PAIR PRODUCTION IN CLASSICAL ELECTRODYNAMICS

A. CARATI
Università di Milano, Dipartimento di Matematica
Via Saldini 50, 20133 Milano (Italy)

ABSTRACT
One of the most relevant features of Quantum Field Theory is the phenomenon of pair production, the existence of which, first suggested by Dirac, was not even suspected in the older theories. On the other hand Feynman, in the spirit of his spatio–temporal approach to quantum mechanics, showed how a description of pair production could be given within classical relativistic kinematics; in fact, he actually exhibited world–lines with the required properties in the framework of a nonlocal modification of classical electrodynamics conceived by Bopp. In the present paper we show how classical world–lines, just of the type required by Feynman to describe the phenomenon of pair production, naturally arise in classical electrodynamics. More precisely, we show that such world–lines occur as solutions of the well known Abraham–Lorentz–Dirac equation, which was originally designed to describe the motion of just a single point charge in selfinteraction with the electromagnetic field.

PACS numbers: 03.50.D, 03.65.B, 02.30.M
Running title: Classical pair production

1. Introduction. In the usual approach to special relativity, the motion of a particle is described by its world–line, i.e. a time–like path $x_\mu(s)$ in space–time parametrized by the proper time s, (by which we mean the arc–length divided by c; obviously, we refer here to the pseudo–riemannian metric, which we take in the form $g_{\mu\nu} = \mathrm{diag}(-c^2, +1, +1, +1)$, c being the speed of light). But the choice of the parametrization does not fix the orientation of the world–line, which can be oriented as the x_0 axis (so that $\dot{x}_0$ is positive) or in the opposite way; the latter choice corresponds to a comoving clock turning counterclockwise. From the physical point of view, obviously both choices are perfectly equivalent, and this corresponds to the fact the equations of motion are unchanged by the inversion $s \to -s$.

From the mathematical point of view, the nonuniqueness in the choice of the world–line orientation is due to the fact that the very definition of the proper time is given by

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = 1 - \frac{v^2}{c^2} \ , \tag{1}$$

which allows for both

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 - \frac{v^2}{c^2}} > 0 \quad \text{i.e.} \quad \dot{x}_0 > 0 \tag{2}$$

and

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\sqrt{1 - \frac{v^2}{c^2}} < 0 \quad \text{i.e.} \quad \dot{x}_0 < 0 \ . \tag{3}$$

Now a problem arises when it occurs that in a single world-line both orientations are present, for example with $\dot{x}_0$ positive in an interval of s and negative in an adjacent one, because in the latter interval the particle travels backward in time. This seems to naturally lead to the conclusion that world–lines of such a type are not physical, and cannot be observed in nature.

On the other hand Feynman showed, in a very simple example that will be recalled below, that world–lines of such a type can occur as solutions of relativistic equations of motions, and interpreted those world–lines, in a way suggested to him by Wheeler (see for example [1], and the review articles [2],[3]), as describing pair production or annihilation. Similar considerations were also made by Stueckelberg [4] and Havas [5]. Feynman's example can be described essentially as follows. Consider a particle on a line (the x_1 axis for example), impinging on a potential barrier $V(x_1)$ of height V_0, and as usual let proper time be initially oriented as time x_0, i.e. let $\dot{x}_0 > 0$. From conservation of energy E (which, we recall, is defined by $E = mc^2\dot{x}_0 + V$, where m is the particle's rest mass) one has

$$mc^2\dot{x}_0^- = mc^2\dot{x}_0^+ + V_0 \ ,$$

i.e.

$$\dot{x}_0^+ = \dot{x}_0^- - \frac{V_0}{mc^2} \ ,$$

where we have denoted by $\dot{x}_0^-$ and $\dot{x}_0^+$ the values of $\dot{x}_0$ before and after collision with the barrier respectively. So, if $V_0 > 2mc^2$ and $\dot{x}_0^-$ is sufficiently small (but obviously greater than 1), one has

$$\dot{x}_0^+ < -1 \ ,$$

i.e. a reversal of the orientation of the world–line at the collision point. Thus, if one has stipulated that, before collision, the particle is moving forward in time, then after collision the particle has to move backward in time, which might appear to be absurd. But Feynman suggests instead to overcome the problem by insisting in considering time always increasing with s; and this naturally leads to consider the pieces of a world–line with negative slope as representing the world–line of another particle (i.e. an "antiparticle"), just moving forward in time with 4–velocity $-\dot{x}_\mu$. In the very words of Feynman, with reference to a path in spacetime from point 1 (at the left of the barrier) to point 2 (right), impinging with the barrier at points P (left) and Q (right) respectively, where the time of P is larger than the time of Q: "*How would such a path appear to someone whose future gradually becomes past through a moving present? He would first see a single particle at* 1*, then at* Q *two new particles would suddenly appear, one moving into the potential to the left, the other out to the right. Later, at* P*, the one moving to the left combines with the original particle at* 1 *and they both disappear. We therefore have a classical description of pair production and annihilation.* " Now, it is clear that in the framework of pure mechanics Feynman's interpretation might appear arbitrary and with no foundation. But the situation changes if one considers the particle in self–interaction with the electromagnetic field. Indeed, suppose one is given a current $e\dot{x}_\mu$ with a given charge e and with $\dot{x}_0 < 0$ in an interval. If, following Feynman, we stipulate that we are describing an antiparticle with velocity $v_\mu = -\dot{x}_\mu$, i.e. with $v_0 > 0$, then this can be done by leaving the current unchanged, if one assigns to the antiparticle a charge $-e$. Thus, in the context of electrodynamics Feynman's interpretation turns out to be not only fully legitimate, but in a sense forced by the theory itself.

Feynman claimed that the "*curious feature*" described above appears (even in presence of a smooth potential) in a nonstandard version of electrodynamics considered by him, where the interaction between charges occurs not along light cones, but over a narrow range about them, as in a theory previously proposed by Bopp [6]. But, in the quoted article, the computations are not explicetly exhibited. Hence it seems to be of a certain interest to check whether kinematical world–lines of the type conceived by Feynman also occur in the framework of standard classical electrodynamics, for example in one of its simplest forms, namely that in which the particle motion is described as a solution of the Abraham–Lorentz–Dirac equation in an external force field. In the present paper we show that this is indeed the case.

Now, in searching for Feynman–type solutions of the Abraham–Lorentz–Dirac equation one meets with a difficulty of an analytical character. Indeed, it is obvious that world–lines describing pair production or annihilation necessarily cannot be smooth and must present angular points (otherwise they would be space–like in an interval of s), while on the other hand the solutions of the Abraham–Lorentz–Dirac equation are regular everywhere the external force is. So one is forced to consider force fields having at least a singular point; in fact, the world–lines presenting a discontinuous derivative at some value of s correspond precisely to solutions falling on the singular point in a finite time. Thus, the mathematical problem arises of exhibiting global solutions to the Abraham–Lorentz–Dirac equation having discontinuous derivatives at some points. Notice that, in principle, one could consider two solutions, one of which springing out from the singular point at a given time x_0, and the other one falling on the singular point at the same time x_0, but the function obtained by considering the first solution for times greater than x_0 and the second one for times smaller than x_0, cannot in general be considered as a global solution to the equation of motions, even though the world line built up in such a way were a continuous one.

The problem of looking for global solutions presenting angular points was rather easy in the nonstandard theory of Bopp considered by Feynman; indeed, such a theory was based on the action principle so that the problem of defining global solutions presenting angular points was reduced to searching for minima of the action integral, in the class of continuous functions with piecewise continuous derivative. But, at least to the author's knowlewdge no variational principle is available for the Abraham–Lorentz–Dirac equation, and so it is not possible to define global solutions through a prescription analogous to that of Feynman. However, use can be made of the property of analytic continuation of the solutions, at least in the case of an analytic external field of forces. We proceed as follows. We take the complex extension of the force field to complex values of its argument in the familiar way, and consider the Abraham–Lorentz–Dirac equation for complex values of the proper time too. As a result, the solution $x_\mu(s)$ of that equation will be an analytic function of the proper time s, having singularities for those values of s for which the particle falls on the singular points of the force field, and being well defined for all other values of s. In particular, the solutions will be well defined for real values of the proper time; thus, if a solution $x_\mu(s)$ will be real for real values of the argument (which is not guaranteed for a generic singularity), it should be considered as a global solution to the Abraham–Lorentz–Dirac equation. Consequently, if the world–line meets Feynman's kinematical requirements, it can be considered as representing a phenomenon of pair production or annihilation.

In this paper we implement such a procedure for a particular potential, by giving the solution as a series expansion in proper time, and showing that such a series has a well defined radius of convergence. The paper is organized as follows. In Section 2 the Abraham–Lorentz–Dirac is put, by a suitable change of variables, in a convenient form. In Section 3 some formal solutions are introduced having the kinematical properties required to describe pair production. Some technical estimates needed to prove that such formal solutions are actual solutions to the Abraham–Lorentz–Dirac equation are deferred to the Appendix. Some further remarks are given in Section 4.

Acknowledgements. I would like to thank prof. L. Galgani for following all phases of this work, with many dicussions and suggestions, together with my friends J. Sassarini, D. Bambusi and D. Noja, who are involved with me in a fascinating study of classical electrodynamics.

This work has been developed under the EC contract ERBCHRXCT940460 for the project " Stability and universality in classical mechanics".

2. The Abraham–Lorentz–Dirac equation. As is well known, in Cartesian coordinates x_μ, with $\mu = 0, \cdots, 3$, where as usual x_0 denotes time and x_k, $k = 1, 2, 3$ are the spatial coordinates, the Abraham–Lorentz–Dirac equation [7] has the form

$$\ddot{x}_\mu = \frac{e}{mc} F_{\mu\,\nu} \dot{x}^\nu + \frac{2e^2}{3mc^3} (\dddot{x}_\mu + \frac{\ddot{x}_\nu \ddot{x}^\nu}{c^2} \dot{x}_\mu) \ . \tag{4}$$

Here $F_{\mu\,\nu}$ represents an external electromagnetic field tensor, while the second term at the r.h.s. is the "force" due to the selfinteraction of the particle with the field. In the rest of the paper we consider a tensor $F_{\mu\,\nu}$ which corresponds to a static external electric field $\mathbf{E}(\mathbf{x})$, so that one can think of $F_{\mu\,\nu}$ as a an external mechanical force. To simplify the discussion we consider the particular case of an electric field $\mathbf{E}$ having a fixed direction, say the x_1 direction; thus, by considering appropriate initial data (i.e. with vanishing $\dot{x}_2$, $\ddot{x}_2$, $\dot{x}_3$ and $\ddot{x}_3$), one can actually consider the system as being spatially one–dimensional, namely having as unknowns only the two functions $x_0(s)$, $x_1(s)$. In place of the full system (4) one has thus the reduced one

$$\begin{aligned} \ddot{x}_0 &= \frac{e}{mc^2}E(x_1)\dot{x}_1 + \frac{2e^2}{3mc^3}(\dddot{x}_0 + \frac{c^2\ddot{x}_0^2 - \ddot{x}_1^2}{c^2}\dot{x}_0) \\ \ddot{x}_1 &= \frac{e}{m}E(x_1)\dot{x}_0 + \frac{2e^2}{3mc^3}(\dddot{x}_1 + \frac{c^2\ddot{x}_0^2 - \ddot{x}_1^2}{c^2}\dot{x}_1) \ . \end{aligned} \tag{5}$$

This system was often investigated also for singular force fields. In particular, for attractive fields (see for example [2], [8],[9], [10] and the references there quoted), the problem was debated whether it is possible for the particle to fall on the singularity or not. In consideration of the fact that, strangely enough, there is no general agreement between different authors on this point, we prefer to concentrate our attention on the case of a repulsive force. In such a case we are able to find solutions which do indeed fall on the center of force, and to express them as suitable power series.

We begin our analysis by recalling the well known fact that system (5) admits the costant of motion $c^2\dot{x}_0^2 - \dot{x}_1^2 = c^2$, as follows from the very definition of proper time; one can then introduce a new variable [9,7] z such that $\dot{x}_0 = \mathrm{Ch}\, z$, $\dot{x}_1 = c\mathrm{Sh}\, z$, so that one is led to the system

$$\begin{aligned} \varepsilon\ddot{z} &= \dot{z} - \frac{eE(x_1)}{mc} \\ \dot{x}_1 &= c\,\mathrm{Sh}\, z \ , \end{aligned}$$

where we have introduced the standard parameter $\varepsilon = \frac{2e^2}{3mc^3}$. By the way, this system closely resembles the nonrelativistic version of the one–dimensional Abraham–Lorentz–Dirac equation, which is in fact obtained by taking the limit $z \to 0$.

The appearance of trascendental functions is avoided if one introduces new variables a and v by $\dot{z} = a$ and $v = e^z$, so that $\mathrm{Sh}\, z = (v - 1/v)/2$ and $\mathrm{Ch}\, z = (v + 1/v)/2$; choosing moreover the electric field so that $\frac{eE(x_1)}{mc} = 1/x_1^3$, the system takes the form which we will actually study, namely

$$\begin{aligned} \varepsilon\dot{a} &= a - \frac{1}{x^3} \\ \dot{v} &= av \\ \dot{x} &= \frac{c}{2}(v - \frac{1}{v}) \ , \end{aligned} \tag{6}$$

where, for notational simplicity, we have denoted x_1 by of x.

Notice that x_0 has actually disapperared from system (6), and could be recovered by just integrating the relation

$$\dot{x}_0 = \frac{1}{2}(v + 1/v) \ ;$$

however this is not required for our purposes, because it is sufficient to remark that the sign of $\dot{x}_0$ is equal to that of v. Thus the solution describes a particle or an antiparticle according to whether v is positive or negative respectively. We can thus conclude that an angular point of the world–line, and consequently an event of pair creation or annihilation, occurs in correspondence with a zero of v or of $1/v$.

3. Special solutions corresponding to a pair production. We thus look for solutions falling on the singularity, and this naturally leads to make the *ansatz* that there exists a solution behaving asymptotically as $x \sim x_0 s^\alpha$, $v \sim v_0 s^\beta$, and $a \sim a_0 s^\gamma$ for $s \to 0$. Inserting such expressions into (6), and retaining only the leading terms in the limit $s \to 0$, the six free parameters α, x_0, β, v_0, γ, a_0 are easily determined, and turn out to be given by two sets: the first one is $\alpha = \frac{2}{3}$, $x_0 = \sqrt[3]{3/\varepsilon}$, $\beta = -\frac{1}{3}$, $v_0 = 4x_0/3c$, $\gamma = -1$ and $a_0 = -1/3$, while the second one is $\alpha = \frac{2}{3}$, $x_0 = \sqrt[3]{-3/\varepsilon}$, $\beta = \frac{1}{3}$, $v_0 = -3c/4x_0$, $\gamma = -1$ and $a_0 = 1/3$. Notice that, if there exists a solution with the considered asymptotic behaviour, it turns out that v in the former case, and $1/v$ in the latter, have a zero at $s = 0$. As mentioned at the end of the previous section, this means that the world line has an angular point at $x = 0$; moreover this is easily checked to correspond to a pair production.

Guided by this *ansatz*, we look for solutions being analytic as functions of $s^{\frac{1}{3}}$; more precisely we write

$$x(s) = \sum_{n=0}^{+\infty} x_n s^{\frac{n+2}{3}} \ , \quad v(s) = \sum_{n=0}^{+\infty} v_n s^{\frac{n-1}{3}} \ , \quad a(s) = \sum_{n=0}^{+\infty} a_n s^{\frac{n-3}{3}} \ . \tag{7}$$

The leading terms of the expansion (7) correspond to the first of the two sets of values for the parameter just determined; in the same way it would be possible to write down an expansion, the leading terms of which are given by the second set of values. In the rest of the paper we concentrate on the expansion (7), but we note that the other expansion gives actually a second solution of (6) which obviously differs from (7), but still represents a phenomenon of pair production.

Multiplying the first equation of (6) by x^3, and inserting into the resulting system the expansions (7) and the corresponding expansions for the derivatives, one finds

$$\begin{aligned}
&\sum_{n=0}^{+\infty} s^{\frac{n}{3}} \sum_{\sum k_i = n} \varepsilon(\frac{k_1}{3} - 1) a_{k_1} x_{k_2} x_{k_3} x_{k_3} - \sum_{n=0}^{+\infty} s^{\frac{n+3}{3}} \sum_{\sum k_i = n} a_{k_1} x_{k_2} x_{k_3} x_{k_3} - 1 = 0 \\
&\sum_{n=0}^{+\infty} s^{\frac{n-4}{3}} \frac{n-1}{3} v_n - \sum_{n=0}^{+\infty} s^{\frac{n-4}{3}} \sum_{k=0}^{n} a_k v_{n-k} = 0 \\
&\sum_{n=0}^{+\infty} s^{\frac{n-2}{3}} \sum_{k=0}^{n} 2\frac{k+2}{3} x_k v_{n-k} - c \sum_{n=0}^{+\infty} s^{\frac{n-2}{3}} \sum_{k=0}^{n} v_k v_{n-k} + c = 0 \ .
\end{aligned}$$

Equating the coefficients of the power of s of the same order, one finds a system of recursive relations from which all unknown coefficients of the series expansions are computed. The recursive relations are easily checked to be

$$
\begin{aligned}
&\varepsilon(\frac{n}{3}-1)a_n x_0^3 - 3\varepsilon a_0 x_0^2 x_n = -\varepsilon \sum_{\sum k_i=n,\, k_i\neq n} (\frac{k_1}{3}-1) a_{k_1} x_{k_2} x_{k_3} x_{k_3} \\
&\qquad\qquad\qquad\qquad\qquad + \sum_{\sum k_i = n-3} a_{k_1} x_{k_2} x_{k_3} x_{k_3} - \delta_{n,0} \\
&(\frac{n-1}{3} - a_0) v_n - v_0 a_n = \sum_{k=1}^{n-1} a_k v_{n-k} \\
&2\frac{n+2}{3} v_0 x_n + (\frac{4}{3} x_0 - 2cv_0) v_n = -\sum_{k=1}^{n-1} (2\frac{k+2}{3} x_k - cv_k) v_{n-k} - c\delta_{n,2} \ ,
\end{aligned}
\tag{8}
$$

where $n > 0$ and $\delta_{n,k}$ is the usual Kronecker symbol. The case $n = 0$ is a bit different, and one finds directly $x_0 = \sqrt[3]{3/\varepsilon}$, $v_0 = 4x_0/3c$ and $a_0 = -1/3$. The r.h.s. of (8) depends only on x_k, v_k and a_k for $k < n$. Thus the recursive relations can be considered as a linear system in the unknowns x_n, v_n and a_n, and the system will admit a unique solution if the matrix of the coefficients has a non vanishing determinant. In other words, the existence of a (formal) expansion for the solution to (6) in powers of $s^{\frac{1}{3}}$ is insured if the determinant of the matrix

$$
\mathcal{A}_n = \begin{pmatrix} (\frac{n}{3}-1)\varepsilon x_0^3 & 0 & -\varepsilon 3 a_0 x_0^2 \\ -v_0 & (\frac{n-1}{3} - a_0) & 0 \\ 0 & (\frac{4}{3}x_0 - 2cv_0) & 2\frac{n+2}{3} v_0 \end{pmatrix}
\tag{9}
$$

is nonvanishing for all n. But this is immediately checked to be the case. Indeed, with the given values for a_0, x_0 and v_0 one finds

$$
\det \mathcal{A}_n = \frac{2\varepsilon v_0 x_0^3}{27} (n(n-3)(n+2)+18) \ ,
$$

so that one has $\inf_{n\geq 1} |\det \mathcal{A}_n| = |20\varepsilon v_0 x_0^3/27| > 0$. Thus the power expansion is, at least formally, well defined. In the appendix it is shown that the expansion is convergent for sufficiently small $|s^{\frac{1}{3}}|$, and this concludes the proof that there exists at least one solution to the Abraham–Lorentz–Dirac equation describing the phenomenon of pair production.

4. Further comments. First we stress that we have chosen the potential $1/x^3$ just for the sake of illustration, but it is clear that the phenomenon described here should be quite general, occurring when the force field diverges at some point. On the other hand, the solution found here does not depend on any free parameter, and so it corresponds to a very particular choice of the initial data. In this respect the phenomenon appears as an exceptional one. The problem of understanding what occurs for generic initial data is still open, and we leave it for future studies.

APPENDIX

We prove here the convergence of the series expansion (6). This will be achieved through a geometric bound on the growth of the coefficients a_n, v_n and x_n. We remark, first of all, that the matrix $\mathcal{A}_n^{-1}$ turns out to be given by

$$\mathcal{A}_n^{-1} = \frac{1}{\det \mathcal{A}_n} \begin{pmatrix} \frac{2v_0}{3} n(n-2) & \frac{4\varepsilon x_0^3}{3} & -\frac{\varepsilon x_0^2 n}{3} \\ -\frac{2v_0^2}{3}(n+2) & \frac{2\varepsilon x_0^2 v_0}{9}(n+2)(n-3) & -\varepsilon v_0 x_0^2 \\ \frac{4v_0 x_0}{3} & \frac{\varepsilon x_0^4}{4}(n-3) & \frac{\varepsilon x_0^3}{9} n(n-3) \end{pmatrix} . \tag{10}$$

On the other hand, the recursive relation (8) can be expressed as $\mathbf{z}_n = \mathcal{A}_n^{-1}\mathbf{f}_n$, where we have denoted (a_n, v_n, x_n) by $\mathbf{z}_n$, and the r.h.s of (6) by $\mathbf{f}_n$. Using the cubic norm in $\mathbf{R}^3$, one obtains $|\mathbf{z}_n| \le ||\mathcal{A}_n^{-1}||\, |\mathbf{f}_n|$, with the estimate $||\mathcal{A}_n^{-1}|| \le K_1/n$, where K_1 ia a suitable constant. Introducing a sequence of positive numbers m_n defined by $m_1 = 1$ and, for $n > 1$, by the recursive relation

$$m_n = 3K_1 (\sum_{\sum k_i = n,\, k_i \ne n} m_{k_1} m_{k_2} m_{k_3} m_{k_4} + \sum_{\sum k_i = n-3} m_{k_1} m_{k_2} m_{k_3} m_{k_4} + 3 \sum_{k=1}^{n-1} m_k m_{n-k} + \delta_{n,2}) , \tag{11}$$

one easily shows, by straightforward algebra, that one has the bound $|\mathbf{z}_n| \le m_n$. Indeed, one easily checks by hand that $\mathbf{z}_1 = 0$, so that obviously $|\mathbf{z}_1| < 1 = m_1$; then one proceeds by induction. Supposing the bounds to be true for $k < n$, one bounds every norm at the r.h.s. of the inequality $|\mathbf{z}_n| \le ||\mathcal{A}_n^{-1}|||\mathbf{f}_n|$ by the corresponding quantity m_k; using the estimate $||\mathcal{A}_n^{-1}|| \le K_1/n$ one thus obtains the r.h.s. of (11), and it is then clear that m_n gives a bound for $|\mathbf{z}_n|$. To prove that the quantity m_n has at most a geometrical growth for increasing n, we use the method of the majorant function, taking as a majorant $M(s) = \sum_{n>1} m_n s^n$. Multiplying (11) by s^n and summing over n, one checks that $M(s)$ is the solution of the equation

$$M - s = 3K_1 \Big((1+s^3)(M^4 + (3x_0 + y_0)M^3 + (3x_0 y_0 + 6x_0^2)M^2) + 3M^2 + s^2 \Big) ,$$

vanishing for $s = 0$. The coefficients of this equation are analytic functions of s; furthermore $M = 0$ is a zero of first order to this equation when $s = 0$. It is a well known result that such a solution depends analytically on s, as s remains in an neighbourhood of $s = 0$. Thus the Taylor coefficients of $M(s)$, which coincide with m_n, grow at most geometrically; this in turn implies that the expansions (6) are convergent. This completes the proof.

REFERENCES

[1] R.P. Feynman, *Phys. Rev.* **74**, 939 (1948).

[2] G.N. Plass, *Rev. Mod. Phys.* **33**, 37 (1961).

[3] T. Erber, *The classical theory of radiation reaction*, Fortschr. der Phys. **9**, 343–392 (1961).

[4] E.C.G. Stueckelberg, *Remarque sur la création de paires de particules en théorie de relativité*, *Helv. Phys. Acta* **14**, 588–594 (1941).
[5] P. Havas *Acta Phys. Austr.* **3**, 342 (1949).
[6] F. Bopp, *Ann. der Phys.* **42**, 573–608 (1943).
[7] P.A.M. Dirac, *Proc. Royal Soc. (London)* A**167**, 148–168 (1938).
[8] C.J. Eliezer, *Proc. Cambridge Phil. Soc.*, **39**,173 (1943).
[9] G. Zin, *Nuovo Cimento*, **6**, 1 (1949).
[10] P. A. Clavier, *Phys. Rev.*, **124**, 616–621 (1961).