On formal aspects of the epistemic approach to paraconsistency *

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Abstract

This paper reviews the central points and presents some recent developments of the epistemic approach to paraconsistency in terms of the preservation of evidence. Two formal systems are surveyed, the basic logic of evidence (*BLE*) and the logic of evidence and truth (LET_J), designed to deal, respectively, with evidence and with evidence and truth. While *BLE* is equivalent to Nelson's logic *N4*, it has been conceived for a different purpose. Adequate valuation semantics that provide decidability are given for both *BLE* and LET_J. The *meanings* of the connectives of *BLE* and LET_J, from the point of view of preservation of evidence, is explained with the aid of an *inferential semantics*. A formalization of the notion of evidence for *BLE* as proposed by M. Fitting is also reviewed here. As a novel result, the paper shows that LET_J is semantically characterized through the so-called *Fidel structures*. Some opportunities for further research are also discussed.

1 Introduction

Paraconsistency is the study of technical and philosophical aspects of formal systems in which the presence of a contradiction does not imply triviality, that is, systems with a non-explosive negation $\neg$ such that a pair of propositions A and $\neg A$ does not (always) lead to trivialization. Differently from classical (and intuitionistic) logic, in paraconsistent logics triviality is not tantamount to contradictoriness. Paraconsistent logics are able to deal with contradictory contexts of reasoning by means of the rejection of the principle of explosion, according to which anything follows from a contradiction.

From the philosophical point of view, maybe the most important question in paraconsistency addresses the nature of the contradictions allowed by paraconsistent logics. The answer to this question, of course, would better have some impacts on the formal systems. There are two basic approaches to this problem. On the one hand, the dialetheists claim that there are some *true contradictions* [**?**, e.g.]]dial.sta. This means that reality is contradictory in the sense that some pairs of contradictory propositions are needed in order to correctly describe reality. On the other hand, the epistemic approach to paraconsistent claims that it is much more plausible to consider that *all* contradictions that occur in real-life contexts of reasoning are epistemic in the sense that they are related to and/or originated in thought and language. The latter is the position endorsed by the authors of this text and has been already presented and defended in some papers (e.g. [Carnielli and Rodrigues(2016b)], [Carnielli and Rodrigues(2015)], [Carnielli and Rodrigues(2016d)]). Our aim here is to review the central points of the epistemic approach on paraconsistency, as well as to present some recent developments.

The remainder of this text is structured as follows. In the section 2, we start by explaining the duality between paraconsistent and paracomplete (so also intuitionistic) logics. We will show that the central point is not really a duality between logics, but rather a duality between *principles of inference* that may be added to a common core, obtaining thus paracomplete or paraconsistent logics. Next, in section 3, we will present the epistemic reading of contradictions in connection with conflicting evidence. Evidence is an epistemic notion, weaker than truth, that means 'reasons for believing/accepting' a proposition as true (or false). In sections 4 and 5 we present two formal systems, the basic logic of evidence (*BLE*) and the logic of evidence and truth (LET_J). *BLE* is a natural deduction system designed to preserve evidence. It can be seen that *BLE* coincides with Nelson's paraconsistent logic *N4*; however, the motivations and interpretations of both systems are different. LET_J is a logic of formal inconsistency and undeterminateness (**LFIU**) that adds to *BLE* means to recover classical logic for formulas that have been established as true (or false). LET_J, thus, is capable of talking simultaneously about preservation of truth and preservation of evidence. Section 6 presents complete and correct valuation semantics for *BLE* and LET_J. Such semantics, however, are better understood as tools to prove technical results than semantics in the sense of providing meanings to the formal system. That the meanings of the expressions in the context of *BLE* and LET_J are given bay the inferences allowed is the topic of Section 7, where an *inferential semantics* is proposed for the logics *BLE* and LET_J. Although the notion of preservation of evidence is defined by the logic *BLE* in a precise way, the notion of evidence presented in section 3 is only intuitively explained. In section 8 we show the formalization of the notion of evidence provided by Melvin Fitting using *justification logics* [Fitting(2016)]. Fitting has shown that *BLE* has both *implicit* and *explicit* evidence interpretations in a strictly formal sense. In section 9 a semantics of Fidel structures is presented for the logics *BLE* and LET_J. In spite of the fact that the algebraizability of LET_J has not been established yet, an 'algebraic-relational semantics' like the one here presented sheds light upon the algebraic aspects of this logic. Finally, in section 10, we will point at some possible topics for further inquiry and philosophical research in the field of paraconsistency.[1]

[1] Some parts of this text draw on other papers by the authors. Parts of sections 2 and 3 have appeared in [Carnielli and Rodrigues(2016d)]. The formal systems presented in sections 4 and 5, as well as the valuation semantics of section 6, appear in [Carnielli and Rodrigues(2016c)]. Section 7 sums up the ideas presented in [Carnielli and Rodrigues(2016a)].

2 On the duality between paraconsistency and paracompleteness

At first sight, it seems to be an easy conclusion that paraconsistent and intuitionistic logics are 'dual', since excluded middle does not hold in the latter and some contradictions are accepted in the former.[2] Indeed, if we take a look at how Newton da Costa devises C_1, the first logic of his C_n hierarchy [**?**, see]p. 499]costa1974, it is not difficult to see that there is a sort of 'informal duality' between C_1 and intuitionistic logic. In the former, excluded middle and introduction of double negation hold, although in the latter non-contradiction and double elimination of negation hold.

However, in our view, this approach to paraconsistency is somewhat misleading. The central point is not that the *logics* are dual, nor that excluded middle and non-contradictions are dual *formulas*, but rather that the *inference rules* excluded middle (*PEM*) and explosion (*EXP*) are dual. This is easily seen in the framework of sequent calculus and multiple-conclusion logic.

$$\frac{}{\Gamma \Rightarrow A, {\sim}A, \Delta}\ PEM \qquad \frac{}{\Gamma, A, {\sim}A \Rightarrow \Delta}\ EXP$$

Indeed, added to the positive fragment of Gentzen's system *LK* [Gentzen(1935)], the axioms above yield classical logic. Notice that although *PEM* and *EXP* are *axioms* of sequent calculus, they express the fact that, classically, $A \vee {\sim}A$ follows from anything, and anything follows from $A \wedge {\sim}A$. From the point of view of classical logic, the invalidity of *PEM* in paracomplete (for instance, intuitionistic) logics and the invalidity of *EXP* in paraconsistent logics are like 'mirror images' of each other.

Now, to see the duality from the semantical viewpoint, let us take a look at the semantic characterization of classical negation $\sim$. A negation is classical if the following conditions hold (for classical $\vee$ and $\wedge$):

$$A \wedge {\sim}A \vDash, \tag{1}$$

$$\vDash A \vee {\sim}A. \tag{2}$$

According to condition (1), there is no model *M* such that $A \wedge {\sim}A$ holds in *M*. (2) expresses the fact that for every model *M*, $A \vee {\sim}A$ holds in *M*. A *paracomplete* negation disobeys (2), and a *paraconsistent* negation disobeys (1). Intuitionistic negation is an example of a paracomplete negation. Each one of the conditions above corresponds to half of the classical semantic clause for negation, respectively:

$$M({\sim}A) = T \text{ only if } M(A) = F; \tag{3}$$

[2] Actually, the invalidity of the principle of non-contradiction is not an essential feature of paraconsistent logics, although the authors of this text share the opinion that both non-contradiction and explosion should be invalid in any paraconsistent logic. An example of a paraconsistent logic where explosion does not hold but non-contradiction is a valid formula is the Logic of Paradox [**?**, see]]priest.lp.

$$M({\sim}A) = F \text{ only if } M(A) = T. \quad (4)$$

The clause (3) above forbids that both A and ${\sim}A$ receive *True*, and the clause (4) forbids that both receive *False*. Given the classical account of logical consequence – *B* follows from *A* iff there is no model *M* such that *A* is true but *B* is false in *M* – from the conditions above it follows that anything is a logical consequence of $A \wedge {\sim}A$ and $A \vee {\sim}A$ is a logical consequence of anything.

A counterexample to the principle of explosion is given by a circumstance such that a pair of propositions A and $\neg A$ hold but a proposition B does not hold ($\neg$ being a paraconsistent negation). Dually, a paracomplete logic requires a circumstance such that both A and $\neg A$ do not hold (now $\neg$ is a paracomplete negation). Notice that neither a paracomplete nor a paraconsistent negation is a contradictory-forming operator, in the sense that applied to a proposition A they do not produce a proposition $\neg A$ such that A and $\neg A$ cannot receive simultaneously the value *F*, nor simultaneously the value *T* – i.e. they do not 'invert' the semantic value of *A*. Besides, neither a paracomplete nor a paraconsistent negation is a 'truth-functional' operator because the semantic value of $\neg A$ is not unequivocally determined by the value of A: in a paraconsistent logic, if A receives *T*, the value of $\neg A$ may be *T* or *F*, and in a paracomplete logic, if A receives *F*, $\neg A$ may be *T* or *F*. It is important to call attention to the fact that we talk about the semantic values *True* and *False* here as a 'façon de parler'. From the epistemic viewpoint proposed here, neither paraconsistent nor paracomplete logics are talking about truth.

3 Epistemic contradictions

We have seen above the duality between the failure of explosion and excluded middle, respectively, in paraconsistent and paracomplete logics. An example of an intuitive motivation for a paracomplete negation is given by intuitionistic logic, where a circumstance such that there is no constructive proof of A nor of $\neg A$ acts as a counterexample for excluded middle. Indeed, the usual proof by cases,

$$A \to B, \neg A \to B \vdash B,$$

cannot be performed in intuitionistic logic. But what would be a justification for a paraconsistent, non-explosive negation?

There are two basic answers to this question. The dialetheist claims that there are true contradictions [Priest and Berto(2013)], what means that contradictions, so to speak, 'belong to the essence of reality'. But since it is not the case that everything holds, a paraconsistent logic is needed in order to describe reality correctly. The other answer, already mentioned here, says that a non-explosive negation should be understood from the epistemic viewpoint.

The acceptance of A and $\neg A$ in some contexts of reasoning does not need to mean, and actually does not mean, that both are true. There are a number of circumstances in which we deal with pairs of propositions A and $\neg A$ such that there are good reasons for accepting and/or believing in both. It does not mean of course that both are true, nor that we actually *believe* that both are true, although we still want to draw inferences in the presence of them. We have already argued elsewhere that a non-dialetheist position in paraconsistency ascribes a property *weaker than truth* to a pair of propositions A and $\neg A$ that 'hold' in a given context [Carnielli and Rodrigues(2016c)]. We propose the notion of *evidence*, understood as 'reasons for believing/accepting a proposition', to play the role of such a property. There may be evidence that A is true even if A is false, and conflicting evidence occurs when there are reasons for accepting A and reasons for accepting $\neg A$, both simultaneous and non-conclusive.[3]

The reading of contradictions as conflicting evidence fits well with the practices of empirical sciences. There are an extensive literature about contradictions in sciences (e.g. [da Costa and French(2003)], [Nickles(2002)]). The notion of contradictions as conflicting evidence is in line with the view that empirical theories are better seen as *tools* to solve problems, rather than *descriptions* of the world (these two approaches are discussed by [Nickles(2002)]). Of course, the occurrence of contradictions is a problem for the descriptive view of theories, since the latter requires that such a representation be correct (i.e. true). Once this non-representational view of scientific work is accepted, contradictions in the empirical sciences are better viewed as originated in limitations of our cognitive apparatus, failure of measuring instruments and/or interactions of these instruments with phenomena, stages in the development of scientific theories or even simply mistakes, to be corrected.

4 *BLE*: the Basic Logic of Evidence

In this section, we present a natural deduction system, the *Basic Logic of Evidence* (*BLE*), suited to the reading of contradictions as conflicting evidence. *BLE* ends up being equivalent to Nelson's logic *N4*, but has been conceived for a different purpose (see Section 9.1). The rules of *BLE* intend to express preservation of evidence in the following sense: supposing the availability of evidence for the truth (or falsity) of the premises, we ask whether an inference rule yields a conclusion for which evidence for its truth (or falsity) is also available. This approach has an analogy to the inference rules for intuitionistic logic, when the latter is understood epistemically as concerned with the availability of a constructive proof. Indeed, the basic idea of the Brouwer-Heyting-Kolmogorov interpretation is that an inference rule is valid if it transforms constructive proofs for one or more premises into a constructive proof of the conclu-

[3] The use we make here of the notion of evidence is close to how evidence in understood in epistemology – see [Kelly(2014)], [Achinstein(2010)] and also [Carnielli and Rodrigues(2016c)].

sion. Natural deduction systems has been presented by [Gentzen(1935)] as formalisms capable of expressing 'natural logical reasoning'. Natural deduction fits our purpose here because we want to express how people actually, and naturally, draw inferences when the criterion is preservation of evidence.

Consider that the falsity of *A* is represented here by $\neg A$. 'Evidence that *A* is true' is understood as 'reasons for accepting/believing in *A*', and 'evidence that *A* is false' means 'reasons for accepting/believing in $\neg A$'. *BLE* is paraconsistent and paracomplete, neither explosion nor excluded middle hold. This is because there may be contexts with conflicting evidence as well as contexts with no evidence at all. In the former both A and $\neg A$ hold, in the latter both A and $\neg A$ do not hold.

DEFINITION 1. ***The basic logic of evidence BLE***
Consider the propositional language L_1 defined in the usual way over the set of connectives $\{\wedge, \vee, \rightarrow, \neg\}$. S_1 is the set of of formulas of L_1. Roman capitals stand for meta-variables for formulas of L_1. The following natural deduction rules define the logic BLE:

$$\frac{A \wedge B}{A} \wedge E \; \frac{A \wedge B}{B} \qquad \frac{A \quad B}{A \wedge B} \wedge I$$

$$\frac{A}{A \vee B} \vee I \; \frac{B}{A \vee B} \qquad \frac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \vee E$$

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \rightarrow I \qquad \frac{A \rightarrow B \quad A}{B} \rightarrow E$$

$$\frac{\neg A}{\neg(A \wedge B)} \neg \wedge I \; \frac{\neg B}{\neg(A \wedge B)} \qquad \frac{\neg(A \wedge B) \quad \begin{matrix}[\neg A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\neg B] \\ \vdots \\ C\end{matrix}}{C} \neg \wedge E$$

$$\frac{\neg A \quad \neg B}{\neg(A \vee B)} \neg \vee I \qquad \frac{\neg(A \vee B)}{\neg A} \neg \vee E \; \frac{\neg(A \vee B)}{\neg B}$$

$$\frac{A \quad \neg B}{\neg(A \rightarrow B)} \neg \rightarrow I \qquad \frac{\neg(A \rightarrow B)}{A} \neg \rightarrow E \; \frac{\neg(A \rightarrow B)}{\neg B}$$

$$\frac{A}{\neg\neg A} \quad DN \quad \frac{\neg\neg A}{A}$$

As an example, let us see how the preservation of evidence works w.r.t. the introduction rules for $\wedge$, $\vee$ and $\rightarrow$. If κ and κ' are evidence, respectively, for A and B, κ and κ' together constitute evidence for $A \wedge B$. Similarly, if κ constitutes evidence for A, then κ is also evidence for any disjunction that has A as one disjunct. For $\rightarrow I$, when the supposition that there is evidence κ for A leads to the conclusion that there is evidence κ' for B, this is evidence for $A \rightarrow B$. The implication, thus, works analogously to both classical and intuitionistic logic. It is not necessary that the contents of *A* and *B* be related.

The rules in which the conclusion is a negation of a conjunction, a disjunction or an implication cannot be obtained from the rules we already have because introduction of negation does not hold.[4] In order to obtain the negative rules we have to ask what would be sufficient conditions for having evidence for the falsity of a conclusion. So, if κ is evidence that *A* is false, κ constitutes evidence that $A \wedge B$ is false – *mutatis mutandis* for B. Thus, we obtain the rule $\neg \wedge I$. Analogous reasoning for disjunction and implication gives the respective introduction rules $\neg \vee I$ and $\neg \rightarrow I$.[5]

It is well-known that the elimination rules for $\wedge$, $\vee$ and $\rightarrow$ may be obtained from the introduction rules with the help of the *inversion principle*, presented by [Prawitz(1965)] as a refinement of the famous Gentzen's remarks that the introductions rules are, so to speak, 'definitions' of the connectives, and the eliminations rules are 'consequences' of these definitions [Gentzen(1935), p. 80]. Analogous reasoning works for the 'negative' elimination rules, $\neg \rightarrow E$, $\neg \wedge E$ and $\neg \vee E$. Suppose an application of the rule $\neg \rightarrow E$ that concludes *A* from $\neg(A \rightarrow B)$. *A* and $\neg B$ together are sufficient conditions for obtaining $\neg(A \rightarrow B)$. So, a derivation of the latter 'already contains' a derivation of *A*.[6] Notice that the negation rules exhibit a 'symmetry' with respect to the corresponding assertion rules for the dual operators.

5 A logic of evidence and truth

The logic *BLE* can express preservation of evidence. But in some contexts of reasoning we deal simultaneously with truth and evidence, that is, with propositions that are

[4] To see that $A \rightarrow B, A \rightarrow \neg B \vdash \neg A$ does not hold, suppose there is conflicting evidence for *B* and $\neg B$, but there is no evidence for $\neg A$. So, both $A \rightarrow B$ and $A \rightarrow \neg B$ hold, but $\neg A$ does not hold.

[5] The idea that natural deduction rules for concluding falsities may be obtained in a way similar to the rules for concluding truths is found e.g. in [López-Escobar(1972)] and also in [Prawitz(1965)]. Instead of asking about the conditions of assertability, the point is to ask about the conditions of *refutability*. This criterion works also for preservation of evidence.

[6] A more detailed account of the natural deduction rules of *BLE* is found in [Carnielli and Rodrigues(2016c)]. Regarding the inversion principle, see [Prawitz(1965), p. 33].

taken as conclusively established as true (or false), as well as others for which only non-conclusive evidence is available. Since preservation of truth is the criterion for a valid inference in classical logic, we get a tool for also dealing with true and false propositions if we can restore classical logic precisely for those propositions.

The *Logics of Formal Inconsistency* (from now on **LFI**s) are a family of paraconsistent logics that encompasses a great number of paraconsistent systems developed within the Brazilian tradition. **LFI**s are able to express the notion of 'consistency' of propositions inside the object language employing a unary connective: $\circ A$ means that A is consistent. Like any other paraconsistent logic, the principle of explosion does not hold in **LFI**s. But **LFI**s are so designed that some contradictions, that we call *consistent contradictions*, lead to triviality. Intuitively, one can understand the notion of a 'consistent contradiction' as a contradiction involving well-established facts, or involving propositions that have been conclusively established as true (or false) – notice that the point is precisely to prohibit consistent contradictions. A logic $\mathcal{L}$ is an **LFI** if the following holds:

For some Γ, *A* and *B*: $\Gamma, A, \neg A \nvdash B$,

For every Γ, *A* and *B*: $\Gamma, \circ A, A, \neg A \vdash B$.

LFIs start from the principle that propositions about the world can be divided into two categories: non-consistent and consistent ones. The latter are subjected to classical logic, and consequently a theory T that contains a pair of contradictory sentences $A, \neg A$ explodes only if A is taken to be a consistent proposition.[7]

The motivation of **LFI**s, restricting some logical property to some propositions, has been extended. In the *Logics of Formal Undeterminedness* (from now on **LFU**s), a class of paracomplete logics introduced in [Marcos(2005)], excluded middle can be restricted, and recovered, in a way analogous to **LFI**s restrict and recover explosion. Propositions can be divided into determined and non-determined ones, and a theory *T* may contain a proposition *A* such that neither *A* nor $\neg A$ hold. In an **LFU** the language is extended by a new unary connective $\star$, where $\star A$ means that A is (in some sense) determined. A logic $\mathcal{L}$ is an **LFU** if the following holds:

For some Γ, *A* and *B*: $\Gamma, A \vdash B, \Gamma, \neg A \vdash B$ but $\Gamma \nvdash B$,

For every Γ, *A* and *B*: if $\Gamma, A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma, \star A \vdash B$.

[7]The idea of expressing a metalogical notion within the object language is found, e.g. in the C_n hierarchy introduced by [da Costa(1963)], through the idea of 'well-behavedness' of a formula. In da Costa's hierarchy, however, this is done employing a definition: in C_1 it is expressed by A°, an abbreviation of $\neg(A \wedge \neg A)$, which makes the 'well-behavedness' of A equivalent to saying that A is non-contradictory. On the other hand, in the **LFI**s, $\circ A$ is introduced in such a way that allows $\circ A$ and $\neg(A \wedge \neg A)$ to be logically independent (non-equivalent). The family of **LFI**s incorporate a wide class of paraconsistent logics, as shown in [Carnielli et al.(2007)Carnielli, Coniglio, and Marcos] and [Carnielli and Coniglio(2016)].

An **LFI** and an **LFU** may be combined in an **LFIU** – a *Logic of Formal Inconsistency and Undeterminateness*. Explosion and excluded middle may be recovered at once with respect to a given formula *A*, and hence the properties of classical negation with respect to *A*. Since here we want to recover consistency and determinateness simultaneously, we use the symbol $\circ$ for both notions. The logic of evidence and truth obtained by extending *BLE*, is an **LFIU**.

DEFINITION 2. ***The logic of evidence and truth*** LET_J.
Consider the propositional language L_2 *defined in the usual way over the set of connectives* $\{\wedge, \vee, \rightarrow, \neg, \circ\}$. S_2 *is the set of of formulas of* L_2. *The logic of formal inconsistency and undeterminedness* LET_J *is defined by adding to BLE the rules below:*

$$\frac{\circ A \quad \begin{array}{c}[A]\\ \vdots\\ B\end{array} \quad \begin{array}{c}[\neg A]\\ \vdots\\ B\end{array}}{B}\ PEM^{\circ} \qquad \frac{\circ A \quad A \quad \neg A}{B}\ EXP^{\circ}$$

From 'outside' of the system, $\circ A$ means the truth-value of *A* has been conclusively established, or that there is *conclusive evidence* with respect to the truth-value of A. So, the fact that a proposition A is true is expressed as $\circ A \wedge A$, and the fact that A is false as $\circ A \wedge \neg A$.

The unary operator, $\circ$ may be called a *classicality* operator because when $\circ A_1, ..., \circ A_n$ hold, classical logic is recovered for all formulas that depend only on $A_1, ..., A_n$ and are formed with $\rightarrow$, $\wedge$, $\vee$ and $\neg$.[8]

6 Valuation semantics for *BLE* and LET_J

The valuation semantics to be presented in this section for *BLE* and LET_J does not intend to be a 'semantics' in the sense of a non-linguistic device that 'explains the meaning' of the corresponding deductive system – like, for example, the truth-tables for classical logic and the possible-worlds semantics for alethic modal logic. In the latter, the semantic clauses 'make sense' independently of the deductive system. On the other hand, the valuation semantics to be presented here is better seen as a mathematical tool capable of *representing* the inference rules in such a way that some technical results may be proved.

Valuation semantics have been proposed for the logics of da Costa's hierarchy C_n as a "generalization of the common semantics of the classical propositional calculus" [da Costa and Alves(1977), p. 622]. Later on, valuation semantics have been proposed also for da Costa's logic C_ω

[8]More details and several technical results that fit the intended intuitive interpretation of *BLE* and LET_J in terms of evidence and truth are to be found in [Carnielli and Rodrigues(2016c)]).

[Loparic(1986)], intuitionistic logic [Loparic(2010)] and several Logics of Formal Inconsistency (**LFI**s) ([Carnielli et al.(2007)Carnielli, Coniglio, and Marcos] and [Carnielli and Coniglio(2016)]). Given a language *L*, valuations are functions from the set of formulas of L to $\{0, 1\}$ in such a way that the semantic clauses are a kind of *representations* of the axioms. Roughly speaking, as we will see, assigning *1* and *0* to a formula *A* means, respectively, that *A* holds and *A* does not hold.

DEFINITION 3. *A semivaluation s for BLE is a function from the set S_1 of formulas to $\{0, 1\}$ such that:*

(i) if $s(A) = 1$ and $s(B) = 0$, then $s(A \to B) = 0$,

(ii) if $s(B) = 1$, then $s(A \to B) = 1$,

(iii) $s(A \wedge B) = 1$ iff $s(A) = 1$ and $s(B) = 1$,

(iv) $s(A \vee B) = 1$ iff $s(A) = 1$ or $s(B) = 1$,

(v) $s(A) = 1$ iff $s(\neg\neg A) = 1$,

(vi) $s(\neg(A \wedge B))$ = 1 iff $s(\neg A) = 1$ or $s(\neg B) = 1$,

(vii) $s(\neg(A \vee B)) = 1$ iff $s(\neg A) = 1$ and $s(\neg B) = 1$,

(viii) $s(\neg(A \to B)) = 1$ iff $s(A) = 1$ and $s(\neg B) = 1$.

DEFINITION 4. *A semivaluation s for LET$_J$ is a function from the set S_2 of formulas to $\{0, 1\}$ that satisfies the clauses (i)-(viii) of Definition 3 plus the following clause:*

(ix) if $s(\circ A) = 1$, then ($s(A) = 1$ if and only if $s(\neg A) = 0$).

DEFINITION 5. *A valuation for BLE/LET$_J$ is a semivaluation for which the condition below holds:*

(Val) For all formulas of the form $A_1 \to (A_2 \to ... \to (A_n \to B)...)$ with B not of the form $C \to D$:
if $s(A_1 \to (A_2 \to ... \to (A_n \to B)...)) = 0$, then there is a semivaluation s' such that for every i, $1 \leq i \leq n$, $s(A_i) = 1$ and $s(B) = 0$.

Logical consequence in *BLE* and *LET$_J$* is defined as usual: $\Gamma \vDash A$ if and only if for every valuation *v*, if $v(B) = 1$ for all $B \in \Gamma$, then $v(A) = 1$. The semantics above is sound and complete, and provides a decision procedure for *BLE* and *LET$_J$* by means of the *quasi-matrices* (see [Carnielli and Rodrigues(2016c)]). Below, as an example, we show how the quasi-matrices work.

EXAMPLE 6. *$p \to (\neg p \to q)$ is invalid in BLE.*

p	*0*							*1*				
$\neg p$	*0*				*1*			*0*			*1*	
q	*0*			*1*	*0*		*1*	*0*		*1*	*0*	*1*
$\neg p \to q$	*0*		*1*	*1*	*0*		*1*	*0*	*1*	*1*	*0*	*1*
$p \to (\neg p \to q)$	*0*	*1*	*1*	*1*	*0*	*1*	*1*	*0*	*1*	*1*	*0*	*1*
	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}

In the example 6 above, the semi-valuation s_{11} turns out to be a valuation that acts as a counter-example. Notice that *BLE* and LET_J are not compositional, in the sense that the semantic value of a complex formula is not always functionally determined by the semantic values of its component parts.

EXAMPLE 7. *$\circ p \to (p \vee \neg p)$ is valid in LET_J.*

p	*0*				*1*		
$\neg p$	*0*		*1*		*0*		*1*
$\circ p$	*0*		*0*	*1*	*0*	*1*	*0*
$p \vee \neg p$	*0*		*1*	*1*	*1*	*1*	*1*
$\circ p \to (p \vee \neg p)$	*0*	*1*	*1*	*1*	*1*	*1*	*1*
	s_1	s_2	s_3	s_4	s_5	s_6	s_7

In the example 7 above, the semi-valuation s_1 is not a valuation, since the clause *Val* of Definition 5 is not satisfied. *BLE*, being coincident with Nelson's logic *N4*, is an extension of positive intuitionistic logic *(PIL)*. Indeed, clauses (i)-(iv) of definition 3 plus clause (Val) of definition 5 give a valuation semantics for *PIL*.[9]

7 Inferential semantics for *BLE* and LET_J

According to the standard view, syntax is concerned with the formal properties of linguistic expressions without regard to their meanings. Syntax, thus, includes formulas, axioms, rules of inference and proofs – in sum: manipulation of symbols according to certain rules. The word 'semantics' in the broad sense has to do with the *meanings* of the linguistic expressions, and such meanings are given by how the expressions are 'related to reality'. However, when a semantics is given to a deductive system, it is not always the case that the respective semantic values 'explain the meanings' of the corresponding expressions. Especially in the case of non-classical logics, it is not uncommon that the semantics, although a useful tool for providing counter-examples,

[9] A more detailed presentation of valuation semantics for *BLE* and LET_J may be found in [Carnielli and Rodrigues(2016c)] and [Carnielli and Rodrigues(2016a)].

decision methods and other relevant results, actually does not give any explanation of the meanings of the expressions, let alone the deductive system as a whole. An example of this situation is precisely the valuation semantics presented above for *BLE* and LET_J. However, the semantics of a deductive system, in the broad sense of an *explanation of meaning*, may be provided by the syntax, that is, by how the system is used to make inferences.

The proof-theoretic – or inferential – semantics is an approach to meaning originated in the natural deduction for intuitionistic logic.[10] Differently from the truth-conditional theory of meaning, inferential semantics provides meanings to the connectives of intuitionistic logic without the need of a semantics in the standard sense, i.e. the attribution of semantic values to formulas. The meanings are given by the deductive system itself, or more precisely, by the inference rules, that in this case do not express preservation of (a transcendent notion of) truth, but rather preservation of the availability of a constructive proof. The 'link to reality', so to speak, is given by the deductive system, more precisely, by the introduction rules. Now, since the meanings no longer depend on the semantics, but have been given by syntax, it becomes clear that valuation semantics for intuitionistic logic are nothing but mathematical representations of the formal system.

The origin of this idea is in [Gentzen(1935)], where the natural deduction system *NJ* for intuitionistic logic is presented. There we find the passage already mentioned in section 4, according to which the introduction rules are 'definitions' of the respective symbols [Gentzen(1935), p. 80]. From this perspective, the meaning of the connective $\vee$, for example, is given by how we *use* it in inferences which are not concerned with preservation of truth, but rather with preservation of availability of a (constructive) proof. The introduction rules for disjunction say that having available a proof of *A* (or a proof of *B*) is a sufficient condition for having a proof of the disjunction $A \vee B$. Intuitionistically, a disjunction cannot be obtained otherwise.

We propose to expand the basic idea of inferential semantics to the paraconsistent logics *BLE* and LET_J. On what regards *BLE*, the point is how we *use* the connectives in inferences that preserve evidence. So, the meanings of the logical connectives is also given by the inference rules, but now in a context where what is at stake is *preservation of evidence*. The same idea applies to LET_J, that is able to deal simultaneously with evidence and truth. In LET_J, classical logic holds for formulas marked with $\circ$. Thus, we can say that for such formulas the meaning of the connectives is classical.[11]

[10] Since we are going to extend the idea of proof-theoretic semantics to paraconsistent logics that are not concerned with 'truth obtained by means of a proof', but rather with 'preservation of evidence', we prefer to use the more general expression 'inferential semantics'.

[11] A more detailed analysis of the natural deduction rules of *BLE* and LET_J regarding preservation of evidence and truth is given in [Carnielli and Rodrigues(2016c)].

8 A calculus for factive and unfactive evidence

The logics *BLE* and *LET*$_J$ define a notion of *preservation of evidence* in a precise way. But the corresponding notion of *evidence* is not formal. Melvin Fitting has provided in [Fitting(2016)] a formal alternative by means of the so-called *justification logics*. Fitting was able to show that *BLE* has both implicit and explicit evidence interpretations in a strictly formal sense. It is convenient to recall that *BLE* is presented through natural deduction rules, where the underlying idea is that rules should preserve evidence for an assertion, rather than its truth. A sequent calculus for the equivalent logic *N4* can be found in [Kamide and Wansing(2012)].

The plan followed in [Fitting(2016)] has a close analogy with the case of intuitionistic propositional logic, which is known since the work of [Gödel(1933)] to be embeddable into the modal logic $S4$. It was proved later (see [Artemov(2001)], [Artemov(2008)] and [Artemov and Fitting(2015)]) that $S4$ in turn embeds into the *strong justification logic* LP,[12] and the latter embeds into arithmetic. The logic LP provides a kind of calculus for certain *justification terms*. These terms can be regarded as representatives of proofs, and instead of $\Box A$ we may write $t : A$, where t is a justification term. The plan is the following:

1. It is first shown that *BLE* embeds into the modal logic $KX4$ explained below, a *logic of implicit uncertain evidence*, in which $\Box A$ can be interpreted as asserting that there is evidence for A, where this evidence can be partial or uncertain, and sometimes even incomplete and contradictory.

2. It is shown, furthermore, that $KX4$ in its turn embeds into the *justification logic* $JX4$, whose terms express pieces of uncertain evidence and are closed under certain operations that perform on such pieces of evidence.

The current axiomatization of the modal logic $S4$, inherited from Kurt Gödel, builds on the idea that $\Box$ has some intrinsic provability properties. The fact that intuitionistic logic embeds into $S4$ justifies the view that 'intuitionistic truth' can be understood as a version of provability *from the viewpoint of a classical mathematician*.

Provability may be considered as 'evidence of the strongest kind'. [Fitting(2016)] points out that provability coincides with the notion of evidence represented implicitly in $S4$, and explicitly in LP. Proofs can be seen as *factive* evidence, that is, evidence that is 'certain and never mistaken'. In contrast, the notion of evidence treated in *BLE* and represented implicitly in $KX4$, and explicitly in $JX4$, is *unfactive* in the sense of being disputable, retrievable or non-conclusive.[13]

[12] It should not be confused with the Logic of Paradox of Priest [Priest(1979)]
[13] We take the liberty to coin the term 'unfactive' due to its enlightening character.

$KX4$ is a normal modal (strict) subsystem of $S4$ obtained by dropping, precisely, the axiom of the factivity
$$\Box A \to A$$
and adding a new axiom schema called $C4$ or X for weaker or erroneous evidence:
$$\Box\Box A \to \Box A.$$
Informally, this schema expresses that evidence for the existence of evidence for A is sufficient to count as evidence for A. The other schemas for $KX4$ are the usual K:
$$\Box(A \to B) \to (\Box A \to \Box B)$$
and 4:
$$\Box A \to \Box\Box A$$
plus *modus ponens*. Obviously, in $KX4$
$$\Box\Box A \equiv \Box A$$
holds, which amounts to saying that evidence for the existence of evidence for some A is the same as evidence for A.

In this way, $KX4$ is an implicit logic of unfactive (or non-factive) evidence, in the same way $S4$ is a logic of provability (the term *implicit* refers to the fact that evidence is not explicitly shown, but just existential, as indicated by the modal operator $\Box$). As remarked in [Fitting(2016)], $KX4$ is complete with respect to frames meeting the conditions of transitivity and denseness.

The notion of 'implicit evidence against A' is also treated. Evidence is understood as something positive. The idea behind the rule $\neg \wedge I$, presented in Definition 1, is the following: if κ is evidence that A is false, κ constitutes evidence that $A \wedge B$ is false. An example given by Fitting illustrates this rule:

> We see that it is not raining, for instance, this is positive evidence that it is false that it is raining, and hence we have positive evidence that it is not both raining and cold. [Fitting(2016)]

This justifies a version of *implicit* evidence for *BLE*.

One of the two main results of [Fitting(2016)] reads:

THEOREM 8. *Theorem on Implicit Evidence for BLE:*
A is a theorem of BLE iff A^f is a theorem of $KX4$, where A^f is an inductively defined translation from the language of BLE into the language of $KX4$ using $\Box$ that reads as 'implicit evidence for A'.
Proof: see [Fitting(2016)].

An *explicit* counterpart to $KX4$, called $JX4$, can be obtained (omitting technical details) in such a way that $JX4$ serves as a justification counterpart of $KX4$ and is connected with it via a realization theorem, just as LP and $S4$ are connected in the sense of [Fitting(2015)].

The *justification formulas* of $JX4$ are built up from propositional letters using the usual propositional connectives, certain justification terms and additional justification formulas of the kind $t : A$, given by the following formation rule: if t is a justification term and A is a justification formula, then $t : A$ is a justification formula. Then it comes to the second main result of [Fitting(2016)]:

THEOREM 9. *Explicit Evidence for BLE:*
A^f is a theorem of $KX4$ if and only if some normal realization of A^f is a theorem of $JX4$.
Proof: see [Fitting(2016)].

The fact that the logic *BLE* embeds into the modal logic $KX4$ (via the Theorem on Implicit Evidence for *BLE*) justifies the view, in analogy with the intuitionistic case, that derivability in *BLE* can be understood as (preservation of) unfactive evidence *from the viewpoint of a classical philosopher*. The Theorem on Explicit Evidence for *BLE* grants that such evidence is rigorous and can be treated in a formal calculus. Several examples are given in [Fitting(2016)], while leaving as an open problem an investigation of *LET*$_J$ in terms of formalized implicit and explicit evidence.

9 An algebraic approach: Fidel structures for *BLE* and *LET*$_J$

9.1 Nelson's logic *N4* and the basic logic of evidence *BLE*: different views under equivalent formalisms

D. Nelson introduced in [Nelson(1949)] a constructible interpretation for the first-order number theory based on intuitionistic logic. Nelson's aims was to overcome what appears to be a non-constructive feature of the intuitionistic negation $\neg$. In Nelson's logic *N* of 1949 some principles valid in the standard intuitionistic logic are not valid – a remarkable example is the principle of non-contradiction $\neg(A \wedge \neg A)$ – and some principles intuitionistically invalid are valid in *N*. In the first-order system N for number theory obtained from Nelson's interpretation, the resulting negation called *strong negation* (here denoted by $-$) satisfies all the properties of a De Morgan negation as well as the following meta-property:

$$\vdash -(A \wedge B) \text{ implies } \vdash -A \text{ or } \vdash -B.$$

Indeed, from the constructive viewpoint, it seems plausible that supposing a formula $A \wedge B$ has been proved false, either a proof of the falsity of A or a proof of the falsity of B should be available.

In 1959 Nelson introduced a system called S based on positive first-order intuitionistic logic (see [Nelson(1959)]) which turned out to be paraconsistent for secondary reasons. Together with A. Almukdad, he later proposed, in 1984, a variant of S called N^- (see [Almukdad and Nelson(1984)]). This system became the standard presentation of Nelson's paraconsistent logic. [Odintsov(2003)] rebaptized N^- as *N4*, proving that it is sound and complete with respect to a class of algebras called *N4*-lattices, as well as with respect to a variant of an algebraic-relational class of structures originally introduced by M. Fidel in [Fidel(1977b)] for da Costa's calculi C_n and afterwards for Nelson's logic N in [Fidel(1980)]. This kind of structures, called *Fidel-structures* or **F**-*structures* in [Odintsov(2003)], will be adapted here (Section 9.3 below) to give a semantical characterization for system *LET*$_J$. It is worth noting that [Odintsov(2004)] also proposed an interesting semantics for *N4* in terms of *twist-structures*, a general semantical framework which was independently proposed by [Fidel(1977a)] and by [Vakarelov(1977)].

As we have mentioned, the logic *BLE* is equivalent to *N4*. However, we must emphasize that *BLE* has been found independently of *N4*, based on a completely different motivation – namely, a logic able to express the deductive behaviour of a notion weaker than truth in order to provide an intuitive and clear interpretation for paraconsistency negation that does not depend on the simultaneous *truth* of a pair of contradictory sentences. Of course, all the technical results valid for *N4* are also valid for *BLE*, but their intended meaning are rather divergent.

9.2 Fidel-structures semantics for *N4/BLE*

From the contemporary perspective, the relationship between logic and algebra comes back to the ideas of A. Lindenbaum and A. Tarski of interpreting the formulas of a given logic with the aid of algebras with operations associated to the logical connectives. This approach was generalized by W. Blok and D. Pigozzi in [Blok and Pigozzi(1989)], in order to encompass a wider range of logics. Afterwards, several generalizations of Blok and Pigozzi's technique were proposed in the literature (see, for instance, [Font and Jansana(2009)] and [Font(2016)]). However, several logic systems lie outside the scope of the general methods of contemporary algebraic logic. For instance, the logics of da Costa's hierarchy C_n are not algebraizable by these methods, and the same holds for most of the **LFI**s studied in the literature (see [Carnielli and Coniglio(2016)]).

In 1977, Manuel Fidel proved, for the first time, the decidability of the calculi C_n using an original algebraic-relational class of semantical structures called C_n-*structures* [**?**, see]]fidel.1977. This kind of structure was called *Fidel-structures* or

F-*structures* in [Odintsov(2003)] (see also [Odintsov(2008)]). Briefly, a C_n-structure is a triple $\langle \mathcal{A}, \{N_a\}_{a\in\mathbf{A}}, \{N_a^{(n)}\}_{a\in\mathbf{A}}\rangle$ such that $\mathcal{A}$ is a Boolean algebra with domain **A** and each N_a and $N_a^{(n)}$ is a non-empty subset of **A**. Intuitively, $b \in N_a$ and $c \in N_a^{(n)}$ means that b and c are possible values for the paraconsistent negation $\neg a$ of a and for the 'well-behavior' (or 'consistency') a° of a, respectively. Because of the previous observations, the use of relations instead of functions for interpreting these two 'non-truth-functional' connectives seems to be appropriate.

As observed in [Odintsov(2008)], the logic *N4* lies in an intermediary stage with regards to algebraizability: the usual equivalence

$$A \leftrightarrow B \overset{\text{def}}{=} (A \to B) \wedge (B \to A)$$

does not define a logical congruence with respect to negation. That is, it is possible that the negations of equivalent formulas are not equivalent. For instance, given a propositional variable p, the formulas $\neg(p \to p)$ and $\neg(p \to (q \to p))$ are not equivalent in this logic, despite $(p \to p)$ and $(p \to (q \to p))$ being both valid (and so equivalent). However, it is possible to define a *strong equivalence*

$$A \Leftrightarrow B \overset{\text{def}}{=} (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$$

which constitutes a logical congruence in *N4*. Because of this, the following *weak replacement property* holds in *N4* (and so in *BLE*):

PROPOSITION 10. *[Odintsov(2008), Proposition 8.1.3] The logic N4 [BLE] satisfies the following weak replacement rule:*

$$\text{if } \vdash A \Leftrightarrow B \text{ then } \vdash C[p/A] \leftrightarrow C[p/B]$$

for every formula C, where $C[p/A]$ (resp., $C[p/B]$) denotes the formula obtained from C by replacing the variable p by the formula A (by the formula B, resp.).

As proved in [Odintsov(2008), Section 8.4], there exists a class of algebraic structures called *N4*-lattices associated to the logic *N4*. The class of *N4*-lattices is a *variety*, that is, it can be axiomatized by a set of equations. As Odintsov has shown, the logic *N4* (and so *BLE*) is algebraizable in the sense of [Blok and Pigozzi(1989)] by means of the variety of *N4*-lattices. Despite this algebraic characterization, Odintsov obtained another characterization of *N4* in respect of **F**-structures, by generalizing the proposal by M. Fidel in 1979 for the original Nelson's system N (see [Fidel(1980)]).

All the results mentioned above, of course, hold also for *BLE*. However, differently of *N4/BLE*, it is not clear whether or not the extension *LET*$_J$ of *BLE* is algebraizable by Blok and Pigozzi's method. Indeed, it is possible to define, in a similar way to *N4*, the following equivalence formula:

$$A \rightleftharpoons B \overset{\text{def}}{=} (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B) \wedge (\circ A \leftrightarrow \circ B).$$

Clearly, it defines a logical congruence in LET_J, and so it induces a weak replacement property for LET_J analogous to that for *N4* stated in Proposition 10. It is an open problem to determine if this congruence is trivial, namely, whether or not it is the case that: if $\vdash A \rightleftharpoons B$ holds in LET_J then $A = B$, for every formulas A and B. More generally, it is a open problem to detemine if LET_J admits non-trivial logical congruences. This question justifies the present semantical approach to LET_J in terms of Fidel-structures, which expands the ones defined by Odintsov for the logic *N4*. The details of the construction will be described in Section 9.3.

Let us recall that an *implicative lattice* is an algebra $\mathcal{A} = \langle \mathbf{A}, \wedge, \vee, \to, 1\rangle$ where $\langle \mathbf{A}, \wedge, \vee, 1\rangle$ is a lattice with top element 1 such that there exists the supremum $\bigvee\{c \in \mathbf{A} \ : \ a \wedge c \leq b\}$ for every $a, b \in \mathbf{A}$. Here, $\leq$ denotes the partial order associated with the lattice, namely: $a \leq b$ iff $a = a \wedge b$ iff $b = a \vee b$; and $\bigvee X$ denotes the supremum of the set $X \subseteq \mathbf{A}$ w.r.t. $\leq$, whenever it exists. In addition, $\to$ is a binary operator (called *implication*) such that $a \to b \stackrel{\text{def}}{=} \bigvee\{c \in \mathbf{A} \ : \ a \wedge c \leq b\}$ for every $a, b \in \mathbf{A}$. It is well-known that, if an implicative lattice has a bottom element 0, then it is a Heyting algebra.

DEFINITION 11. *Fidel-structures for BLE (N4)*
A Fidel-structure for *BLE (or an* **F***-structure for BLE) is a pair*

$$\mathcal{E} = \left\langle \mathcal{A}, \{N_a\}_{a\in\mathbf{A}} \right\rangle$$

such that $\mathcal{A} = \langle \mathbf{A}, \wedge, \vee, \to, 1\rangle$ is an implicative lattice and $\{N_a\}_{a\in\mathbf{A}}$ is a family of nonempty subsets of **A** *where, for every $a, b, c, d \in \mathbf{A}$, the following holds:*

(1) if $c \in N_a$, then $a \in N_c$;

(2) if $c \in N_a$ and $d \in N_b$, then $c \wedge d \in N_{a\vee b}$;

(3) if $c \in N_a$ and $d \in N_b$, then $c \vee d \in N_{a\wedge b}$;

(4) if $d \in N_b$, then $a \wedge d \in N_{a\to b}$.

Intuitively, $c \in N_a$ means that c is a 'possible negation' $\neg a$ of a.

DEFINITION 12. *A valuation over an* **F***-structure $\mathcal{E} = \left\langle \mathcal{A}, \{N_a\}_{a\in\mathbf{A}} \right\rangle$ for BLE is a mapping v from the language L_1 to* **A** *satisfying the following:*

(1) $v(\neg p) \in N_{v(p)}$, for every propositional letter p;

(2) $v(A\#B) = v(A)\#v(B)$ for $\# \in \{\wedge, \vee, \to\}$;

(3) $v(\neg(A \wedge B)) = v(\neg A) \vee v(\neg B)$;

(4) $v(\neg(A \vee B)) = v(\neg A) \wedge v(\neg B)$;

(5) $v(\neg(A \to B)) = v(A) \wedge v(\neg B)$*;*

(6) $v(\neg\neg A) = v(A)$.

Let $\mathbf{P}$ be the set of propositional letters of L_1. A valuation is completely determined by its values over the set $\mathbf{P} \cup \{\neg p \ : \ p \in \mathbf{P}\}$. It is immediate to prove the following:

PROPOSITION 13. *Let v be a valuation over an $\mathbf{F}$-structure $\mathcal{E}$ for BLE. Then $v(\neg A) \in N_{v(A)}$ for every formula A.*

The semantical consequence relation associated with **F**-structures is defined in a natural way:

DEFINITION 14. *Let $\Gamma \cup \{A\} \subseteq L_1$ and let $\mathcal{E}$ be a Fidel-structure for BLE. Then, A follows from Γ in $\mathcal{E}$, written as $\Gamma \models^{\mathcal{E}}_{\mathbf{F}} A$, if, for every valuation v over $\mathcal{E}$, $v(A) = 1$ whenever $v(B) = 1$ for every $B \in \Gamma$. We say that A is a semantical consequence of Γ (w.r.t. Fidel-structures for BLE, denoted by $\Gamma \models^{\mathbf{F}}_{BLE} A$, if $\Gamma \models^{\mathcal{E}}_{\mathbf{F}} A$ for every $\mathbf{F}$-structure $\mathcal{E}$ for BLE.*

Then, the following holds (see [Odintsov(2008)]):

THEOREM 15. *Adequacy of BLE (N4) w.r.t. Fidel-structures Let $\Gamma \cup \{A\}$ be a set of formulas such that Γ is non-trivial in BLE. Then:*

$$\Gamma \vdash_{BLE} A \ \textit{ iff } \ \Gamma \models^{\mathbf{F}}_{BLE} A.$$

9.3 Fidel-structures semantics for *LET*$_J$

Recall that the logic *LET*$_J$ is an extension of *BLE* in the language L_2 obtained by adding the rules PEM° and EXP° to the latter (Definition 2). Given the adequacy of *BLE* w.r.t. **F**-structures (Theorem 15), it is natural to consider extensions of these **F**-structures, in order to capture semantically the logic *LET*$_J$.

DEFINITION 16. *Fidel-structures for LET$_J$*
A Fidel-structure for *LET*$_J$ *(or an* **F***-structure for LET$_J$) is a triple*

$$\mathcal{E} = \big\langle \mathcal{A}, \{N_a\}_{a \in \mathbf{A}}, \{O_a\}_{a \in \mathbf{A}} \big\rangle$$

where $\mathcal{A} = \langle \mathbf{A}, \wedge, \vee, \to, 0, 1 \rangle$ is a Heyting algebra, $\big\langle \mathcal{A}, \{N_a\}_{a \in \mathbf{A}} \big\rangle$ is a Fidel-structure for BLE (N4), and $\{O_a\}_{a \in \mathbf{A}}$ is a family of nonempty subsets of $\mathbf{A}$ such that, for every $a, b \in \mathbf{A}$, the following holds:

(FJ) *if $b \in N_a$ then $BD_{ab} \cap BC_{ab} \neq \emptyset$, where*

$$BD_{ab} = \big\{c \in O_a \ : \ c \to (a \vee b) = 1\big\}$$

and

$$BC_{ab} = \big\{c \in O_a \ : \ a \wedge b \wedge c = 0\big\}.$$

REMARK 17. *Let $\mathcal{A}$ be a Heyting algebra, and let $\div$ be the intuitionistic negation in $\mathcal{A}$, which is defined as $\div a = a \to 0$ for every $a \in \mathbf{A}$. For each $a \in \mathbf{A}$ let $a{\downarrow}$ be the set $\{b \in \mathbf{A} \ : \ b \leq a\}$. Observe that $a \wedge c = 0$ if and only if $c \in (\div a){\downarrow}$, and $c \to a = 1$ iff $c \in a{\downarrow}$, for every $a, c \in \mathbf{A}$. Then, condition (FJ) states that, if $b \in N_a$, then $O_a \cap (a \vee b){\downarrow} \cap (\div(a \wedge b)){\downarrow} \neq \emptyset$. Equivalently, (FJ) requires that, if $b \in N_a$, then $O_a \cap ((a \vee b) \wedge \div(a \wedge b)){\downarrow} \neq \emptyset$.*

Intuitively, $b \in N_a$ means that b is a 'possible negation' $\neg a$ of a, while $c \in O_a$ means that c is a 'possible recovery value' $\circ a$ of a coherent with a given $b \in N_a$. This is supported by the following definition:

DEFINITION 18. *A* valuation *over an* $\mathbf{F}$*-structure* $\mathcal{E} = \langle \mathcal{A}, \{N_a\}_{a \in \mathbf{A}}, \{O_a\}_{a \in \mathbf{A}} \rangle$ *for LET$_J$ is a map v from L_2 to $\mathbf{A}$ satisfying the clauses (2)-(6) of Definition 12, plus the following properties, for every formula A:*

(1) $v(\neg A) \in N_{v(A)}$;

(2) $v(\circ A) \in BD_{v(A)v(\neg A)} \cap BC_{v(A)v(\neg A)}$.

REMARK 19. *Given that $B \wedge \neg B \wedge \circ B$ is a bottom (that is, trivializing) formula in LET$_J$, for any formula B, then $\div A$ can be represented in LET$_J$ by $A \to (B \wedge \neg B \wedge \circ B)$. Being so, $v(\div A) = \div v(A)$ for every valuation v over an $\mathbf{F}$-structure $\mathcal{E}$ for LET$_J$.*

EXAMPLE 20. *Let $\mathbb{R}$ be the set of real numbers endowed with the usual topology generated by the open intervals of the form (a, b), $(-\infty, a)$ and $(a, +\infty)$. It is well-known that the set of open subsets of $\mathbb{R}$ constitutes a Heyting algebra $\Omega(\mathbb{R})$ where $1 = \mathbb{R}$, $0 = \emptyset$ and, for every $X, Y \in \Omega(\mathbb{R})$: $X \vee Y = X \cup Y$; $X \wedge Y = X \cap Y$; and $X \to Y = Int((\mathbb{R} \setminus X) \cup Y)$, where $Int(Z)$ denotes the* interior *of a subset Z of $\mathbb{R}$ (that is, the greatest open contained in Z). Hence $\div X = Int(\mathbb{R} \setminus X)$. Consider an $\mathbf{F}$-structure $\mathcal{E}$ over $\Omega(\mathbb{R})$ such that $(1, 3) \in N_{(0,2)}$. Let A, B two formulas and let v be a valuation v over $\mathcal{E}$ such that $v(A) = (0, 2)$ and $v(B) = (1, 3)$. Then $v(A \vee B) = v(A) \cup v(B) = (0, 3)$; $v(A \wedge B) = v(A) \cap v(B) = (1, 2)$; and $v(\div(A \wedge B)) = \div v(A \wedge B) = (-\infty, 1) \cup (2, +\infty)$. Thus, by Remark 17, the element $v(\circ A)$ of $O_{(0,2)}$ must be an open subset of $v(A \vee B) \cap v(\div(A \wedge B)) = (0, 1) \cup (2, 3)$.*

The next step is to prove that the proposed semantics for *LET$_J$* is adequate, that is, the logic *LET$_J$* is sound and complete w.r.t. Fidel-structures. The proof will be similar to the one obtained by Odintsov for *N4* (see [Odintsov(2008)]) and the adaptation to *mbC* given in [Carnielli and Coniglio(2016), ch. 6].

Let Γ be a non-trivial theory in *LET$_J$*, that is, a set of formulas such that $\Gamma \nvdash_{LET_J} A$ for some formula A. Define the following relation $\equiv_\Gamma$ between the formulas of L_2:

$$A \equiv_\Gamma B \ \text{ iff } \ \Gamma \vdash_{LET_J} A \to B \ \text{ and } \ \Gamma \vdash_{LET_J} B \to A.$$

It is immediate to prove that $\equiv_\Gamma$ is an equivalence relation. Moreover, $\equiv_\Gamma$ is a congruence w.r.t. the connectives in the language of positive intuitionistic logic (*PIL*). Denote by $[A]_\Gamma$ the equivalence class of each formula A and let

$$\mathbf{A}_\Gamma \stackrel{\text{def}}{=} L_2/_{\equiv_\Gamma} = \{[A]_\Gamma \ : \ A \in L_2\}$$

be the set of all the equivalence classes. From the observation above, it is possible to define the following operations:

$$[A]_\Gamma \# [B]_\Gamma \stackrel{\text{def}}{=} [A \# B]_\Gamma \qquad \text{for } \# \in \{\wedge, \vee, \to\}.$$

All these operations are well-defined, that is, they do not depend upon the representative chosen for each equivalence class. This means that

$$\mathcal{A}_\Gamma \stackrel{\text{def}}{=} \langle \mathbf{A}_\Gamma, \wedge, \vee, \to, 0_\Gamma, 1_\Gamma \rangle$$

(where $0_\Gamma \stackrel{\text{def}}{=} [p_1 \wedge \neg p_1 \wedge \circ p_1]_\Gamma$ and $1_\Gamma \stackrel{\text{def}}{=} [p_1 \to p_1]_\Gamma$) is a Heyting algebra, given that 0_Γ is a bottom element of the underlying implicative lattice. It is now possible to define from here an $\mathbf{F}$-structure for *LET*$_J$ by considering

$$N_{[A]_\Gamma} \stackrel{\text{def}}{=} \{ [\neg B]_\Gamma \ : \ B \in [A]_\Gamma \}$$

and

$$O_{[A]_\Gamma} \stackrel{\text{def}}{=} \{ [\circ B]_\Gamma \ : \ B \in [A]_\Gamma \}$$

for every $[A]_\Gamma \in \mathbf{A}_\Gamma$. This structure will be called *the Lindenbaum* $\mathbf{F}$*-structure for LET*$_J$ over Γ. Observe that this is coherent with the intuitive reading for the sets N_a and O_a given above.

PROPOSITION 21. *Let Γ be a non-trivial theory in LET$_J$, and let $\mathcal{A}_\Gamma$ and $\mathbf{A}_\Gamma$ as above. Then, the triple*

$$\mathcal{E}_\Gamma = \langle \mathcal{A}_\Gamma, \{N_a\}_{a \in \mathbf{A}_\Gamma}, \{O_a\}_{a \in \mathbf{A}_\Gamma} \rangle$$

is an $\mathbf{F}$-structure for LET$_J$.

Proof. The pair $\mathcal{E}_\Gamma = \langle \mathcal{A}_\Gamma, \{N_a\}_{a \in \mathbf{A}_\Gamma} \rangle$ is an $\mathbf{F}$-structure for *BLE*(*N4*) (see [Odintsov(2008)]). It remains to prove that the family $\{O_a\}_{a \in \mathbf{A}_\Gamma}$ satisfies the requirement (FJ) of Definition 16. Thus, let $[\neg B]_\Gamma \in N_a$ (for a given $a \in \mathbf{A}_\Gamma$). Then, $B \in a$ and so $a = [B]_\Gamma$. From this, $[\circ B]_\Gamma \in O_a$ satisfies:

$$[\circ B]_\Gamma \to \big(a \vee [\neg B]_\Gamma\big) = [\circ B]_\Gamma \to \big([B]_\Gamma \vee [\neg B]_\Gamma\big) = [\circ B \to (B \vee \neg B)]_\Gamma = 1_\Gamma$$

since $\circ B \to (B \vee \neg B) \equiv_\Gamma p_1 \to p_1$. In an analogous way it is proved that

$$a \wedge [\neg B]_\Gamma \wedge [\circ B]_\Gamma = [B]_\Gamma \wedge [\neg B]_\Gamma \wedge [\circ B]_\Gamma = [B \wedge \neg B \wedge \circ B]_\Gamma = 0_\Gamma$$

since $B \wedge \neg B \wedge \circ B \equiv_\Gamma p_1 \wedge \neg p_1 \wedge \circ p_1$. This means that condition (FJ) is satisfied. $\square$

We thus arrive at the desired result:

THEOREM 22 (Adequacy of LET_J w.r.t. Fidel-structures)**.** *Let $\Gamma \cup \{A\}$ be a set of formulas such that Γ is non-trivial in LET_J. The following conditions are equivalent:*

(1) $\Gamma \vdash_{LET_J} A$;

(2) $\Gamma \models_{\mathbf{F}} A$;

(3) $\Gamma \models_{\mathbf{F}}^{\mathcal{E}_\Gamma} A$.

Proof. (1) $\Rightarrow$ (2): This is the Soundness theorem, which can be proved in a straightforward way as usual. Indeed, it is enough to prove that all the rules of LET_J are valid w.r.t. **F**-structures.

(2) $\Rightarrow$ (3): It is an immediate consequence of Definition 14.

(3) $\Rightarrow$ (1): Let $v : L_2 \to L_2/_{\equiv_\Gamma}$ be the canonical mapping given by $v(B) = [B]_\Gamma$. By the very definition of $\mathcal{A}_\Gamma$, it follows that v is a valuation over $\mathcal{E}_\Gamma$ satisfying the following: $v(B) = 1_\Gamma$ iff $\Gamma \vdash_{LET_J} B$, for every formula B. Hence, $v(B) = 1_\Gamma$ for every $B \in \Gamma$, which, by hypothesis, implies that $v(A) = 1_\Gamma$. That is, $\Gamma \vdash_{LET_J} A$. $\square$

10 Final remarks

This paper reviewed the main points of the approach to paraconsistent with reference to preservation of evidence. The ideas presented also suggests a promising approach to the issue of logical pluralism. The difference between classical, intuitionistic and paraconsistent logics, the last two understood from the epistemic point of view, is what is being preserved – respectively, truth, availability of a constructive proof and availability of evidence. Notice that there is a kind of informal duality in this reading of these three logics, since proof is a notion stronger (and evidence weaker) than truth. This helps to understand that the pluralist perspective is perfectly coherent, and in principle nothing prevents these three logics to be combined in some kind of 'general approach to rationality'.

It is also worth noting that the formalization of the notion of evidence provided by M. Fitting, as surveyed in Section 8, is yet another indication that we have taken the correct path basing the epistemic approach on the (formal and informal) duality between paraconsistency and paracompleteness. Actually, there are several 'convergences' in our approach. As it has been mentioned, the logic *BLE* has been conceived independently of Nelson's *N4*, although they are equivalent. The 'evidence interpretation' of *BLE* is endorsed by the fact that *BLE* is related to justification logics, as Fitting has shown. The paper also proves in Section 9 that both *BLE* and LET_J are semantically characterized through **F**-structures, a kind of algebraic-relational semantic structures. There are, however, several points yet to be developed and investigated. The first is to check how much **F**-structures can help to solve the algebraizability problem for LET_J, an open problem by now (*BLE*, being equivalent to *N4*, is algebraizable in the sense of Blok and Pigozzi). The second problem was raised by Fitting: how to formalize the notions of implicit and explicit evidence for LET_J (as it was done for *BLE* in [Fitting(2016)]). The third, philosophically more ambitious, is how to frame the classical, intuitionistic and paraconsistent paradigms in terms of preservation of levels of evidence. This would be a leap towards a better understanding of logical pluralism.

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