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Minimal extensions of Π^0_1 classes

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A minimal extension of a Π^0_1 class P is a Π^0_1 class Q such that $P \subset Q$, $Q - P$ is infinite, and for any Π^0_1 class R, if $P \subset R \subset Q$, then either $R - P$ is finite or $Q - R$ is finite; Q is a nontrivial minimal extension of P if in addition P and Q have the same Cantor-Bendixson derivative. We show that for any class P which has a single limit point A, and that point of degree $\leq \mathbf{0}'$, P admits a nontrivial minimal extension. We also show that as long as P is infinite, then P does not admit any decidable nontrivial minimal extension Q.

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1 Introduction

The structure of the lattice $\mathcal{E}_\Pi$ of Π^0_1 classes under inclusion has been studied in several recent papers. Here a Π^0_1 class is simply an effectively closed set of real numbers and, specifically, a Π^0_1 class of sets is a subset of the Cantor set $\{0,1\}^{\mathbb{N}}$. A Π^0_1 class may also be viewed as the set $[T]$ of infinite paths through a computable tree T. An important theme has been to compare and contrast this lattice with the well-known lattice $\mathcal{E}$ of computably enumerable sets. Key issues include the definability and complexity of various properties, automorphisms of the lattice and orbits under automorphisms, and the analysis of certain substructures of the lattice.

Here is an example. Given two Π^0_1 classes $P \subset Q$, the interval $[P,Q] = \{R : P \subseteq R \subseteq Q\}$, and in particular, $[\emptyset, Q]$ is an *initial segment* of $\mathcal{E}_\Pi$. A Π^0_1 class P is said to be *thin* if $[\emptyset, Q]$ is a Boolean algebra. P is *perfect* if every element of P is a limit point. Cholak, Coles, Downey and Herrmann [8] have shown that the family of all perfect thin classes is in certain ways analogous to the hyper-hypersimple c. e. sets. That is, any two perfect thin classes are automorphic in $\mathcal{E}_\Pi$, the family of perfect thin classes is definable in $\mathcal{E}_\Pi$ and the degrees of perfect thin classes are exactly the c. e. array noncomputable degrees. (Here the degree of $P = [T]$ is the degree of the set of nodes of T which have an extension in P.)

An infinite Π^0_1 class P is *minimal* if every Π^0_1 subclass of P is either finite or cofinite in P. For any lattice $\mathcal{L}$, let $\mathcal{L}^*$ be the quotient lattice of $\mathcal{L}$ modulo finite difference. Then P is minimal if and only if $[0,P]^*$ is the trivial Boolean algebra. Cenzer, Downey, Jockusch and Shore [2] first constructed a minimal thin class. Cenzer and Nies [4] characterized the order types of the finite intervals of $\mathcal{E}^*_\Pi$ as finite distributive lattices with the dual reduction property. In particular, this means that there are intervals (in fact, initial segments) of order type n for any finite ordinal n. This contrasts with the classic result that finite intervals of $\mathcal{E}^*$ are all Boolean algebras. However, it is shown in [4] that for any *decidable* Π^0_1 class P, if $[0,P]^*$ is finite, then it must be a Boolean algebra.

Recently, Lawton [10] introduced the notion of *minor superclasses of* Π^0_1 *classes*, as an analogue of major subsets of c. e. sets and gave a characterization of the Π^0_1 classes which have strong minor superclasses.

The present paper continues the research into lattice of Π^0_1 classes. We define some notions of a *minimal extension* of a Π^0_1 class analogous to the notion of a c. e. subset maximal in another c. e. set. We show the existence of minimal extensions under certain conditions and also prove a splitting theorem for Π^0_1 classes which shows that decidable minimal extensions are in general not possible. The various notions of minimal extension

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are compared to Lawton's notion of a minor superclass. An embedding of the lattice $\mathcal{E}$ of c. e. sets into the lattice $\mathcal{E}_\Pi$ is presented and gives rise to natural examples of extensions of Π^0_1 classes.

Some definitions are needed.

Let $\{0,1\}^{<\omega}$ be the set of binary *strings*, that is, functions from a finite initial segment of ω into $\{0,1\}$. We write $\sigma \preceq \tau$ for $\sigma \subseteq \tau$. For $x \in \{0,1\}^{\mathbb{N}}$ and $n \in \omega$, let $x\lceil n = (x(0), x(1), \dots, x(n-1))$. Let $\sigma \prec x$ if $\sigma = x\lceil n$ for some n. We write $|\sigma|$ for the length n of the string $\sigma = (\sigma(0), \sigma(1), \dots, \sigma(|\sigma|-1))$.

A subset T of $\{0,1\}^{\mathbb{N}}$ is a *tree* if whenever $\tau \in T$ and $\sigma \preceq \tau$, then $\sigma \in T$. For any tree T, $[T]$ denotes the set of infinite paths through T, that is, $[T] = \{x \in \{0,1\}^{\mathbb{N}} : (\forall n)\,[x\lceil n \in T]\}$. The set of *extendible nodes of* T is defined by $\mathrm{Ext}(T) = \{\sigma : (\exists x \in [T])\,[\sigma \prec x]\}$.

The usual product topology on the space $\{0,1\}^{\mathbb{N}}$ has a sub-basis of *intervals* $I(\sigma) = \{x : \sigma \prec x\}$. With this topology, the closed subsets of $\{0,1\}^{\mathbb{N}}$ are exactly those of the form $[T]$ for some tree T. The clopen subsets of $\{0,1\}^{\mathbb{N}}$ are just the finite unions of intervals.

$P \subset \{0,1\}^{\mathbb{N}}$ is a Π^0_1 class if $P = [T]$ for some computable tree T. It is easily seen that an equivalent definition is obtained by requiring T to be primitive recursive, or only co-c. e., instead of computable. This leads to an effective enumeration of the Π^0_1 classes as $\{P_e\}_{e\in\omega}$, where $P_e = [T_e]$ and T_e is the e'th primitive recursive tree. See Cenzer and Remmel [6, 7] for details.

A Π^0_1 class is called *decidable* if it has the form $[T]$ for some tree T such that $\mathrm{Ext}(T)$ is computable.

An element x of a Π^0_1 class P is said to be *isolated* if there is some σ such that $P \cap I(\sigma) = \{x\}$. The Cantor-Bendixson derivative $D(P)$ is the set of nonisolated points of P.

For Π^0_1 classes $P \subseteq Q$, $[P,Q]$ denotes the lattice of Π^0_1 classes R such that $P \subseteq R \subseteq Q$, and $[P,Q]^*$ denotes the quotient lattice of $[P,Q]$ modulo finite difference. We also write $A =^* B$ to mean that the symmetric difference of two sets A and B is finite.

It was observed by Herrmann that the $\mathcal{E}_\Pi$ has the *dual reduction property*, that is, for any two Π^0_1 classes P and Q, there exist Π^0_1 classes $P_1 \supseteq P$ and $Q_1 \supseteq Q$ such that $P_1 \cup Q_1 = \{0,1\}^{\mathbb{N}}$ and $P_1 \cap Q_1 = P \cap Q$. It follows that the lattices $[P,Q]$ and $[P,Q]^*$ also have the dual reduction property for any P, Q. The following result from [4] will be needed later.

Theorem 1.1 *For any finite lattice $\mathcal{L}$ with the dual reduction property, there is a Π^0_1 class P such that $\mathcal{L}$ is isomorphic to $[0,P]^*$.*

For more on Π^0_1 classes and the dual concept of c. e. ideals of computable Boolean algebras, see the survey papers by Cenzer [1] and Cenzer and Remmel [6].

Here is an outline of the paper.

In Section 2, we define notions of minimal and r-minimal extensions and minor superclass as natural analogues of classical notions from the lattice of c. e. sets. Some simple implications are shown between these notions. In Section 3, a natural embedding of lattice $\mathcal{E}$ of c. e. sets into the lattice $\mathcal{E}_\Pi$ of Π^0_1 classes is defined. This produces examples of extensions of Π^0_1 classes and refines the connections between our notions. In Section 4, we show that if the Π^0_1 class P has a single limit point A, and $A \leq_{\mathrm{T}} \mathbf{0}'$, then P admits a non-trivial minimal extension. In Section 5, we prove a Π^0_1 class version of the Splitting Theorem of Owings and use it to show that no infinite Π^0_1 class P can have a *decidable* non-trivial minimal extension Q.

2 Minimal extensions

In this section, we define various notions of minimal extension and relate them to each other and to Lawton's notion of a minor superclass. These concepts are all analogues of notions from the lattice of c. e. sets, or rather *dual* analogues.

Recall that a c. e. set A is said to be *maximal* if there is no c. e. set B such that $A \subset B$ and both $B - A$ and $\omega - B$ are infinite. The analogue for Π^0_1 classes is a *minimal* Π^0_1 class, where P is minimal if there is no Π^0_1 class $Q \subset P$ such that both Q and $P - Q$ are infinite. A minimal class can be pictured as consisting of a single limit point together with a sequence of isolated points which converge to the limit. The role of minimal Π^0_1 classes in the lattice $\mathcal{E}_\Pi$ of Π^0_1 classes has been studied in several recent papers [3, 4, 5].

For two c.e. sets $A \subset C$, C is said to be a *maximal extension of* A if there is no intermediate c. e. set B such that $A \subset B \subset C$ and both $C - B$ and $B - A$ are infinite. There are several variations of the dual notion of minimal extension for Π^0_1 classes. Examples will be given below.

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Definition 2.1 Q is said to be a *minimal extension* of P if $P \subset Q$, $Q - P$ is infinite, and for any Π^0_1 class R, if $P \subset R \subset Q$, then either $R - P$ is finite or $Q - R$ is finite.

Note that Q is a minimal extension of P if and only if $[P, Q]^* = \{P, Q\}$.

Now every Π^0_1 class has a trivial minimal extension obtained by adding a copy of a minimal class M to some interval disjoint from P. We can exclude this possibility by not allowing any new limit points in Q.

Definition 2.2 A minimal extension Q of P is *non-trivial* if $D(Q) = D(P)$; otherwise Q is a *trivial minimal extension*.

It is easy to see that for any minimal extension Q of P, $D(Q) - D(P)$ can contain at most one point. That is, if A and B are both in $D(Q) - D(P)$, then there are disjoint clopen sets U and V such that $A \in U$, $B \subset V$ and $P \cap (U \cup V) \subset \{A\}$. It follows that $Q - U$ and $Q - V$ are distinct sets in $[P, Q]^*$.

A non-trivial minimal extension might still arise from P by adding a minimal class M with the limit point already in P.

Definition 2.3 Q is a *good minimal extension of* P if Q is a minimal extension of P and there is no minimal Π^0_1 class M such that $Q = P \cup M$.

There are good minimal extensions which are non-trivial. That is, the linear ordering with 3 elements is a lattice with the dual reduction property and hence there exists a Π^0_1 class P such that $[0, P]^*$ has 3 elements, $\emptyset$, P and some minimal class $P_1 \subset P$. Then P is a minimal extension of P_1 and there can be no Π^0_1 class M such that $P = P_1 \cup M$, since M would be in $[0, P]^*$ but incomparable with P_1.

We need to consider the notion of a complement for a Π^0_1 class.

Definition 2.4

1. Π^0_1 classes P and Q in $[0, R]$ are *complements in* R if $P \cup Q = R$ and $P \cap Q = \emptyset$; P is *complemented in* R if it has a Π^0_1 complement in R.

2. Π^0_1 classes P and Q in $[0, R]$ are *almost complements in* R if $P \cup Q = R$ and $P \cap Q$ is finite; P is *almost complemented in* R if it has a Π^0_1 almost complement.

The notion of an almost complement was introduced by Lawton in [10]. Note that this is equivalent to having $P \cup Q =^* R$ and $P \cap Q$ finite, since every element of the difference $R - (P \cup Q)$ is computable and can be put into Q.

Proposition 2.5 *Let Q be a minimal extension of P.*

1. *Q is a trivial minimal extension of P if and only if P is complemented in Q, that is, if and only if $Q - P$ is a Π^0_1 class.*

2. *Q is a good minimal extension of P if and only if P is not almost complemented in Q.*

Proof.

1. If $R = Q - P$ is a Π^0_1 class, then R is infinite and hence must contain a limit point which will be in $D(Q) - P$. Thus Q is a trivial minimal extension. Suppose now that Q is a trivial extension and let $A \in D(Q) - P$. Since $A \notin P$, there is a clopen set V such that $A \in V$ and $V \cap P = \emptyset$. Then $Q \cap V = R$ is infinite and $R \cap P = \emptyset$, so that $P \cup R =^* Q$ by minimality. Let F be the finite set $Q - R$. Then $F \subset Q - P$ and $A \notin F$, so each element of F is isolated in Q and therefore computable. Thus F is a Π^0_1 class and P has complement $R \cup F$.

2. Suppose that P is almost complemented in Q by the Π^0_1 class M, so that $P \cup M = Q$ and $M \cap P$ is finite. It is clear that M must be minimal, so that Q is not a good minimal extension. Next suppose that $Q = P \cup M$ with M minimal. Then $M - P$ is infinite since Q is a proper extension of P and therefore $M \cap P$ must be finite. Hence M is an almost complement of P in Q. □

This has an immediate corollary.

Corollary 2.6 *Any good minimal extension is non-trivial.* □

There is a stronger possible notion of minimal extension.

Definition 2.7 Q is a *proper minimal extension of* P if, for any $R \subset Q$, either $Q - R$ is finite or $R - P$ is finite; equivalently, $[0, Q]^* = [0, P]^* \cup \{Q\}$.

Equivalently, Q is a proper minimal extension of P if $[0, Q]^* = [0, P]^* \cup \{Q\}$. In [4] Π^0_1 classes P were constructed, for each finite n, such that $[0, P]^*$ is isomorphic to $(\{0, 1, \ldots, n\}, <)$. Thus there is a chain of Π^0_1

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classes $P_0 = \emptyset \subset P_1 \subset \cdots \subset P_n = P$ such that $[0, P]^* = \{P_0, \ldots, P_n\}$ and hence P_{k+1} is a proper minimal extension of P_k for all $k < n$.

It is clear that any proper minimal extension is good, since if $Q = P \cup M$, then M is also a subclass of Q but not a subclass of P.

Not every good minimal extension is proper. To see this, let $\mathcal{L} = \{0, a, b, c, 1\}$ where 0 is the least element, 1 is the greatest element, $b \wedge c = a$ and $b \vee c = 1$ and let $P = P_1$ be a Π^0_1 class such that $[0, P]^* = \{0, P_a, P_b, P_c, P_1\}$ is isomorphic to $\mathcal{L}$. Then P_1 is a good minimal extension of P_b but is not proper since $[0, P_a]^* = \{0, P_a, P_b\}$ and $[0, P_1]^*$ has two additional subclasses, P_1 and also P_c.

An infinite c. e. set A is said to be *r-maximal* if the complement is not split into two infinite sets by any computable set. Note that any maximal set is r-maximal but there is an example of an r-maximal set which is not maximal (see Soare [11, p. 191]). Also, an arbitrary set C is said to be *r-cohesive* if there is no computable set R such that both $C \cap R$ and $C - R$ are infinite.

There are two possible analogues of r-maximality for Π^0_1 classes. To say that there is no clopen set V such that both $P \cap V$ and $P - V$ are infinite is equivalent to saying that P has a unique limit point. We will say that P is r-minimal if there do not exist infinite Π^0_1 subclasses R_1 and R_2 such that $R_1 \cup R_2 = P$ and $R_1 \cap R_2$ is finite. Thus for example, $\{0^\omega\} \cup \{0^n 1^\omega : n \in \mathbb{N}\}$ is not r-cohesive as demonstrated by $R_1 = \{0^\omega\} \cup \{0^{2n} 1^\omega : n \in \mathbb{N}\}$ and $R_2 = \{0^\omega\} \cup \{0^{2n+1} 1^\omega : n \in \mathbb{N}\}$. It is clear that any minimal Π^0_1 class is also r-minimal.

Definition 2.8 Q is an *r-minimal extension of* P if there are no infinite Π^0_1 classes R_1 and R_2 such that $P \subset R_i \subset Q$ for $i = 1, 2$ and $R_1 \cup R_2 = Q$ and $(R_1 \cap R_2) - P$ is finite.

Equivalently, Q is an r-minimal extension of P if $[P, Q]^*$ has no pair of complements (other than P and Q). It is clear that any minimal extension is also r-minimal. The reverse implication does not hold, since P_n is an r-minimal extension of P_0 for any chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of minimal extensions.

A c. e. set A is *hypersimple* if the family of c. e. supersets of A is complemented (this is not the original definition, but is equivalent). A Π^0_1 class P is said to be *thin* if $[0, P]$ (the lattice of Π^0_1 subclasses of P) is complemented, that is, whenever $Q \subset P$, there exists a complement R such that $Q \cap R = \emptyset$ and $Q \cup R = P$. Thin classes have been studied in [2, 8]. In particular, we note that a minimal Π^0_1 class is thin if and only if the unique element of $D(P)$ is not computable. We will also say that P is *almost thin* if $[0, P]^*$ is complemented. For example, if P is a minimal Π^0_1 class with the limit point computable, then P is almost thin but not thin.

Definition 2.9

1. Q is a *thin extension of* P if $[P, Q]$ is a Boolean algebra, that is, whenever $P \subset R \subset Q$, there exists a complement S such that $R \cap S = P$ and $R \cup S = Q$.

2. Q is an *almost thin extension of* P if $[P, Q]^*$ is a Boolean algebra, that is, whenever $P \subset R \subset Q$, there exists a complement S such that $R \cap S =^* P$ and $R \cup S = Q$.

Proposition 2.10 *If Q is an almost thin, r-minimal extension of P, then Q is a minimal extension of P.*

P r o o f. Suppose that Q is an almost thin, but not minimal extension of P and let $R \subset Q$ with $R - P$ and $Q - R$ both infinite. Since $[P, Q]^*$ is complemented, there exists R_2 so that $R \cup R_2 = Q$ and $R \cap R_2 =^* P$. Clearly R_2 must be infinite, so that Q is not an r-minimal extension of P. □

Given c. e. sets $A \subset B$, A is a *major subset of* B if $B - A$ is infinite and, for every c. e. set W, if $\overline{B} \subseteq^* W$, then $\overline{A} \subseteq^* W$. (The condition $\overline{B} \subseteq W$ is equivalent here.) If A is infinite, then A is a major subset of B if and only if, for any computable set R, $R \subset B$ implies that $R - A$ is finite. Lachlan proved that every noncomputable c. e. set has a major c. e. subset. (See Soare [11, pp. 190ff] for details.)

This leads naturally to the following (slightly extended) notions of Lawton [10].

Definition 2.11

1. Q is a *minor superclass of* P if $P \subset Q$, $Q - P$ is infinite and, for every Π^0_1 R, if $R \cap P = \emptyset$, then $R \cap Q$ is finite.

2. Q is a *strong minor superclass of* P if $P \subset Q$, $Q - P$ is infinite and, for every Π^0_1 R, if $R \cap P$ is finite, then $R \cap Q$ is finite.

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It is clear that no clopen set can have a minor superclass and Lawton shows that every non-clopen Π^0_1 class has a minor superclass. More interestingly, she shows that P has a strong minor superclass if and only if P is not almost complemented.

There exist strong minor superclasses which are not minimal extensions. For example, let $P_0 \subset P_1 \subset P_2$ be given by Theorem 1.1 such that, modulo finite difference, $[0, P_1] = \{0, P_0, P_1, P_2\}$. Then P_2 is a strong minor superclass of P_0 but is clearly not a minimal extension.

Proposition 2.12

1. *If Q is a good minimal extension of P, then Q is a strong minor superclass of P.*
2. *If Q is a non-trivial minimal extension of P, then Q is a minor superclass of P.*

Proof.

1. Suppose that Q is a minimal extension of P but is not a strong minor superclass of P. Then there is some Π^0_1 class R such that $R \cap Q$ is infinite but $R \cap P$ is finite. We may assume that $R \subset Q$ by taking $R \cap Q$ if necessary. Since Q is a minimal extension, it follows that $Q = P \cup R$. R must be minimal, since $P \cup R_0 \subseteq Q$ for all $R_0 \subset R$. Thus Q is not a good extension of P.

2. Suppose that Q is a minimal extension of P but is not a strong minor superclass of P. Then there is some infinite Π^0_1 class $R \subset Q$ such that $R \cap P = \emptyset$. It follows that R has a limit point $B \notin P$ and thus $D(Q) - D(P) \neq \emptyset$. □

3 An embedding of $\mathcal{E}$ into $\mathcal{E}_\Pi$

In this section we present a natural embedding of the lattice $\mathcal{E}$ of c. e. sets into the lattice $\mathcal{E}_\Pi$ of Π^0_1 classes and use this to give some examples of minimal classes and minimal extensions which are not proper. For any c. e. set $A \subset \omega$, let $P_A = \{0^\omega\} \cup \{0^n 1^\omega : n \notin A\}$. Then P_A is always a Π^0_1 class and clearly for two c. e. sets A, B, we have that $A \subset B$ if and only if $P_B \subset P_A$. Hence there is a natural (reverse) embedding of $\mathcal{E}$ into $\mathcal{E}_\Pi$. We state this formally.

Theorem 3.1 *The mapping taking the c. e. set A to the Π^0_1 class P_A is a (reverse) lattice embedding of $\mathcal{E}$ into $\mathcal{E}_\Pi$.* □

If A is a maximal c. e. set, then P_A is a natural candidate to be a minimal Π^0_1 class and this will follow from the next proposition. Recall that for c. e. sets A, B, A is *maximal in* B if for any c. e. set D with $A \subset D \subset B$, either $D - A$ is finite or $B - D$ is finite.

Proposition 3.2 *Let A, B be c. e. sets such that A is maximal in B. Then P_A is a minimal extension of P_B.*

Proof. Suppose that $P_B \subset Q \subset P_A$ and let $C = \{n : 0^n 1^\omega \notin Q\}$. Then C is a c. e. set and $A \subset C \subset B$. Thus either $C - A$ is finite, in which case $P_A - Q$ is finite, or $B - C$ is finite, in which case $Q - P_B$ is finite. □

Notice that in general P_A is a decidable Π^0_1 class if and only if A is computable and, of course, if A is maximal in B, then A cannot be computable.

It is an easy application of the existence of a maximal c. e. set that for every infinite c. e. set B, there exists a c. e. set A such that A is maximal in B. Of course P_A and P_B have the same unique limit point 0^ω. Thus we have the following

Corollary 3.3 *For every infinite c. e. set B, the Π^0_1 class P_B has a minimal extension.* □

Proposition 3.4 *There is an r-minimal Π^0_1 class which is not minimal.*

Proof. Let A be a c. e. set which is r-maximal but not maximal and let $B \supset A$ be a coinfinite c. e. set such that $B - A$ is infinite. Then P_A is not minimal, since it has a proper Π^0_1 subclass P_B. Suppose that $Q_1 \cup Q_2 = P$ and $Q_1 \cap Q_2$ is finite and let $R = \{n : 0^n 1^\omega \in Q_1\}$. Then R is clearly co-c. e. and, modulo finite, $\omega - R = \{n : n \notin A \text{ or } 0^n 1^\omega \in Q_2\}$ is also co-c. e.. Thus R is computable and clearly $\omega - A$ is split by R into two infinite subsets. □

Lachlan proved that every noncomputable c. e. set has a major c. e. subset. We apply this to derive the existence of strong minor superclasses for classes of the form P_A.

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Proposition 3.5 *For every noncomputable c. e. set A, the Π^0_1 class P_A has a strong minor superclass.*

Proof. Let B be a major subset of A; we claim that P_B is a strong minor superclass of P_A. Suppose therefore that that R is a Π^0_1 class with $R \cap P_A$ finite and let $W = \{n : 0^n 1^\omega \notin R\}$. Then W is a c. e. set and $W \cup A$ is cofinite, that is, if $n \notin W \cup A$, then $0^n 1^\omega \in R \cap P_A$. Since B is a major subset of A, $W \cup B$ is also cofinite. Now if $0^n 1^\omega \in R \cap P_B$, then $n \notin W \cup B$, so that $R \cap B$ is also finite. □

We will see in Section 5 that there can be no decidable proper minimal extensions.

The existence of maximal c. e. sets can be applied more generally.

Theorem 3.6 *Any non-complemented Π^0_1 class P has a non-trivial minimal extension. Furthermore, if P contains no computable boundary points, then P has a good minimal extension.*

Proof. Let $M = \{m_0, m_1, \dots\}$ be a maximal c. e. set and let $M_s = \{m_0, \dots, m_s\}$. Let $P = [T]$ and define the computable set $S = \{\sigma : \sigma \notin T \,\&\, \sigma\lceil|\sigma| - 1 \in T\}$. S is infinite since P is not complemented. Observe that any two elements of S are incompatible. Let $S = \{\sigma_0, \sigma_1, \dots\}$. Now define the minimal extension Q of P by $Q = P \cup \{\sigma_i{}^\frown 0^\omega : i \notin M\}$. Q is a Π^0_1 class since it is the set of infinite paths through the tree T_Q, where $\sigma \in T_Q$ if either $\sigma \in T$ or $\sigma = \sigma_i{}^\frown 0^s$ (with i and s unique) such that $i \notin M_s$. Suppose that $P \subset R \subset Q$ and let $A = \{i : \sigma_i{}^\frown 0^\omega \notin R\}$. Then $M \subset A$ so that either $A =^* M$ or $A =^* \omega$. In the first case, $R =^* Q$ and in the second case $R =^* P$. Thus Q is a minimal extension. For any element $B = \sigma_i{}^\frown 0^\omega \in Q - P$, B is the unique element of $Q \cap I(\sigma)$ and is therefore not a limit point of Q. Hence A is a nontrivial minimal extension.

Note that $Q - P$ has a limit point (being an infinite set) which is not in $Q - P$ since Q is a nontrivial minimal extension. Furthermore this limit point must be unique, by the same argument as given above that $D(Q) - D(P)$ contains at most one point. Let A be the unique limit point of $Q - P$ and suppose that Q is not a good minimal extension. Then there is a minimal Π^0_1 class R such that $P \cup R = Q$ and $P \cap R$ is finite. Since $A \in R \cap P$, it follows that A is computable and in fact that $Q \cup \{A\}$ is Π^0_1 class which can be obtained from R by removing finitely many isolated points. □

It follows that P has a non-trivial minimal extension if and only if P is not complemented. Finally, we consider proper minimal extensions.

Proposition 3.7 *If Q is a proper minimal extension of P, then P has exactly one limit point.*

Proof. If P has no limit points, then P is finite, so that Q is minimal and is not a proper minimal extension of P. If P has two limit points, say A and B, let U be a clopen set with $A \in U$ and $B \notin U$ and let $Q_1 = Q - U$ and $Q_2 = Q \cap U$. $Q_1 \notin [0, P]^*$, since $A \notin Q_1$ and A is a limit point of P and likewise $Q_2 \notin [0, P]^*$ since $B \notin Q_2$. Similarly, $Q_1 \neq^* Q_2$. Thus $[0, Q]^*$ contains at least two Π^0_1 classes not in $[0, P]^*$ whereas there is only one new class (Q) if Q is a proper extension. □

4 Nontrivial minimal extensions of Π^0_1 classes

In this section, we demonstrate the existence of nontrivial minimal extensions of certain Π^0_1 classes.

We need to use the limit lemma, that any function f that is computable in $0'$ has a uniformly computable approximation $\{f_s\}_{s\in\omega}$ such that $\lim_{s\to\infty} f_s(x) = f(x)$ (see Soare [11, p. 57]).

Theorem 4.1 *Each Π^0_1 class P with a single limit point A with $A \leq \mathbf{0}'$, admits a nontrivial minimal extension.*

Proof. Let S be a computable tree such that $P = [S]$ and let $D(P) = \{A\}$. Let A^s be the uniformly computable approximation given by the Limit Lemma. Since $A \in P = [S]$, we may assume that $A^s\lceil s \in S$ for all s. If it is not, simply find the longest initial segment α of $A^s\lceil s$ which is in S and replace $A^s\lceil s$ with any extension τ of α which is in S and has length s. For any fixed n, there exists an m such that $A\lceil n \prec A^s\lceil s$ for all $s \geq m$ and it follows that the modified version of A^s also extends $A\lceil n$ for all $s \geq m$, so that we still get A as the limit of the sequence A^s.

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The minimal extension Q of the class P is obtained by adding to P an infinite sequence B_n of new isolated paths which satisfy the following requirement:

$\mathrm{R}_{0,n}$ For each n, $A\lceil n \prec B_n$.

This immediately ensures that Q can have no new limit paths, so that Q will be a nontrivial minimal extension of P if we can show that it is a minimal extension.

To ensure that Q is in fact a minimal extension, we need to satisfy the additional requirement:

$\mathrm{R}_{e+1,n}$ For all $m \geq n \geq e$, if $B_n \in P_e$, then $B_m \in P_e$.

Given this requirement, there are two possibilities for a class P_e such that $P \subseteq P_e \subseteq Q$. First, we may have $B_n \notin P_e$ for all $n \geq e$, so that $P_e - P \subset \{B_0, \dots, B_{e-1}\}$ and is finite. Second, we may have $B_n \in P_e$ for some $n \geq e$, so that by the requirement, $\{B_n, B_{n+1}, \dots\} \subset P_e$ which means that $Q - P_e \subset \{B_0, \dots, B_{n-1}\}$ and is finite.

The construction of the tree T is in stages s, using a priority argument. We will define a computable sequence of threshold numbers $n(s)$. At each stage s, we also define the tree $T^s = T \cap \{0,1\}^{n(s)}$, and for $i < s$, the s-approximation β_i^s of B_i, chosen so that $\beta_i^s \notin S$. T will be a computable tree since for $\sigma \in \{0,1\}^s$, we have that $\sigma \in T$ if and oly if $\sigma \in T^s$. The new isolated paths B_i will be defined by $B_i = \lim_{s\to\infty} \beta_i^s$.

Requirement $\mathrm{R}_{0,i}$ requires attention at stage $s+1$ when β_i^s does not extend $A^{s+1}\lceil i$.

Before describing the actions to be taken for this requirement we note that since $P = [S]$ has only one limit path, every node $\sigma \in S$ has an extension which is not in S. The action for this and the other requirements all require defining nodes α_s and γ_s as follows. Let α_s be the shortest and then lexicographically least extension of $A^{s+1}\lceil n(s)^\frown(1 - A^{s+1}(n(s)))$ which is not in S and let γ_s be the shortest and then lexicographically least extension of $A^{s+1}\lceil(n(s)+1)$ which is not in S. We will define $n(s+1)$ to be the maximum of $\{|\alpha_s|, |\gamma_s|\}$.

The action to be taken when $\mathrm{R}_{0,i}$ requires attention is the following. We need to redefine β_i and also define β_s for the first time. Define β_i^{s+1} to have length $n(s+1)$ and extend α_s by a string of 0's and similarly define β_s^{s+1} to have length $n(s+1)$ and extend γ_s by a string of 0's. For each $j < s$ different from i, let β_j^{s+1} have length $n(s+1)$ and extend β_j^s by a string of 0's. The tree T^{s+1} contains all nodes from S of length $n(s+1)$ as well as the nodes β_k^{s+1} for all $k \leq s$.

Requirement $\mathrm{R}_{e+1,n}$ requires attention at stage $s+1$ when there exists $m \leq s$ such that $m > n$ and such that

(i) $A^{s+1}\lceil i \prec \beta_m^{s\frown}0$,

(ii) $\beta_m^{s\frown}0 \notin T_e$ and $\beta_n^{s\frown}0 \in T_e$, and

(iii) for all $d < e$, if $\beta_n^{s\frown}0 \notin T_d$, then $\beta_m^{s\frown}0 \notin T_d$.

Conditions (i) and (iii) ensure that action taken on requirement $\mathrm{R}_{e+1,n}$ will respect higher priority requirements.

The action to be taken when $\mathrm{R}_{e+1,n}$ requires attention as above is the following. We let β_m^{s+1} be the sequence of length $n(s+1)$ which extends $\beta_n^{s\frown}0$ by a string of 0's, we let β_n^{s+1} be the sequence of length $n(s+1)$ which extends α_s by a string of 0's and we let β_s^{s+1} be the sequence of length $n(s+1)$ which extends γ_s by a string of 0's. For each $k < s$ different from m and n, let β_k^{s+1} have length $n(s+1)$ and extend β_k^s by a string of 0's. The tree T^{s+1} is defined as above to contain all nodes from S of length $n(s+1)$ as well as the nodes β_k^{s+1} for all $k \leq s$.

Here are the details of the construction.

S t a g e 0. Let $n(0) = 0$ and let $T^0 = \{\emptyset\}$.

Note that at Stage 1, we have $T^1 = \{\emptyset, (0), (1)\}$ and $\beta_0^1 = (1 - A^1(0))$.

S t a g e $s+1$. Find the least $n \leq s$ such that some requirement $\mathrm{R}_{e,n}$ needs attention and take action on $\mathrm{R}_{e,n}$ as described above, where e is the least possible. If no requirement needs attention, let $n(s+1) = |\alpha_s|$, let $\beta_s^{s+1} = \alpha_s$ and for each $j < s$, let β_j^{s+1} have length $n(s+1)$ and extend β_j^s by a string of 0's. The tree T^{s+1} is defined as usual.

It is clear from the construction that for each s, $\beta_0^s, \beta_1^s, \dots, \beta_s^s$ and $A^s\lceil n(s)$ are all distinct nodes of length $n(s)$ in T^s.

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We have to show that each requirement is eventually satisfied and that for each s, $\lim_{s\to\infty}\beta_i^s = B$ exists and belongs to $Q = [T]$.

A key concept in showing this convergence is the notion of the e-state of a finite or infinite path. An infinite path B has e-state $(c_0, c_1, \dots)$ where $c_e = 1$ if $B \in P_e$ and $c_e = 0$ otherwise. For the finite path β_i^s, we define the e-state to be $(c_0, c_1, \dots, c_i)$ where $c_e = 1$ if and only if $\beta \in T_e$. In either definition, the e-states are ordered lexicographically, so that $(c_0, \dots, c_i)$ is lower than $(c_0', \dots, c_i')$ if $c_e < c_e'$ where e is the least such that they are different.

An important observation is that whenever we take action on any requirement $\mathrm{R}_{e+1,n}$ at stage $s+1 > i$, we lower the e-state of β_n, that is, the e-state of β_n^{s+1} is lower than the e-state of β_n^s. The action taken changes c_e itself from 1 to 0 and, for any $d < e$, c_d can only decrease because of clause (iii) in the definition of requiring attention. On the other hand, if we do not take action on $\mathrm{R}_{e+1,n}$ at stage $s+1$, then the e-state of β_n cannot increase, since in that case β_n^{s+1} is an extension of β_n^s.

Claim 1 *Each requirement only requires attention at a finite number of stages.*

P r o o f. Suppose by induction that all higher priority requirements than $\langle e, n\rangle$ require attention at only a finite number of stages. Let s_0 be large enough so that no higher priority requirements ever need attention after stage s_0, and also such that $A\lceil n = A\lceil n^t$ for all $t \geq s_0$. There are two cases.

If $e = 0$ and $\mathrm{R}_{0,n}$ ever requires attention at some stage $t > s_0$, then we take action and get $A^t\lceil n \prec \beta_n^t$. It follows from the construction that action taken on any requirement of lower priority will preserve $A^t\lceil n \prec \beta_n^t$.

For the second case, suppose that $\mathrm{R}_{e+1,n}$ requires attention at some stage $t > s_0$. Then we take action to get $\beta_n^t \notin T_e$ and action taken later on any lower priority requirements can never put $\beta_n \in T_e$ at a later stage. □

This demonstrates the following claim.

Claim 2 *For each n, the e-state of β_n^t converges to a limit $(c_{n,0}, c_{n,1}, \dots, c_{n,n})$.* □

Claim 3 *For each n, the sequence β_n^s converges to an infinite path $B_n \in Q$.*

P r o o f. By the previous claims, we may take s large enough so that the e-stage of β_n has converged and so that for all $e \leq n$, $\mathrm{R}_{e+1,n}$ no longer requires attention. Then after stage s, any action taken only extends β_n^s by a string of 0's. Thus $B_n = \beta_n^s{}^\frown 0^\omega$ and $B_n \in Q$. □

Now we can determine the structure of our Π_1^0 class $Q = [T]$.

Claim 4 $Q = P \cup \{B_n : n < \omega\}$.

P r o o f. Consider an arbitrary path $B \in Q$. For each s, $B\lceil n(s) \in T^s$, and is thus either in S or equal to some β_i^s. If the former happens infinitely often, then $B \in P$, thus we may assume without loss of generality that $B\lceil n(s) = \beta_{i(s)}^s$ for some sequence $i(s)$. Now if there is a fixed i such that $i(s) = i$ for infinitely many s, then $B = B_i$, as desired. Otherwise, there must be infinitely many stages $s+1$ such that $i = i(s+1) \neq j = i(s)$. However, this means that $\beta_j^s \prec \beta_i^{s+1}$, which can only happen when we act on some requirement $\mathrm{R}_{e+1,i}$ with $e \leq i < j$. But this makes $i(s+1) < i(s)$, which, by assumption, can only happen finitely often. □

Next we check that the B_i will approach A in the limit.

Claim 5 *Requirement $\mathrm{R}_{0,n}$ is satisfied for all n, that is, $A\lceil n \prec B_n$.*

P r o o f. By Claim 3, $B_n = \beta_n^s{}^\frown 0^\omega$, where s is large enough so that $\mathrm{R}_{0,n}$ never requires attention after stage s and, by the argument in Claim 2, large enough so that $A^s\lceil n = A\lceil n$. It follows that $A\lceil n = A^s\lceil n \prec B_n$. □

We now consider a stronger notion of convergence of the e-states, necessary to obtain the minimality condition. That is, we prove that for each e, $\lim_n c_{n,e}$ exists.

Claim 6 *For each e, there exists $k = k(e)$ such that for any $n \geq k$, $B_n \in P_e$ if and only if $B_k \in P_e$.*

P r o o f. The sequence $k(e)$ is defined inductively, beginning with the definition of $k(0)$.

C a s e I. For all k, $B_k \notin P_0$. Then let $k(0) = 0$.

C a s e II. There exists k such that $B_k \in P_0$. Then let $k(0)$ be the least such k and let $n > k$. Let s be large enough such that neither $\mathrm{R}_{1,k}$ nor any requirements of higher priority ever require attention after stage s and such

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that the 0-states of both B_n and B_k have converged by stage s. By the argument in Claim 3, we may assume that, for all $i \leq n$, $B_i = \beta_i^s{}^\frown 0^\omega$ and that for all $t > s$, $\beta_i^s \prec \beta_i^t$. Suppose by way of contradiction that $B_n \notin P_e$. Then for some $t > s$, $\beta_n^t \notin T_e$, whereas of course $\beta_n^s \in T_e$. According to the construction, requirement $\mathrm{R}_{1,k}$ would need attention at stage $t+1$, a contradiction.

Here is the definition of $k(e+1)$:

Case I. For all $k \geq k(e)$, $B_k \notin P_{e+1}$. Then let $k(e+1) = k(e)$.

Case II. For some $k \geq k(e)$, $B_k \in P_{e+1}$. Then let $k(e+1)$ be the least such k and let $n > k$. As above, let s be large enough such that the $(e+1)$-states of B_k and B_n have converged, such that neither $\mathrm{R}_{e+1,k}$ nor any requirements of higher priority ever require attention after stage s, and such that, for all $t > s$, $\beta_i^s \prec \beta_i^t$ for $i \leq n$. By induction on e, we know that B_n and B_k have the same e-state and therefore for all $t > s$ and all $d \leq e$, $\beta_n^t \in T_d$ if and only if $\beta_k^t \in T_d$. Suppose by way of contradiction that $B_n \notin P_e$. Then for some $t > s$, $\beta_n^t \notin T_e$, whereas of course $\beta_n^s \in T_e$. According to the construction, requirement $\mathrm{R}_{e+1,k}$ would need attention at stage $t+1$, a contradiction. □

Finally, we can demonstrate that Q is a minimal extension of P.

Claim 7 *For any e, if $P \subset P_e \subset Q$, then either $P_e - P$ is finite or $Q - P_e$ is finite.*

Proof. Suppose that $P_e - P$ is infinite and let $k(e)$ be given by Claim 6 so that for $m > n \geq k(e)$, $B_m \in P_e$ if and only if $B_n \in P_e$. Then there is some $n \geq k(e)$ such that $B_n \in P_e$ and therefore $B_m \in P_e$ for all $m \geq n$. Thus $Q - P_e$ is finite as desired. □

Claim 8 $D(P) = D(Q)$.

Proof. Let $C \neq A$ be an element of Q and let U be a clopen set such that $C \in U$ and $A \notin U$. Since A is the unique limit point of P, it follows that $U \cap P$ is finite. Since $A = \lim_n B_n$, it follows that only finitely many of the B_n can belong to U. Thus $U \cap Q$ is also finite and $C \notin D(Q)$. □

This completes the proof of Theorem 4.1. □

5 The splitting property

In this section, we establish an analogue of the Owings Splitting Theorem for c. e. sets and use it to show that there are no decidable (nontrivial) minimal extensions.

The original Splitting Theorem of Friedberg-Muchnik states that any nonrecursive c. e. set B can be split into disjoint nonrecursive c. e. sets A_0 and A_1. The Owings Splitting Theorem states that whenever $C \subset B$ are two c. e. sets such that $B - C$ is not co-c. e., then B can be split into disjoint c. e. sets A_0 and A_1 such that $A_i - C$ is not co-c. e. for $i = 0, 1$. (We omit some of the further properties of A_i.) It follows from this result that for any c. e. set A, if the lattice $\mathcal{S}(A)^*$ of c. e. supersets of A (modulo finite) is not a Boolean algebra, then it must be infinite. Thus in particular $\mathcal{S}(A)^*$ may not be a chain of 3 sets.

The following result plays the role of the Owings Splitting Theorem for the lattice $\mathcal{E}_\Pi$. (There is a stronger version due to R. Weber [12].)

Theorem 5.1 (Splitting Theorem for decidable Π^0_1 classes) *Let P be a decidable Π^0_1 class and let $P_0 \in [0, P]$ be non-complemented. Then there exists $P_1 \in [P_0, P]$ such that $P - P_1$ and $P_1 - P_0$ are both infinite, and furthermore P_1 is non-complemented.*

Proof. Let $P = [T]$, $P_0 = [T_0]$ for some computable tree T with no dead ends, such that P_0 has no complements in $[0, P]$. This implies that $P - P_0$ is infinite. Let $S = \{\tau \in T - T_0 : \tau\lceil(|\tau| - 1) \in T_0\}$. Then S is a computable set and we note that any two distinct elements of S must be incompatible. It is clear that $P - P_0 = \{x \in P : (\exists \sigma \in S)\, \sigma \prec x\}$. If S were finite, then this would be a Π^0_1 definition of $P - P_0$, contradicting the assumption that P_0 is not complemented in $[0, P]$. Hence S must be an infinite set; let $S = \{\sigma_n : n \in \omega\}$.

Since T has no dead ends, $I(\sigma_n) \cap (P - P_0)$ is nonempty for each n. Now let

$$P_1 = P - \bigcup_{n\in\omega} I(\sigma_{2n}) \ \text{ and } \ P_2 = P - \bigcup_{n\in\omega} I(\sigma_{2n+1}).$$

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Then P_1 and P_2 are Π^0_1 classes and it is clear that $P_1 \cap P_2 = P_0$ while $P_1 \cup P_2 = P$. $P_1 - P_0$ is infinite since it contains an element of $I(\sigma_{2n+1})$ for each n and $P - P_1$ is infinite since it contains an element of $I(\sigma_{2n})$ for each n. Similarly both $P - P_2$ and $P_2 - P_0$ are infinite. Now if P_1 had complement Q_1 so that $P_1 \cup Q_1 = P$ and $P_1 \cap Q_1 = \emptyset$, and P_2 had complement Q_2, then $P_0 = P_1 \cap P_2$ would have complement $Q_1 \cup Q_2$. It follows that at least one of the two sets P_1 and P_2 has no complement in $[0, P]$. □

This has a number of consequences for the lattice $\mathcal{E}_\Pi$.

Theorem 5.2 *Let P be an infinite Π^0_1 class. Then P does not admit a decidable non-trivial minimal extension.*

P r o o f. Suppose that Q is a decidable Π^0_1 class and is a minimal extension of the Π^0_1 class P. Consider the following two cases:

C a s e I. Suppose that P is complemented in $[P, Q]$. Then there is a Π^0_1 class R such that $P \cup R = Q$ and $P \cap R = \emptyset$. It follows that R is a minimal Π^0_1 class and Q is a trivial minimal extension of P.

C a s e II. Suppose that P is not complemented in Q. Then by the Splitting Theorem 5.1 there exists a Π^0_1 class P_1 with $P \subset P_1 \subset Q$ with $P - P$ and $Q - P_1$ both infinite, so that Q is not a minimal extension of P. □

Theorem 5.3 *If P is a decidable Π^0_1 class and P is not thin, then $[0, P]^*$ is infinite.*

P r o o f. Let P be a decidable class with a proper subclass P_0 which is not complemented. By the Splitting Theorem 5.1, there is a subclass P_1 with $P_0 \subset P_1 \subset P$ with P_1 also not complemented. Applying the theorem again, we get P_2 between P_1 and P and by repeating the process, we get $P_0 \subset P_1 \subset P_2 \subset \dots \subset P$, with each inclusion proper. □

We note that this theorem was shown by Cenzer and Nies in [4] using a direct proof. We have seen that the point 0^ω is the unique limit point of infinitely many different Π^0_1 classes. However, it follows from Theorem 5.3 that they cannot be decidable.

Proposition 5.4 *If P is a decidable, minimal Π^0_1 class and $D(P) = \{A\}$, then A is not computable.*

P r o o f. Let $D(P) = \{A\}$. It was shown in [2] that P is thin if and only if A is noncomputable. Thus if A is computable, then P is not thin, so that P is not minimal by Theorem 5.3. □

The family of *decidable* Π^0_1 classes may be also be viewed as a lattice, since the union and intersection of decidable classes is decidable (although of course a Π^0_1 subclass of a decidable class is not necessarily decidable). Theorem 5.3 shows that the structure of the lattice of *decidable* Π^0_1 classes behaves more like $\mathcal{E}$. This raises the natural conjecture that this structure is elementarily equivalent or perhaps isomorphic to $\mathcal{E}$.

Theorem 5.3 has an interesting corollary as follows. Recall that a nontrivial minimal extension Q of a class P may be obtained by adjoining a minimal class M whose limit point is already in P. If P itself is minimal, then P and M are two distinct minimal classes with the same limit point. This cannot be done with decidable minimal classes by the following

Corollary 5.5 *Let P_1 and P_2 be any two decidable minimal classes. If $D(P_1) = D(P_2)$, then $P_1 =^* P_2$.*

P r o o f. Suppose that P_1 and P_2 are decidable minimal Π^0_1 classes, that $D(P_1) = D(P_2) = \{A\}$, and that, by way of contradiction, $P_1 \neq^* P_2$. It follows that $P_1 \cup P_2$ is a minimal extension of P_1 and is decidable. Furthermore, $D(P_1 \cup P_2) = D(P_1) \cup D(P_2) = \{A\}$, so that $P_1 \cup P_2$ is a nontrivial minimal extension. This contradicts Theorem 5.2. □

This corollary implies that, modulo finite, there is at most one decidable minimal Π^0_1 class P with limit point A for any A. We conclude with the following question.

Problem 5.6 *If there is a minimal Π^0_1 class with limit point A, then is there always a decidable minimal Π^0_1 class with limit point A ?*

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