Inconsistency of $\mathbb{N}$ with the set union operation

Enrico P. G. Cadeddu

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Abstract

Considering the axiom of infinity, then $\mathbb{N}$ and Peano axioms, together a list of $\mathbb{N}$ subsets, inclusion relation and union operation, a contradiction is obtained.

1 Introduction

The issue of infinity, in particular the actual infinity, leads us to write this article, as other previous ones [1] [2].

We consider the axiom of infinity [4] [5] [8] , then the existence of $\mathbb{N}$ and Peano axioms [3]. Sets are considered with the usual graphical-symbolic notation $\{0, 1, 2, ...n\}$ (see also [6] [7]). We starting with the sets-list $\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together}$. Each set is shown to be finite, then using inclusion relation and union operation, we obtain a contradiction about the presence of $\mathbb{N}$ in the list.

2 Iconsistency of $\mathbb{N}$

We consider an infinite list of sets defined by:

$$\{x|x \leq y\} \quad \forall y \in \mathbb{N} \quad \textit{with all y taken together} \tag{1}$$

They are subsets of $\mathbb{N}$. In agreement with the axiom of infinity, the set of all natural numbers $\mathbb{N}$ exists, together its subsets.

We highlight the greatest number of each set, y, and there is the list of these numbers, which are all numbers of ℕ.

{ 0 }	0
{ 0, 1 }	1
{ 0, 1, 2 }	2
{ 0, 1, 2, 3 }	3
{ 0, 1, 2, 3, 4 }	4
.	.
.	.
.	.
{ 0, 1, 2, 3, 4, ... ? }	?

Immediately we deduce that **there isn't a set equal to ℕ in the sets-list**, because each set is a finite set (but in the list there are all natural numbers). In fact each y is a natural number (a finite number) that does have a successor for Peano axioms; so each set doesn't have all numbers, unlike ℕ. So we have:

$$\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \{0,1,2,3\} \subset \{0,1,2,3,4\} \subset ... \subset \mathbb{N} \qquad (2)$$

Each set is a proper subset of other sets that follow. Relation (2) tells us that ℕ contains numbers not contained in the others sets of (1). Then, given $I_y = \{0, 1, 2, ...y\}$, the union of all I_y is a subset of ℕ, the union of all sets being the set including all numbers of all sets:

$$\bigcup_{y\in\mathbb{N}} I_y \subset \mathbb{N} \qquad (3)$$

ℕ is an excluded limit "value" (set). We also note that (by proof of induction): $I_0 \cup I_1 \cup I_2 \cup ... \cup I_y \cup I_{y+1} = I_{y+1} \;\; \forall y \in \mathbb{N}$ (obviously with $y + 1 \in \mathbb{N}$) and because in $\bigcup_{y\in\mathbb{N}} I_y$ there are only $y \in \mathbb{N}$ (there aren't terms like I_∞), the previous inductive union applies simultaneously on all terms in $\bigcup_{y\in\mathbb{N}} I_y$, justifying $\bigcup_{y\in\mathbb{N}} I_y \subset \mathbb{N}$.

But all numbers of the list (1) form the set $\mathbb{N}$ (all y taken together) and this is given by $\bigcup_{y\in\mathbb{N}} I_y$ (the collection of all numbers of sets), that is:

$$\bigcup_{y\in\mathbb{N}} I_y = \mathbb{N} \tag{4}$$

So, we have a contradiction with (3) and (4): $\bigcup_{y\in\mathbb{N}} I_y \subset \mathbb{N}$ and $\bigcup_{y\in\mathbb{N}} I_y = \mathbb{N}$.

We want to clarify the relation (3) further. The infinite union of a set A, finite or infinite, is A again: $A \cup A \cup A \cup A \cup ... = A$ (infinite "$\cup$" and A objects). But if we have a subset of A, $B \subset A$, then: $B \cup B \cup B \cup B \cup ... \subset A$. So, considering subsets of A, $B_y \subset A \;\; \forall y \in \mathbb{N}$, all included between them as in (2), then: $B_1 \cup B_2 \cup B_3 \cup B_4 \cup ... \subset A$, $\bigcup_{y\in\mathbb{N}} B_y \subset A$.

We can summarize and visualize what was said above about (3) by considering a set-representation of $\mathbb{N}$ and its all proper subsets, all included between them (as in (2)). Then:

$$\{\ \{\ \ 0\},1\},2\},3\},4\},...\ ?\ \}\ \ ,\ ...\ ?,\ ...\ \}_{\mathbb{N}} \tag{5}$$

Blue brackets represent the sets of the list (1) (for simplicity we have only considered a blue left bracket).

So, to describe the fact that **all sets of the list are proper subsets of $\mathbb{N}$ and simultaneously they observe relation (2)**, in the points "$\}, ...?, ...\}_{\mathbb{N}}$" of (5) there are some numbers without blue brackets; that is there are numbers of $\mathbb{N}$ which don't belong to any set of the list. This is a symbolic-graphical model for (2) and (3).

Another approach to the contradiction is the following.

We are referring to (1) and (6).

$$\{\ \{\ \ 0\},1\},2\},3\},4\},...\ \}\quad \}_{\mathbb{N}} \tag{6}$$

$\mathbb{N}$ includes all sets (blue brackets) defined by list (1) (they are subsets of $\mathbb{N}$). These sets contain each natural number y (as defined in (1)). In this way we obtain (6). Then we see a blue bracket including **all natural numbers** (and all sets). So, there is a set (a blue bracket) including all natural numbers (let's keep in mind that all numbers are in (6)).

But no set on the list (1) includes all numbers; each number having a successor (for Peano axioms) and then each set having a "successor-set". So there is a contradiction.

3 Conclusion

Inconsistency appears clear considering diagram (5) (then relations (2) and (3)) and relation (4).

This proof leads us to ask some questions.

To what extent is this proof valid?

What consequences might this have on other theories that include $\mathbb{N}$?

Without the axiom of infinity, are Peano axioms coherent?

The axiom of infinity implies a time-independent approach (unlike the ambiguous potential infinity). So, is the concept of time necessary to achieve coherence? Does this have a physical meaning?

Is this inconsistency a demonstration of a concrete, finite physical reality?

References

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E-mail address, E. Cadeddu: cadeddu.e@gmail.com