THE JOURNAL OF SYMBOLIC LOGIC
Volume 56, Number 4, Dec. 1991

ON THE WEAK KLEENE SCHEME IN KRIPKE'S THEORY OF TRUTH

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Abstract. It is well known that the following features hold of $AR + T$ under the strong Kleene scheme, regardless of the way the language is Gödel numbered:

1. There exist sentences that are neither paradoxical nor grounded.
2. There are $2^{\aleph_0}$ fixed points.
3. In the minimal fixed point the *weakly definable sets* (i.e., sets definable as $\{n \mid A(n)$ is true in the minimal fixed point$\}$, where $A(x)$ is a formula of $AR + T$) are precisely the Π^1_1 sets.
4. In the minimal fixed point the *totally defined sets* (sets weakly defined by formulae all of whose instances are true or false) are precisely the Δ^1_1 sets.
5. The closure ordinal for Kripke's construction of the minimal fixed point is ω_1^{CK}.

In contrast, we show that under the weak Kleene scheme, depending on the way the Gödel numbering is chosen:

1. There may or may not exist nonparadoxical, ungrounded sentences.
2. The number of fixed points may be any positive finite number, $\aleph_0$, or $2^{\aleph_0}$.
3. In the minimal fixed point, the sets that are weakly definable may range from a subclass of the sets 1-1 reducible to the truth set of AR to the Π^1_1 sets, including intermediate cases.
4. Similarly, the totally definable sets in the minimal fixed point range from precisely the arithmetical sets up to precisely the Δ^1_1 sets.
5. The closure ordinal for the construction of the minimal fixed point may be ω, ω_1^{CK}, or any successor limit ordinal in between.

In addition we suggest how one may supplement $AR + T$ with a function symbol interpreted by a certain primitive recursive function so that, irrespective of the choice of the Gödel numbering, the resulting language based on the weak Kleene scheme has the five features noted above for the strong Kleene language.

Preliminaries. We presuppose familiarity with Kripke's theory of truth as presented in [4].[1] (A detailed technical exposition can be found in [1]. See also [2].) We consider the language $L = AR + \{T(x)\}$ of arithmetic, with $(0, ', +, \times)$ interpreted in the usual way, plus the uninterpreted one-place predicate $T(x)$. The predicate $T(x)$ is given a partial interpretation by a pair of disjoint sets (S_1, S_2), S_1 and S_2 being respectively the extension and antiextension of $T(x)$. More complex formulae will be interpreted in accordance with the weak valuation rules of

Received May 7, 1990; revised December 13, 1990.

[1]The authors would like to acknowledge their debt to Professor Saul Kripke, whose lectures on truth in Spring 1987, 1988 and 1989 have stimulated their ideas on the subject.

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0022-4812/91/5604-0021/$02.70

Kleene (cf. [3, §64]), henceforth referred to as the *weak Kleene* valuation scheme. For definiteness, we describe the valuation rules briefly as follows. Given a partial interpretation $\Theta = (S_1, S_2)$ of $T(x)$ we let $\mathbf{val}(\Theta, A)$ be one of $\{T, F, u\}$ depending on whether the sentence A is valuated as true, false or "undefined" (i.e. as lacking truth-value) under the weak Kleene scheme. We thus have

$$\mathbf{val}(\Theta, \neg A) = u \quad \text{if } \mathbf{val}(\Theta, A) = u,$$
$$\mathbf{val}(\Theta, (B \,\&\, C)) = u \quad \text{if } \mathbf{val}(\Theta, B) = u \text{ or } \mathbf{val}(\Theta, C) = u,$$
$$\mathbf{val}(\Theta, (\exists x)B) = u \quad \text{if for some } m, \mathbf{val}(\Theta, B(\underline{m})) = u.$$

The remaining logical operations are defined in the usual way, and in all other respects **val** agrees with the classical valuation scheme.

Following Kripke's terminology let $\mathscr{L}(S_1, S_2)$ be the language L with $T(x)$ interpreted by (S_1, S_2). Let $\Phi_{\mathrm{WK}}(S_1, S_2) = (S_3, S_4)$, where S_3 is the set of Gödel numbers of true sentences of $\mathscr{L}(S_1, S_2)$ under the weak Kleene scheme, and S_4 is the set of Gödel numbers of false sentences of $\mathscr{L}(S_1, S_2)$ and integers that are not Gödel numbers of any sentences. Let ${}^+(S_1, S_2) = S_1$ and ${}^-(S_1, S_2) = S_2$. Let $\Phi^0_{\mathrm{WK}}(S_1, S_2) = (S_1, S_2)$, $\Phi^{\alpha+1}_{\mathrm{WK}}(S_1, S_2) = \Phi_{\mathrm{WK}}(\Phi^{\alpha}_{\mathrm{WK}}(S_1, S_2))$, and, for limit λ, $\Phi^{\lambda}_{\mathrm{WK}}(S_1, S_2) = (S_3, S_4)$, where $S_3 = \bigcup_{\beta<\lambda} {}^+\Phi^{\beta}_{\mathrm{WK}}(S_1, S_2)$ and $S_4 = \bigcup_{\beta<\lambda} {}^-\Phi^{\beta}_{\mathrm{WK}}(S_1, S_2)$. Letting AR^+ (AR^-) be the set of Gödel numbers of truths (falsehoods) of AR, we define $\mathscr{L}_\alpha = \mathscr{L}(\Phi^{\alpha}_{\mathrm{WK}}(AR^+, AR^- \cup \text{Nonsent}))$. (Kripke's construction has $\mathscr{L}_\alpha = \mathscr{L}(\Phi^{\alpha}_{\mathrm{WK}}(\Lambda, \Lambda))$, but to do so here would make our later exposition more awkward. For $\alpha \geq \omega$ the extension and antiextension for the $\mathscr{L}_\alpha$'s of the two constructions are the same.) Let $S_{1,\beta}$ be the extension of $T(x)$ in $\mathscr{L}_\beta$, and $S_{2,\beta}$ the antiextension of $T(x)$ in $\mathscr{L}_\beta$. In Kripke's theory the predicate $T(x)$ can be interpreted under certain conditions as a truth-predicate for the language L. This happens when the partial interpretation (S_1, S_2) of $T(x)$ is a fixed point of the operation on sets of integers associated with a given valuation scheme. (Φ_{WK} is such an operation for the weak Kleene scheme.) A minimal fixed point exists as long as the operation is monotone, which holds for the usual valuation schemes. We say that the construction of the minimal fixed point *closes off at* α if $\mathscr{L}_\alpha = \mathscr{L}_{\alpha+1}$ and, for $\beta < \alpha$, $\mathscr{L}_\beta \neq \mathscr{L}_{\beta+1}$. We say that an L sentence A is *declared* true (false) in $\mathscr{L}_\alpha$ if $\mathrm{gn}(A) \in S_{1,\alpha}$ (resp. $S_{2,\alpha}$), and A is *valuated* true (false) in $\mathscr{L}_\alpha$ if $\mathbf{val}((S_{1,\alpha}, S_{2,\alpha}), A) = T$ (resp. F). We shall sometimes write "$\mathscr{L}_\alpha \models A$" (resp. "$\mathscr{L}_\alpha \dashv A$") to indicate that A is valuated true (resp. false) in $\mathscr{L}_\alpha$.

Kripke calls a sentence A *paradoxical* (with respect to a given valuation scheme) iff $\mathrm{gn}(A) \in (S_1 \cup S_2)$ for no fixed point (S_1, S_2) of $\mathscr{L}$. (We assume that an effective Gödel numbering gn of L is given in advance.) A sentence A is *grounded* (with respect to a given valuation scheme) iff $\mathrm{gn}(A) \in (S_1 \cup S_2)$, where (S_1, S_2) is the minimal fixed point of $\mathscr{L}$. Also, a set $S \subseteq N$ is *weakly defined* in a fixed point (S_1, S_2) by some L formula $A(x_1)$ with x_1 as its sole free variable iff $S = \{m \mid \mathrm{gn}(A(\underline{m})) \in S_1\}$. Then S is *weakly definable* in (S_1, S_2) if some L formula weakly defines it there. On the other hand, S is *strongly definable* in (S_1, S_2) if both S and $-S$ are weakly definable in (S_1, S_2). Finally, an L formula $A(x_1)$ *totally defines* S in (S_1, S_2) iff $A(x_1)$ weakly defines S in (S_1, S_2) and, in addition, for all m, $\mathrm{gn}(A(\underline{m})) \in S_1 \cup S_2$.

§1. We motivate our discussion by highlighting some differences between the strong Kleene and the weak Kleene valuation schemes. (See [4] for the definition of the former.) It is well known that under the strong Kleene scheme the language $\mathscr{L}$ has the following features irrespective of the choice of the Gödel numbering:

—There exist sentences that are neither paradoxical nor grounded.

—There are $2^{\aleph_0}$ fixed points.

These properties follow from the fact that the strong Kleene language contains denumerably many truth-tellers. Consider an infinite enumeration of distinct truth-tellers of the form

$$(\exists x)(Q_n(x) \mathbin{\&} T(x) \mathbin{\&} \underline{\mathrm{n}} = \underline{\mathrm{n}}),$$

where in each case $Q_n(x)$ is an arithmetical formula true of just the Gödel number of the very formula $(\exists x)(Q_n(x) \mathbin{\&} T(x) \mathbin{\&} \underline{\mathrm{n}} = \underline{\mathrm{n}})$. For any assignment of T, F, or u to these various truth-tellers there exists a fixed point in which the truth-tellers have the truth values given in the assignment. Under the strong Kleene scheme the truth-tellers will be neither grounded nor paradoxical. Since there are $2^{\aleph_0}$ possible assignments, there will be at least that many fixed points. Therefore, given that the language is countable, there are exactly $2^{\aleph_0}$ fixed points.

Consider this argument in relation to the weak Kleene scheme. In any fixed point of any valuation scheme some instance of the formula $T(x)$ will be undefined—that follows from Tarski's theorem on the undefinability of truth in (sufficiently strong) classical languages. But under the weak Kleene scheme if an instance of $T(x)$ lacks truth value (i.e. is assigned u), $(\exists x)(Q_n(x) \mathbin{\&} T(x) \mathbin{\&} \underline{\mathrm{n}} = \underline{\mathrm{n}})$, the would-be truth-teller, also lacks truth-value. So the method of generating (nonparadoxical) truth-tellers we used for the strong Kleene languages fails here.

This does not rule out the possibility of (nonparadoxical) truth-tellers. Direct self-reference provides one way of obtaining them, e.g.

(1) $$T(1).$$

Even without direct self-reference we *may* have (nonparadoxical) truth-tellers. Suppose $Q(x)$ is an arithmetical formula true of just the Gödel number of

(*) $$(\exists x)(Q(x + x) \mathbin{\&} T(x + x)).$$

If, say, (*) has an even Gödel number and the only other sentences with even Gödel numbers are arithmetical, then (*) will be a (nonparadoxical) truth-teller. Note too that, although (*) will be nonparadoxical, the sentence

$$(\exists x)(Q(x) \mathbin{\&} T(x))$$

will remain paradoxical. In fact, the difference between a sentence's being grounded, paradoxical or neither grounded nor paradoxical may simply be a matter of how we "describe" an object. If we extend the present example, letting 2 be the Gödel number of $0 = 0$, then the following sentences:

$$T(0''), \qquad (\exists x)(x + x = 0'' \mathbin{\&} T(x + x)), \qquad (\exists x)(x = 0'' \mathbin{\&} T(x))$$

will be, respectively, (a) grounded, (b) neither paradoxical nor grounded, and (c) paradoxical.

Generally when we consider an extension of the language AR under some effective Gödel numbering, we think of the language as having adequate resources for self-reference. If $G(x)$ uniquely holds of an object we can usually express "the G is F" by "$(\exists x)(G(x) \;\&\; F(x))$" or "$(\forall x)(G(x) \supset F(x))$". But the above considerations show that this does not necessarily hold in the weak Kleene language $AR + T$. This raises the problem whether the weak Kleene language $AR + T$ has adequate "descriptive" resources to talk about its own syntax. We formulate the issue more precisely in terms of definability in the minimal fixed point and study the resulting phenomena in greater detail.

In general, we shall see that the theory of the weak Kleene language $AR + T$ is not invariant with respect to the choice of Gödel numbering. This is not the case with the theory of the strong Kleene language, where, speaking in terms of the convenient analogy mentioned in [5, p. 22], the use of a particular Gödel numbering is much like the use of a particular coordinate system to obtain coordinate-free results in geometry. In fact, the exact expressive strength of the weak Kleene language $AR + T$ depends on the way its syntax is numerically coded within the language. Among the consequences of this is that the weak Kleene language enables one to draw a sharp distinction between the functions and the relations (which include the graphs of the functions) expressible in it. (See the remarks in §5.) On the other hand, if the two valuation schemes are compared in terms of relative strength, the picture is more complicated than one might initially suppose. It is not difficult to see that the strong Kleene language $AR + T$ of the minimal fixed point "contains" the weak Kleene language of the minimal fixed point under a certain translation. We show (see the remarks in §4) that under a suitable Gödel numbering the logical operations of the strong Kleene language can in a certain sense be defined in the weak Kleene language of the minimal fixed point. And if one considers a slightly richer vocabulary that includes a function symbol for a particular primitive recursive function, then, regardless of the choice of Gödel numbering, the weak Kleene and the strong Kleene languages of the minimal fixed points turn out to be of the same expressive power. The resulting weak Kleene language becomes in an appropriate sense invariant for different choices of Gödel numberings.

One might try to cope with this situation by formulating restrictions on what may count as a "natural" Gödel numbering of the vocabulary $AR + T$. We do not regard this approach as promising. A more natural way to address the problem would be to expand the vocabulary $AR + T$ with a definite description operator or additional function symbols. Nor do we discuss which one of the two valuation schemes better accords with pre-theoretic intuitions from ordinary language or is in general "more useful". Our primary goal is to describe the noninvariance phenomenon peculiar to the weak Kleene scheme and attempt to gauge its extent by means of comparison with the well-known properties of the strong Kleene scheme.

§2. By *the values of term t*, we mean the set of values taken by t when assignments of values are made to all variables occurring in the term. Every term of L has either (1) a single value, or (2) an infinite number of values all of which are even, or (3) an infinite number of odd values (with either no even values or infinitely many even values). Furthermore, it is decidable into which category a term falls; we can effectively enumerate the terms of category (3), and for an enumeration

$t_1, t_2, t_3, \ldots$ of the terms of category (3) there is a recursive function val(i, j) which for each i enumerates the values of t_i.

Throughout the paper we shall assume that a Gödel numbering is a 1-1 effective map from formulae (or sentences) of L of the kind described in general terms in [3, p. 300]. In defining a *special Gödel numbering* for L we will make use of an effective enumeration of C_1, C_2, $C_3 \ldots$ of a set C of sentences, which we choose so that each C_i turns out to be paradoxical. As noted earlier, $(\exists x)T(x)$ is paradoxical in any weak Kleene language. So let $C_i = (\exists x_i)T(x_i)$. We let E be a decidable set of sentences disjoint from C, and effectively assign even Gödel numbers to them. We call sentences of E *special*. Let D be the remaining set of formulae. This will be a decidable set, which we may enumerate as: $D_1, D_2, D_3, \ldots$.

For any length, m, of a sequence of symbols of L, one can effectively compute the maximum number that is a value for a term that is of that length or less and does not have infinitely many values; call this number max(m). We assign Gödel numbers to the remaining formulae, first to C_1, then D_1, then C_2, then D_2, etc. using the following procedure: Assign to C_n (D_n) the first member of the sequence val$(n, 1)$, val$(n, 2)$, val$(n, 3), \ldots$ which is odd, has not yet been assigned as a Gödel number and is greater than max(length(C_n)) (resp. max(length(D_n))).

Note that this Gödel numbering is effective and has the following two properties:

(∗) Every term with infinitely many odd values has the Gödel number of a paradoxical sentence (viz., C_i) among its values.

(∗∗) The Gödel number of any nonspecial formula is greater than any value of any term that (a) is of the length of that formula or less and (b) does not have infinitely many values.

The following lemma holds for any weak Kleene language based on L, regardless of the Gödel numbering. We shall call a term t a *T-term of A* iff $T(t)$ is a subformula of A. We let V_t be the set of values taken by term t.

LEMMA 1. *For any sentence A of L, A is valuated at a given partial interpretation* (S_1, S_2) *of T iff, for every T-term t of A,* $V_t \subseteq S_1 \cup S_2$.

COROLLARY. *If any T-term of a sentence A has a value that is the Gödel number of a paradoxical sentence B, then A is paradoxical.*

LEMMA 2. *Suppose L has a special Gödel numbering. Then for any sentence A of L, if A contains a T-term with infinitely many odd values, then A is paradoxical.*

Say that $Q \subseteq N$ is *fixed at stage* α just in case, for any $n \in Q$, $n \in S_{i,\alpha}$ iff $n \in S_i^*$, for $i = 1, 2$, where (S_1^*, S_2^*) is the minimal fixed point. We will speak of a set of sentences being *fixed at* α if the set of their Gödel numbers is fixed at stage α. Say that a set Q of (Gödel numbers) of sentences is *minimally fixed in* (S_1, S_2) if there is a smallest fixed point above (S_1, S_2) and, for all $x \in Q$, $x \in S_1$ (S_2) iff x is in the extension (antiextension) of T in the minimal fixed point above (S_1, S_2).

LEMMA 3. *Suppose L has a special Gödel numbering.*

(a) *Distinct fixed points differ in their valuations of the special sentences.*

(b) *Suppose the special sentences are minimally fixed in* (S_1, S_2). *Then for any sentence A there is a finite n such that* $\{\mathrm{GN}(A)\}$ *is minimally fixed in* $\Phi^n_{\mathrm{WK}}(S_1, S_2)$, *and so the minimal fixed point above* (S_1, S_2) *is* $\Phi^\omega_{\mathrm{WK}}(S_1, S_2)$.

(c) *If the special sentences under a given Gödel numbering become fixed at some stage, then L has only one fixed point.*

(d) *Suppose the special sentences are fixed at stage* α. *Then for every odd m which is not the Gödel number of a paradoxical sentence, there is a finite k such that* $m \in S_{1,\alpha+k}$ *or* $m \in S_{2,\alpha+k}$.

PROOF. (a) Suppose otherwise. Let (S_1, S_2) and (S_1^*, S_2^*) be distinct fixed points that do not differ in their valuations of the special sentences. Let A be the sentence with the least Gödel number which is valuated differently in the two fixed points. A must be valuated true or false in at least one of the two fixed points, so by Lemma 1 all values of its T-terms are in the extension or antiextension of that fixed point. Since all these values that are odd are less than $\mathrm{GN}(A)$ (the Gödel number of A), we have by hypothesis that, for any n which is a value of a T-term of A, $n \in S_1$ iff $n \in S_1^*$, and $n \in S_2$ iff $n \in S_2^*$. But then $\mathscr{L}(S_1, S_2)$ and $\mathscr{L}(S_1^*, S_2^*)$ do not differ in their valuation of A after all.

(b) Suppose otherwise. Let A be the sentence with least Gödel number such that $\mathrm{GN}(A)$ is in the minimal fixed point (S_1^*, S_2^*) above (S_1, S_2) but is not in $\Phi_{\mathrm{WK}}^{\omega}(S_1, S_2)$. (By a straightforward monotonicity argument, $\Phi_{\mathrm{WK}}^{\omega}(S_1, S_2) \subseteq (S_1^*, S_2^*)$.)

Consider any value, n, of a T-term of A. We have $n \in S_1^* \cup S_2^*$ by Lemma 1. If n is even, then $n \in S_1 \cup S_2$ since the special sentences are minimally fixed in (S_1, S_2). On the other hand, if n is odd, $n < \mathrm{GN}(A)$ and so there is a finite j for which $n \in {}^{+}\Phi_{\mathrm{WK}}^{j}(S_1, S_2) \cup {}^{-}\Phi_{\mathrm{WK}}^{j}(S_1, S_2)$. Furthermore, there are only finitely many odd values of T-terms of A by Lemma 2. But then there is a finite j such that all values of the T-terms of A are in ${}^{+}\Phi_{\mathrm{WK}}^{j}(S_1, S_2) \cup {}^{-}\Phi_{\mathrm{WK}}^{j}(S_1, S_2)$, and so, by Lemma 1, A is valuated true or false in $\mathscr{L}(\Phi_{\mathrm{WK}}^{j}(S_1, S_2))$.

(c) and (d) follow from (a) and (b). Q.E.D.

PROPOSITION 1. (a) *If the set of special sentences is empty, then the construction of the minimal fixed point closes off at* ω *and* $\mathscr{L}_\omega$ *is the only fixed point of the language.*

(b) *For any finite n there is a Gödel numbering of L for which there are exactly* $n + 1$ *fixed points. There are also Gödel numberings having exactly* $\aleph_0$ *and* $2^{\aleph_0}$ *fixed points.*

PROOF. (a) That $\mathscr{L}_\omega$ is the minimal fixed point follows from Lemma 3(c), (d). That the construction of the fixed point does not close off earlier follows from the fact that $\mathrm{GN}(T^{n+1}(0 = 0)) \notin S_{1,n}$, but $\mathrm{GN}(T^{n+1}(0 = 0)) \in S_{1,n+1}$. (Here we let $T^0(A) = A$ and $T^{n+1}(A) = T(\ulcorner T^n(A) \urcorner)$, where $\ulcorner A \urcorner$ is the numeral for $\mathrm{GN}(A)$.)

(b) Part (a) shows us that there is a Gödel numbering with only one fixed point. Consider the following enumeration of sentences: $T(0) \vee \neg T(0)$, $T(0) \vee T(2) \vee \neg T(2)$, $T(2) \vee T(4) \vee \neg T(4)$, $T(4) \vee T(6) \vee \neg T(6)$, etc. To construct a Gödel numbering in which there are exactly $n + 1$ fixed points (for $n > 0$), for each $m < n$ let $2m$ be the Gödel number of the $(m + 1)$st sentence in the above enumeration, and for $m \geq n$ let $2m$ not be the Gödel number of any sentence. For any initial segment of the sequence of these special sentences (possibly empty, possibly including all the special sentences) there is a fixed point in which all sentences in the initial segment are true and all the remaining special sentences lack truth-value. Furthermore, every fixed point makes an initial segment of the sequence of special sentences all true and the rest undefined. Since there are exactly $n + 1$ initial segments of the sequence of special sentences, by Lemma 3(a) there are exactly $n + 1$ distinct fixed points. To cover the case where there are $\aleph_0$ fixed points we let all the sentences in the above enumeration be special sentences. The same argument works, since a fixed point still makes an initial segment of the sequence of special sentences true and the rest undefined. To get an example where there are $2^{\aleph_0}$ fixed points, let each even number m Gödel number $T(\underline{m})$. Now there will be a fixed point for any of the $2^{\aleph_0}$ ways of assigning truth values to the special sentences. Q.E.D.

REMARK. The last-mentioned Gödel numbering can be used to show that *any* set $S \subseteq N$ is totally defined in some fixed point by the formula $T(2x_1)$. Hence there is no bound on the complexity of sets definable in a fixed point.

We will consider one useful way of Gödel numbering the special sentences. Let $[n, m]$ abbreviate $2 \times ((n + m) \times (n + m) + n)$. This expresses an effective 1-1 map from pairs of integers (n, m) to integers, and its inverse is also effective. (We include the factor 2 so that we may use even numbers to code special sentences.) To code ordered triples we use $[[i, j], k]$, which we abbreviate as $[i, j, k]$. This may be repeated to obtain codings of longer sequences.

Let $R_1, R_2, R_3, \ldots$ be a recursive enumeration of 2-place primitive recursive relations. We say that a special Gödel numbering is *J-determined* if the following conditions are satisfied: J is recursive and, for $j \in J$, $[j, n_1, n_2]$ codes $(x)T([\underline{j}, \underline{n}_2, x])\ \&\ \underline{n}_1 = \underline{n}_1$ if $R_j(n_2, n_1)$ holds, and codes $[\underline{j}, \underline{n}_1, \underline{n}_2] = [\underline{j}, \underline{n}_1, \underline{n}_2]$ otherwise; and furthermore, all special sentences are coded by Gödel numbers of the form $[j, n_1, n_2]$ where $j \in J$. (We add the conjunct $\underline{n}_1 = \underline{n}_1$ to ensure that the coding is 1-1 effective. We shall, however, generally omit writing out this part of the sentence coded, and when we speak of an "instance" of a special sentence we shall mean an instance of the first conjunct.)

Let R be any relation on the natural numbers. Say that m_0 is *grounded in R* iff there is no infinite sequence $m_1, m_2, m_3, \ldots$ (m_i, m_j not necessarily distinct for $i \neq j$) such that, for each $i \geq 0$, $R(m_{i+1}, m_i)$. Let

$$G_R = \{m \in \mathrm{Fld}_R \mid m \text{ is grounded in } R\} \quad \text{and} \quad U_R = \mathrm{Fld}_R \cap -G_R.$$

(Thus, R is well-founded on G_R in the usual sense.) We define, for each ordinal α,

$$R^{(\alpha)} = \{m \in \mathrm{Fld}_R \mid \text{for all } n, \text{ if } R(n, m), \text{ then } n \in \textstyle\bigcup_{\beta < \alpha} R^{(\beta)}\}.$$

Let $\mathrm{Ord}(R) = \mu\alpha(R^{(\alpha)} = R^{(\alpha+1)})$. If $m \in G_R$ we let $|m|_R = \mu\alpha(m \in R^{(\alpha)})$. For $j \in J$ let R_j be the relation indexed by j, and abbreviate $|n|_{R_j}$ as $|n|_j$ and G_{R_j} as G_j.

LEMMA 4. *Suppose the Gödel numbering of special sentences is J-determined. Consider the construction of the minimal fixed point starting from* $\mathscr{L}_0 = \mathscr{L}(AR^+, AR^- \cup \mathrm{Nonsent})$. *The following hold*:

(a) *If* $\neg R_j(n_2, n_1)$, *then* $[j, n_1, n_2] \in S_{1,0}$.

(b) *If* $R_j(n_2, n_1)$, *then the sentence with Gödel number* $[j, n_1, n_2]$ *first becomes valuated true in* $\mathscr{L}_{|n_2|_j}$ *if* $n_2 \in G_j$ (*i.e.*, n_2 *is not on an infinite descending chain in* R_j); *otherwise the sentence is ungrounded.*

(c) *If* $n_1 \in G_j$, *then* $(x)T([\underline{j}, \underline{n}_1, x])$ *is first valuated true in* $\mathscr{L}_\alpha$, *where* $\alpha = \mathrm{l.u.b.}\{|n_2|_j \mid R_j(n_2, n_1)\}$; *and otherwise* $(x)T([\underline{j}, \underline{n}_1, x])$ *is ungrounded.*

(d) *The set of sentences with Gödel numbers of the form* $[j, n_1, n_2]$, *for a given j, first becomes fixed at* $\mathrm{Ord}(R_j)$.

(e) *If* $\alpha = \mathrm{l.u.b.}_{j \in J}\mathrm{Ord}(R_j)$, *the special sentences are fixed at stage* α.

(f) *A special sentence is grounded iff it is true in the minimal fixed point.*

(g) *There is a fixed point in which every special sentence is true.*

PROOF. (a)–(f) are straightforwardly proved from the definitions. For (g), let $S_1 = \{[j, n_1, n_2] \mid j \in J,\ n_1, n_2 \in N\}$. Then, for each $j \in J$ and $n_2 \in N$, $\mathscr{L}(S_1, \Lambda) \models (x)T([\underline{j}, \underline{n}_2, x])$. Hence, for each $j \in J$ and $n_1, n_2 \in N$, $[j, n_1, n_2] \in {}^{+}\Phi_{\mathrm{WK}}(S_1, \Lambda)$. So $S_1 \subseteq {}^{+}\Phi_{\mathrm{WK}}(S_1, \Lambda)$ and (S_1, Λ) can be extended to a fixed point M. (By Lemma 3(a), M is a maximal fixed point.) Hence every special sentence is true in M. Q.E.D.

We can now establish the following.

PROPOSITION 2. *The closure ordinals for the inductive construction of the minimal fixed point include ω, ω_1^{CK}, and any successor limit ordinal in between.*

PROOF. ω was treated in Proposition 1. Any successor limit ordinal below ω_1^{CK} is of the form $\alpha + \omega$, where α is a successor ordinal $< \omega_1^{\mathrm{CK}}$. There is an R_j such that $\mathrm{Ord}(R_j) = \alpha$. If we let the special sentences be $\{j\}$-determined, then they become fixed at α (by Lemma 4(d)), and thus the construction of the minimal fixed point closes off by $\alpha + \omega$ at the latest (by Lemma 3(b)). That the construction does not close off before $\alpha + \omega$ follows from the fact that the special sentences first become fixed at α, a successor ordinal, and thus, for some A, A first is thrown into the extension of T at $\mathscr{L}_\alpha$. But then all the sentences of the sequence $T(\ulcorner A \urcorner)$, $T(\ulcorner T(\ulcorner A \urcorner) \urcorner), \ldots$ will eventually be put in the extension of T, but not before stage $\alpha + \omega$.

Next we show that the construction of the minimal fixed point can close off at ω_1^{CK}. If we let the Gödel numbering be N-determined (where N is the set of natural numbers) then the special sentences become fixed at $\mathrm{l.u.b.}_{n \in N} \mathrm{Ord}(R_n) = \omega_1^{\mathrm{CK}}$. So the closure ordinal $\geq \omega_1^{\mathrm{CK}}$. On the other hand, the minimal fixed point of the weak Kleene language is the smallest fixed point of an arithmetical monotone operation on (pairs of) sets of integers. By a theorem of Spector, the closure ordinal of such an operation is $\leq \omega_1^{\mathrm{CK}}$. Thus the closure ordinal $= \omega_1^{\mathrm{CK}}$. Q.E.D.

Question. Can the inductive construction of the minimal fixed point close at (any? all?) limit limit ordinals λ where $\omega < \lambda < \omega_1^{\mathrm{CK}}$?

§3. We now proceed to study definability in weak Kleene languages with J-determined Gödel numberings. Call A^* a *simplification* of the formula A if A^* is the result of replacing a single formula of the form $T(t)$ in A, where t has a single odd value, by (a) the sentence $0 = 0'$, if the value of t is not the Gödel number of a sentence, or by (b) the sentence whose Gödel number is the value of t. Let A_t^x be the result of replacing all free occurrences of x in A with t. Call formulae with free variables *equivalent in a fixed point* iff they are satisfied by the same infinite sequences and falsified by the same infinite sequences in that fixed point. The following lemma is easily established by induction on the complexity of formulae.

LEMMA 5. (a) *If A^* is a simplification of a formula A, then A^{*x}_t is a simplification of A_t^x.*

(b) *If A^* is a simplification of the sentence A, then in every fixed point A and A^* have the same truth value (T, F or u).*

(c) *If A^* is a simplification of a formula A, then $A^*(x_1, \ldots, x_n)$ and $A(x_1, \ldots, x_n)$ are equivalent in any fixed point and thus define (weakly or totally) the same n-ary relation in any fixed point.*

Say that a formula A^{**} is the *reduction* of the formula A if it can be obtained by applying the following procedure to A.

Step 1. Check whether the formula in question contains any T-term with infinitely many odd values. If it does, let $A^{**} =$ the conjunction of that formula with $\underline{n} = \underline{n}$ (where n is GN(A)), and stop; otherwise apply Step 2 to the formula in question.

Step 2. Check whether the formula in question has any T-term with a single odd value. If not, let $A^{**} =$ the conjunction of that formula with $\underline{n} = \underline{n}$, and stop. If it does, apply Step 1 to the formula obtained from the formula in question by replacing the leftmost subformula of the form $T(t)$ such that t has a single odd value $\geq$ the value of any other single odd valued T-term in the formula with (a) $0 = 0'$ if that value is not the Gödel number of a sentence, or with (b) the sentence whose Gödel number is the value of t if the value of t is the Gödel number of a sentence.

This reduction procedure determines a 1-1 effective mapping, and the resulting formulae either contain a T-term with infinitely many odd values, or they contain no T-terms with any odd value. We see that the mapping is completely defined (since the procedure eventually halts for any formula) as follows. The procedure must halt unless Step 2 is repeated indefinitely. Suppose this happens. Take the least m such that there is a formula B for which the process does not terminate and m is the maximum value of an odd valued T-term in B. (There must be such an m if there is a formula for which the process does not terminate, for if a formula has no maximum odd T-term value then either it has infinitely many odd T-term values, and thus at least one T-term with infinitely many odd values, making the process halt at Step 1, or it has no odd T-term values and so the process halts.) Call a T-term with a single value m an m-T-term. Suppose B has n m-T-terms. By the second condition on the Gödel numbering, each time a subformula with an m-T-term is replaced by a sentence, that new sentence has no T-term with a single odd value that is $\geq m$. So after n applications of Step 2 we are left with a sentence with no single odd-valued T-terms with a value $\geq m$. Thus by hypothesis this process will terminate after finitely many steps.

From Lemma 5 we have:

LEMMA 6. *Under a special Gödel numbering in any fixed point of the weak Kleene scheme, the reduction of any formula A is equivalent to A.*

PROPOSITION 3. *If the Gödel numbering of $AR + T$ is J-determined, then there is an effective procedure which* (1) *determines whether or not a given sentence is paradoxical, and* (2) *maps* 1-1 *the nonparadoxical sentences of $AR + T$ into sentences of $AR + T$ in which all values of T-terms are even.*

PROOF. We claim that, for any sentence A, A is paradoxical iff A^{**} contains a T-term with (infinitely many) odd values.

By Lemma 2, A^* is paradoxical if it contains a T-term with infinitely many odd values, whence by Lemma 6 so is A. Conversely, suppose A is paradoxical. By Lemma 6, A^{**} is paradoxical. Since by Lemma 4(g) there is a fixed point in which all special sentences are true, it follows by Lemma 1 that A^{**} contains an odd-valued T-term. But then by the reduction procedure A^{**} contains a T-term with infinitely many odd values. Q.E.D.

We can now set a bound on the maximal complexity of sets weakly definable in a weak Kleene language based on a J-determined Gödel numbering.

LEMMA 7. *Suppose the Gödel numbering of $AR + T$ is J-determined and that the* *special sentences are fixed at stage α. If a set S is weakly definable in the minimal fixed point, then*

$$S \leq_1 {}^{+}\Phi_{\mathrm{WK}}(S_{1,\alpha} \cap \text{Even}, \text{Nonsent} \cap \text{Even}).$$

PROOF. Suppose the special sentences are fixed at α. Consider any formula $A(x)$ of $AR + T$. For any n, the reduction procedure takes us from $A(\underline{n})$ to a sentence B_n such that if $A(\underline{n})$ is paradoxical, B_n contains a T-term with odd values; otherwise no T-term in B_n has any odd values. Let $\theta(n) = \mathrm{GN}(B_n)$. θ is 1-1 recursive. $A(\underline{n})$ is true in the minimal fixed point iff B_n is true in the minimal fixed point, iff (by Lemma 3(b)) $\mathrm{GN}(B_n) \in {}^{+}\Phi_{\mathrm{WK}}(\Phi^{\omega}_{\mathrm{WK}}(S_{1,\alpha}, S_{2,\alpha}))$, iff

$$\mathrm{GN}(B_n) \in {}^{+}\Phi_{\mathrm{WK}}({}^{+}\Phi^{\omega}_{\mathrm{WK}}(S_{1,\alpha}, S_{2,\alpha}) \cap \text{Even}, {}^{-}\Phi^{\omega}_{\mathrm{WK}}(S_{1,\alpha}, S_{2,\alpha}) \cap \text{Even})$$

(since the T-terms in B_n have only even values if B_n is not paradoxical), iff

$$\mathrm{GN}(B_n) \in {}^{+}\Phi_{\mathrm{WK}}(S_{1,\alpha} \cap \text{Even}, S_{2,\alpha} \cap \text{Even})$$

(since the special sentences are fixed at α), iff

$$\mathrm{GN}(B_n) \in {}^{+}\Phi_{\mathrm{WK}}(S_{1,\alpha} \cap \text{Even}, \text{Nonsent} \cap \text{Even})$$

(since by Lemma 4(f) the special sentences are never false in the minimal fixed point). Thus for a given $A(x)$ there is a recursive function θ such that $n \in \{m \mid A(\underline{m})$ is true in the minimal fixed point$\}$ iff $\theta(n) \in {}^{+}\Phi_{\mathrm{WK}}(S_{1,\alpha} \cap \text{Even}, \text{Nonsent} \cap \text{Even})$. Q.E.D.

Using Lemma 7, we can establish the following (we omit the proof):

PROPOSITION 4. *Suppose that under a J-determined Gödel numbering the special sentences are fixed at α. Then any set weakly definable in the minimal fixed point is* 1-1 *reducible to the smallest fixed point of a monotone arithmetical operation whose closure ordinal is* $\leq \alpha + \omega$.

REMARKS. (1) Consider the following special case. Suppose the set of special sentences is empty. (Thus $\alpha = 0$.) Then for any $A(x)$ we have $n \in \{m \mid A(\underline{m})$ is true in the minimal fixed point$\}$ iff $\theta(n) \in {}^{+}\Phi_{\mathrm{WK}}(\Lambda, \text{Nonsent} \cap \text{Even})$. However, ${}^{+}\Phi_{\mathrm{WK}}(\Lambda, \text{Nonsent} \cap \text{Even})$ is of the same complexity as AR^{+}.

(2) By a theorem of Kleene, any Π^1_1 set Q of integers can be defined in the form $Q = \{m \mid [m, 1] \in G_R\}$ for some primitive recursive relation R depending on Q. In particular, if Q is also Δ^1_1, we have that $\mathrm{Ord}(R) < \omega_1^{\mathrm{CK}}$. For a fixed Q, let $R = R_j$. Then, by Lemma 4(c),

$$Q = \{m \mid \mathscr{L}^* \models (y)T([\underline{j}, [\underline{m}, 1], y])\},$$

where $\mathscr{L}^*$ is the language of the minimal fixed point. Furthermore, if $\mathrm{Ord}(R_j) = \alpha$, then Q is weakly definable in the minimal fixed point above $\mathscr{L}_\alpha$ (at $\alpha + \omega$). Hence any Δ^1_1 set is weakly definable in some weak Kleene language $\mathscr{L}_\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$. Also, every Π^1_1 set is weakly definable in some weak Kleene language. In fact, in the minimal fixed point language based on the N-determined Gödel numbering all Π^1_1 sets are weakly definable.

Let us now turn to total definability in weak Kleene languages based on J-determined Gödel numberings.

PROPOSITION 5. *If the Gödel numbering of AR + T is J-determined, then the sets totally definable in the minimal fixed point are precisely the arithmetical sets.*

PROOF. Let $A(x)$ be a totally defined formula in the minimal fixed point, and let $A^*(x)$ be the reduction of $A(x)$. Since $A(x)$ is totally defined, $A^*(x)$ contains only even valued T-terms (by the reduction procedure). By Lemma 5(c), $A(x)$ and $A^*(x)$ totally define the same set, since definability is preserved under simplification and $A^*(x)$ is obtained from $A(x)$ by successive applications of simplification. Call $A'(x)$ the *AR-reduction* of $A(x)$ if it is the result of replacing all subformulae of the form $T(t)$ in $A^*(x)$ with

$$(\exists x)(\exists y)(\exists z)(t = [x, y, z] \;\&\; x \in J)$$

(where $x_1 \in J$ is an AR formula defining the set J, which is recursive, and x, y, and z are the first variables in the list $x_1, x_2, x_3, \ldots$ not in t or bound in $x_1 \in J$). We have in effect replaced occurrences of $T(t)$ in $A^*(x)$ with formulae which "say" "t is a special sentence". By Lemma 4(f) all special sentences are either true in the minimal fixed point or are neither true nor false there. Since $A^*(x)$ is totally defined in the minimal fixed point, any T-term in $A^*(x)$ must have as its values only Gödel numbers of grounded sentences (which are true in the minimal fixed point) or numbers which do not Gödel number any sentence and thus not of the form $[x, y, z]$ for $x \in J$. Hence $T(t)$ and its replacement will be equivalent in the minimal fixed point, and $A^*(x)$ and $A'(x)$ totally define the same set in the minimal fixed point. So $A(x)$ and its AR-reduction totally define the same set in the minimal fixed point. $A'(x)$ is a formula of AR; therefore $A(x)$ defines an arithmetical set. Since all arithmetical sets are totally definable in the minimal fixed point, the totally definable sets in the minimal fixed point are precisely the arithmetical sets. Q.E.D.

It is instructive to see where the above argument breaks down if we do not assume that $A(x)$ is a totally defined formula.

REMARKS. (1) In particular, in the N-determined Gödel numbering all Π^1_1 sets are weakly definable and the totally definable sets are just the arithmetical sets. Here for any nonarithmetical Δ^1_1 set Q, both Q and $-Q$ are weakly definable, but Q is not totally defined by any formula.

(2) Recall that in the J-determined Gödel numberings there is a maximal fixed point in which all special sentences are true. (Cf. the proof of Lemma 4(g).) Suppose the Gödel numbering were slightly altered by appending to all the special sentences the clause "$\vee\ 0 = 0$". Then (S_1, Λ) from the proof of Lemma 4(g) is an intrinsic point in Kripke's sense (see [1] for the definition), since under the modified Gödel numbering no sentence A with GN(A) of the form $[j, n_1, n_2]$ can be false in any fixed point. Consequently the smallest fixed point M that extends (S_1, Λ) is also intrinsic. Then it follows by Lemma 3(a) that M is actually the largest fixed point and hence also the largest intrinsic fixed point. (Compare this to the strong Kleene scheme, where there is no largest fixed point and the largest intrinsic fixed point is the largest fixed point contained in the intersection of all maximal fixed points. See [2].)

(3) Since the special sentences in (2) are minimally fixed in (S_1, Λ), we have $M = \Phi^{\omega}_{\mathrm{WK}}(S_1, \Lambda)$ by Lemma 3(b). We can now apply the AR-reduction procedure as in the proof of Proposition 5 to show that the truth set ^+M of M is 1-1 reducible to AR^+ and, since clearly $AR^+ \leq_1 {}^+M$, is recursively isomorphic to AR^+. Given any sentence A, observe that since all special sentences are true in M, a subformula $T(t)$ of A^*, where t is not odd-valued, and its replacement in the AR-reduction are

equivalent in M. It follows that any set weakly definable in M is 1-1 reducible to AR^+. (Contrast this with stronger valuation schemes where the class of sets definable in the largest intrinsic fixed point properly includes the sets definable in the minimal fixed point. For the strong Kleene scheme, the sets weakly definable in the largest intrinsic fixed point are Σ^1_1-in-a-Π^1_1 parameter, a proper subclass of Δ^1_2. Cf. [1].) Also, the argument for total definability from the proof of Proposition 5 applies without any changes. So the totally definable sets in M are precisely the arithmetical sets.

§4. We have seen that under J-determined Gödel numberings the totally definable sets are arithmetical. A variant style Gödel numbering will allow us to define Δ^1_1 sets totally. Call a special Gödel numbering (J, K)-*determined* iff the following conditions are satisfied: J and K are recursive sets and, for $j \in J$ and $k \in K$, $[j, m_1, m_2, k, n_1, n_2]$ codes:

$$[\underline{\mathrm{j}}, \underline{\mathrm{m}}_1, \underline{\mathrm{m}}_2, \underline{\mathrm{k}}, \underline{\mathrm{n}}_1, \underline{\mathrm{n}}_2] = 0'$$

(a falsehood) if $\neg R_j(m_2, m_1)$ or $\neg R_k(n_2, n_1)$, and codes

$$(x)(R_j(x, \underline{\mathrm{m}}_2) \supset (\exists y)T([\underline{\mathrm{j}}, \underline{\mathrm{m}}_2, x, \underline{\mathrm{k}}, \underline{\mathrm{n}}_2, y])) \;\&\; \underline{\mathrm{m}}_1\underline{\mathrm{n}}_1 = \underline{\mathrm{m}}_1\underline{\mathrm{n}}_1$$

otherwise. In the second case we have added the conjunct $\underline{\mathrm{m}}_1\underline{\mathrm{n}}_1 = \underline{\mathrm{m}}_1\underline{\mathrm{n}}_1$ to ensure that the coding is 1-1 effective; we will not generally write out this conjunct, and when we speak of an "instance" of such a special sentence we will mean an instance of its left conjunct.

LEMMA 8. *Suppose that a Gödel numbering is (J, K)-determined, $j \in J$, $k \in K$, $R_j(m_2, m_1)$ and $R_k(n_2, n_1)$. Then the following hold under both the strong and weak Kleene schemes*:

(a) *$m_2 \in U_j$ and $n_2 \in U_k$ iff $[j, m_1, m_2, k, n_1, n_2]$ is ungrounded.*

(b) *Suppose $m_2 \in G_j$ and $n_2 \in G_k$. Then*

$$|m_2|_j \leq |n_2|_k \quad \textit{iff} \quad [j, m_1, m_2, k, n_1, n_2] \textit{ is true in } \mathscr{L}_\alpha,$$

where $\alpha = |m_2|_j$, and $\{[j, m_1, m_2, k, n_1, n_2]\}$ is fixed at $|m_2|_j + 1$; and

$$|m_2|_j > |n_2|_k \quad \textit{iff} \quad [j, m_1, m_2, k, n_1, n_2] \textit{ is false in } \mathscr{L}_\alpha,$$

where $\alpha = |n_2|_k$, and $\{[j, m_1, m_2, k, n_1, n_2]\}$ is fixed at $|n_2|_k + 1$.

(c) *If $m_2 \in G_j$ but $n_2 \in G_k$, then $[j, m_1, m_2, k, n_1, n_2]$ is true in $\mathscr{L}_\alpha$, where $\alpha = |m_2|_j$, and is fixed at $|m_2|_j + 1$.*

(d) *If $n_2 \in G_k$ but $m_2 \in U_j$, then $[j, m_1, m_2, k, n_1, n_2]$ is false in $\mathscr{L}_\alpha$, where $\alpha = |n_2|_k$, and is fixed at $|n_2|_j + 1$.*

(e) *Furthermore, if $(S_1^{\mathrm{SK}}, S_2^{\mathrm{SK}})$ is the minimal fixed point of the strong Kleene scheme and $(S_1^{\mathrm{WK}}, S_2^{\mathrm{WK}})$ is the minimal fixed point of the weak Kleene scheme, then $S_1^{\mathrm{SK}} \cap$ Even $= S_1^{\mathrm{WK}} \cap$ Even and $S_2^{\mathrm{SK}} \cap$ Even $= S_2^{\mathrm{WK}} \cap$ Even.*

PROOF. (a)–(d) are proved by transfinite induction; (e) is immediate from the fact that the special sentences are valuated in the same way under both valuation schemes. Q.E.D.

We see that if the Gödel numbering is (J, K)-determined, for $k \in K$, R_k is well-founded on N, and $\text{l.u.b.}_{k \in K}\text{Ord}(R_k) = \alpha$, then Even $\subseteq S_{1,\alpha+1} \cup S_{2,\alpha+1}$, all special sentences have a truth value in $\mathscr{L}_\alpha$, and $\mathscr{L}_{\alpha+\omega}$ is the only fixed point of the language.

We say that a formula $A(x)$ defines a pair of sets (S_1, S_2) in $\mathscr{L}_\alpha$ iff $S_1 = \{n \mid \mathscr{L}_\alpha \models A(\underline{n})\}$ and $S_2 = \{n \mid \mathscr{L}_\alpha \Dashv A(\underline{n})\}$. This extends naturally to formulae with n free variables for $n \geq 1$.

PROPOSITION 6. *For any disjoint pair of Π^1_1 sets (S_1, S_2) there is a (J, K)-determined Gödel numbering and a formula $A(x)$ which defines (S_1, S_2) in the minimal fixed point.*

PROOF (Due to Kripke). Let (S_1, S_2) be a disjoint pair of Π^1_1 sets. Then there exist primitive recursive relations R_j and R_k such that

$$m \in S_1 \quad \text{iff} \quad [m, 1] \in G_j,$$
$$m \in S_2 \quad \text{iff} \quad [m, 1] \in G_k,$$

and we have that, for each m, $[m, 1] \in \text{Fld}_{R_j} \cap \text{Fld}_{R_k}$. Since S_1 and S_2 are disjoint, it follows that

(1) $$\text{if } [m, 1] \in G_j, \text{ then } [m, 1] \in U_k$$

and

(2) $$\text{if } [m, 1] \in G_k, \text{ then } [m, 1] \in U_j.$$

We consider the language with $(\{j\}, \{k\})$-determined Gödel numbering. We let $A(x)$ be the formula

$$(y)(R_j(y, [x, 1]) \supset (\exists z)T([\underline{j}, [x, 1], y, \underline{k}, [x, 1], z])).$$

Then, if $\mathscr{L}$ is the language of the minimal fixed point, $\mathscr{L} \models A(\underline{m})$ iff (by (1) and Lemma 8) $[m, 1] \in G_j$ and $[m, 1] \in U_k$ iff $m \in S_1$. Similarly, using (2) and Lemma 8, we have

$$\mathscr{L} \Dashv A(\underline{m}) \quad \text{iff} \quad m \in S_2.$$

Hence $A(x)$ defines (S_1, S_2) in $\mathscr{L}$. Q.E.D.

COROLLARY. (a) *Any Π^1_1 set is weakly definable in the minimal fixed point of some language with a $(\{j\}, \{k\})$-determined Gödel numbering, for appropriate j and k. Furthermore, for any Δ^1_1 set S there is a language with a $(\{j\}, \{k\})$-determined Gödel numbering in which S is totally definable in the minimal fixed point.*

(b) *The pairs of sets definable in the minimal fixed point on the (N, N)-determined Gödel numbering are precisely the disjoint pairs of Π^1_1 sets.*

(c) *In the (N, N)-determined Gödel numbering the sets weakly definable in the minimal fixed point are precisely the Π^1_1 sets, and the totally definable sets are all Δ^1_1 sets.*

(d) *Let T_{SK} and F_{SK} be the extension (resp. antiextension) of $T(x)$ in the minimal fixed point of the strong Kleene scheme under any Gödel numbering. Then the pair (T_{SK}, F_{SK}) is definable in the weak Kleene language with (N, N)-determined Gödel numbering.*

PROOF. (a) follows from the proof of the proposition by taking, for a given Π^1_1 set S, the pair (S, Λ). For a Δ^1_1 set S, we consider the pair $(S, -S)$.

(b) and (c). That the disjoint pairs of Π^1_1 sets are definable in the minimal fixed point follows as in the proof of the proposition. And clearly only disjoint pairs of Π^1_1 sets are definable in the minimal fixed point.

(d) follows from the fact, established by Kripke, that (T_{SK}, F_{SK}) is a disjoint pair of Π^1_1 sets. Q.E.D.

REMARKS. (1) Part (d) of the corollary extends to the pair (T_{VF}, F_{VF}), where T_{VF} and F_{VF} are the extension and antiextension of $T(x)$ in the minimal fixed point under the van Fraassen supervaluations scheme. (Cf. [1].)

(2) Let $\mathscr{L}_{(N,N)}$ be the language of the minimal fixed point of the weak Kleene scheme based on the (N,N)-determined Gödel numbering. We show that it is possible to define in $\mathscr{L}_{(N,N)}$ *pseudoconnectives* that correspond to the strong Kleene disjunction and existential quantification in the language of the minimal fixed point under any Gödel numbering. (This concept is a modified version of a concept introduced by Kripke.) Let $\mathrm{fv}(y)$ be the maximum index i of a free variable x_i in the formula with Gödel number y. We assume that a standard coding scheme of finite sequences of integers is given in which every integer codes some sequence. Let $\mathscr{L}(S_1, S_2)$ be the strong Kleene language of the minimal fixed point under an arbitrary Gödel numbering. Say that a formula $\mathrm{disjSK}(x,y,z)$ defines a *strong Kleene pseudodisjunction for* $\mathscr{L}(S_1, S_2)$ iff it defines a pair of relations (R_1, R_2) such that

$R_1(m,n,k)$ iff $m = \mathrm{GN}(A)$ and $n = \mathrm{GN}(B)$ for some formulae A and B and $\mathrm{length}(k) \geq \max(\mathrm{fv}(m), \mathrm{fv}(n))$, and $\mathscr{L}(S_1, S_2) \models A[k]$ or $\mathscr{L}(S_1, S_2) \models B[k]$,

where $A[k]$ is the sentence obtained by replacing the free variables x_j in A by the numerals for the jth member $(k)_j$ of the sequence coded by k, respectively, and

$R_2(m,n,k)$ iff $\neg\mathrm{Fmla}(m)$ or $\neg\mathrm{Fmla}(n)$ or $\mathrm{length}(k) < \max(\mathrm{fv}(m), \mathrm{fv}(n))$, or $\mathrm{Fmla}(m)$ and $\mathrm{Fmla}(n)$ and $\mathrm{length}(k) \geq \max(\mathrm{fv}(m), \mathrm{fv}(n))$, and $\mathscr{L}(S_1, S_2) \mathrel{=\!\!\!|} A[k]$ and $\mathscr{L}(S_1, S_2) \mathrel{=\!\!\!|} B[k]$.

Similarly, we say that a formula $\mathrm{EquantSK}(x,y,z)$ defines a *strong Kleene pseudoexistential quantification* iff it defines a pair (R_1, R_2) where

$R_1(m,n,k)$ iff $m = \mathrm{GN}(A)$ for some formula A and, for some k' such that $\mathrm{fv}(m) \leq \mathrm{length}(k')$ and $k' =_n k$, $\mathscr{L}(S_1, S_2) \models A[k']$,

letting $k' =_n k$ hold iff the sequences coded by k' and k differ at most at nth place, and

$R_2(m,n,k)$ iff $\neg\mathrm{Fmla}(m)$ or for all k' such that $\mathrm{fv}(m) \leq \mathrm{length}(k')$ and $k' =_n k$, $\mathscr{L}(S_1, S_2) \mathrel{=\!\!\!|} A[k']$.

It is easily seen that since the extension and antiextension of $T(x)$ in $\mathscr{L}(S_1, S_2)$ are both Π^1_1 sets, (R_1, R_2) are disjoint pairs of Π^1_1 relations. Hence the strong Kleene pseudodisjunction and pseudoexistential quantification for $\mathscr{L}(S_1, S_2)$ are both definable in the minimal fixed point of $\mathscr{L}_{(N,N)}$.

§5. We saw that what sets are definable in the minimal fixed point of $AR + T$ under the weak Kleene scheme is highly dependent on the choice of the Gödel numbering of the vocabulary. This variability vanishes if the language is sufficiently expanded to include new function symbols or a definite description operator.

We recall the primitive recursive function $U(x)$ and predicate $T_1(x, y, z)$ of [3, §58]. Let Ψ_e be the effective enumeration of 1-place partial recursive functions defined by

$$\Psi_e(x) \simeq U(\mu y T_1(e, x, y)).$$

Suppose the vocabulary of L is extended with a single primitive function symbol f to be interpreted by the 4-place function ϕ such that for any e, m, n, k,

$$\phi(e, m, n, k) = \begin{cases} U(n) & \text{if } n = (\mu y < n')T_1(e, m, y), \\ k & \text{otherwise.} \end{cases}$$

(Cf. [3] for the definition of the $\mu y < z$ operation.) Call the resulting vocabulary L'.

PROPOSITION 7. *Let $*$ and $^{\#}$ be any Gödel numberings of L', and let $\mathscr{L}^*$ and $\mathscr{L}^{\#}$ be the weak Kleene languages of the minimal fixed point defined relative to $*$ and $^{\#}$, respectively. If a pair (S_1, S_2) is definable in $\mathscr{L}^*$, then it is definable in $\mathscr{L}^{\#}$.*

PROOF. Let $k = \#(0 = 0')$. Using the recursion theorem, we define a recursive function $\psi(x) = \Psi(e_0, x)$ with Kleene index e_0 satisfying the following conditions:

$$\begin{aligned} \psi(*(t_1 = t_2)) &= \#((t_1 = t_2) \,\&\, (t_1 = t_2)), \\ \psi(*T(t)) &= \#(\exists z)T(f(\underline{e}_0, t, z, \underline{k})), \\ \psi(*\neg A) &= \text{neg}^{\#}(\psi(*A)), \\ \psi(*(A \,\&\, B)) &= \text{conj}^{\#}(\psi(*A), \psi(*B)), \\ \psi(*(\exists x_i)A) &= \text{equant}^{\#}(i, \psi(*A)). \end{aligned}$$

Here, $\text{equant}^{\#}(i, x)$ is the recursive function which yields $\#(\exists x_i)A$ whenever $x = \#A$, for any formula A of L'. The functions $\text{neg}^{\#}$ and $\text{conj}^{\#}$ are appropriate recursive functions similarly associated with $\neg$ and &. We assume that the variable z in $\psi(*T(t))$ does not occur in t. And whenever n is not of the form $*A$ for any L' formula A, we let $\psi(n) = \#(\underline{n} = \underline{n}')$. Thus ψ is 1-1. (The conjunction in the first clause of the definition of ψ is intended to ensure this.) We now claim that, for any L' sentence A,

$$(1) \qquad *A \in S^*_{1,\alpha} \text{ (resp. } S^*_{2,\alpha}) \quad \text{iff} \quad \psi(*A) \in S^{\#}_{1,\alpha} \text{ (resp. } S^{\#}_{2,\alpha})$$

for all α. We argue by transfinite induction on α.

The case where $\alpha = 0$ is trivial. Assume, as the induction hypothesis, that (1) holds for β; we show the same for $\beta + 1$. We proceed by induction on the complexity of sentences of L'. Suppose A is of the form $T(t)$, where $\text{val}(t) = *B$ for some L' sentence B. Then

$$\begin{aligned} &*T(t) \in S^*_{1,\beta+1} \text{ iff } *B \in S^*_{1,\beta} \\ &\quad \text{iff } \psi(*B) \in S^{\#}_{1,\beta} \quad \text{[by the main induction hypothesis]} \\ &\quad \text{iff } \phi(e_0, \text{val}(t), n, k) \in S^{\#}_{1,\beta} \text{ for some } n \\ &\qquad \text{and } \phi(e_0, \text{val}(t), j, k) \in S^{\#}_{1,\beta} \cup S^{\#}_{2,\beta} \text{ for each } j \\ &\qquad\qquad \text{[by definition of } \phi \text{, since } e_0 \text{ is a Kleene index of } \psi] \\ &\quad \text{iff } \mathscr{L}^{\#}_{\beta} \models (\exists z)T(f(\underline{e}_0, t, z, \underline{k})) \text{ iff } \psi(*T(t)) \in S^{\#}_{1,\beta+1}. \end{aligned}$$

An analogous argument shows that

$$*T(t) \in S^*_{2,\beta+1} \quad \text{iff} \quad \psi(*T(t)) \in S^\#_{2,\beta+1}.$$

The case where val(t) is not a Gödel number (under $*$) of any sentence of L' is straightforward, as are the remaining cases of the induction on the complexity of sentences of L'. The case when α is a limit ordinal follows immediately from the definition of limit stages. This completes the proof of (1).

Let α^* and $\alpha^\#$ be the ordinals of $\mathscr{L}^*$ and $\mathscr{L}^\#$, respectively. From (1) we have that $\alpha^* \leq \alpha^\#$. Suppose $A(x_1)$ defines a pair (S_1, S_2) in $\mathscr{L}^*$. Then, for any m,

$$\begin{aligned} *A(\underline{m}) \in S^*_{1,\alpha^*} &\Rightarrow \psi(*A(\underline{m})) \in S^\#_{1,\alpha^*} \Rightarrow \psi(*A(\underline{m})) \in S^\#_{1,\alpha^\#} \\ &\Rightarrow *A(\underline{m}) \in S^*_{1,\alpha^\#} \Rightarrow *A(\underline{m}) \in S^*_{1,\alpha^*}, \end{aligned}$$

since α^* is the ordinal of the minimal fixed point. Therefore,

$$m \in S_1 \quad \text{iff} \quad *A(\underline{m}) \in S^*_{1,\alpha^*} \quad \text{iff} \quad \psi(*A(\underline{m})) \in S^\#_{1,\alpha^\#}.$$

An analogous argument shows that

$$m \in S_2 \quad \text{iff} \quad *A(\underline{m}) \in S^*_{2,\alpha^*} \quad \text{iff} \quad \psi(*A(\underline{m})) \in S^\#_{2,\alpha^\#}.$$

For fixed $A(x_1)$, let e_1 be such that, for any m, $\Psi(e_1, m) = \psi(*A(\underline{m}))$. Then $(\exists y)T(f(\underline{e}_1, x_1, y, \underline{k}))$ defines (S_1, S_2) in $\mathscr{L}^\#$. Q.E.D.

COROLLARY 1. *Let* $*$ *and* $\#$ *be any Gödel numberings of* L', *and let* (S^*_1, S^*_2) *and* $(S^\#_1, S^\#_2)$ *be the minimal fixed points of* $\mathscr{L}^*$ *and* $\mathscr{L}^\#$, *respectively. Then* (S^*_1, S^*_2) *and* $(S^\#_1, S^\#_2)$ *are recursively isomorphic.*

This follows immediately from the proof of Proposition 7.

COROLLARY 2. *Let* $*$ *and* $\#$ *be any Gödel numberings of* L'. *For any set* S, S *is weakly (strongly, totally) definable in* $\mathscr{L}^*$ *iff* S *is weakly (strongly, totally) definable in* $\mathscr{L}^\#$.

REMARKS. (1) In stark contrast to classical languages as well as the partially interpreted language $\mathscr{L}$ of the minimal fixed point based on stronger valuation schemes, the sets definable in the augmented language with the function symbol f for the primitive recursive function ϕ may be different from those definable in the unaugmented weak Kleene language. This holds despite the well-known fact that within the language AR, whose vocabulary is contained in L', one can eliminate function symbols for any function χ whose graph $\chi(x_1, \ldots, x_n) = y$ is an arithmetical relation.

(2) Using the fact that the arithmetical part of L contains a term $[x, y, z, w]$ that determines a 1-1 primitive recursive coding of $N \times N \times N \times N$ by N, it is easy to modify the proof of Proposition 7 to show that the preceding remark holds when L is instead extended with a single function symbol for a 1-place primitive recursive function ϕ' defined so that $\phi'([x, y, z, w]) = \phi(x, y, z, w)$.

(3) Therefore, the weak Kleene language $\mathscr{L}$ of the minimal fixed point, obtained by expanding the vocabulary L with function symbols for any denumerable set $\mathscr{F}$ of total 1-place functions such that $\phi' \in \mathscr{F}$, is "well-defined" in the sense that the interpretation of $T(x)$ is recursively invariant (in the sense of [5]) under different choices of Gödel numbering.

(4) Proposition 7 still holds if we let $\mathscr{L}^*$ be the language of the minimal fixed point based on the original vocabulary $L = AR + T$ and Gödel numbering *. Hence the ordinal of the minimal fixed point of the weak Kleene language $\mathscr{L}'$ is ω_1^{CK}, the weakly definable sets are the Π_1^1 sets, and strongly definable = totally definable $= \Delta_1^1$.

(5) Furthermore, it can be shown that the addition of a definite description operator to L, or expanding L' with function symbols for total functions that are not primitive recursive, will not increase the strength of the resulting weak Kleene language (in terms of the sets definable in the minimal fixed point) so long as the enumeration of these functions is weakly definable in the minimal fixed point of the strong Kleene language. The reason for this is that the expanded language can still be embedded in the strong Kleene language by an effective truth-value preserving translation. In particular, this holds when the expanded vocabulary contains function symbols for all hyperarithmetical functions.

(6) To see that the cardinality of the set of fixed points of L' is the same for different choices of Gödel numberings—namely $2^{\aleph_0}$, as for the strong Kleene language—let * and $^\#$ be two Gödel numberings of L' and $\psi(x)$ a recursive function with index e_0 chosen as in the proof of Proposition 7. It is easily verified that if $T(t)$ is a "truth-teller" in L' under *—that is, $* T(t) = \text{den}(t)$—then its ψ-image is a "truth-teller", $(\exists z)T(f(\underline{e}_0, t, z, \underline{k}))$, under $^\#$, which is true just as long as $\psi(\text{den}(t))$ $[=(\exists z)T(f(\underline{e}_0, t, z, \underline{k}))]$ is true. We can then apply the argument of Proposition 1(b) to conclude that under *any* Gödel numbering $\mathscr{L}'$ must have exactly $2^{\aleph_0}$ fixed points.

(7) The primitive recursive orderings R referred to in Remark (2) following Proposition 4 are in fact elementary in the sense of Kalmár (see [3, §57] for the definition). Hence instead of the enumeration of primitive recursive relations considered in the definition of a J-determined Gödel numbering we can consider a primitive recursive enumeration of elementary relations. The arguments of the paper will not be affected, and all Gödel numberings considered will be primitive recursive.

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