Rice and Rice-Shapiro Theorems for Transfinite Correction Grammars

John Case[1] and **Sanjay Jain**[2, *]

[1] Department of Computer and Information Sciences, University of Delaware, Newark, DE 19716-2586, United States of America. Email: case@cis.udel.edu
[2] Department of Computer Science, National University of Singapore, Singapore 117417, Republic of Singapore. Email: sanjay@comp.nus.edu.sg

Key words Recursion theory, Rice and Rice-Shapiro theorems, Transfinite correction grammars, Decidability
Subject classification 03D99

Hay and, then, Johnson extended the classic Rice and Rice-Shapiro Theorems for computably enumerable sets, to analogs for all the higher levels in the *finite* Ershov Hierarchy. The present paper extends their work (with some motivations presented) to analogs in the *transfinite* Ershov Hierarchy. Some of the transfinite cases are done for all transfinite notations in Kleene's important system of notations, $\mathcal{O}$. Other cases are done for all transfinite notations in a very natural, proper subsystem $\mathcal{O}_{\mathrm{Cantor}}$ of $\mathcal{O}$, where $\mathcal{O}_{\mathrm{Cantor}}$ has at least one notation for each constructive ordinal. In these latter cases it is open as to what happens for the entire set of transfinite notations in $(\mathcal{O} - \mathcal{O}_{\mathrm{Cantor}})$.

1 Introduction and motivation

In Section 1.1 we motivate and informally describe our correction grammar approach to indexing the sets in the levels, (finite and transfinite) of the Ershov Hierarchy [1, 2, 3]. A more formal describing and other mathematical preliminaries are in Section 2.

Then, in Section 1.2, first we describe the ordinary Rice and Rice-Shapiro Theorems for sets of grammars or indices for the computably (or recursively) enumerable (c.e.) sets [4]. Secondly, we describe their generalizations from Hay [5] and later Johnson [6] up into the *finite* Ershov Hierarchy whose levels are each based on bounded finite iteration of *differences* of c.e. sets [1]. Lastly, we briefly, informally describe our further generalizations herein up into the *transfinite* Ershov Hierarchy [2, 3].

The formal statements of our main new results are in Section 4. The proofs of our main results are based on various hardness results proven in Section 3.

1.1 Motivating and describing correction grammars

Burgin [7] suggested that a human knowing a language L may involve his/her storing a representation of the language in terms of *two* grammars, say g_1 and g_2: g_2 is used to "edit" errors of (make corrections to) g_1. In set-theoretic terms, the language L is represented as the difference $(L_1 - L_2)$, where L_i is the language generated by the grammar g_i. The pair $\langle g_1, g_2\rangle$ can thus be seen as a single description of (or "grammar" for) the language L. Burgin called these grammars *grammars with prohibition*. We prefer, as in [8, 9], to call them *correction grammars*. These correction grammars may be seen as modeling the self-correcting behavior of humans.[1] In our computability-theoretic context herein we first think of p as being a *correction grammar* for L if and only if $p = \langle i, j\rangle$ and $L = (W_i - W_j)$.[2] Hence, such a correction

* Supported in part by NUS grant number C252-000-087-001.
[1] For more discussion, see [9].
[2] W_i is the i-th c.e. set, where i codes a program for generating or for accepting W_i [4]; see formal definition in Section 2.

grammar $\langle i, j\rangle$ is an *index* from [5] for the *difference of c.e. sets*, i.e., for the *d.c.e. set*, $(W_i - W_j)$. The concept of correction grammar, then, provides a motivation for studying the indices of d.c.e. sets.

Extensions [8, 9], of the correction grammars paradigm are natural. The concept is generalizable to descriptions of languages as finite, fixed differences of c.e. languages. This idea formalizes the concept of a finite, fixed number of successive editings for errors. For integer $n > 0$, an *n-correction grammar* for a language L is a p such that $p = \langle i_1, \ldots, i_n\rangle$ and $L = (W_{i_1} - (W_{i_2} - \cdots (W_{i_{n-1}} - W_{i_n})\cdots))$. This extension may model humans' tendency to correct, bounded iteratively, their linguistic utterances. Clearly, the n-correction grammar $\langle i_1, \ldots, i_n\rangle$ can be thought of as an *index* from [6] for the *n-c.e. set*, $(W_{i_1} - (W_{i_2} - \cdots (W_{i_{n-1}} - W_{i_n})\cdots))$ from the Σ^{-1}_n level in the finite Ershov Hierarchy [1]. Hence, n-correction grammars provide a motivation for studying these indices of n-c.e. sets.

As we will spell out in more detail below in Section 2, the formal concept of a *general* correction grammar can be equivalently expressed in terms of algorithms that initially exclude all candidate items but are allowed, for each candidate item, a finite number of mind-changes about whether to include in or exclude from the language that candidate item (in other words, correction grammars are algorithms for limiting computable functions — that initially output 0 for exclusion). For example, a correction grammar $\langle g_1, g_2\rangle$ for $L = L_1 - L_2$ can be equivalently thought of as an algorithm that initially excludes each item x and, then, can change its mind about x's inclusion or exclusion up to *twice* on the way to giving its final, correct answer as to whether x is included or excluded in L.[3] More generally, for $n > 0$, an n-correction grammar is equivalent to an algorithm that initially excludes each item x and, then, it can change its mind about x's inclusion or exclusion up to n times on the way to giving its final, correct answer as to whether x is included or excluded.

A next mathematical step is to extend the notion of correction grammars into the *constructive transfinite*. We explain briefly. We will assume herein that the reader already knows the material about ordinals, including constructive ordinals, from [4]. Intuitively, constructive ordinals are equivalent to those that have a program, called a *notation*, which specifies how to build them algorithmically or lay them out end-to-end [4]. Our concept of a general correction grammar is based on the idea of using constructive ordinal notations to bound the number of corrections that the grammar can make. To do this rigorously, we provide below in Section 2 the concept of (algorithmic) counting-down from notations for (countable, constructive) transfinite ordinals.

For example, it can be shown that counting down corrections allowed from any notation for ω is equivalent to declaring algorithmically, at the time a first correction is made, a finite number bound on the number of *further* corrections to be allowed. This is more powerful than just initially setting a *fixed, finite* number of corrections allowed. As another example consider using a notation for the ordinal $\omega + \omega$ (two copies of ω laid end to end), also constructive and transfinite: in this case, at the time the first correction is made, the algorithm declares a finite bound on the number of further corrections it is going to make; this bound is, however, allowed to be changed once, at a later time. For a notation for the constructive ordinal $\omega + \omega + \omega$, the algorithm is allowed to update the bound twice. For at least natural notations for the constructive ordinal ω^2, the algorithm is allowed to make a finite number of changes to the bound, where the maximum number of changes allowed to the bound is announced at the time the algorithm makes the first correction!

In the present paper we primarily (but not exclusively) employ Kleene's important general system $\mathcal{O}$ (coded as a proper subset of the set of natural numbers) [10, 11, 12, 4, 13]. This system has at least one notation for each constructive ordinal and comes with Kleene's standard, useful order relation $<_{\mathcal{O}}$ on the notations in $\mathcal{O}$ and naturally embedding into the ordering of the corresponding constructive ordinals.

We also employ a very natural, proper subsystem of Kleene's $\mathcal{O}$ which we call $\mathcal{O}_{\mathrm{Cantor}}$. In Section 2.2 below we discuss computable operations which, on $\mathcal{O}$-notations, provide notational analogs of $+, \times$, and exponentiation for ordinals. $\{^{\cdot}\}_{\mathcal{O}}$ is the computable operation which, on $\mathcal{O}$-notations, is analogous to ordinal exponentiation. Fix an $\mathcal{O}$-notation w for ω. Suppose $v \in \mathcal{O}$. Suppose v is a notation for

[3] Of course, an ordinary grammar (c.e. index) g for a c.e. language L can be thought of as an algorithm that initially excludes each item x and, then, it can change its mind about x's inclusion or exclusion up to *once* on the way to giving its final, correct answer as to whether x is included or excluded in L.

the (constructive) ordinal β. Then the $\mathcal{O}$-notation $\{w^v\}_{\mathcal{O}}$ is a natural $\mathcal{O}$-notation for the ordinal ω^β. $\mathcal{O}_{\text{Cantor}} \stackrel{\text{def}}{=} \{x \in \mathcal{O} \mid (\exists v \in \mathcal{O})[x \leq_{\mathcal{O}} \{w^v\}_{\mathcal{O}}]\}$. Corollary 2.4 in Section 2.2 below provides a very pleasant normal form for the notations in $\mathcal{O}_{\text{Cantor}}$. $\mathcal{O}_{\text{Cantor}}$ also has at least one notation for each constructive ordinal.

For notations $u \in \mathcal{O}$ for constructive ordinals (finite or transfinite), our *u-correction grammars* are (by definition) each algorithms for counting down corrections from u. Essentially, then, from [14, 8, 9], a u-correction grammar p can be thought of as an *index* for the corresponding *u-c.e. set* from the Σ_u^{-1} level in the general Ershov Hierarchy [1, 2, 3]. Hence, u-correction grammars provide a motivation for studying these indices of u-c.e. sets.[4]

For notation u in $\mathcal{O}$ and for one of our u-correction grammars p, we let W_p^u denote the u-c.e. set defined by p. It is essentially shown in [8, 9] how to handle defining the u-correction grammars so that $W_p^u, p = 0, 1, \ldots$ defines an acceptable programming system (numbering) of the u-c.e. sets, i.e., of the class Σ_u^{-1}.[5]

By definition, the *acceptable* programming systems for a class are those: which contain a universal simulator for the class and into which all other universal programming systems for the class can be compiled. Acceptable systems for a class are characterized as the universal systems for the class with an algorithmic substitutivity principle called S-m-n [4, 24, 25, 8, 9]. Acceptable systems also satisfy convenient self-reference principles such as Recursion Theorems [4, 24, 25, 8, 9].

For non-negative integers n we identify n with the finite well-ordering $\{0 < 1 < 2 < \cdots < (n-1)\}$. Kleene's partially computable and unique *$\mathcal{O}$-notation* for the well-ordering n is not n itself but some other number which we will write as $\underline{n}$.[6] Hence, in Section 1.2 just below where we spell out the previous work of Johnson [6] on Rice and Rice-Shapiro Theorems in the *finite* Ershov Hierarchy, where she indexes (or would index) by n, we'll index by $\underline{n}$. Also, in our informal discussion above of the *finite* levels of the Ershov Hierarchy, we indexed by n, but from now on we'll index those levels instead by $\underline{n}$. This will simplify our discussion re our extending herein Johnson's work up into the *transfinite* Ershov Hierarchy.

1.2 Introduction to Rice theorems

The Rice Theorem for c.e. sets is as follows [26, 4].

Theorem 1.1 (Rice Theorem) *Suppose $\mathcal{C}$ is a class of c.e. sets. Then, $\{i : W_i \in \mathcal{C}\}$ is computable iff, $\mathcal{C}$ is empty or the entire class of c.e. sets.*

We let D_q denote the finite subset of non-negative integers with *canonical* index q [4].

A theorem, called the Rice-Shapiro Theorem, conjectured by Rice [26] and independently proved by Shapiro, Myhill, and McNaughton (see [27, 4]) can be formulated (as in [5]) as follows.

Theorem 1.2 (Rice-Shapiro Theorem) *Suppose $\mathcal{C}$ is a class of c.e. sets. Then, $\{i : W_i \in \mathcal{C}\}$ is c.e. iff, $\mathcal{C}$ is empty or there exists a computable function f such that $\mathcal{C} = \{W_i : (\exists u)[D_{f(u)} \subseteq W_i]\}$.*

---

[4] Even if u is a notation for a transfinite constructive ordinal, count-downs from u are finite since there are no infinite descending chains of notations under u.

Ones employing notations to perform *algorithmic* count-down from transfinite (constructive) ordinals is widely used in Proof Theory (e.g., to measure the strength of formal systems and classify their provably total functions) [15, 16], and in Term Rewriting (e.g., to prove termination of rewrite systems) [17, 18]. In Computational Learning Theory, this idea was introduced by Freivalds and Smith in [19].

We have not considered herein notations or programs for non-well orderings, with, for example, no computable infinite descending chains [2, 20, 21, 22].

[5] This was originally to make sure such systems existed, and it was shown how to get them uniformly algorithmically in u. Herein, to simplify exposition, we'll imagine ourselves to be working in these particular W^u-systems. However, fortunately, we'll be able to do so without having to know in extreme detail how they actually work. Furthermore, our results are all independent of which acceptable system for the Σ_u^{-1}s are actually employed.

The reader should note, for example, that for our above fixed $w \in \mathcal{O}$, a notation for ω, while our w-c.e. sets are the Σ_w^{-1} sets, they are not at all the same as the well known ω-c.e. sets, the sets for which there is an also initially empty, effective approximation of their elements but where the number of mind changes are bounded by some computable function [23]. The ω-c.e. sets all belong to the more restricted Ershov level $\Delta_w^{-1} = (\Sigma_w^{-1} \cap \Pi_w^{-1})$.

[6] Herein we won't need to spell out in detail Kleene's exact numerical coding for his notations in his $\mathcal{O}$, but the interested reader can consult, e.g., [4].

Hay [5] lifted these results to classes of d.c.e. sets, i.e., to the $\underline{2}$-c.e. sets, essentially as follows, where we first present her analog of the Rice Theorem, and, then, her analog of the Rice-Shapiro Theorem.

Theorem 1.3 (Hay [5]) *Suppose $\mathcal{C}$ is a class of $\underline{2}$-c.e. sets. Then, $\{i : W_i^{\underline{2}} \in \mathcal{C}\}$ is c.e. iff, $\mathcal{C}$ is empty or the entire class of $\underline{2}$-c.e. sets.*

Theorem 1.4 (Hay [5]) *Suppose $\mathcal{C}$ is a class of $\underline{2}$-c.e. sets. Then, $\{i : W_i^{\underline{2}} \in \mathcal{C}\}$ is $\underline{2}$-c.e. iff, $\mathcal{C}$ is empty or there exists a finite set S such that $\mathcal{C} = \{W_i^{\underline{2}} : S \subseteq W_i^{\underline{2}}\}$.*

Johnson [6], then, for each natural number $n > 2$, lifted these results (respectively, analogs of the Rice Theorem and the Rice-Shapiro Theorem) to classes of $\underline{n}$-c.e. sets as follows.

Theorem 1.5 (Johnson [6]) *Suppose $\mathcal{C}$ is a class of $\underline{n}$-c.e. sets for natural number $n > 2$. Then: $\{i : W_i^{\underline{n}} \in \mathcal{C}\}$ is $(\underline{n-1})$-c.e. iff, $\mathcal{C}$ is empty or the entire class of $\underline{n}$-c.e. sets.*

Theorem 1.6 (Johnson [6]) *Suppose $\mathcal{C}$ is a class of $\underline{n}$-c.e. sets for natural number $n > 2$. Then: $\{i : W_i^{\underline{n}} \in \mathcal{C}\}$ is $\underline{n}$-c.e. iff, $\mathcal{C}$ is empty, or $\mathcal{C} = \Sigma_{\underline{n}}^{-1}$, or there exists a non-negative integer a such that $\mathcal{C} = \{W_i^{\underline{n}} : a \in W_i^{\underline{n}}\}$.*

Our first main result, Corollary 4.2, implies an extension of Johnson's analog of the Rice Theorem (Theorem 1.5 above) to all transfinite $u \in \mathcal{O}$: for any u' such that $\underline{1} \leq u' <_{\mathcal{O}} u$, for $\mathcal{C} \subseteq \Sigma_u^{-1}$, we have, $\{i : W_i^u \in \mathcal{C}\}$ is u'-c.e. iff, $\mathcal{C} = \emptyset$ or $\mathcal{C} = \Sigma_u^{-1}$.

Our second main result, Corollary 4.5, implies an extension of Johnson's analog of the Rice-Shapiro Theorem (Theorem 1.6 above) to all $u \in \mathcal{O}$, *where* u is for a transfinite *successor* ordinal: for $\mathcal{C} \subseteq \Sigma_u^{-1}$, we have, $\{i : W_i^u \in \mathcal{C}\}$ is u-c.e. iff, $\mathcal{C} = \emptyset$, or $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.

Our next and last two main results extend Rice-Shapiro for notations for limit ordinals, but they are for limit ordinal notations in $\mathcal{O}_{\text{Cantor}}$.

It is open as to what happens for every limit ordinal notation in $(\mathcal{O} - \mathcal{O}_{\text{Cantor}})$.

Our third main result, Corollary 4.10, looks like an an extension of Hay's analog of Rice-Shapiro (Theorem 1.4 above): for every *limit* ordinal ($\mathcal{O}_{\text{Cantor}}$) notation $u = w^v$ (then $\underline{0} <_{\mathcal{O}} v$), for $\mathcal{C} \subseteq \Sigma_u^{-1}$, we have, $\{i : W_i^u \in \mathcal{C}\}$ is u-c.e. iff, either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.

Our last main result, Corollary 4.14, looks like an an extension of Johnson's analog of Rice-Shapiro (Theorem 1.6 above): for every *limit ordinal* notation u *in* $\mathcal{O}_{\text{Cantor}}$, where, for all $v \in \mathcal{O}$, u is *not* of the form w^v, for $\mathcal{C} \subseteq \Sigma_u^{-1}$, we have $\{i : W_i^u \in \mathcal{C}\}$ is u-c.e. iff, either $\mathcal{C} = \emptyset$, $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.

2 Mathematical preliminaries

This section contains basic terminology for the rest of the paper (Section 2.1), information and terminology re Kleene's ordinal notation system $\mathcal{O}$ as well as the subsystem $\mathcal{O}_{\text{Cantor}}$ (Section 2.2), and general background on the Ershov Hierarchy (Section 2.3).

2.1 Basic terminology

Any unexplained recursion theoretic notation is from [4]. We let $\mathbb{N}$ denote the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, and $\supset$, respectively denote empty set, subset, proper subset, superset, and proper superset. Cardinality of a set S is denoted by $|S|$.

$\langle \cdot, \cdot \rangle$ denotes a computable and bijective mapping from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ (see [4]). We assume without loss of generality that $\langle \cdot, \cdot \rangle$ is increasing in both its arguments. One can extend the pairing function to encoding of n-tuples, for $n \geq 3$, by taking $\langle x_1, x_2, x_3, \ldots, x_n \rangle = \langle \langle x_1, x_2, \ldots, x_{n-1} \rangle, x_n \rangle$.

By φ we denote a fixed *acceptable* programming system for the partial computable functions mapping $\mathbb{N}$ to $\mathbb{N}$ [4]. φ_i denotes the partial computable function computed by the program number i in this system. For functions over two or more arguments, one can assume that the input to the functions are encoded using the pairing functions as above, and we will assume so without explicitly mentioning it. Φ denotes a fixed Blum complexity measure [28, 29] for the φ-system. Intuitively, $\Phi_i(x)$ can be thought of as the number of steps needed to compute $\varphi_i(x)$.

$$\varphi_{i,s} = \begin{cases} \varphi_i(x), & \text{if } x < s \text{ and } \Phi_i(x) < s; \\ \uparrow, & \text{otherwise.} \end{cases}$$

A language is a subset of $\mathbb{N}$. We let L, with or without decorations, range over languages. Symbol $\mathcal{E}$ denotes the set of all computably enumerable (recursively enumerable) languages. $\overline{L} = \mathbb{N} - L$, and χ_L is the characteristic function for L; it is 1 on L and 0 off L. Let $W_i = \mathrm{dom}(\varphi_i)$. In other words, W_i is the language accepted by φ_i. Let $W_{i,s} = \mathrm{dom}(\varphi_{i,s}) = \{x < s : \Phi_i(x) < s\}$. $\leq_{\mathrm{m}}$ and $\leq_1$ respectively denote many-one and 1–1 reducibility among sets. K denotes the halting problem, $\{i : i \in W_i\}$. K' denotes the halting problem relative to K, that is, when one allows the programs access to oracle K. Note that K' is $\leq_{\mathrm{m}}$-hard for Σ_2, the sets computably enumerable using the oracle K. Furthermore, $K' \leq_{\mathrm{m}} \{i : W_i \text{ is finite}\}$.

For a function f of two variables, we let $f(x, \infty) = \lim_{t \longrightarrow \infty} f(x,t)$. A function F is *partial limiting computable* if there exists a computable f such that $f(x, \infty) = F(x)$ for each $x \in \mathbb{N}$. If partial limiting computable function F is total, then we call it *limiting computable*. Program p is a limiting computable program for a total function F if for all x, $\lim_{t \to \infty} \varphi_p(x,t) = F(x)$.

2.2 Kleene's system $\mathcal{O}$ and the subsystem $\mathcal{O}_{\mathrm{Cantor}}$

As above, a system of notations is a collection of programs each of which specifies how to build a constructive ordinal. In the present paper, we will be employing two such systems, (primarily) Kleene's $\mathcal{O}$ [10, 11, 12, 4, 13], but, also, a very natural (proper) subsystem which we call $\mathcal{O}_{\mathrm{Cantor}}$ (see Definition 2.3 below). Each of these systems will provide at least one notation for each constructive ordinal.

We let u, v range over notations in the system $\mathcal{O}$. We let $|v|_{\mathcal{O}}$ be the constructive ordinal with notation v.

As noted above, we will not go into the details regarding the notation system $\mathcal{O}$, but refer the reader to [10, 11, 12, 4, 13] and to Remark 2.1 below.

As noted above, there is a partial order $<_{\mathcal{O}}$ on the notations $\mathcal{O}$. We let $u \leq_{\mathcal{O}} v$ iff, $u = v$ or $u <_{\mathcal{O}} v$. Similarly: $u >_{\mathcal{O}} v$ iff $v <_{\mathcal{O}} u$; and $u \geq_{\mathcal{O}} v$ iff $v \leq_{\mathcal{O}} u$.

Regarding $\mathcal{O}$, we have the following.

Remark 2.1

(a) There is a partial computable function *pred* such that, if x is a notation for successor ordinal, then $pred(x)$ is a notation for its predecessor. That is, $|pred(x)|_{\mathcal{O}} + 1 = |x|_{\mathcal{O}}$.

(b) There is a computably enumerable set Z such that, for each $v \in \mathcal{O}$, $\{u : \langle u, v \rangle \in Z\} = \{u : u <_{\mathcal{O}} v\}$.

It is useful to discuss the above mentioned very basic *computable* operations (for $\mathcal{O}$-notations), $+_{\mathcal{O}}$, $\times_{\mathcal{O}}$, and $\{\cdot\}_{\mathcal{O}}$. These operations on $\mathcal{O}$ naturally embed (by $|\cdot|_{\mathcal{O}}$) into the corresponding (constructive) ordinal operations of addition, multiplication, and exponentiation, respectively.[7] Just as Kleene essentially changed his definition of $+_{\mathcal{O}}$ between [10] and [12] to obtain some auxiliary, useful properties, in [30] an additional base case is added for each of $+_{\mathcal{O}}$ and $\times_{\mathcal{O}}$ over what would be needed merely to get the embedding into the corresponding ordinal operations. We omit the details herein, but these extra base cases importantly guarantee the *Notational* Cantor Normal Form Theorem from [30] (reproduced herein as Theorem 2.2 just below).[8] Below '=' means has the same numerical value, so, when comparing notations from $\mathcal{O}$, it means the two sides are the very same notation. This is much stronger than that the two sides merely denote the same constructive ordinal.

Theorem 2.2 (Notational CNF Theorem [30]) *For each $v \in \mathcal{O}$, for all $w \in \mathcal{O}$ for ω, for all $x >_{\mathcal{O}} \underline{0}$,*

[7] Intuitively, for $u, v \in \mathcal{O}$, $u +_{\mathcal{O}} v$ is a *program* for laying out the ordinal $|u|_{\mathcal{O}}$ followed by the ordinal $|v|_{\mathcal{O}}$ to form the ordinal $|u +_{\mathcal{O}} v|_{\mathcal{O}}$. $\times_{\mathcal{O}}$, and $\{\cdot\}_{\mathcal{O}}$ are programs for standardly iterating this action of $+_{\mathcal{O}}$.

[8] In unparenthesized expressions involving these computable operations, $\{\cdot\}_{\mathcal{O}}$ has higher priority than $\times_{\mathcal{O}}$ which in turn has higher priority than $+_{\mathcal{O}}$.

$$x <_{\mathcal{O}} \{w^v\}_{\mathcal{O}} \Leftrightarrow \begin{cases} \textit{there exists a unique } k \in \mathbb{N}, \textit{ unique } n_0, n_1, \ldots, n_k \in \mathbb{N}^+, \\ \textit{and unique } v_0, v_1, \ldots, v_k \textit{ where } v >_{\mathcal{O}} v_0 > v_1 \cdots >_{\mathcal{O}} v_k, \\ \textit{such that } x = \{w^{v_0}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{n_0} +_{\mathcal{O}} (\cdots +_{\mathcal{O}} (\{w^{v_k}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{n_k}) \cdots). \end{cases}$$

Furthermore, for the left to right direction of the $\Leftrightarrow$ *statement just above, the values of* $k, n_0, \ldots, n_k$*, and* $v_0, \ldots, v_k$ *can be algorithmically obtained from* v, w*, and* x*.*

We call the above unique representation of $x <_{\mathcal{O}} \{w^v\}_{\mathcal{O}}$, for v, w, and x as in the above theorem, the *Notational CNF of* x *with respect to* $\{w^v\}_{\mathcal{O}}$. Where it is clear from the context, we will drop the phrase "with respect to $\{w^v\}_{\mathcal{O}}$" — when referring to the Notational CNF of such an x.[9]

As in Section 1.1 above, we have

Definition 2.3 Fix a $w \in \mathcal{O}$ for ω.
Let $\mathcal{O}_{\text{Cantor}} = \{x \in \mathcal{O} \mid (\exists v \in \mathcal{O})[x \leq_{\mathcal{O}} \{w^v\}_{\mathcal{O}}]\}$.

Hence, by Theorem 2.2 just above, we have

Corollary 2.4 $\mathcal{O}_{\text{Cantor}} = \{\underline{0}\} \cup \{\{w^{v_0}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{n_0} +_{\mathcal{O}} (\{w^{v_1}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{n_1} +_{\mathcal{O}} (\cdots +_{\mathcal{O}} (\{w^{v_k}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{n_k}) \cdots)) \mid k \in N \ \wedge\ n_0, n_1, \ldots, n_k \in \mathbb{N}^+ \ \wedge\ v_0 >_{\mathcal{O}} v_1 >_{\mathcal{O}} \cdots >_{\mathcal{O}} v_k\}$.

Since, by contrast with ordinal addition, $+_{\mathcal{O}}$ is not associative, the parenthesizations above are needed [30].

We note that for every ordinal $\alpha > 0$, there exists a (unique) ordinal β such that $\omega^\beta \leq \alpha < \omega^{\beta+1}$ [31, Theorem 2, Chapter XIV.18, page 321]. Therefore, the above Notational CNF Theorem is sufficiently general to apply to notations for as large a constructive ordinal as we may choose, and we have, then, for each constructive ordinal β, at least one notation in $\mathcal{O}_{\text{Cantor}}$ for β.[10]

As in [30], for all $x, y <_{\mathcal{O}} \{w^v\}_{\mathcal{O}}$, we define the useful algorithmic *natural sum* of x and y and denote the operation by $(+)_{\mathcal{O}}$; this operation is a notational analog of the *natural sum of ordinals* from [32, Chapter VII, Section 7, pages 259-260].[11]

If $x = \underline{0}$ or $y = \underline{0}$,

$$x(+)_{\mathcal{O}}y \stackrel{\text{def}}{=} x +_{\mathcal{O}} y;$$

Otherwise, write x, y (as per Theorem 2.2 above) as follows.

$$\begin{aligned} x &= \{w^{v_0}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_0'} +_{\mathcal{O}} (\{w^{v_1}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_1'} +_{\mathcal{O}} (\cdots +_{\mathcal{O}} (\{w^{v_k}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_k'}) \cdots)), \\ y &= \{w^{v_0}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_0''} +_{\mathcal{O}} (\{w^{v_1}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_1''} +_{\mathcal{O}} (\cdots +_{\mathcal{O}} (\{w^{v_k}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{m_k''}) \cdots)), \end{aligned}$$

where $m_0', \ldots, m_k'$ and $m_0'', \ldots, m_k''$ are possibly 0, and $v >_{\mathcal{O}} v_0$.

Then $x(+)_{\mathcal{O}}y \stackrel{\text{def}}{=}$

$$\{w^{v_0}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{(m_0' + m_0'')} +_{\mathcal{O}} (\cdots +_{\mathcal{O}} (\{w^{v_k}\}_{\mathcal{O}} \times_{\mathcal{O}} \underline{(m_k' + m_k'')}) \cdots).$$

Clearly, $(+)_{\mathcal{O}}$ is an algorithmic operation. It is also commutative and associative, *un*like $+_{\mathcal{O}}$ [30].

2.3 The Ershov hierarchy

We now present the Ershov Hierarchy [1, 2, 3], somewhat formally and including the transfinite levels.

Definition 2.5 (Count-Down Functions) A computable function $F : \mathbb{N} \times \mathbb{N} \to \mathcal{O}$ is called a *count-down function* iff, for all x and t, $F(x, t + 1) \leq_{\mathcal{O}} F(x, t)$.

---

[9] The Cantor Normal Form Theorem itself ([31, Theorem 2, Chapter XIV.19, page 323] and [32, Theorems 2 and 5, Chapter VII, Section 7]) states: for any *ordinal* $\beta > 0$, there exists a unique k, unique $n_0, n_1, \ldots, n_k \in \mathbb{N}^+$, and unique *ordinals* $\alpha_0, \alpha_1, \ldots, \alpha_k$, where $\alpha_0 > \alpha_1 > \cdots > \alpha_k$, such that $\beta = \omega^{\alpha_0} \times n_0 + \omega^{\alpha_1} \times n_1 + \cdots + \omega^{\alpha_k} \times n_k$.

[10] We expect that, beyond the present paper and [30], $\mathcal{O}_{\text{Cantor}}$ is a generally useful notation system for the entire class of constructive ordinals. It is ostensibly free of the problematic, pathological notations found in $\mathcal{O}$, for example, for each constructive ordinal $\geq \omega^2$ [33, 34, 30].

[11] N.B. the notational $(+)_{\mathcal{O}}$ depends on the choice of the pre-given v and w.

Definition 2.6 (Ershov Hierarchy) Suppose $A \subseteq \mathbb{N}$. $A \in \Sigma_u^{-1}$ iff, there exists a computable function $E : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ and a count-down function F such that, for all $x, t \in \mathbb{N}$,

(i) $E(x,\infty) = \chi_A(x)$,

(ii) $E(x,0) = 0$ and $F(x,0) \leq_{\mathcal{O}} u$,

(iii) $E(x,t+1) \neq E(x,t) \Rightarrow F(x,t+1) <_{\mathcal{O}} F(x,t)$.

In this case we say that E and F *witness* $A \in \Sigma_u^{-1}$. As above, a member of Σ_u^{-1} is also called a u-c.e. set.

Note that $E(x,0) = 0$, and thus, $\Sigma_{\underline{0}}^{-1} = \{\emptyset\}$. It is well known that, $u <_{\mathcal{O}} v \Rightarrow \Sigma_u^{-1} \subset \Sigma_v^{-1}$. Hence, the Ershov Hierarchy is strict.

The formal W^u-system itself has *implicit* functions like E, F in Definition 2.6 above for witnessing sets A are in Σ_u^{-1}. In fact, there exist functions $E_p^u(\cdot,\cdot)$ and $F_p^u(\cdot,\cdot)$, computable uniformly in p, which witness that $W_p^u \in \Sigma_u^{-1}$. Below, we'll drop the superscript u from $E_p^u(\cdot,\cdot)$, $F_p^u(\cdot,\cdot)$ (since u will be understood) and will write these functions instead as $E_p(\cdot,\cdot)$, $F_p(\cdot,\cdot)$.

From here on in the present paper, as in [9], when we are working with our system W^u, many times we will *also* describe functions like the computable functions E, F witnessing $A \in \Sigma_u^{-1}$ (from Definition 2.6 above) *in*formally, *but*, then, we'll implicitly invoke W^u's acceptability, and imagine our informal descriptions are actually compiled into the W^u-system for use in that system. When we do this, we'll use, instead of the symbols E, F, the respective symbols H, G (many times with program subscripts, and, then, the corresponding A will have the same program subscript).

Suppose we give an algorithmic description of a computable $\{0,1\}$-valued function $H_\cdot(\cdot,\cdot)$ and count-down function $G_\cdot(\cdot,\cdot)$, such that (i) for all p, x, $\lim_{t\to\infty} H_p(x,t)$ converges, and (ii) for all p, $H_p(\cdot,\cdot)$ and $G_p(\cdot,\cdot)$ witness that $A_p \in \Sigma_u^{-1}$, where $A_p = \{x : \lim_{t\to\infty} H_p(x,t) = 1\}$.

Informally, then, the S-m-n Theorem for the W^u-system provides that there exists a computable function f such that $W_{f(p)}^u = A_p$, for all p, and the Kleene Recursion Theorem for the W^u-system provides that there exists an e, which, in effect, creates a self-copy and uses it to make $W_e^u = A_e$. In this Kleene Recursion Theorem application, as in [9], the e is a formal program in the W^u-system and we implicitly invoke the acceptability of W^u to obtain a translation of the informal descriptions of $H_\cdot(\cdot,\cdot), G_\cdot(\cdot,\cdot)$ into the W^u-system to get what e really does in the W^u-system with its self-copy — so as to make $W_e^u = A_e$.

We let $K_u = \{i : i \in W_i^u\}$. It can be shown that K_u is $\leq_{\mathrm{m}}$ complete for Σ_u^{-1}.

For $\mathcal{C} \subseteq \Sigma_u^{-1}$, for brevity, we write $\Theta_u(\mathcal{C}) = \{i : W_i^u \in \mathcal{C}\}$. That is, $\Theta_u(\mathcal{C})$ is the u-index set for $\mathcal{C}$ in the W^u-system.

3 Basic hardness lemmas and their corollaries

As above, $u \in \mathcal{O}$. We fix an arbitrary $\mathcal{C} \subseteq \Sigma_u^{-1}$.

Lemma 3.1 *Suppose W_i^u is infinite. Then, there exists a computable 1–1 function f such that,*
(a) if W_y is infinite, then $W_{f(y)}^u = W_i^u$;
(b) if W_y is finite, then $W_{f(y)}^u$ is a finite subset of W_i^u.

P r o o f. By the S-m-n theorem for u-c.e. sets, there exists a computable function f such that $W_{f(y)}^u = A_y$, where $A_y \in \Sigma_u^{-1}$ as witnessed by H_y and G_y defined below. For all x, t,
$H_y(x,t) = E_i(x,t)$, if $x \leq |W_{y,t}|$; $H_y(x,t) = 0$, if $x > |W_{y,t}|$.
$G_y(x,t) = F_i(x,t)$, if $x \leq |W_{y,t}|$; $G_y(x,t) = u$, if $x > |W_{y,t}|$.
It is easy to verify that f witnesses the lemma. □

We next prove some hardness results for $\Theta_u(\mathcal{C})$.

The following three lemmas allow us to deduce in Corollary 3.5 that if $\Theta_u(\mathcal{C})$ is u-c.e., then $\mathcal{C} = \{W_i^u :$ for some finite set $S \in \mathcal{C}$, $W_i^u \supseteq S\}$.

The first lemma shows that if $\mathcal{C}$ contains some infinite u-c.e. set, but not any finite subset of it, then $K' \leq_1 \overline{\Theta_u(\mathcal{C})}$.

Lemma 3.2 *Suppose $\mathcal{C}$ contains some infinite u-c.e. set, but no finite subset of it, then $K' \leq_1 \overline{\Theta_u(\mathcal{C})}$.*

Proof. Let W_i^u be the infinite u-c.e. set which is in $\mathcal{C}$, but no finite subset of W_i^u is in $\mathcal{C}$.

Let f be as in Lemma 3.1 (corresponding to above i). Then, f witnesses that W_y is infinite iff $f(y) \in \Theta_u(\mathcal{C})$. Thus, $K' \leq_1 \overline{\Theta_u(\mathcal{C})}$. □

The next lemma shows that if $\mathcal{C}$ contains some finite set S, every finite superset of S, but not some u-c.e. superset of S, then $K' \leq_1 \Theta_u(\mathcal{C})$.

Lemma 3.3 *If $\mathcal{C}$ contains every finite superset of a finite set S, but not some u-c.e. superset S' of S, then $K' \leq_1 \Theta_u(\mathcal{C})$.*

Proof. Suppose $W_i^u = S'$.

Let f be as in Lemma 3.1 (for above i). Let f' be 1–1 computable function such that $W_{f'(x)}^u = W_{f(x)}^u \cup S$.

Then, W_x is infinite implies $W_{f'(x)}^u = W_i^u$, and thus $f'(x) \notin \Theta_u(\mathcal{C})$. On the other hand, if W_x is finite, then $W_{f'(x)}^u$ is a finite superset of S, and thus $f'(x) \in \Theta_u(\mathcal{C})$. Thus, $K' \leq \Theta_u(\mathcal{C})$. □

The next lemma shows that if $\mathcal{C}$ contains S but not some c.e. (in particular finite) superset S' of S, then $K_u \leq_1 \overline{\Theta_u(\mathcal{C})}$, thus $\Theta_u(\mathcal{C})$ cannot be u-c.e.

Lemma 3.4 *Suppose $u \geq_{\mathcal{O}} 1$. If $\mathcal{C}$ contains a finite set S, but not some c.e. superset S' of S, then $K_u \leq_1 \overline{\Theta_u(\mathcal{C})}$.*

Proof. Suppose $W_i^u = K_u$. Let z be such that $W_z = S' - S$.

By the S-m-n Theorem for u-c.e. sets, there exists a 1–1 computable function f such that $W_{f(y)}^u = A_y$, where $H_y(\cdot,\cdot), G_y(\cdot,\cdot)$ below witness that A_y is in Σ_u^{-1}.

For all x, y:

$$H_y(x, 0) = 0 \text{ and } G_y(x, 0) = u.$$

For $x \in S$ and $t > 0$:

$$H_y(x, t) = 1 \text{ and } G_y(x, t) = 0.$$

For $x \notin S$ and $t > 0$:

$$
\begin{aligned}
&H_y(x,t) = 1, \text{ if } E_i(y,t) = 1 \text{ and } x \in W_{z,t};\\
&H_y(x,t) = 0, \text{ if } E_i(y,t) = 0 \text{ or } x \notin W_{z,t};\\
&G_y(x,t) = u, \text{ if for all } t' \leq t,\ H_y(x,t') = 0;\\
&G_y(x,t) = F_i(y,t), \text{ if for some } t' \leq t,\ H_y(x,t') = 1.
\end{aligned}
$$

Note that $S \subseteq W_{f(y)}^u$, for all y. Furthermore, if $x \notin W_z \cup S$, then $x \notin W_{f(y)}^u$, and if $x \in W_z$, then $x \in W_{f(y)}^u$ iff $y \in W_i^u = K_u$. Therefore, if $y \in K_u$, then $W_{f(y)}^u = W_z \cup S = S'$ and if $y \notin K_u$, then $W_{f(y)}^u = S$. Thus, $y \in K_u$ iff $f(y) \in \overline{\Theta_u(\mathcal{C})}$.

The Lemma follows. □

As a corollary to Lemmas 3.2, 3.3, and 3.4, we have the following.

Corollary 3.5 *Suppose $u \geq_{\mathcal{O}} \underline{1}$. If $\Theta_u(\mathcal{C})$ is u-c.e., then $\mathcal{C} = \{W_i^u$: for some finite set $S \in \mathcal{C}$, $W_i^u \supseteq S\}$.*

If $u \geq \underline{2}$ and $\Theta_u(\mathcal{C})$ is u-r.e., then the following lemma, along with the above corollary, allows us to conclude that either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.

Lemma 3.6 *Suppose $u \geq_{\mathcal{O}} \underline{2}$. Suppose $S, S' \in \mathcal{C}$ but $S \cap S' \notin \mathcal{C}$, where S, S' are finite. Then, $\Theta_u(\mathcal{C})$ is not u-c.e.*

P r o o f. Suppose by way contradiction z is an u-c.e. index for $\Theta_u(\mathcal{C})$.

Then by the Kleene's Recursion Theorem for W^u, there exists an e such that $W^u_e = A_e$, where $H_e(\cdot,\cdot)$ and $G_e(\cdot,\cdot)$ below witness that $A_e \in \Sigma^{-1}_u$.

For all x:

$$H_e(x,0) = 0 \text{ and } G_e(x,0) = u.$$

For all $x \in S \cap S'$ and for all $t > 0$:

$$H_e(x,t) = 1 \text{ and } G_e(x,t) = 0.$$

For $x \notin S' \cup S$ and all t,

$$H_e(x,t) = 0 \text{ and } G_e(x,t) = u.$$

For $x \in S - S'$ and $t > 0$:

$$H_e(x,t) = 1 \text{ and } G_e(x,t) = 1, \text{ if for all } t' < t,\ E_z(e,t') = 0;$$
$$H_e(x,t) = 0 \text{ and } G_e(x,t) = 0, \text{ if for some } t' < t,\ E_z(e,t') \neq 0.$$

For $x \in S' - S$ and $t > 0$:

$$H_e(x,t) = 0, \text{ if } E_z(e,t) = 1 \text{ or, for all } t' < t,\ E_z(e,t') = 0.$$
$$H_e(x,t) = 1, \text{ if } E_z(e,t) = 0 \text{ and, for some } t' < t,\ E_z(e,t') = 1.$$
$$G_e(x,t) = u, \text{ if for all } t' < t,\ H_e(x,t') = H_e(x,t);$$
$$G_e(x,t) = F_z(e,t), \text{ if for some } t' < t,\ H_e(x,t') \neq H_e(x,t).$$

If $E_z(e,t) = 0$ for all t, then clearly, $W^u_e = S$, and thus, z is not an u-c.e. index for $\Theta_u(\mathcal{C})$. On the other hand, if $E_z(e,t) = 1$ for some t, then $W^u_e = S'$ if $E_z(e,\cdot)$ converges to 0 and $W^u_e = S \cap S'$ if $E_z(e,\cdot)$ converges to 1. It follows that z is not an u-c.e. index for $\Theta_u(\mathcal{C})$ in this case also. □

The following Lemma strengthens Lemma 3.6 in the case that $u \geq_0 \underline{3}$ and $|u|_\mathcal{O}$ is *not* a limit ordinal.

Lemma 3.7 *Suppose $u \geq_\mathcal{O} \underline{3}$ and $|u|_\mathcal{O}$ is not a limit ordinal. Suppose $\mathcal{C}$ contains S but no proper subset of S, and S contains at least 2 elements. Then $\Theta_u(\mathcal{C})$ cannot be u-c.e.*

P r o o f. Suppose by way of contradiction that z is an u-c.e. index for $\Theta_u(\mathcal{C})$. Then by the Kleene's Recursion Theorem for W^u, there exists an e such that $W^u_e = A_e$, where $H_e(\cdot,\cdot)$ and $G_e(\cdot,\cdot)$ below witness that $A_e \in \Sigma^{-1}_u$.

Let $a, b \in S$, where a, b are distinct.

For all x, let

$$H_e(x,0) = 0 \text{ and } G_e(x,0) = u.$$

For $x \in S - \{a,b\}$, and $t > 0$, let

$$H_e(x,t) = 1 \text{ and } G_e(x,t) = 0.$$

For $x \notin S$, and $t > 0$, let

$$H_e(x,t) = 0 \text{ and } G_e(x,t) = 0.$$

Let $H_e(a,t) = H_e(b,t) = 1$ and $G_e(a,t) = G_e(b,t) = pred(u)$, for $1 \leq t \leq t'_e$, where t'_e is the least number (if any) such that $E_z(e,t'_e) = 1$.

Let $H_e(a,t) = 1$, $H_e(b,t) = 0$, $G_e(a,t) = pred(u)$ and $G_e(b,t) = 1$, for $t'_e < t \leq t''_e$, where t''_e is the least number $> t'_e$, if any, such that $E_z(e,t''_e) = 0$.

Let $H_e(b,t) = 1$, $G_e(b,t) = 0$, for $t > t''_e$.

For $t > t''_e$, let $H_e(a,t) = 1$, if $E_z(e,t) = 0$, and $H_e(a,t) = 0$, if $E_z(e,t) = 1$. Let $G_e(a,t) = F_z(a,t)$, for $t > t''_e$.

Clearly, if t'_e is not defined, then $W^u_e = S$, but $e \notin W^u_z$. Similarly, if t'_e is defined but t''_e is not defined, then $W^u_e = S - \{b\}$, but $e \in W^u_z$. On the other hand, if t''_e is defined, then $W^u_e = S$ if $e \notin W^u_z$ and $W^u_e = S - \{a\}$ if $e \in W^u_z$. Thus, in any of the cases, $W^u_z \neq \Theta_u(\mathcal{C})$. □

4 Results

Theorem 4.1 *Suppose $u \geq_{\mathcal{O}} \underline{2}$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. If $\Theta_u(\mathcal{C})$ is u-c.e., then either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.*

Proof. Follows from Corollary 3.5 and Lemma 3.6. □

As a corollary to the just prior theorem (Theorem 4.1), we get the following analog of Rice's Theorem and the first of our main results. New to the present paper are the cases where u is for a transfinite ordinal.

Corollary 4.2 *Suppose $u \geq_{\mathcal{O}} \underline{2}$. Suppose $\underline{1} \leq u' <_{\mathcal{O}} u$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then $\Theta_u(\mathcal{C})$ is u'-c.e. iff, $\mathcal{C} = \emptyset$ or $\mathcal{C} = \Sigma_u^{-1}$. Furthermore, $\Theta_u(\mathcal{C})$ is computable iff, $\mathcal{C} = \emptyset$ or $\mathcal{C} = \Sigma_u^{-1}$.*

Theorem 4.3 *Suppose $u \geq_{\mathcal{O}} 3$, and $|u|_{\mathcal{O}}$ is not a limit-ordinal. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. If $\Theta_u(\mathcal{C})$ is u-c.e., then either $\mathcal{C} = \emptyset$, $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.*

Proof. Follows from Corollary 3.5, Lemma 3.6, and Lemma 3.7. □

Theorem 4.4 *Let $a \in \mathbb{N}$, and $\mathcal{C} = \{W_i^u : a \in W_i^u\}$. Then, for $u \geq_{\mathcal{O}} \underline{1}$, $\Theta_u(\mathcal{C})$ is u-c.e.*

Proof. Let $H(i,t) = E_i(a,t)$, and $G(i,t) = F_i(a,t)$. Then $\Theta_u(\mathcal{C}) \in \Sigma_u^{-1}$ as witnessed by H and G. □

The just above two theorems together characterize when $\Theta_u(\mathcal{C})$ is u-c.e., for non-limit ordinal notations $u \geq_{\mathcal{O}} 3$. The corresponding corollary immediately below is, then, the second of our main results, and it provides, for notations in $\mathcal{O}$ for transfinite successor ordinals, a new analog of the Rice-Shapiro Theorem.

Corollary 4.5 *Suppose $u \geq_{\mathcal{O}} 3$, and $|u|_{\mathcal{O}}$ is not a limit-ordinal, and $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, $\mathcal{C} = \emptyset$, or $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.*

However, a result similar to Theorem 4.3 does not hold for at least some limit ordinals, as the following Theorem shows.

Theorem 4.6 *Suppose u is a notation for ω. Suppose S is a finite set. Let $\mathcal{C}$ consist of all u-c.e. sets which contain S. Then, $\Theta_u(\mathcal{C})$ is u-c.e.*

Proof. Given a fixed S, define H and G as follows.
$H(e,t) = 0$, if $E_e(x,t) = 0$ for some $x \in S$.
$H(e,t) = 1$, if $E_e(x,t) = 1$ for all $x \in S$.
Let t'_e be the least t such that $H(e,t) = 1$ (here, if no such t exists, then we take t'_e to be ∞).
$G(e,t) = u$, for $t < t'_e$.
$G(e,t) = \sum_{x \in S} F_e(x,t)$, for $t \geq t'_e$.
(Note that, in the above, $\sum$ is the $+_{\mathcal{O}}$ summation over the finite ordinals $F_e(x,t)$).
It is easy to verify that H and G witness that $\Theta_u(\mathcal{C})$ is a Σ_u^{-1} set. □

Note that the above Theorem along with Theorem 4.1 characterizes when $\Theta_u(\mathcal{C})$ is u-r.e., for u being a notation for ω.

Corollary 4.7 *Suppose u is a notation for ω. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.*

The proof of Theorem 4.6 easily generalizes to give the following theorem.

Theorem 4.8 *Suppose u is a notation for a limit ordinal. Suppose there is a partial computable function* fplus *such that* fplus *is strictly monotonic with respect to $<_{\mathcal{O}}$ in each of its arguments $<_{\mathcal{O}} u$, and* $\mathrm{fplus}(u_1, u_2) <_{\mathcal{O}} u$*, for all $u_1, u_2 <_{\mathcal{O}} u$ (thus,* $\mathrm{fplus}(u_1, \mathrm{fplus}(u_2, \mathrm{fplus}(u_3, \ldots \mathrm{fplus}(u_k, u_{k+1})))) <_{\mathcal{O}} u$*, for any $u_1, u_2, \ldots, u_k, u_{k+1} <_{\mathcal{O}} u$). Suppose S is a finite set. Let $\mathcal{C}$ consist of all u-c.e. sets which contain S. Then, $\Theta_u(\mathcal{C})$ is u-c.e.*

Corollary 4.9 *Suppose u is a notation for a limit ordinal. Suppose there is a partial computable function* fplus *such that* fplus *is strictly monotonic with respect to $<_{\mathcal{O}}$ in each of its arguments $<_{\mathcal{O}} u$, and* $\mathrm{fplus}(u_1, u_2) <_{\mathcal{O}} u$*, for all $u_1, u_2 <_{\mathcal{O}} u$ (thus,* $\mathrm{fplus}(u_1, \mathrm{fplus}(u_2, \mathrm{fplus}(u_3, \ldots \mathrm{fplus}(u_k, u_{k+1})))) <_{\mathcal{O}} u$*, for any $u_1, u_2, \ldots, u_k, u_{k+1} <_{\mathcal{O}} u$). Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.*

The just above corollary (Corollary 4.9) applies to notations (in $\mathcal{O}_{\mathrm{Cantor}}$) of the form w^v, $v >_{\mathcal{O}} \underline{0}$, where we take $\mathrm{fplus} = (+)_{\mathcal{O}}$. Hence, we obtain immediately below our third main result which provides an analog of the Rice-Shapiro Theorem for such notations.

Corollary 4.10 *Suppose $v >_{\mathcal{O}} \underline{0}$. Suppose $u = w^v$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, either $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{W_i^u : S \subseteq W_i^u\}$, for some finite set S.*

Lemma 4.11 *Suppose u is a notation for a limit ordinal. Suppose for some $u_1 <_{\mathcal{O}} u$, $u_2 <_{\mathcal{O}} u$ and for some $f(\cdot)$, a partial computable function strictly monotonic with respect to $<_{\mathcal{O}}$ on any argument $<_{\mathcal{O}} u_2$, we have, for all u' satisfying $u_1 \leq_{\mathcal{O}} u' \leq_{\mathcal{O}} u$, there is a $u_2' <_{\mathcal{O}} u_2$ such that $f(u_2') = u'$.*

Suppose $\mathcal{C}$ contains S but no proper subset of S, and S contains at least 2 elements. Then $\Theta_u(\mathcal{C})$ cannot be u-c.e.

Proof. Suppose by way of contradiction that z is an u-c.e. index for $\Theta_u(\mathcal{C})$. Then by the Kleene's Recursion Theorem for W^u, there exists an e such that $W_e^u = A_e$, where $H_e(\cdot, \cdot)$ and $G_e(\cdot, \cdot)$ below witness that $A_e \in \Sigma_u^{-1}$.

Let $a, b \in S$, where a, b are distinct.

For all x, let

$$H_e(x, 0) = 0 \text{ and } G_e(x, 0) = u.$$

For $x \in S - \{a, b\}$, and $t > 0$, let

$$H_e(x, t) = 1 \text{ and } G_e(x, t) = 0.$$

For $x \notin S$, and $t > 0$, let

$$H_e(x, t) = 0 \text{ and } G_e(x, t) = 0.$$

Let $H_e(a, 1) = H_e(b, 1) = 1$ and $G_e(a, 1) = u_1$, $G_e(b, 1) = u_2$. Let $t_e' > 1$ be least, if any, such that $F_z(e, t_e') <_{\mathcal{O}} u_1$ (if there is no such t_e', then we take $t_e' = \infty$ for the following).

For $1 < t < t_e'$, we let $H_e(a, t) = 1, G_e(a, t) = u_1$, $H_e(b, t) = 1 - E_z(e, t)$, and $G_e(b, t)$ be the successor of $f^{-1}(F_z(e, t))$. For $t \geq t_e'$, we let $H_e(b, t) = 1$, $G_e(b, t) = 0$, $H_e(a, t) = 1 - E_z(e, t)$, and $G_e(a, t) = F_z(e, t)$.

Now, clearly, $W_e^u = A_e$ is u-c.e. as witnessed by H_e and G_e. Furthermore, if $\lim_{t \to \infty} E_z(e, t) = 0$ then $W_e^u = S$, and if $\lim_{t \to \infty} E_z(e, t) = 1$ then $W_e^u = S - \{a\}$ or $W_e^u = S - \{b\}$, based on whether $\lim_{t \to \infty} F_z(e, t) < u_1$ or not. Thus, if $e \in W_z^u$ then W_e^u is a proper subset of S, and if $e \notin W_z^u$ then $W_e^u = S$. Thus, in any of the cases, $W_z^u \neq \Theta_u(\mathcal{C})$. □

From the just above lemma, as a generalization of Theorem 4.3, we get the following.

Theorem 4.12 *Suppose u is a notation for a limit ordinal. Suppose for some $u_1 <_{\mathcal{O}} u$, $u_2 <_{\mathcal{O}} u$ and for some $f(\cdot)$, a partial computable function strictly monotonic with respect to $<_{\mathcal{O}}$ on any argument $<_{\mathcal{O}} u_2$, we have, for all u' satisfying $u_1 \leq_{\mathcal{O}} u' \leq_{\mathcal{O}} u$, there is a $u_2' <_{\mathcal{O}} u_2$ such that $f(u_2') = u'$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. If $\Theta_u(\mathcal{C})$ is u-c.e., then either $\mathcal{C} = \emptyset$, $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.*

Proof. Follows from Corollary 3.5, Lemma 3.6, and Lemma 4.11. □

As a corollary, using Theorem 4.4, we get

Corollary 4.13 *Suppose u is a notation for a limit ordinal. Suppose for some $u_1 <_{\mathcal{O}} u$, $u_2 <_{\mathcal{O}} u$ and for some $f(\cdot)$, a partial computable function strictly monotonic with respect to $<_{\mathcal{O}}$ on any argument $<_{\mathcal{O}} u_2$, we have, for all u' satisfying $u_1 \leq_{\mathcal{O}} u' \leq_{\mathcal{O}} u$, there is a $u_2' <_{\mathcal{O}} u_2$ such that $f(u_2') = u'$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, either $\mathcal{C} = \emptyset$, $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.*

We will show, as a consequence of the just prior corollary (Corollary 4.13), our fourth and last main result immediately below. It handles the remaining case of limit ordinal notations $u \in \mathcal{O}_{\mathrm{Cantor}}$, the case where u is *not* of the form w^v.

Corollary 4.14 *Suppose $u \in \mathcal{O}_{\mathrm{Cantor}}$ is for a limit ordinal but, for all v, $u \neq w^v$. Suppose $\mathcal{C} \subseteq \Sigma_u^{-1}$. Then, $\Theta_u(\mathcal{C})$ is u-c.e. iff, either $\mathcal{C} = \emptyset$, $\mathcal{C} = \Sigma_u^{-1}$, or $\mathcal{C} = \{W_i^u : a \in W_i^u\}$, for some $a \in \mathbb{N}$.*

P r o o f. Suppose the hypotheses. We first reason about such $u \in \mathcal{O}_{\mathrm{Cantor}}$.

Below in this proof, it is to be understood that *terms connected by $+_{\mathcal{O}}$ are associated to the right.*

By Theorem 2.2 above, u must be of the form $w^{v_1} \times_{\mathcal{O}} \underline{n_1} +_{\mathcal{O}} w^{v_2} \times_{\mathcal{O}} \underline{n_2} +_{\mathcal{O}} \cdots +_{\mathcal{O}} w^{v_k} \times_{\mathcal{O}} \underline{n_k}$, where $k > 0$, $n_1, n_2 \ldots, n_k \in \mathbb{N}$, $v_1 >_{\mathcal{O}} v_2 >_{\mathcal{O}} \cdots >_{\mathcal{O}} v_k >_{\mathcal{O}} \underline{0}$, and either (i) $n_1 > 1$ or (i) $n_1 = 1$ and some $n_m > 0$ where $2 \leq m \leq k$,

In each of cases (i) and (ii) about u just above, we choose below corresponding u_1, u_2. For each of our cases, for its corresponding choice of u_1, u_2, for each $x <_{\mathcal{O}} u_2$, we take $f(x) = u_1 (+)_{\mathcal{O}} x$. In each of our cases, we suppose $u_1 \leq_{\mathcal{O}} u' \leq_{\mathcal{O}} u$, and we need to provide a $u_2' <_{\mathcal{O}} u_2$ so that $f(u_2') = u'$.

In case (i), choose $u_1 = w^{v_1} \times \underline{n_1 - 1}$, and $u_2 = (w^{v_1} \times_{\mathcal{O}} \underline{1} +_{\mathcal{O}} w^{v_2} \times_{\mathcal{O}} \underline{n_2} +_{\mathcal{O}} \cdots +_{\mathcal{O}} w^{v_k} \times_{\mathcal{O}} \underline{n_k}) (+)_{\mathcal{O}} w^{\underline{0}} \times_{\mathcal{O}} \underline{1}$. Hence, $u_2 = w^{v_1} \times_{\mathcal{O}} \underline{1} +_{\mathcal{O}} w^{v_2} \times_{\mathcal{O}} \underline{n_2} +_{\mathcal{O}} \cdots +_{\mathcal{O}} w^{v_k} \times_{\mathcal{O}} \underline{n_k} +_{\mathcal{O}} w^{\underline{0}} \times_{\mathcal{O}} \underline{1}$, and $w^{\underline{0}} \times_{\mathcal{O}} \underline{1} = \underline{1}$.

In this case, u' must, then, be of the form $u_1 (+)_{\mathcal{O}} u_2'$, for some $u_2' <_{\mathcal{O}} u_2$. Then, clearly, $f(u_2') = u'$.

In case (ii), choose $u_1 = w^{v_1} \times \underline{1}$, and $u_2 = (w^{v_2} \times_{\mathcal{O}} \underline{n_2} +_{\mathcal{O}} \cdots +_{\mathcal{O}} w^{v_k} \times_{\mathcal{O}} \underline{n_k}) (+)_{\mathcal{O}} w^{\underline{0}} \times_{\mathcal{O}} \underline{1}$. Hence, $u_2 = w^{v_2} \times_{\mathcal{O}} \underline{n_2} +_{\mathcal{O}} \cdots +_{\mathcal{O}} w^{v_k} \times_{\mathcal{O}} \underline{n_k} +_{\mathcal{O}} w^{\underline{0}} \times_{\mathcal{O}} \underline{1}$, and $w^{\underline{0}} \times_{\mathcal{O}} \underline{1} = \underline{1}$.

In this case too, u' must, then, be of the form $u_1 (+)_{\mathcal{O}} u_2'$, for some $u_2' <_{\mathcal{O}} u_2$. Then, clearly, $f(u_2') = u'$. □

5 Acknowledgements

We thank Frank Stephan for many helpful discussions and comments.

References

[1] Y. Ershov, *A hierarchy of sets I*, Alg. Log. **7**, 23–43 (1968).
[2] Y. Ershov, *A hierarchy of sets II*, Alg. Log. **7**, 212–232 (1968).
[3] Y. Ershov, *A hierarchy of sets III*, Alg. Log. **9**, 20–31 (1970).
[4] H. Rogers, Theory of Recursive Functions and Effective Computability (McGraw Hill, New York, 1967), Reprinted, MIT Press, 1987.
[5] L. Hay, *Rice theorems for d.r.e. sets*, Can. J. Math. **27**, 352–365 (1975).
[6] N. Johnson, *Rice theorems for Σ_n^{-1} sets*, Can. J. Math. **29**, 794–805 (1977).
[7] M. Burgin, Grammars with prohibition and human-computer interaction, in: Proceedings of the 2005 Business and Industry Symposium and the 2005 Military, Government, and Aerospace Simulation Symposium, edited by J. Hill (Society for Modeling and Simulation, 2005), pp. 143–147.
[8] J. Case and J. Royer, Program size complexity of correction grammars, 2010, Working draft.
[9] L. Carlucci, J. Case, and S. Jain, *Learning correction grammars*, J. Symb. Log. **74**(2), 489–516 (2009).
[10] S. Kleene, *Notations for ordinal numbers*, J. Symb. Log. **3**, 150–155 (1938).
[11] S. Kleene, *On the forms of predicates in the theory of constructive ordinals*, Am. J. Math. **66**, 41–58 (1944).
[12] S. Kleene, *On the forms of predicates in the theory of constructive ordinals (second paper)*, Am. J. Math. **77**, 405–428 (1955).
[13] G. Sacks, Higher Recursion Theory, Perspectives in Mathematical Logic, Vol. 2 (Springer-Verlag, Berlin, 1990).
[14] C. Ash and J. Knight, *Recursive structures and Ershov's hierarchy*, Math. Log. Q. **42**, 461–468 (1996).
[15] G. Takeuti, Proof Theory, second edition (North-Holland, 1987).
[16] M. Rathjen, The realm of ordinal analysis, in: Sets and Proofs, edited by S. Cooper and J. Truss, London Mathematical Society Lecture Notes Series (Cambridge University Press, 1999), pp. 219–279.
[17] W. Buchholz, *Proof-theoretic analysis of termination proofs*, Ann. Pure Appl. Log. **75**, 57–65 (1995).
[18] A. Weiermann, Proving termination for term rewriting systems, in: Computer Science Logic, 5th Workshop, edited by E. Börger, G. Jäger, H. Büning, and M. Richter, Lecture Notes in Computer Science Vol. 626 (Springer, 1992), pp. 419–428.

[19] R. Freivalds and C. Smith, *On the role of procrastination in machine learning*, Inform. Comput. **107**(2), 237–271 (1993).
[20] A. Sharma, F. Stephan, and Y. Ventsov, *Generalized notions of mind change complexity*, Inform. Comput. **189**(2), 235–262 (2004).
[21] A. Ambainis, R. Freivalds, and C. Smith, *Inductive inference with procrastination: Back to definitions*, Fundam. Inform. **40**, 1–16 (1999).
[22] J. Case and T. Kötzing, Dynamically delayed postdictive completeness and consistency in learning, in: 19th International Conference on Algorithmic Learning Theory (ALT'08), edited by T. Freund, L. Györfi, G. Turán, and T. Zeugmann, Lecture Notes in Artificial Intelligence Vol. 5254 (Springer, 2008), pp. 389–403.
[23] R. Soare, *Turing oracle machines, online computing, and three displacements in computability theory*, Ann. Pure Appl. Log. **160**, 368–399 (2009).
[24] M. Machtey and P. Young, An Introduction to the General Theory of Algorithms (North Holland, New York, 1978).
[25] J. Royer and J. Case, Subrecursive Programming Systems: Complexity and Succinctness, Progress in Theoretical Computer Science (Birkhäuser Boston, 1994).
[26] H. Rice, *Class of recursively enumerable sets and their decision problems*, Transactions of the American Mathematical Society **74**, 358–366 (1953).
[27] J. Myhill, *A fixed point theorem in recursion theory*, J. Symb. Log. **20**, 205 (1955).
[28] M. Blum, *A machine independent theory of the complexity of recursive functions*, J. ACM **14**, 322–336 (1967).
[29] J. Hopcroft and J. Ullman, Introduction to Automata Theory Languages and Computation (Addison-Wesley Publishing Company, 1979).
[30] A. Ambainis, J. Case, S. Jain, and M. Suraj, *Parsimony hierarchies for inductive inference*, J. Symb. Log. **69**, 287–328 (2004).
[31] W. Sierpiński, Cardinal and ordinal numbers (PWN –Polish Scientific Publishers, 1965), Second revised edition.
[32] K. Kuratowski and A. Mostowski, Set Theory, Studies in Logic and The foundations of Mathematics, Vol. 86 (North-Holland, 1967).
[33] R. Epstein, R. Haas, and R. Kramer, Hierarchies of sets and degrees below $\mathbf{0}'$, in: Logic Year 1979–80, edited by M. Lerman, J. H. Schmerl, and R. I. Soare, Lecture Notes in Mathematics, Vol. 859 (Springer–Verlag, Heidelberg, 1981), pp. 32–48.
[34] V. Selivanov, *On a hierarchy of limiting computations*, Sib. Math. Zh. **25**(5), 146–156 (1984), In Russian (English translation in *Sib. Math. J.*, 25:798-806, 1984).