Computable Symbolic Dynamics

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May 15, 2008

Abstract

We investigate computable subshifts and the connection with effective symbolic dynamics. It is shown that a decidable Π^0_1 class P is a subshift if and only if there is a computable function F mapping $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ such that P is the set of itineraries of elements of $2^{\mathbb{N}}$. Π^0_1 subshifts are constructed in $2^{\mathbb{N}}$ and in $2^{\mathbb{Z}}$ which have no computable elements. We also consider the symbolic dynamics of maps on the unit interval.

1 Introduction

Computable analysis studies the effective content of theorems and constructions in analysis. In this paper, we study computable dynamical systems and symbolic dynamics associated with computable functions on the Cantor space $2^{\mathbb{N}}$, $2^{\mathbb{Z}}$ and the unit interval $[0, 1]$.

The papers of Gregorczyk [14] and Lacombe [17] which initiated the study of computable analysis provide the starting point of our study since those papers provide careful definitions of computably closed sets of reals and computable real functions. Computable real dynamical systems have been studied by Cenzer [3], where the Julia set of a computably continuous real function is shown to be a Π^0_1 class and Ko [16], who examined fractal dimensions and Julia sets. The computability of complex dynamical systems have recently been investigated Rettinger and Weihrauch [20], and by Braverman and Yampolsky [2]. Weihrauch [23] has provided a comprehensive foundation for computability theory on various spaces, including the space of compact sets and the space of continuous real functions.

This research was partially supported by NSF grants DMS 0532644 and 0554841.
Keywords: Computability, symbolic dynamics, Π^0_1 Classes
This is an expanded version of the paper [5].

Effectively closed sets (Π^0_1 classes) occur naturally in the application of computability to many areas of mathematics. See the recent surveys of Cenzer and Remmel [7, 8] for many examples. In particular, the computability of a closed set K in a computable metric space (X, d) may be defined in terms of the distance function d_K, where $d_K(x)$ is the infimum of $\{d(x, y) : y \in K\}$. K is a Π^0_1 class if and only if d_K is upper semi-computable and K is a *decidable* (or *computable*) closed set if d_K is computable. One important example in Euclidean space is the set of zeroes of a computably continuous function. This leads easily to related examples such as the set of fixed points or the set of extrema of a computably continuous function. That is, for any continuous function F, it is easy to see that the set of zeroes of F, the set of fixed points of F, and the set of points where F attains an extremum, are all closed sets. For a computably continuous function F, the corresponding closed sets are all Π^0_1 classes. In fact, Nerode and Huang [22] showed that any Π^0_1 class of reals may be represented as the set of zeroes of a computably continuous function. Ko extended the Nerode-Huang results [15] to show that any Π^0_1 class may be represented as the set of zeroes of a polynomial time computable function. Thus Π^0_1 classes also appear naturally in the theory of polynomial time computable functions on the reals.

The outline of this paper is as follows. Section 2 contains definitions and preliminaries. In section 3, we construct a subsimilar Π^0_1 class with no computable element. The symbolic dynamics of effective dynamical systems on the Cantor space $2^{\mathbb{N}}$ and $2^{\mathbb{Z}}$ is studied in section 4. For any finite k, the *shift* function on $\{0, 1, \ldots, k\}^{\mathbb{Z}}$ ($\{0, 1, \ldots, k\}^{\mathbb{N}}$) is defined by $\sigma(x) = y$, where $y(n) = x(n + 1)$. A closed set $Q \subseteq \{0, 1, \ldots, k\}^{\mathbb{N}}$ ($\{0, 1, \ldots, k\}^{\mathbb{N}}$) is said to be a *subshift* if it is closed under the shift function; for $Q \subseteq 2^{\mathbb{Z}}$, Q is a subshift if it is closed under σ and under σ^{-1}. We will refer to a Π^0_1 class which is also a subshift as a *subsimilar* Π^0_1 class.

Fix a finite alphabet Σ, let $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be a computable function and let a partition $\{U_0, U_1, \ldots, U_k\}$ of $\Sigma^{\mathbb{N}}$ into clopen sets be given. The *itinerary* of a point $x \in \Sigma^{\mathbb{N}}$ is the sequence $It(x) \in \{0, 1, \ldots, k\}^{\mathbb{N}}$ where

$$It(x)(n) = i \iff F^n(x) \in U_i.$$

Now let $IT[F] = \{It(x) : x \in \Sigma^{\mathbb{N}}\}$. We will show that $IT[F]$ is a decidable subsimilar Π^0_1 class and that, for any decidable subsimilar Π^0_1 class $Q \subseteq \Sigma^{\mathbb{N}}$, there exists a computable F such that $Q = IT[F]$. We also consider itineries in $\Sigma^{\mathbb{Z}}$ for points x such that inverse ${}^{-n}(x)$ are defined. The topology on $\Sigma^{\mathbb{N}}$ has a basis of *intervals*, which are clopen sets of the form

$$J[w] = \{x : w \prec x\}.$$

Similarly the topology on $\Sigma^{\mathbb{Z}}$ has a basis of *intervals*, which are clopen sets of the form

$$J[w] = \{x : w \between x\}.$$

A subset of $\Sigma^{\mathbb{N}}$ or $\Sigma^{\mathbb{Z}}$ is clopen if and only if it is a finite union of basic intervals.

A *tree* T over Σ^* is a set of finite strings from Σ^* which contains the empty string λ and which is closed under initial segments. We say that $w \in T$ is an *immediate successor* of $v \in T$ if $w = va$ for some $a \in \Sigma$.

A *bi-tree* T over Σ^* is a set of finite strings of odd length from Σ^* which also contains the empty string and is closed under central segments. $w \in T$ is an *immediate successor* of $v \in T$ if $w = bva$ for some a and b in Σ.

We will assume that $\Sigma \subseteq \mathbb{N}$, so that $T \subseteq \mathbb{N}^*$.

For any tree T, an *infinite path* through T is a sequence $(x(0), x(1), \ldots)$ such that $x\lceil n \in T$ for all n.

And similarly for any bi-tree T, an *bi-infinite path* through T is a sequence $(\ldots, x(-1), x(0), x(1), \ldots)$ such that $x\lceil[-n, n] \in T$ for all n.

We let $[T]$ denote the set of infinite or bi-infinite paths through T. It is well-known that a subset Q of $\Sigma^{\mathbb{N}}(\Sigma^{\mathbb{Z}})$ is closed if and only if $Q = [T]$ for some tree(bi-tree) T. A subset P of $\mathbb{N}^{\mathbb{N}}(\mathbb{N}^{\mathbb{Z}})$ is a Π^0_1 *class* (or *effectively closed set*) if $P = [T]$ for some computable tree(bi-tree) T. A node $w \in T$ is *extendible* (*bi-extendible*) if there exists $x \in [T]$ such that $w \prec x$ ($w \between x$). The set of extendible (bi-extendible)nodes forms a tree T_P which is a co-c.e. subset of Σ^* but is not in general computable. P is said to be *decidable* (or *computable*) if T_P is a computable set.

A tree $T \subseteq \Sigma^*$ said to be *subsimilar* if for every v and w, $vw \in T$ implies $w \in T$. Similarly a bi-tree $T \subseteq \Sigma^*$ said to be *subsimilar* if for every v and every w of odd length, $vw \in T$ implies $w \in T$.

The closed set P is subsimilar (or a subshift) if T_P is subsimilar.

A function $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ is computable (or *computably continuous*) if there exists a computable approximating function $f : \Sigma^* \to \Sigma^*$ such that, for all $x \in \Sigma^{\mathbb{Z}}$ and all $v, w \in \Sigma^*$:

(i) $v \between w \longrightarrow f(v) \between f(w)$.

(ii) $(\forall m)(\exists n)(\forall v \in \Sigma^n)|f(v)| \geq m$.

(iii) $F(x) = \bigcup_n f(x\lceil[-n, n])$.

Similarly, a function $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ is computable if there exists a computable approximating function $f : \Sigma^* \to \Sigma^*$ such that, for all $x \in \Sigma^{\mathbb{N}}$ and all $v, w \in \Sigma^*$:

(i) $v \prec w \longrightarrow f(v) \preceq f(w)$.

(ii) $(\forall m)(\exists n)(\forall v \in \Sigma^n)|f(v)| \geq m$.

(iii) $F(x) = \bigcup_n f(x\lceil n)$.

Note that in both cases (iii) implies (ii) by compactness.

A function $F : [0, 1] \to [0, 1]$ is computable if there exists a computable approximating sequence $\langle f_n \rangle$ of functions $f_n : D \to D$ (where D is the set of dyadic rationals in $[0, 1]$) and a computable modulus function $M : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, all $d \in D$ and all $x \in [0, 1]$,

(i) $|x - d| < 2^{-M(n)} \longrightarrow |F(x) - f_n(d)| < 2^{-n}$.

2 Undecidable subshifts

In this section, we construct a subsimilar Π^0_1 class with no computable element. We will give the construction in $2^{\mathbb{N}}$ and $2^{\mathbb{Z}}$, but it can be generalized to $\Sigma^{\mathbb{N}}$ and $\Sigma^{\mathbb{Z}}$ for any finite Σ. Now every decidable Π^0_1 class has a computable element (in fact, the leftmost path is computable). Hence we have an undecidable subsimilar Π^0_1 class.

For any set S of strings, we may define a closed set P_S, where $x \in P_S$ if and only if, for all n and all $w \in S$, w is not a factor of $x\lceil n$. If the set P_S is nonempty, then S is said to be *avoidable*. For this section, we restrict ourselves to $\Sigma = \{0, 1\}$

Lemma 1. *Given any sequence $x_0, x_1, \ldots$ of elements of $2^{\mathbb{N}}$, there is a nonempty subshift containing no x_i.*

Proof. Define the sequence $l_0, l_1, \ldots$ by

$$l_n = 3(2^{n(n+3)}).$$

This will imply that $l_{n+1} = 2^{2n+4} l_n$. Now let $w_n = x_n\lceil 2l_n$ for each n and define subshift P to consist of all x which do not contain any w_n as a factor. Clearly $x_i \notin P$ for all i. It remains to show that P is nonempty, that is, $\{w_n : n \in \mathbb{N}\}$ is avoidable.

It is important to notice that given any word w of length $2k$, it has at most $k + 1$ distinct factors of length k. Since there are 2^k words of length k, for k large enough so that $2^k > k + 1$, there are words of length k that do not appear as a factor of w. With this in mind, we construct recursively two sequences of words $< A_n >_{n\in\mathbb{N}}$ and $< B_n >_{n\in\mathbb{N}}$ such that, for all n:

1. $|A_n| = |B_n| = l_n$;
2. $A_n \neq B_n$;
3. A_n and B_n are not factors of w_n; this is possible for $n = 0$ since $|w_0| = 6$ so w_0 has at most 4 distinct factors of length 3.
4. A_{n+1} and B_{n+1} are taken from $\{A_n, B_n\}^*$, have A_n as a prefix, and have length l_{n+1}, so each is a concatenation of $m = 2^{2n+4} = l_{n+1}/l_n$ copies of A_n, B_n. This is possible since there are 2^{m-1} such words, but there are at most $l_{n+1} + 1$ factors of length l_{n+1} in w_{n+1} and $2^{m-1} \geqq l_{n+1} + 1 + 2$. To check this, note that $l_{n+1} + 3 = 3(2^{n^2+3n}) + 3 \leq 2^{n^2+3n+2} \leq 2^{m-1}$, since $m - 1 = 2^{2n+4} - 1 \geq 2^{2n+3} \geq n^2 + 3n + 2$ for all n.

Now let $x = lim_n A_n$. This exists since each $A_n \prec A_{n+1}$. We claim that $x \in P$. Suppose by way of contradiction that some w_n is a factor of x. We can view x as an infinite concatenation of blocks length l_n, where each block is either A_n or B_n. Since w_n has length 2 l_n, it must completely contain one of the blocks, which would imply that either A_n or B_n is a factor of w_n. This contradiction shows that $x \in P$. $\square$

We need to improve this lemma in two ways. First, we need an effective version. Second, we may allow for only a subset of words w_k of length $2l_{n_k}$.

Proposition 2. *$P \subseteq 2^{\mathbb{N}}$ is a Π^0_1 subshift if and only if there is a c.e. set S such that $P = P_S$, and similarly for $P \subseteq 2^{\mathbb{Z}}$.*

Proof. First suppose that S is a c.e. set of strings with effective enumeration $v_0, v_1, \dots$. Then $P_S = [T]$ where a string w of length n is in T if and only if none of $v_0, \dots, v_n$ is a factor of w; this works equally well in $2^{\mathbb{N}}$ and in $2^{\mathbb{Z}}$. Next suppose that $P \subseteq 2^{\mathbb{N}}$ is a subsimilar Π^0_1 class, so that the set T_P is a Π^0_1 set and let $S = \{0,1\}^* - T_P$. If $x \notin P$, then for some n, $x\lceil n \in S$, so that $x \notin P_S$. On the other hand, suppose that $x \notin P_S$ and let $x = x\lceil n^\frown w^\frown y$ for some $n < \omega$ and some $w \in S$. Then $w^\frown y \notin P$ and hence, since P is subsimilar, $x \notin P$. A similar argument using the bi-tree T_P works in for $P \subseteq 2^{\mathbb{Z}}$. Here we let S be the set of odd length strings not in T_P. If $x \notin P$, then for some n, $x\lceil[-n,n] \in S$, so that $x \notin P_S$. If $x \notin P_S$, then for some $w \in T_P$ of length $2k+1$, and for some m, $w = x\lceil[m, m+2k]$. Then for $y = \sigma^{-m-k}(x)$, $y\lceil[-k,k] = w$, so that $y \notin P$ and thus $x \notin P$ since P is subsimilar. ☐

Proposition 3. *There is a recursive sequence of natural numbers $l_0, l_1, \dots$ such that if for any subsequence $< l_{n_k} >_{k\in\mathbb{N}}$ and any set $S = \{v_k : k \in \mathbb{N}\}$ of words such that $|v_k| = 2l_{n_k}$, S is avoidable. Furthermore, if ϕ is a partial computable function such that $\phi(n_k) = v_k$, then there is a nonempty subsimilar Π^0_1 class P such that no element of P contains any factor v_k.*

Proof. For the first part, just let $w_k = v_{n_k}$ for n_k in the subsequence and observe that in the proof of Lemma 1, the construction goes through if there is no word w_n to avoid. The second part follows from Proposition 2. ☐

Theorem 4. *There is a nonempty subsimilar Π^0_1 class $P \subseteq 2^{\mathbb{N}}$ with no computable element.*

Proof. Let the sequence $< l_n >$ be given as in Lemma 1. Let Let $\phi_0, \phi_1, ..., \phi_e, \dots$ be an enumeration of partial computable functions. Now define the partial recursive function ϕ by

$$\phi(k) = \begin{cases} \phi_k\lceil 2l_k, & \text{if } \phi_k(i)\downarrow \text{ for all } i < 2l_k; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

By Theorem 3, there is a nonempty subsimilar Π^0_1 class P such that no element of P has any word $\phi(k)$ as a factor. Now let y be any computable element of $2^{\mathbb{N}}$. Then $y = \phi_k$ for some k such that ϕ_k is a total function. Thus $\phi(k) = \phi_k\lceil k$ is defined and is not a factor of any $x \in P$ and hence certainly $\phi_k \notin P$. ☐

Thus far, we have constructed a nonempty Π^0_1 subsimilar subclass of $2^{\mathbb{N}}$ that has no computable elements. We would like to establish the same for the space $2^{\mathbb{Z}}$. Some modifications to the above lemmas and theorems are needed to achieve this.

Lemma 5. *For any set S of strings, S is avoidable in $2^{\mathbb{N}}$ if and only if S is avoidable in $2^{\mathbb{Z}}$.*

Proof. Suppose first that S is avoidable in $2^{\mathbb{Z}}$, so that some bi-infinite word $x \in 2^{\mathbb{Z}}$ avoids all $w \in S$. Then the word $(x(0), x(1), \dots) \in 2^{\mathbb{N}}$ also avoids S. Next suppose that S is not avoidable in $2^{\mathbb{Z}}$. Then P_S is empty in $2^{\mathbb{Z}}$, so that by compactness, the bi-tree T of strings of odd length which do not contain a factor from S is finite. Thus there exists n such that every string of length $2n+1$ contains a factor from S and it follows that P_S is empty in $2^{\mathbb{N}}$. $\square$

Theorem 6. *There is a nonempty subsimilar Π^0_1 class $P_S \subseteq 2^{\mathbb{Z}}$ with no computable element.*

Proof. Have the sequence $< m_n = 6(2^{n(n+3)} : n \geqq 0 >$. Let Let $\phi_0, \phi_1, ..., \phi_e, \dots$ be an enumeration of partial computable functions whose domain is a subset of $\mathbb{Z}$. Now let

$$S = \{w_k = \phi_k \lceil [-m_k/2, m_k/2], \text{if } \phi_k(i) \downarrow \text{ for all } -m_k/2 < i < m_k/2$$

Since the set, S, of $\phi(k)$s is a computable enumerable set of forbidden words whose elements have lengths which compose a subset of $< m_n >_{n \in \omega}$, it follows that P_S is a nonempty subsimilar Π^0_1 class such that no element of P has any word $\phi(k)$ as a factor. Then for any computable element ϕ_k of $2^{\mathbb{Z}}$, $w_k = \phi_k \lceil [-l_k/2, l_k/2]$ is defined and is not a factor of any $x \in P$, and hence $\phi_k \notin P$. $\square$

3 Symbolic Dynamics for Functions on $\Sigma^{\mathbb{N}}$ and $\Sigma^{\mathbb{Z}}$

Fix a finite alphabet $\Sigma = \{0, 1, \dots, k\}$, let $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be a computable function and let a partition $\{U_0, U_1, \dots, U_k\}$ of $\Sigma^{\mathbb{N}}$ into clopen sets be given. The *itinerary* of a point $x \in \Sigma^{\mathbb{N}}$ is the sequence $It(x) \in \{0, 1, \dots, k\}^{\mathbb{N}}$ where

$$It(x)(n) = i \iff F^n(x) \in U_i.$$

Now let $IT[F] = \{It(x) : x \in \Sigma^{\mathbb{N}}\}$. We observe that $IT[F]$ is a subshift. That is, suppose $y = It(x) \in IT[F]$. Then $\sigma(y) = It(F(x))$, so that $\sigma(y) \in IT[F]$ as well. The function It is continuous and hence $IT[F]$ is a closed set, as seen by the proof of the following lemma.

Lemma 7. *The function from $\Sigma^{\mathbb{N}} \to \{0, 1, \dots, k\}^{\mathbb{N}}$ mapping x to $It(x)$ is computable.*

Proof. Given clopen sets $U_0, \dots, U_k$, there exists a finite j and a finite subset W of $\{0,1\}^j$ such that each U_i is a finite union of intervals $J[w]$ for some set of $w \in W$. Thus one can determine from $y \lceil j$ the unique i for which $y \in U_i$. Given $x \in \Sigma^{\mathbb{N}}$, let $y = It(x)$. To compute $y(n)$, it suffices to find the first j values of $F^n(x)$, which can be computed uniformly from x and n. $\square$

Theorem 8. *Let $F : \Sigma^{\mathbb{N}}$ to $\Sigma^{\mathbb{N}}$ be computable and let $\{U_0, U_1, \dots, U_k\}$ be a partition of $\Sigma^{\mathbb{N}}$ into clopen sets. Then*

(a) For any computable $x \in \Sigma^{\mathbb{N}}$, the itinerary $It(x)$ is computable.

(b) The set $IT[F]$ of itineraries is a decidable, subsimilar Π^0_1 class.

Proof. Part (a) follows from the well-known result that computable functions map computable points to computable points and (b) follows from the fact that the image of a decidable Π^0_1 class under a computable function is a decidable Π^0_1 class. See [7, 8]. □

Next we prove the converse. Note that $F^0(x) = x$ for all $x \in \Sigma^{\mathbb{N}}$ and therefore $IT[F]$ meets every U_i. Note that if Q is a subshift and Q does not meet $J[i]$, then $Q \subseteq \{0, 1, \dots, i-1, i+1, \dots, k\}^{\mathbb{N}}$.

Theorem 9. *Let $\Sigma = \{0, 1, \dots, k\}$ be a finite alphabet and let $Q \subseteq \Sigma^{\mathbb{N}}$ be a decidable, subsimilar Π^0_1 class which meets $J[i]$ for all i. Then there exists a partition $\{U_0, \dots, U_k\}$ and a computable $F : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ such that $Q = IT[F]$. Similarly, there exists a partition $\{U_0, \dots, U_k\}$ and a computable $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ such that $Q = IT[F]$.*

Proof. We will use the partition given by $U_i = J[i]$. Since Q is decidable, we can define a function $G : \Sigma^{\mathbb{N}} \to Q$ such that $G(x) = x$ for all $x \in Q$. Let $Q = [T]$ where T is a computable tree without dead ends. The approximating function g for G is defined as follows. For any $w \in \{0, 1, \dots, k\}^n$, find the longest initial segment v such that $v \in T$ and let $g(v)$ be the lexicographically least (or leftmost) extension of v which is in $T \cap \{0, 1, \dots, k\}^n$; this exists since T has no dead ends. Now let $F(x) = \sigma(G(x))$. We claim that $IT(F) = Q$.

For any $x \in Q$, we have $F(x) = \sigma(x)$ and $\sigma(x) \in Q$, since Q is a subshift. Hence $F^n(x) = \sigma^n(x)$, so that $F^n(x)(0) = x(n)$, so that $F^n(x)$ belongs to the set $U_{x(n)}$. Thus the itinerary $I(x) = x$. This shows that $Q \subseteq IT[F]$.

Next consider any $x \in \Sigma^{\mathbb{N}}$. We will show by induction that $F^n(x) = \sigma^n(G(x))$ for all $n > 0$. For $n = 1$, this is the definition. Then

$$F^{n+1}(x) = \sigma(G(F^n(x))) = \sigma(G(\sigma^n(G(x)))),$$

by induction. But $G(x) \in Q$, so that $\sigma^n(G(x)) \in Q$ by subsimilarity and therefore $G(\sigma^n(G(x))) = \sigma^n(G(x))$ and finally $F^{n+1}(x) = \sigma^{n+1}(G(x))$, as desired. It follows that for $n > 0$, $It(x)(n) = G(x)(n)$. But for $n = 0$, the assumption that Q meets $J[x(0)]$ implies that $G(x)(0) = x(0)$ and hence $It(x)(0) = x(0) = G(x)(0)$ as well. Therefore $It(x) \in Q$ as desired.

To extend this to functions on $\Sigma^{\mathbb{Z}}$, similarly let $G(x)$ follow the central segment of x until this goes out of T and continue the argument as above. □

Next we consider the possible itineraries of F in $\Sigma^{\mathbb{Z}}$. Let $X = \Sigma^{\mathbb{N}}$ or $X = \Sigma^{\mathbb{Z}}$ and let $y \in IT^Z(F)$ if and only if there exists a bi-infinite sequence $(\dots, x_{-1}, x_0, x_1, \dots) \in X^{\mathbb{Z}}$ such that $F(Fx_i) = x_{i+1}$ and $y(i) = x_i(i)$ for all $i \in \mathbb{Z}$.

Theorem 10. *For any computable function $F : X \to X$, $IT^Z(F)$ is a decidable, subsimilar Π^0_1 class.*

Proof. We give the argument for $X = 2^{\mathbb{N}}$ and leave the case of $2^{\mathbb{Z}}$ to the reader. Let f be a computable approximating function for F and define the computable tree T to consist of all finite sequences $(y(0), \ldots, y(n-1)$ such that there exists $w_0, w_1, \ldots, w_{n-1}, w_n$ such that for $i < n$,

(i) $f(w_i) \prec w_{i+1}$;

(ii) $w_i(0) = y(i)$.

Note that T is computable since there will be a bound on the length of the w_i. It follows from the definition of T that $y \in [T]$ if and only if, for each n, there exists finite sequence $(w_{-n}, \ldots, w_{-1}, w_0, w_1, \ldots, w_n)$ such that $f(w_i) \prec w_{i+1}$ and $w_i(0) = y(i)$ $i \in \{-n, \ldots, n-1\}$. Certainly $IT^Z(F) \subseteq [T]$. The other inclusion follows from the compactness of $X^{\mathbb{Z}}$. That is, fix $y \in [T]$ and let K_n be the set of bi-infinite sequences $(\ldots, x_{-1}, x_0, x_1, \ldots) \in X^Z$ such that for $-n \leq i < n$, $F(x_i) = x_{i+1}$ and $x_i(0) = y(i)$. Then each K_n is closed and nonempty, so that by compactness $\bigcap_n K_n$ is also nonempty. Thus there exists $(\ldots, x_{-1}, x_0, x_1, \ldots) \in X^Z$ which has itinerary y. $\square$

Theorem 11. *Let $\Sigma = \{0, 1, \ldots, k\}$ be a finite alphabet and let $Q \subseteq \Sigma^{\mathbb{Z}}$ be a decidable, subsimilar Π^0_1 class which meets $J[i]$ for all i. Then there exists a partition $\{U_0, \ldots, U_k\}$ and a computable $F : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ such that $Q = IT^Z[F]$.*

Proof. The proof is a slight modification of theproof of Theorem 9. We define $G : \Sigma^{\mathbb{Z}} \to Q$ such that $G(x)$ is the nearest element in Q to x. That is, the approximating function g maps $x\lceil[-n, n]$ to $x\lceil[-n, n]$ as long as $x\lceil[-n, n] \in T_Q$ and when n is the least such that $x\lceil[-n-1, n+1] \notin T_Q$, $g([-n-k-1, n+k+1])$ is the lexicographically least extension of $x\lceil[-n, n]$ which is in T_Q. Then we let $F(x) = \sigma(G(x))$. Once again for $x \in Q$, we have $F(x) = \sigma(x)$, so that $(\ldots, \sigma^{-1}(x), x, \sigma(x), \sigma^2(x), \ldots)$ has itinerary x and therefore $Q \subseteq IT^Z(F)$. On the other hand, $F(x) \in Q$ for all x, so that for any bi-infinite sequence $z = (\ldots, x_{-1}, x_0, x_1, \ldots)$ with $F(x_i)x_{i+1}$ for all $i \in Z$, we must have each $x_i \in Q$ and therefore the itinerary of z is $x_0 \in Q$. Thus $IT^Z(F) \subseteq Q$ as well. $\square$

The function given in Theorem 11 above is in general not one-to-one or onto, and we don't know if a one-to-one and/or onto function can be given.

4 Unimodal Maps

In this section, we consider symbolic dynamics for mappings on the unit interval, which is much more complicated. We recall some definitions and facts of [21].

Definition 12. *A function $F : [0, 1] \to [0, 1]$ is a unimodal map with critical point c if*

1. *$F(c)$ is the unique absolute maximum of F;*

2. *F is strictly increasing over the interval $[0, c)$ and is strictly decreasing over $(c, 1]$.*

3. *F(0)=0=F(1).*

The value of the critical point is not essential for this discussion, so for simplicity c is taken to be $\frac{1}{2}$. Given any $x \in [0, 1]$ the itinerary of x under F, $I(x)$, is defined as follows:

$$I(x)_i = \begin{cases} 1 & F^i(x) > \frac{1}{2}, \\ C & F^i(x) = \frac{1}{2}, \\ 0 & F^i(x) < \frac{1}{2} \end{cases}$$

Hence the space of itineraries, $IT[F]$, of the elements of the interval $[0, 1]$ is a symbolic subspace of $X = \{0, 1, C\}^\omega$. In contrast to the symbolic dynamics of $\Sigma^{\mathbb{N}}$, the function taking x to $It(x)$ need not be continuous and the set of itineraries need not be a closed set. We are interested in the subset of itineraries in $2^{\mathbb{N}}$, that is,

$$I[F] = IT[F] \cap 2^{\mathbb{N}}.$$

The most important itinerary of a unimodal map F is its *kneading sequence*, $KS(F)$, which is $I(F(\frac{1}{2}))$. There is a connection between the kneading sequence and the set $I[F]$ by way of the following well-known linear ordering on $\{0, 1, C\}^*$. Here C represents the critical point $\frac{1}{2}$.

A word $w \in \{0, 1, C\}^*$ is said to be *even* (respectively, *odd*) if it has an even (odd) number of ones. The ordering is defined as follows.

- $0 < C < 1$
- For any w_1 and w_2,
  - (i) If $w_1 \preceq w_2$, then $w_1 \leq w_2$ and vice versa.
  - (ii) Otherwise, let u be the largest common prefix of w_1 and w_2 and let $|u| = m$. If u is even, then $w_1 < w_2$ if and only if $w_1(m) < w_2(m)$ and if u is odd, then $w_1 < w_2$ if and only if $w_1(m) > w_2(m)$.

This ordering can be extended to $\{0, 1, C\}^{\mathbb{N}}$ just using clause (ii). For finite words, this is clearly a computable linear ordering.

A finite word w is called *shift-maximal* if $\sigma^i(w) \leq w$ for all i with $1 \leq i \leq |w| - 1$. Similarly an infinite word x is shift-maximal iff $\sigma^i(x) \leq x$ for all i.

It is well-known that the kneading sequence $KS(F)$ for a unimodal map F is shift-maximal [21].

Given a shift-maximal $x \in \{0, 1, C\}^{\mathbb{N}}$, $y \in 2^{\mathbb{N}}$ is said to be *admissible* with respect to x if $\sigma^i(y) \leq x$ for all i. Let $Adm(x)$ be the set of all admissible sequences with respect to x.

Theorem 13. *For any computable shift-maximal sequence $x \in \{0, 1, C\}^{\mathbb{N}}$, $Adm(x)$ is a decidable, subsimilar Π^0_1 class.*

Proof. It is immediate from the definition that $Adm(x)$ is a subshift. For the effectiveness, we have $Adm(x) = [T]$, where $w \in T \iff (\forall i < |w|)\sigma^i w \leq x\lceil|w|$. $[T]$ is decidable since, for any $w \in T$, either $w^\frown 0$ or $w^\frown 1$ in T. $\square$

Theorem 14. *Given any decidable, subsimilar Π^0_1 subclass Q of the cantor space, there exists a shift maximal sequence, $x \in 2^\omega$, such that $Q \subseteq Adm(x)$.*

Proof. Let T be a computable tree without dead ends such that $Q = [T]$ and such that $w \in T$ implies $\sigma(w) \in T$. Let $x(0) = 1$, since clearly (1) is shift-maximal. Suppose we have defined the shift-maximal word $s = x\lceil n \in T$ such that for all $w \in T \cap \{0,1\}^{\leq n}$, $w \leq x\lceil n$. Let $w \in T$ be the maximal word of length $n+1$. We claim that $s \prec w$. To see this, let $w = v^\frown i$ for $v \in T$ and suppose by way of contradiction that $s \neq v$. Since T has no dead ends, $s^\frown j \in T$ for some j. Since s is shift-maximal, $v < s$, so that $w = v^\frown i < s^\frown j$, contradicting the assumption that w is maximal in T. Now let $x(n) = w(n)$ so that $x\lceil n+1 = w$. To see that w is shift-maximal, let $u = \sigma^i(w)$ for some i. Then $u \in T$ since Q is subsimilar and thus $u \leq w$ by maximality of w in $T^{\leq n+1}$. Proceeding in this fashion, we construct $x \in T$ such that $x\lceil n$ is shift-maximal for all n, and hence x is shift-maximal. Also x is maximal in Q, so that, for any $y \in Q$, $y \leq x$. $\square$

It is not the case that every decidable, subsimilar Π^0_1 class Q *equals* $Adm(x)$ for the maximal element x of Q. For example, if $x = 10^\omega$, then $Adm(x) = 2^{\mathbb{N}}$. However, for $S = \{111\}$ and $Q = P_S$, we have $x \in Q$ and thus x is the maximal element of Q, but certainly $Q \neq Adm(x) = 2^{\mathbb{N}}$.

We return to the analysis of unimodal maps. The connection between the kneading sequence and the itineraries is given by the following [10]. For a continuous function F, let $Adm(F)$ denote $Adm(KS(F))$.

Proposition 15. *For any unimodal map $F : [0,1] \to [0,1]$ with kneading sequence $KS(F)$, the set $Adm(F)$ is the closure of the set $I[F]$ of itineraries of F in $2^{\mathbb{N}}$.*

Next we consider computable unimodal maps.

Theorem 16. *For any computable unimodal map $F : [0,1] \to [0,1]$ and any computable real $x \in [0,1]$, I(x) is a computable sequence. In particular, the kneading sequence $KS(F)$ is computable.*

Proof. Note that, for any computable x, $L(x) = \{n : F^n(x) < \frac{1}{2}\}$ and $R(x) = \{n : F^n(x) > \frac{1}{2}$ are both c. e. sets. Suppose first that $F^n(x) \neq \frac{1}{2}$ for any n. Then $L(x)$ and $R(x)$ are complements and hence both sets are computable. Then $I(x)(n) = 0 \iff n \in L(x)$, so that $I(x)$ is computable.

For the other case, we first consider the kneading sequence. If $F^n(\frac{1}{2}) = \frac{1}{2}$ for some n, then $KS(F)$ is periodic and certainly computable. Thus $KS(F)$ is computable in any case. Now for arbitrary $x \in [0,1]$ such that $F^n(x) = \frac{1}{2}$ for some n, then $I(x) = I(x)\lceil n+1^\frown KS(F)$ and is therefore computable since $KS(F)$ is computable. $\square$

We have the following corollary to Theorems 13 and 16.

Corollary 17. *For any computable unimodal map $F : [0,1] \to [0,1]$, $Adm(F)$ is a decidable, subsimilar Π^0_1 class.*

The remaining goal is to find a converse to this result, that is, given a decidable subsimilar Π^0_1 class, to find a computable unimodal map F with $Adm(F) = Q$. We will make some progress in this direction.

For the rest of the section, we confine the discussion of unimodal maps to the quadratic maps $F_\mu(x) = \mu x(1-x)$. These form a so-called *full family*, so that, by [10], we have

Proposition 18. *For any $\mu_0 < \mu_1$ and for every shift-maximal $y \in \{0,1,C\}^{\mathbb{N}}$, there exists a parameter $\mu \in [\mu_0, \mu_1]$ such that $KS(F_\mu) = y$.*

For $\mu > 3$, the unimodal map $F_\mu = \mu x(1-x)$ has $F_\mu(\frac{1}{2}) > \frac{1}{2}$, so that there exist points $x_0 \in (0, \frac{1}{2})$ and $x_1 \in (\frac{1}{2}, 1)$ such that $F_\mu(x_0) = \frac{1}{2} = F_\mu(x_1)$ and therefore $0C \prec I(x_0)$ and $1C \prec I(x_1)$. In general, F_μ may have kth order inverses of $\frac{1}{2}$ for all k.

For the surjective quadratic map $G(x) = 4x(1-x)$, it is clear that G has 2^k k-th order inverses of $\frac{1}{2}$ for all k and hence for any $w \in 0,1^*$, there exists x such that $wC \prec I(x)$; we will say that this x is the *coordinate* of the path wC. For this map, we have the following.

Lemma 19. *Let $F(x) = 4x(1-x)$ and let $I(x)$ be the itinerary of x under F. Then I is one-to-one, $I[F] = 2^{\mathbb{N}}$, and the inverse of I, restricted to $2^{\mathbb{N}}$, is computable.*

For the surjective quadratic map $F = F_4$, we say that wC is a *legal inverse path (l.i.p.)* if the coordinate $r \in [0,1]$ of the path is the greatest numerical value of any point on the path, that is, if $F^n(r) < r$ for $n \leq |w|$. Metropolis et al [21] provides the following crucial fact.

Proposition 20. *There is a one-to-one correspondence between the set of periodic kneading sequences of the full family of the quadratic maps and the the set of legal inverse paths of F_4. In particular, $(wC)^\omega$ is a periodic kneading sequence for some μ if and only if wC is a legal inverse path for F_4. Moreover, if x_w is the coordinate of wC and μ_w has kneading sequence $(wC)^\omega$, then in general $x_v < x_w$ if and only if $\mu_v < \mu_w$.*

Consequently if $< w_n C >$ is a sequence of l.i.p.'s such that the corresponding sequence $< x_{w_n} >_{n<\omega}$ converges, then the sequence $< \mu_{w_n} >_{n<\omega}$ also converges.

We next give a condition for a finite word $w \in \{0,1\}^*$ which will imply that wC is shift-maximal if w is shift-maximal. Suppose that w is shift-maximal but wC is not shift-maximal. Then some $\sigma^i(wc) > wc$; let $\sigma^i(wc) = uC$ and let $v = w\lceil i$, so that $w = vu$. Then $wC < uC$. But we know that $u < w$, which implies that $uC < wC$ *unless* u is a prefix of w. With this in mind, we say that for finite words u and w, $u \neq w$ is a *proper prefix-suffix* (in short PS) of w, if there exists words t and v such that $w = uv$ and $w = tu$. Call a PS u of w *trivial* if $|u| = 1$; in this case wC will be shift-maximal. Note that given any shift-maximal sequence $x \in 2^{\mathbb{N}}$, $x \lceil n$ is a shift-maximal word for all n, but it might have PS factors. Call $x \in 2^{\mathbb{N}}$ *strongly shift-maximal* if there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that, for all n, $x\lceil f(n)$ has at most a trivial proper PS. For example, fix any m and let

$$x = 10^m 110^m 1110^m \dots$$

Then for each n, $10^m 11 \dots 0^m 1^n C$ is shift-maximal.

The argument above has proved the following.

Lemma 21. *If the finite word w has at most a trivial proper PS and is shift-maximal, then wC is shift-maximal.*

Theorem 22. *Given any (computable) strong shift-maximal sequence $x \in 2^{\mathbb{N}}$, there is a (computable) unimodal map F with kneading sequence x.*

Proof. Let $x \in 2^{\mathbb{N}}$ be a strong shift-maximal sequence. Then by Lemma 21, we can find a subsequence $< w_n >_{n<\omega}$ of the initial segments of x such that $w_n C$ is shift-maximal word for each n. It follows from Proposition 18 that each $w_n C$ is a legal inverse path and hence there exist μ_n such that F_{μ_n} has kneading sequence $(w_n C)^\omega$. Since $lim_n w_n C = x$, it follows from Proposition 20 that $lim_n \mu_n = \mu$ exists and that $KS(F_\mu) = x$.

If x is computable, then we can compute the sequence w_n (since testing to see if $w_n C$ is shift-maximal is computable) and then compute μ_n from w_n. By Proposition 18, we may assume that μ_{n+2} is between μ_n and μ_{n+1}, so that the limite μ is also computable. ☐

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