Inconsistency of $\mathbb{N}$ and the question of infinity

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Abstract

In the article ”Inconsistency of $\mathbb{N}$ from a not-finitist point of view” we have shown the inconsistency of $\mathbb{N}$, going through a denial. Here we delete this indirect step and essentially repeat the same proof. Contextually we find a contradiction about natural number definition. Then we discuss around the rejection of infinity.

1 Introduction

To justify the presence of $\mathbb{N}$ in inequalities, for all y, in [1] we have considered an indirect approach, denying the presence of the same $\mathbb{N}$. But the presence of $\mathbb{N}$ is a direct consequence of considering all n, that is all natural numbers.

To understand this we consider the list of all natural numbers, each n in one-to-one correspondence with sets $n+1=\{0,1,2,...n\}$. The list of sets is $\{x|x \leq y\}$ $\forall y \in \mathbb{N}$.

(0)	**{ 0 }**
(1)	**{ 0, 1 }**
(2)	**{ 0, 1, 2 }**
(3)	**{ 0, 1, 2, 3 }**
(4)	**{ 0, 1, 2, 3, 4 }**

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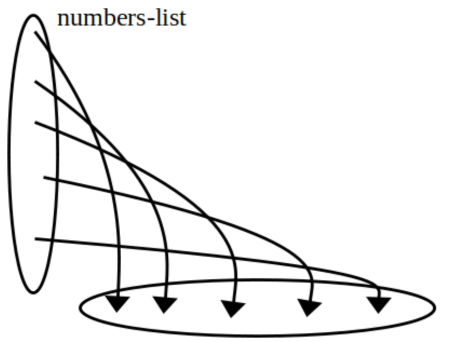
$$(?) \qquad \mathbf{\{\ 0,\ 1,\ 2,\ 3,\ 4,\ ...\ ?\ \}} = \mathbb{N}$$

Each number of the numbers-list is included in the corresponding set and all successor sets of the sets-list, because each number and then each set has a successor for Peano axiom. There is a one-to-one correspondence between the numbers-list and a set of the sets-list. So all numbers of the numbers-list are included in a set of the sets-list, that consequently is equal to $\mathbb{N}$. Then on the sets-list there is $\mathbb{N}$. To take all numbers together is in agreement with the axiom of infinity (that is, there is a set with all natural numbers, $\mathbb{N}$) [3] [4] [7].

Figure 1:



It should be noted that this one-to-one correspondence is equivalent to that one between n and $n + 1 \equiv \{0, 1, 2, ...n\}$, for number definition [5] [6]. So also at this level a contradiction appears. In fact rewriting the list in this way:

$$\begin{array}{cc} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{4} & \mathbf{5} \end{array}$$

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$$\cdot$$

$$\textbf{(?)} \qquad \textbf{(? + 1)}$$

We can see that $\mathbb{N}$ cannot appear on the list, because $\mathbb{N}$ isn't a natural number. This in contradiction with the previous argument about all numbers in the list. First of all this is related to Von Neumann's number definition.

2 Presence of $\mathbb{N}$ on the list

To take simultaneously all sets of the list $\{x|x \leq y\}$ $\forall y \in \mathbb{N}$, we can consider the number notation of sets and the sets-list becomes: $\{1, 2, 3, 4, ...\}$. So doesn't $\mathbb{N}$ appear on the list? When we take great, indeterminate, sets, the number definition with sets coincides with $\mathbb{N}$, in particular for the list: $\{\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}, ...\{0, 1, 2, 3, ...\}\}$. And $\{0, 1, 2, 3, ...\}$ represents $\mathbb{N}$.

To clarify the issue, let's reconsider the sets-list, representing all infinite sets, together the information that all sets have the same form ($\{x|x \leq y\}$ $\forall y \in \mathbb{N}$ and all y taken together in agreement with the axiom of infinity):

$$\begin{array}{l} \mathbf{\{\ 0\ \}} \\ \mathbf{\{\ 0,\ 1\ \}} \\ \mathbf{\{\ 0,\ 1,\ 2\ \}} \\ \mathbf{\{\ 0,\ 1,\ 2,\ 3\ \}} \\ \mathbf{\{\ 0,\ 1,\ 2,\ 3,\ 4\ \}} \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ \mathbf{\{\ 0,\ 1,\ 2,\ 3,\ 4,\ ...\ \}} \end{array}$$

A "flat representation" (an orthogonal projection) of the list is:

$\mathbf{\{\ 0\ ,\ \}\ 1\ ,\ \}\ 2\ ,\ \}\ 3\ ,\ \}\ 4\ ,\ \}\ ...\ \}}$, containing all natural numbers, all y. To consider a specific set we detect a specific right bracket and delete all the other right brackets. It is clear that on the list there is the set $\{0, 1, 2, 3, 4, ...\} = \mathbb{N}$, obtained keeping the last right bracket and deleting all the others.

In a more exhaustive manner we can say the following. In agreement with the axiom of infinity we can consider all y, $\{0, 1, 2, 3, 4, ...\}_{\mathbb{N}}$. From definition, $\{x|x \leq y\}$ $\forall y \in \mathbb{N}$, we know that each y is included, together all previous y, in a set of the list. So all y are included in a set of the list: $\mathbf{\{\ \{0, \}1, \}2, \}3, \}4, \}...\ \}\ \}_{\mathbb{N}}}$. We don't determine all numbers but we know that each one is included in a set of the list; that is a precise number cannot be adjacent to "$\}_{\mathbb{N}}$", but we know: "$\}\}_{\mathbb{N}}$" . Then there is a set of the list such that $\{0, 1, 2, 3, 4, ...\} = \mathbb{N}$.

So it is clear that a strong argument establishing the presence of $\mathbb{N}$, on the list, exists (independently of other arguments).

At this point we propose again the proof exposed in [1].

3 Proof of inconsistency

Two sets containing different elements are different, so we have, as it is simple to proof by induction (then for all y):

$\{x|x \leq y\} \neq \{x|x \leq y+1\} \quad \forall y \in \mathbb{N} \quad with \quad y+1 = S(y) \in \mathbb{N}$ (one set for each y), that is:

$$\begin{array}{rl}
\mathbf{\{\ 0\ \}} \neq & \mathbf{\{\ 0\ ,\ 1\ \}} \\
\mathbf{\{\ 0\ ,\ 1\ \}} \neq & \mathbf{\{\ 0\ ,\ 1\ ,\ 2\ \}} \\
\mathbf{\{\ 0\ ,\ 1\ ,\ 2\ \}} \neq & \mathbf{\{\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ \}} \\
\mathbf{\{\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ \}} \neq & \mathbf{\{\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ 4\ \}} \\
. & \\
. & \\
. & \\
\mathbf{\{\ 0\ ,\ 1\ ,\ 2\ ,\ 3,\ ...\ \}} \neq & \mathbf{\{\ 0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ 4,\ ...\ \}}
\end{array}$$

Each set in the list contains all the y of previous sets starting from " 0 ". So, considering all the y in the list (then all y together) implies that **in the list there is a set** $=\mathbb{N}$, as we have explained in the previous section and in the introduction.

The set to the right of "$\neq$" contains "0" and all the successors of the set to the left. The set to the left of "$\neq$" contains all the predecessors of the set to the right. At the same time with Peano axiom [2] [4] we have:

$\{x|x \in \mathbb{N}\} = \{0, \{S(x)|x \in \mathbb{N}\}\} = \mathbb{N}$, that is "0" and all successors (of $\mathbb{N}$) $\in \mathbb{N}$.

Then for all y we have:

$$\begin{aligned} \{\,0\,\} &\neq \{\,0\,,1\,\} \\ \{\,0\,,1\,\} &\neq \{\,0\,,1\,,2\,\} \\ \{\,0\,,1\,,2\,\} &\neq \{\,0\,,1\,,2\,,3\,\} \\ \{\,0\,,1\,,2\,,3\,\} &\neq \{\,0\,,1\,,2\,,3\,,4\,\} \\ &. \\ &. \\ &. \\ \mathbb{N} &\neq \mathbb{N} \end{aligned}$$

So $\mathbb{N} \neq \mathbb{N}$, a contradiction.

4 The question of infinity

We could have doubts about this proof of inconsistency, above all whether we consider meta-theoretical arguments. But giving a concrete, physical meaning, which is mostly what interests us, seems to remove any doubt. For example we could think to numbered objects inside boxes, instead of elements and sets, and to assuming the existence of infinite objects in our world.

An alternative is to admit that not all numbers in $\mathbb{N}$ are attainable, but this means that particular numbers, not equivalent to the others, have to exist; or infinity exists but it isn't attainable. A position not properly in line with the axiom of infinity. It could be said that there are two realities (one ideal and one concrete) and that therefore the finite physical reality has been deduced in this way. This would be a no small result.

A finite set wouldn't admit Peano axioms, in particular $\forall x(S(x))$ with $S(x) \in \mathbb{N}$, because the greatest number doesn't have a successor into the finite set (the successor couldn't even be an empty set, because zero doesn't have a predecessor, in agreement with an axiom). So Peano axioms implies a not-finite set. But concretely what does not-finite mean? Infinity? Potential infinity? The meaning of the letter is ambiguous. Anyway the axiom of infinity was adopted, to allow the existence of all natural numbers; in fact this is not possible inductively. But as we have shown, Peano axioms and the axiom of infinity appear incompatible. We should consider finite sets and partially modify Peano axioms to avoid inconsistency. Or else not to consider the axiom of infinity?

Infinity idea is very rooted in us. It is more natural and simple to think to the existence of infinity rather than its not-existence. Putting an end to counting or to thinking something beyond something else seems impossible. It is a time dependent idea that isn't an actual infinity, but imagines it as limit. We think that this difficulty to avoid infinity is principally due to our limited, slow computing ability, compared with large memory spaces, human and physical ones. But imagining to complete with very great numbers the greatest, but supposed finite, space memory of the entire universe, we would no longer be able to count, then processing data or thinking. So the problem would vanish and the idea of finite appears possible to us.

5 Conclusion

This work has been done to determine the existence or not of infinity (and consequently infinitesimals), or its undecidability. The argument is very important in philosophy, logic, mathematics and physics. We have shown an inconsistency regarding the assumption of the axiom of infinity in the context of Peano axioms. It is reasonable to think to a finite, not infinite (then nor with infinitesimals) reality. It would also be appropriate to verify to what extent this result can affect theories that include $\mathbb{N}$.

References

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