(b))$.*

PROOF. Let $a \neq b \in \mathbf{N}$. There exists $s \in E$ with $s \neq 1$ such that $a(s) \neq b(s)$. Write s as a concatenation $e_{i_1} \cdot \ldots \cdot e_{i_n}$, where $i_1, \ldots, i_n \in \{1, 2\}$. For each $j = 1, \ldots, n$, there is an associated unary term u_j such that, for all $w \in E$,

$$(u_{i_j}(a))(w) = (a \circ \gamma_{i_j})(w) = a(w \cdot e_{i_j}).$$

Write $u_{i_n} \circ \ldots \circ u_{i_1}$ as u_s. Then $u_s(c)(1) = c(s)$ for all $c \in \mathbf{N}$ and hence

$$(\omega \circ u_s)(a) = u_s(a)(1) = a(s) \neq b(s) = u_s(b)(1) = (\omega \circ u_s)(b).$$

∎

LEMMA 5.5. *If $a \not\preccurlyeq b$ in $\underset{\sim}{\mathbf{N}}^\flat$, then there exists a unary term function t of $\mathbf{N}$ such that $\omega(t(a)) = 1$ and $\omega(t(b)) = 0$.*

PROOF. We have

$$a \not\preccurlyeq b \iff \exists s \in E \begin{cases} a(s) = 1 \ \& \ b(s) = 0 & \text{if } |s| \text{ is even,} \\ a(s) = 0 \ \& \ b(s) = 1 & \text{if } |s| \text{ is odd.} \end{cases}$$

When $|s|$ is even, $\omega(t_s(a)) = t_s(a)(1) = a(s) = 1$ and $\omega(t_s(b)) = t_s(b)(1) = b(s) = 0$. Similarly, if $|s|$ is odd, $\omega(t_s(a)) = t_s(a)(1) = c \circ a(s) = 1 - a(s) = 1$ and $\omega(t_s(b)) = t_s(b)(1) = c \circ b(s) = 1 - b(s) = 0$. ∎

THEOREM 5.6. (Strong Duality Theorem for Double Ockham Algebras) *Let $\mathbf{N} = (\{0,1\}^E; \vee, \wedge, f, g, 0, 1)$ and $\underset{\sim}{\mathbf{N}} = (\{0,1\}^E; u_1, u_2, \preccurlyeq, \mathcal{T})$ be as defined*

above. Let $\omega \in \mathbf{\mathcal{D}}(\mathbf{N}^\flat, \mathbf{2}) \cap \mathbf{\mathcal{P}}(\underset{\sim}{\mathbf{N}}^\flat, \underset{\sim}{\mathbf{2}})$ *be given by evaluation at* 1, *the identity of the monoid* E. *Let* $\mathsf{D}\colon \mathsf{d\mathcal{O}} \to \mathbf{\mathcal{Y}}$ *and* $\mathsf{E}\colon \mathbf{\mathcal{Y}} \to \mathsf{d\mathcal{O}}$ *be the hom-functors:* $\mathsf{D} \coloneqq \mathsf{d\mathcal{O}}(-, \mathbf{N})$ *and* $\mathsf{E} \coloneqq \mathbf{\mathcal{Y}}(-, \underset{\sim}{\mathbf{N}})$. *Then* $\underset{\sim}{\mathbf{N}}$ *strongly dualises* $\mathbf{N}$, *that is,* D *and* E *establish a strong duality between* $\mathsf{d\mathcal{O}}$ *and* $\mathbf{\mathcal{Y}}$. *Moreover*

$$\mathsf{D}(\mathbf{A})^\flat \cong \mathsf{H}(\mathbf{A}^\flat) \ \textit{in} \ \mathbf{\mathcal{P}} \quad \textit{and} \quad \mathsf{E}(\mathbf{Y})^\flat \cong \mathsf{K}(\mathbf{Y}^\flat) \ \textit{in} \ \mathbf{\mathcal{D}},$$

for $\mathbf{A} \in \mathsf{d\mathcal{O}}$ *and* $\mathbf{Y} \in \mathbf{\mathcal{Y}}$, *where the isomorphisms are set up by* $\Phi_\omega^{\mathbf{A}}\colon x \mapsto \omega \circ x$, *for* $x \in \mathsf{D}(\mathbf{A})$, *and* $\Psi_\omega^{\mathbf{Y}}\colon \alpha \mapsto \omega \circ \alpha$, *for* $\alpha \in \mathsf{E}(\mathbf{Y})$.

PROOF. We simply need to confirm that the conditions of [13, Theorem 4.4] are satisfied. We have everything set up to ensure that all the functors work as the theorem requires. In addition Lemmas 5.3–5.5 tell us that Conditions (1)–(3) in the theorem are satisfied. ∎

Some remarks are in order here. We stress that it is critical that we could find a map ω which acts as a morphism both on the algebra side and on the dual side, and has the separation properties set out in Lemmas 5.4 and 5.5. We also observe that for our application of [13, Theorem 4.4], its Condition (3) is met in a simpler way than the theorem allows for: the special form of the f, g (*viz.* dual endomorphisms with respect to the bounded lattice operations) that forces $(\omega, \omega)^{-1}(\leqslant)$ to contain just one maximal subalgebra.

We should comment too on how our natural duality for $\mathsf{d\mathcal{O}}$ relates to a Priestley-style duality for $\mathsf{d\mathcal{O}}$. The latter can be set up in just the same way as that for $\mathcal{O}$ originating in [28]. This duality is an enrichment of that between $\mathbf{\mathcal{D}}$ and $\mathbf{\mathcal{P}}$, whereby f and g are captured on the dual side via a pair of order-reversing continuous maps p and q, and morphisms are required to preserve these maps. Theorem 5.6 tells us that, for any $\mathbf{A} \in \mathsf{d\mathcal{O}}$, there is an isomorphism between the Priestley space reduct $\mathsf{D}(\mathbf{A})^\flat$ of the natural dual of $\mathbf{A} \in \mathsf{d\mathcal{O}}$ and the Priestley dual $\mathsf{H}(\mathbf{A}^\flat)$ of the $\mathbf{\mathcal{D}}$-reduct of $\mathbf{A}$. Both these Priestley spaces carry additional structure: u_1 and u_2 in the former case and p and q in the latter. When the reducts of the natural and Priestley-style dual spaces of the algebras are identified these pairs of maps coincide. Thus the two dualities for $\mathsf{d\mathcal{O}}$ are essentially the same and one may toggle between them at will. We have a new example here of a 'best of both worlds' scenario, in which we have both the advantages of a natural duality and the benefits, pictorially, of a duality based on Priestley spaces. See [7, Sect. 3], [8, Sect. 6] and [13, Sect. 4] for earlier recognition of occurrences of this phenomenon: other varieties for which it arises are De Morgan algebras and Ockham algebras. In general it is not hereditary: it fails to occur for Kleene algebras, for example.

Combining our results we arrive at our duality for the variety $\mathbf{\mathcal{DB}}_-$.

THEOREM 5.7. (Strong Duality Theorem for Bounded Distributive Bilattices with Generalised Conflation) *Let* $\underset{\sim}{\mathbf{N}} = (\{0,1\}^E; u_1, u_2, \preccurlyeq, \mathfrak{T})$ *be as in Theorem* 5.6. *Then* $\underset{\sim}{\mathbf{N}} \times \underset{\sim}{\mathbf{N}}$ *yields a strong duality on* $\mathcal{DB}_-$. *Moreover the dual category for this duality is* $\mathcal{Y} := \mathbb{IS}_c \mathbb{P}^+(\underset{\sim}{\mathbf{N}})$ *which may, in turn, be identified with the category* $\mathcal{P}_{\mathbf{dO}}$ *of double Ockham spaces.*

To illustrate the rewards derived from a natural duality for $\mathsf{F}_{\mathcal{DB}_-}$, we highlight the simple description of free objects that follows from Theorem 5.7: for a non-empty set S, the free algebra $\mathsf{F}_{\mathcal{DB}_-}(S)$ on S has $(\underset{\sim}{\mathbf{N}}^2)^S$ as its natural dual space. Hence $\mathsf{F}_{\mathcal{DB}_-}(S)$ can be identified with the family of continuous structure-preserving maps from $(\underset{\sim}{\mathbf{N}}^2)^S$ into $\underset{\sim}{\mathbf{N}}^2$, with the operations defined pointwise. (Recall the remark on free algebras in Sect. 3.)

6.   Dualities for Pre-bilattice-based Varieties

In this final section we consider dualities for pre-bilattice-based varieties. Here we call on the adaptation of the product representation theorem given in [9, Theorem 9.1]. We first recall how that theorem differs from Theorem 2.1. We start from a base class $\mathbb{V}(\mathcal{N})$, where $\mathcal{N}$ is a class of algebras over a common language Σ. Let Γ and $\mathsf{P}_\Gamma(\mathcal{N})$ be as in Sect. 2. Negation in a product bilattice links the two factors, and condition (P) from the definition of duplication by Γ reflects this. In the absence of negation, (P) is dropped and the following condition is substituted:

(D) for $(t_1, t_2) \in \Gamma$ with $n_{(t_1, t_2)} = n$, there exist n-ary Σ-terms r_1 and r_2 such that $t_1(x_1, \ldots, x_{2n}) = r_1(x_1, x_3, \ldots, x_{2n-1})$ and $t_2(x_1, \ldots, x_{2n}) = r_2(x_2, x_4, \ldots, x_{2n})$.

A product algebra associated with Γ now takes the form

$$\mathbf{P} \odot_\Gamma \mathbf{Q} = (P \times Q; \{[t_1, t_2]^{\mathbf{P} \odot_\Gamma \mathbf{Q}} \mid (t_1, t_2) \in \Gamma\}),$$

where $\mathbf{P}, \mathbf{Q}$ belong to the base variety $\mathcal{B} = \mathbb{V}(\mathcal{N})$. This construction is used to define a functor $\odot_\Gamma \colon \mathcal{B} \times \mathcal{B} \to \mathcal{A}$ as follows:

on objects: $(\mathbf{P}, \mathbf{Q}) \mapsto \mathbf{P} \odot_\Gamma \mathbf{Q},$

on morphisms: $\odot_\Gamma (h_1, h_2)(a, b) = (h_1(a), h_2(b)).$

THEOREM 6.1. [9, Theorem 9.3] *Let* $\mathcal{N}$ *be a class of* Σ-*algebras and let* Γ *a set of pairs of* Σ-*terms satisfying* (L), (M) *and* (D). *Let* $\mathcal{B} = \mathbb{V}(\mathcal{N})$. *Then the functor* $\odot_\Gamma \colon \mathcal{B} \times \mathcal{B} \to \mathcal{A}$, *sets up a categorical equivalence between* $\mathcal{B} \times \mathcal{B}$ *and* $\mathcal{A} = \mathbb{V}(\{\mathbf{P} \odot_\Gamma \mathbf{Q} \mid \mathbf{P}, \mathbf{Q} \in \mathbb{V}(\mathcal{N})\})$.

Hitherto in this paper we have worked with dualities for prevarieties of the form $\mathbb{ISP}(\mathbf{M})$, thereby encompassing dualities for many classes of interest in the context of bilattices. However when we drop negation and so move from bilattices to pre-bilattices the situation changes and we encounter classes of the form $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$, where $\boldsymbol{\mathcal{M}}$ is a finite set of algebras over a common language. For example, for distributive pre-bilattices $\boldsymbol{\mathcal{M}}$ consists of a pair of two-element algebras, one with truth and knowledge orders equal, the other with these as order duals. Fortunately a form of natural duality theory exists which is applicable to classes of the form $\mathbb{ISP}(\boldsymbol{\mathcal{M}})$; this makes use of multisorted structures on the dual side. In barest outline, the construction goes as follows. For a class $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\boldsymbol{\mathcal{M}})$, where $\boldsymbol{\mathcal{M}}$ is a finite set of finite algebras, we seek an alter ego $\underset{\sim}{\boldsymbol{\mathcal{M}}}$ is a structure whose universe is the disjointified union of the sets M, for $\mathbf{M} \in \boldsymbol{\mathcal{M}}$ (the *sorts*), which carries sets R and G of relations and operations, and which is equipped with the discrete topology. Here an n-ary relation in R is the universe of a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_n$, where $\mathbf{M}_1, \ldots \mathbf{M}_n$ are drawn from $\boldsymbol{\mathcal{M}}$, and similarly for operations in G. A multisorted topological prevariety $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\boldsymbol{\mathcal{M}}})$ is then constructed in the expected way and morphisms between members of $\boldsymbol{\mathcal{X}}$ are those continuous maps which preserve the sorts and the structure. The dual space of $\mathbf{A} \in \mathbb{ISP}(\boldsymbol{\mathcal{M}})$ is the disjoint union of the hom-sets $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$, for $\mathbf{M} \in \boldsymbol{\mathcal{M}}$. A self-contained summary of the rudiments of multisorted duality theory can be found in [8, Sect. 9] or in [10, Chap. 7].

In [8, Sect. 10] we set up multisorted dualities for the varieties of pre-bilattices, with and without bounds. In this section we shall consider multisorted dualities for pre-bilattice-based varieties arising by duplication. For simplicity we shall first assume that the base variety $\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{N})$ has a single-sorted duality with alter ego $\underset{\sim}{\mathbf{N}} = (N; G, R, \mathcal{T})$. We need ti to determine a set of generators for $\boldsymbol{\mathcal{A}}$ as a prevariety. We denote the trivial algebra by $\mathbf{T}$. For $\mathbf{C} \in \boldsymbol{\mathcal{B}}$ let $f_C^* \colon C \to T$ be the unique homomorphism from $\mathbf{C}$ into $\mathbf{T}$.

LEMMA 6.2. *If* $\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{N}) = \mathbb{V}(\mathbf{N})$ *for some algebra* $\mathbf{N}$, *then*

$$\boldsymbol{\mathcal{A}} = \mathbb{V}(\{\mathbf{P} \odot_\Gamma \mathbf{Q} \mid \mathbf{P}, \mathbf{Q} \in \mathbb{ISP}(\mathbf{N})\}) = \mathbb{ISP}(\mathbf{N} \odot_\Gamma \mathbf{T}, \mathbf{T} \odot_\Gamma \mathbf{N}).$$

PROOF. Let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ and $a \neq b \in A$. By Theorem 6.1, we may assume that there exist $\mathbf{B}, \mathbf{C} \in \boldsymbol{\mathcal{B}}$ such that $\mathbf{A} = \mathbf{B} \odot_\Gamma \mathbf{C}$. Let $a_1, b_1 \in B$ and $a_2, b_2 \in C$ such that $a = (a_1, a_2)$ and $b = (b_1, b_2)$. By symmetry we may assume that $a_1 \neq b_1$. Then there exists a homomorphism $h \colon \mathbf{B} \to \mathbf{N}$ such that $h(a_1) \neq h(b_1)$. Now $h \odot_\Gamma f_C^* \colon \mathbf{B} \odot_\Gamma \mathbf{C} \to \mathbf{N} \odot_\Gamma \mathbf{T}$ is such that

$$(h \odot_\Gamma f_C^*)(a) = (h(a_1), f_C^*(a_2)) \neq (h(b_1), f_C^*(b_2)) = (h \odot_\Gamma f_C^*)(b).$$

∎

Let $\boldsymbol{\mathcal{M}} = \{\mathbf{N} \odot_\Gamma \mathbf{T}, \mathbf{T} \odot_\Gamma \mathbf{N}\}$. We now 'double up' $\underset{\sim}{\mathcal{N}}$ in the obvious way. Let $\underset{\sim}{\mathcal{N}} \uplus \underset{\sim}{\mathcal{N}} = (N_1 \dot{\cup} N_2; G_1, G_2, R_1, R_2, \mathcal{T})$, based on disjointified universes N_1 and N_2, such that $(N_i; G_i, R_i, \mathcal{T}|_{N_i})$ is isomorphic to $\underset{\sim}{\mathcal{N}}$ for $i = 1, 2$. Identify N_1 with $N \times T$ and N_2 with $T \times N$ and define $\underset{\sim}{\boldsymbol{\mathcal{M}}} = \underset{\sim}{\mathcal{N}} \uplus \underset{\sim}{\mathcal{N}}$.

We now present our transfer theorem for natural dualities associated with Theorem 6.1 (the single-sorted case). Its proof is largely a diagram-chase with functors. Below, $\mathsf{Id}_{\mathcal{C}}$ denotes the identity functor on a category $\mathcal{C}$ and $\cong$ is used to denote natural isomorphism.

THEOREM 6.3. *Let* $\mathbf{N}$ *be a* Σ-*algebra and assume that* Γ *satisfies* (L), (M) *and* (D) *relative to* $\mathbf{N}$. *Assume that* $\underset{\sim}{\mathbf{N}} = (N; G, R, \mathcal{T})$ *yields a duality on* $\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{N}) = \mathbb{V}(\mathbf{N})$ *with dual category* $\boldsymbol{\mathcal{Y}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$. *Let* $\boldsymbol{\mathcal{M}}$ *and* $\underset{\sim}{\boldsymbol{\mathcal{M}}}$ *be defined as above. Then* $\underset{\sim}{\boldsymbol{\mathcal{M}}}$ *yields a multisorted duality for* $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\boldsymbol{\mathcal{M}}) = \mathbb{V}(\mathbf{P} \odot_\Gamma \mathbf{Q} \mid \mathbf{P}, \mathbf{Q} \in \mathbb{V}(\underset{\sim}{\mathcal{N}}))$ *for which the dual category is* $\boldsymbol{\mathcal{X}} \cong \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Y}}$. *If the duality for* $\boldsymbol{\mathcal{B}}$ *is full, respectively strong, then the same is true of that for* $\boldsymbol{\mathcal{A}}$.

PROOF. Let $(\mathbf{X}_1, \mathbf{X}_2) \in \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Y}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}}) \times \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{N}})$. We identify this structure with $\mathbf{X}_1 \uplus \mathbf{X}_2 = (X_1 \dot{\cup} X_2; G_1, G_2, R_1, R_2, \mathcal{T})$, where as before $\dot{\cup}$ denotes disjoint union and the topology $\mathcal{T}$ is the union of $\mathcal{T}_1$ and $\mathcal{T}_2$. Morphisms in $\boldsymbol{\mathcal{X}}$ are maps $f \colon X_1 \dot{\cup} X_2 \to Y_1 \dot{\cup} Y_2$ that respect the structure and are such that $f(x) \in Y_i$ when $x \in X_i$ and $i \in \{1, 2\}$. Hence the assignment:

on objects: $(\mathbf{X}_1, \mathbf{X}_2) \mapsto \mathbf{X}_1 \uplus \mathbf{X}_2$,

on morphisms: $(f, g) \mapsto f \dot{\cup} g$

sets up a categorical equivalence, $\uplus$. Let $\mathsf{F} \colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} \times \boldsymbol{\mathcal{Y}}$ denote its inverse.

Identify $\mathbf{N} \odot_\Gamma \mathbf{T}$ and $\mathbf{T} \odot_\Gamma \mathbf{N}$ with N_1 and N_2 respectively. One sees that $\underset{\sim}{\boldsymbol{\mathcal{M}}} \coloneqq \underset{\sim}{\mathcal{N}} \uplus \underset{\sim}{\mathcal{N}} = (N_1 \dot{\cup} N_2; G_1, G_2, R_1, R_2, \mathcal{T})$ is a legitimate alter ego for $\boldsymbol{\mathcal{M}}$. Let $\mathsf{D}_{\boldsymbol{\mathcal{B}}} \colon \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{Y}}$ and $\mathsf{E}_{\boldsymbol{\mathcal{B}}} \colon \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{B}}$, and $\mathsf{D}_{\boldsymbol{\mathcal{A}}} \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}}$ and $\mathsf{E}_{\boldsymbol{\mathcal{A}}} \colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}}$ removed, be the hom-functors determined by $\underset{\sim}{\mathcal{N}}$ and $\underset{\sim}{\boldsymbol{\mathcal{M}}}$ respectively. By Theorem 6.1, there exists a functor $\mathsf{C} \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{B}} \times \boldsymbol{\mathcal{B}}$ that together with $\odot_\Gamma \colon \boldsymbol{\mathcal{B}} \times \boldsymbol{\mathcal{B}} \to \boldsymbol{\mathcal{A}}$ determines a categorical equivalence. Take $\mathbf{A}, \mathbf{B} \in \boldsymbol{\mathcal{B}}$ and let

$$\mathsf{D}_{\boldsymbol{\mathcal{A}}}(\mathbf{A} \odot_\Gamma \mathbf{B}) = (X_1 \dot{\cup} X_2; G_1, G_2, R_1, R_2, \mathcal{T}).$$

Again by Theorem 6.1,

$$X_1 = \boldsymbol{\mathcal{A}}(\mathbf{A} \odot_\Gamma \mathbf{B}, \mathbf{N} \odot_\Gamma \mathbf{T}) = \{(h_A, f_B^*) \mid h_A \in \boldsymbol{\mathcal{B}}(\mathbf{A}, \mathbf{N})\} = \boldsymbol{\mathcal{B}}(\mathbf{A}, \mathbf{N}) \times \{f_B^*\},$$

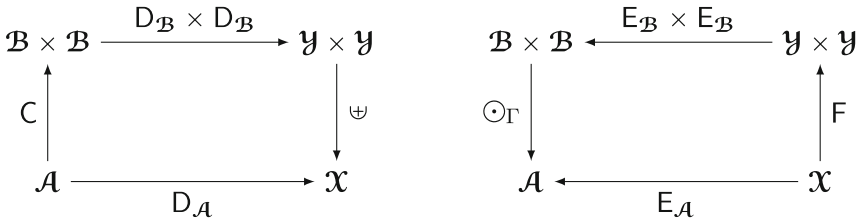and likewise $X_2 = \{f_A^*\} \times \boldsymbol{\mathcal{B}}(\mathbf{B}, \mathbf{N})$.

Figure 1. Natural duality by duplication

For an n-ary relation $r \in R$, let $r_i^{\mathbf{A} \odot_\Gamma \mathbf{B}}$ be the corresponding relation in $R_i^{\mathbf{A} \odot_\Gamma \mathbf{B}} \subseteq X_i^n$ ($i = \{1, 2\}$). So $(h_1, \ldots, h_n) \in r_1^{\mathbf{A} \odot_\Gamma \mathbf{B}}$ if and only if $h_i = (g_i, f_B^*) \in \mathcal{B}(\mathbf{A}, \mathbf{N}) \times \{f_B^*\}$ for $i \in \{1, \ldots, n\}$ and $(g_1, \ldots, g_n) \in r^{\mathbf{A}}$. Similarly, a tuple $(h_1, \ldots, h_n)$ belongs to $r_2^{\mathbf{A} \odot_\Gamma \mathbf{B}}$ if and only if $h_i = (f_A^*, g_i) \in \{f_A^*\} \times \mathcal{B}(\mathbf{B}, \mathbf{N})$ for $i \in \{1, \ldots, n\}$ and $(g_1, \ldots, g_n) \in r^{\mathbf{B}}$. The same argument applied to G proves that $(X_1; G_1, R_1, \mathcal{T}{\upharpoonright}_{X_1})$ and $(X_2; G_2, R_2, \mathcal{T}{\upharpoonright}_{X_2})$ are isomorphic to $\mathsf{D}_{\mathcal{B}}(\mathbf{A})$ and $\mathsf{D}_{\mathcal{B}}(\mathbf{B})$, respectively. Thus $\mathsf{F}(\mathsf{D}_{\mathcal{A}}(\mathbf{A} \odot_\Gamma \mathbf{B}))$ is isomorphic to $(\mathsf{D}_{\mathcal{B}}(\mathbf{A}), \mathsf{D}_{\mathcal{B}}(\mathbf{B}))$ in $\mathcal{Y} \times \mathcal{Y}$. Moreover, it is easy to see that the assignment $\mathsf{F}(\mathsf{D}_{\mathcal{A}}(\mathbf{A} \odot_\Gamma \mathbf{B})) \mapsto (\mathsf{D}_{\mathcal{B}}(\mathbf{A}), \mathsf{D}_{\mathcal{B}}(\mathbf{B}))$ determines a natural isomorphism between $\mathsf{F} \circ \mathsf{D}_{\mathcal{A}} \circ \odot_\Gamma$ and $\mathsf{D}_{\mathcal{B}} \times \mathsf{D}_{\mathcal{B}} \colon \mathcal{B} \times \mathcal{B} \to \mathcal{X} \times \mathcal{X}$.

Similarly, for each $(\mathbf{X}, \mathbf{Y}) \in \mathcal{X} \times \mathcal{X}$,

$$\mathsf{E}_{\mathcal{A}}(\mathbf{X} \uplus \mathbf{Y}) = (\mathsf{E}_{\mathcal{B}}(\mathbf{X}) \odot_\Gamma \mathbf{T}) \times (\mathbf{T} \odot_\Gamma \mathsf{E}_{\mathcal{B}}(\mathbf{Y}))$$
$$\cong (\mathsf{E}_{\mathcal{B}}(\mathbf{X}) \times \mathbf{T}) \odot_\Gamma (\mathbf{T} \times \mathsf{E}_{\mathcal{B}}(\mathbf{Y})) \cong \mathsf{E}_{\mathcal{B}}(\mathbf{X}) \odot_\Gamma \mathsf{E}_{\mathcal{B}}(\mathbf{Y}).$$

Moreover, the assignment $\mathsf{E}_{\mathcal{A}}(\mathbf{X} \uplus \mathbf{Y}) \mapsto \mathsf{E}_{\mathcal{B}}(\mathbf{X}) \odot_\Gamma \mathsf{E}_{\mathcal{B}}(\mathbf{Y})$ is natural in $\mathbf{X}$ and $\mathbf{Y}$, that is, $\mathsf{E}_{\mathcal{A}} \circ \uplus \cong (\mathsf{E}_{\mathcal{B}} \times \mathsf{E}_{\mathcal{B}}) \circ \odot_\Gamma$.

So (up to natural isomorphism) the diagrams in Figure 1 commute. A symbol-chase now confirms that $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$ because $\underset{\sim}{\mathbf{N}}$ dualises $\mathbf{N}$:

$$\mathsf{E}_{\mathcal{A}} \circ \mathsf{D}_{\mathcal{A}} \cong \odot_\Gamma \circ (\mathsf{E}_{\mathcal{B}} \times \mathsf{E}_{\mathcal{B}}) \circ \mathsf{F} \circ \uplus \circ (\mathsf{D}_{\mathcal{B}} \times \mathsf{D}_{\mathcal{B}}) \circ \mathsf{C}$$
$$= \odot_\Gamma \circ (\mathsf{E}_{\mathcal{B}} \times \mathsf{E}_{\mathcal{B}}) \circ (\mathsf{D}_{\mathcal{B}} \times \mathsf{D}_{\mathcal{B}}) \circ \mathsf{C} \cong \odot_\Gamma \circ (\mathsf{Id}_{\mathcal{B}} \times \mathsf{Id}_{\mathcal{B}}) \circ \mathsf{C}$$
$$= \odot_\Gamma \circ \mathsf{C} \cong \mathsf{Id}_{\mathcal{A}}.$$

Assume that $\underset{\sim}{\mathbf{N}}$ yields a full duality. Then the diagram in Figure 2 commutes. We can easily prove that $\mathsf{D}_{\mathcal{A}} \circ \mathsf{E}_{\mathcal{A}} \cong \mathsf{Id}_{\mathcal{X}}$, that is, $\underset{\sim}{\mathcal{M}}$ yields a full duality. Moreover, if $\underset{\sim}{\mathbf{N}}$ is injective in $\mathcal{Y}$ then $(\underset{\sim}{\mathbf{N}}, \underset{\sim}{\mathbf{N}})$ is injective in $\mathcal{Y} \times \mathcal{Y}$, or equivalently $\underset{\sim}{\mathcal{M}} = \underset{\sim}{\mathbf{N}} \uplus \underset{\sim}{\mathbf{N}}$ is injective in $\mathcal{X}$. Hence $\underset{\sim}{\mathcal{M}}$ yields a strong duality if $\mathcal{N}$ does. ∎

Theorem 6.3 applies to the variety $\mathsf{p}\mathcal{DB}_u$ of (unbounded) distributive pre-bilattices. Its members are algebras $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k)$ for which

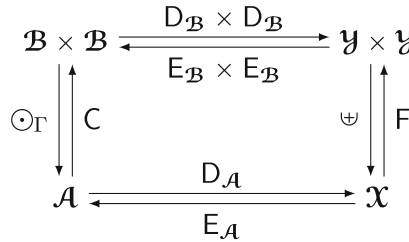Figure 2. Full duality by duplication

$(A; \vee_t, \wedge_t) \in \boldsymbol{\mathcal{D}}_u$ and $(A; \vee_k, \wedge_k) \in \boldsymbol{\mathcal{D}}_u$. The well-known product represen-
tation for $\mathsf{p}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{B}}_u$ comes from the observation that the set

$$\Gamma_{\mathsf{p}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{B}}_u} = \{(\vee_{13}^4, \wedge_{24}^4), (\wedge_{13}^4, \vee_{24}^4), (\vee_{13}^4, \vee_{24}^4), (\wedge_{13}^4, \wedge_{24}^4)\}$$

satisfies (L), (M) and (D) [9, Sect. 9]. Since $\boldsymbol{2}_u$ strongly dualises $\boldsymbol{\mathcal{D}}_u$, the
structure $\boldsymbol{2}_u \uplus \boldsymbol{2}_u$ determines a multisorted strong duality for $\mathsf{p}\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{B}}_u$. This
was established by different techniques in [8, Theorem 10.2].

Theorem 6.3 also yields dualities for distributive trilattices, well-studied
classes to which duality methods have not hitherto been applied. Our pur-
pose in mentioning these examples, in [9] and in this paper, is twofold.
Firstly, in the literature, trilattices are largely treated on their own whereas
we seek more overtly to integrate them within the wider family of lattice-
based algebras. Secondly, the way in which various trilattice varieties relate
to pre-bilattices illuminates both the duplication process and the structure
of trilattices. We illustrate by considering the variety $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{T}}_{-_t}$ of (unbounded)
distributive trilattice with t-involution. This variety is the class of algebras
$(A; \vee_t, \wedge_t, \vee_f, \wedge_f, \vee_i, \wedge_i, -_t)$ for which $(A; \vee_t, \wedge_t)$, $(A; \vee_f, \wedge_f)$ and $(A; \vee_i, \wedge_i)$
are distributive lattices and $-_t$ is an involution that preserves the f- and
i-lattice operations and reverses $\vee_t$ and $\wedge_t$. We showed in [9, Example 9.4]
how $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{T}}_{-_t}$ arises as a duplicate of $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{B}}_u$. In Sect. 4, we used Theorem 3.1 to
prove that $(\boldsymbol{2}_u)^2$ yields a strong duality on $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{B}}_u$. Now Theorem 6.3 implies
that $(\boldsymbol{2}_u)^2 \uplus (\boldsymbol{2}_u)^2$ gives a multisorted strong duality for $\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{T}}_{-_t}$. Here we
have an instance of a duality obtained by a 2-stage transfer.

We can easily adapt our results to cater for a base variety which admits
a multisorted duality rather than a single-sorted one. Predictably this leads
to multisortedness at the duplicate level. In the case of Theorem 3.1, one
obtains the required alter ego by squaring the base level alter ego, sort by
sort; as before, the base variety and its duplicate have the same dual cate-
gory. The extension of Theorem 6.3 employs two disjoint copies of each sort
of the base-level alter ego. The proofs of these results involve only minor

modifications of those for the single-sorted case. As an example, the multi-sorted version of Theorem 6.3 combined with the results in [9, Example 9.4] leads to a strong duality for unbounded distributive trilattices which has four sorts, obtained from the two-sorted duality for $\mathsf{p}\mathcal{DB}_u$.

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