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Decidability by Filtrations for Graded Normal Logics (Graded Modalities V)

Abstract. We prove decidability for all of the main graded normal logics, by a notion of filtration suitably conceived for this environment.

1. Introduction

Graded normal logics $(GNLs)$ are the extensions of modal normal logics to a language with graded modalities. The interpretation in usual Kripke models of a formula $\Diamond_n A$ $(n < \omega)$, whose main operator is a graded possibility, is there are *more than* n accessible worlds where A is true.

When graded modalities were introduced (in [7], independently rediscovering a former idea of [8]), the purpose was to offer axiomatizations for the graded versions of the normal modal logics, and then to show completeness, compactness and decidability theorems for the fifteen main $GNLs$ between K° and $S5^\circ$. Completeness and compactness were fully proved in several steps ([8], [7], [5], [6], [2]), while the decidability had only partial answers in [7], [9], [1], [10], [11]. We prove decidability for all of the main $GNLs$ by a suitable version of the notion of filtration, completing the basic investigation of graded modalities.

Usually, when talking about filtrations one has in mind to start from a given model, then to obtain a quotient model with respect to a certain kind of set of formulas, and finally to suitably arrange the accessibility relation ([12], [13], [14], [3]). On the contrary, we reduce a model to a finite one by combining generated models with usual filtration techniques and with controls on the grades of modalities, respecting in a quite natural way the properties of the accessibility relation.

Finally, as a corollary, since graded modalities really extend the usual ones, our notion of filtration can be used also for the usual normal modal logics (by restricting the attention to the formulas that contain as modal operators only $\Box_0$ and $\Diamond_0$), avoiding the problems that usual filtrations offer to respect the properties of the accessibility relation ([3]).

2. Usual filtrations fail

Filtrations are used when, having in mind to prove the decidability of a modal system S, one proves that S has the finite model property ($f.m.p.$), i.e. that it is complete with respect to the class of its finite models. Then, by soundness, this amounts to proving that if a formula A is not an S-theorem then there exists a finite S-model where A is not valid. By completeness, if a formula A is not an S-theorem then there exists an S-model (in general not finite) where A is not valid; so, to have the $f.m.p.$ we must only prove that if there exists an S-model where A is not valid then there exists a finite S-model where A is not valid.

But actually, usual filtrations assure a stronger result: for any model $\mathfrak{A} = \langle W, R, V \rangle$ and for any finite set of formulas Γ, closed under subformulas, there exists an equivalence relation $\equiv$ induced by Γ and a finite model $\mathfrak{A}^* = \langle W^*, R^*, V^* \rangle$, where $W^* = W/\equiv = \{[w] : w \in W\}$, such that $\mathfrak{A} \models_w B$ iff $\mathfrak{A}^* \models_{[w]} B$, for each $w \in W$ and $B \in \Gamma$; as an immediate corollary if Γ is the set of the subformulas of A, then A is valid in $\mathfrak{A}$ iff A is valid in $\mathfrak{A}^*$. But in [2] a peculiar model was exhibited for the symmetric systems $\boldsymbol{KB^\circ}$, $\boldsymbol{KBD^\circ}$, $\boldsymbol{KBT^\circ}$ that validates the formula $(A \rightarrow \Diamond!_1 B \wedge \Diamond!_2 C) \wedge (B \rightarrow \Diamond!_1 A \wedge \Diamond!_1 C) \wedge (C \rightarrow \Diamond!_1 A \wedge \Diamond!_1 B) \wedge (A \vee B \vee C)$ while every finite model of those systems does not validate it. So for such symmetric systems the usual filtrations fail. To prove the $f.m.p.$ we allow filtrations preserving the value of at least one formula in at least one world.

In the usual filtrations ([12], [13], [14], [3]) we easily respect the value of modal formulas whose main operator is the possibility, by making two worlds accessible in the filtration just when plausible; the usual modalities do not count accessible worlds, so that to have one accessible world or n accessible worlds is the same thing. On the contrary, graded modalities count accessible worlds, so we must use sharper controls when setting the accessibility relation.

In [9] (as in [1], [10], [11]) it was also suggested that the explosion of copies of equivalent worlds could be controlled by fixing as an upper bound the maximum of the indexes of the graded modalities occurring in the formula, plus one; but this control also fails for symmetric systems: e.g., the formula $F = \Diamond!_2(A \wedge \Diamond!_2(B \wedge \neg A \wedge \Diamond!_1 A \wedge \Diamond!_0 B))$ forces a $\boldsymbol{KB^\circ}$ symmetric model to have at least 4 worlds (that agree on the subformulas of F) where B is true, while the maximum of the indexes is 2 and the suggested upper bound of copies is 3. However, we use the indexes to establish a local control of the number of copies, abandoning, as we have done for canonical models (see [2]), the attempt to find an algorithmic global control that works well in every case.

3. General filtrations

First for sake of simplicity we rewrite formulas using only the graded possibilities, transforming when necessary each formula into an equivalent one in accordance with $\Box_n B = \neg \Diamond_n \neg B$ and $\Diamond!_n B = \Diamond_{n-1} B \wedge \neg \Diamond_n B$.

Then we introduce some technical notations about particular sets of subformulas that will be useful when proving the *f.m.p.*: given a model $\mathfrak{A} = \langle W, R, V \rangle$, when determining the value of a formula A in a world w we are not interested in the local value (i.e. in w) of occurrences of subformulas of A that are in the scope of modal operators (in fact, we must take care only of their values in the accessible worlds). So we consider the set of local subformulas of A, $LS(A)$, defined as:

$A \in LS(A)$;

if $B \in LS(A)$ and $B = C \wedge D$ $(C \vee D, C \to D, C \leftrightarrow D)$ then $C, D \in LS(A)$;

if $B \in LS(A)$ and $B = \neg C$ then $C \in LS(A)$.

As an example, if $A = C \to \neg \Diamond_3 (D \vee E)$ then $LS(A) = \{ C \to \neg \Diamond_3 (D \vee E), C, \neg \Diamond_3 (D \vee E), \Diamond_3 (D \vee E) \}$ while $(D \vee E), D, E \notin LS(A)$.

Furthermore, denoted as $\mathbf{P}$ the set of the propositional symbols of the language, let:

$$
\begin{aligned}
P(A) &= \mathbf{P} \cap LS(A) \\
M(A) &= \{ \Diamond_n B : \Diamond_n B \in LS(A) \} \\
I(A) &= \{ B : \Diamond_n B \in LS(A) \}
\end{aligned}
$$

Thus $P(A)$ is the set of the atomic formulas in $LS(A)$, $M(A)$ is the set of the formulas of $LS(A)$ whose main connectives are the modal operators, and $I(A)$ $(= \{ \Diamond_n B : \Diamond_n B \in M(A) \})$ is the set of those formulas of $LS(A)$ under the scope of a modal operator.

In the above example $P(A) = \{C\}$, $M(A) = \{ \Diamond_3 (D \vee E) \}$ and $I(A) = \{D \vee E\}$. We remark that those sets have not to be disjoint: e.g. when $A = C \wedge \Diamond_2 C$, $LS(A) = \{ C \wedge \Diamond_2 C, C, \Diamond_2 C \}$, $P(A) = \{C\}$, $M(A) = \{ \Diamond_2 C \}$, $I(A) = \{C\}$.

Clearly, every formula of $LS(A)$ is a propositional combination of formulas of $P(A)$ and $M(A)$; the value of the formulas of $P(A)$ is completely determined in w, while the value of the formulas of $M(A)$ depends on the value of the formulas of $I(A)$ in the accessible worlds.

Finally, we extend the above notations to sets of formulas: let Γ be a set of formulas, we define $LS(\Gamma) = \bigcup \{ LS(A) : A \in \Gamma \}$ and in a similar way we define $P(\Gamma)$, $M(\Gamma)$ and $I(\Gamma)$.

Now, after these technical premises, we are ready to prove the *f.m.p.* for *GNLs*; first we start with $K°$:

let $\mathfrak{A} = \langle W, R, V \rangle$ be a $K°$–model, v be a world and A be a formula; we construct a finite submodel of $\mathfrak{A}$, namely $\mathfrak{A}^*$, such that $\mathfrak{A} \models_v A$ iff $\mathfrak{A}^* \models_v A$, i.e. the *f.m.p.* holds. We construct $\mathfrak{A}^*$ in three steps:

1) first we restrict the domain to those worlds so near to v to influence the value of modal subformulas of formulas of v; namely, we recognize the maximal number of nested modalities in A, m, and restrict the domain to the worlds accessible from v at most in m steps;

2) then we reduce the accessibility relation so that every world has only a finite number of accessible worlds; we respect only the value of some selected modal subformulas of A: they obviously form a finite set of formulas and each of them requires a finite number of accessible worlds;

3) finally, we restrict the domain again to the worlds accessible in a finite number of steps from v by the new relation; since every world has only a finite number of accessible worlds (by 2) and we can reach worlds accessible from v at most in m steps (by 1), we obtain a finite model; moreover, we take care of the value of some selected modal subformulas of A so that we easily prove $\mathfrak{A}^*$ preserves the value of A in v, i.e. the *f.m.p.* is proved.

1) We define the degree of modal complexity of formulas, ∂, as:

$$\begin{aligned} \partial(P) &= 0 \\ \partial(\Diamond_n C) &= \partial(C) + 1 \\ \partial(C \wedge D) = \partial(C \vee D) = \partial(C \to D) = \partial(C \leftrightarrow D) &= max.\{\partial(C), \partial(D)\} \\ \partial(\neg C) &= \partial(C) \end{aligned}$$

and

$$\partial(\Gamma) = max.\{\partial(C) : C \in \Gamma\}$$

where Γ is a finite set of formulas (we really consider only subsets of the set of the subformulas of A, that are finite). Obviously, we have $\partial(\Gamma) = \partial(LS(\Gamma)) = \partial(M(\Gamma))$.

Now we set:

$$\begin{aligned} W_0' &= \{v\}, LS_0 = LS(A), P_0 = P(A), M_0 = M(A), I_0 = I(A); \\ W_1' &= \{w \in W : vRw\}, \\ LS_1 &= LS(I_0), P_1 = P(I_0), M_1 = M(I_0), I_1 = I(I_0); \\ &\quad \cdots \\ W_{n+1}' &= \{w \in W : \text{ there exists } w' \in W_n' \text{ such that } w'Rw\}, \\ LS_{n+1} &= LS(I_n), P_{n+1} = P(I_n), M_{n+1} = M(I_n), I_{n+1} = I(I_n); \\ &\quad \cdots \quad . \end{aligned}$$

Since $\partial(LS(\Gamma)) = \partial(M(\Gamma)) = \partial(I(\Gamma)) + 1 = \partial(\ LS(I(\Gamma))) + 1$ (by definition) the degree of LS_i decreases by 1 at each step; so after $m = \partial(LS(A))$ steps $\partial(LS_m) = 0$ so that $M_m = I_m = \emptyset$ and $LS_m = P_m$: here we stop the definition of the W_i'.

The restriction of the model $\mathfrak{A}$ to the union of the W_i' $(i \leq m)$ is a (not still necessarily finite) submodel that respects the value of A in v. In fact, let $W' = \bigcup\{W_i' : i \leq m\}$ and $\mathfrak{A}' = \mathfrak{A}|_{W'} = \langle W', R', V' \rangle$, where $R' = R|_{W'} = R \cap (W' \times W')$ and $V' = V|_{W'}$ (i.e. V' is a valuation on W' such that $V'(w, P) = V(w, P)$ for every $w \in W'$ and $P \in \mathbb{P}$, while the value for the other formulas is defined in the usual way). Then

a) in every world of $\mathfrak{A}'$ the values of atomic formulas are the same as in $\mathfrak{A}$;

b) for every $w \in \bigcup\{W_i' : i < m\}$ the accessible worlds are exactly the same as in $\mathfrak{A}$;

c) each $w \in W'$ could belong to more than one W_i', so that we define $\rho(w) = \{i \leq m : w \in W_i'\}$.

Every $w \in W'$ preserves the values (in $\mathfrak{A}'$) of the formulas of LS_j, P_j, M_j for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$; in fact:

(o) in every world of W' the values of the formulas of P_j are the same as in $\mathfrak{A}$ (by a);

(oo) since $LS_m = P_m$, in every world of W_m' the values of the formulas of LS_m are the same as in $\mathfrak{A}$ (by o);

(ooo) for each $i < m$, if for every world of W_{i+1}' the values of the formulas of $LS_{i+1}(\supseteq I_i)$ are the same as in $\mathfrak{A}$, then for every world of W_i' the values of the formulas of M_i are the same as in $\mathfrak{A}$ (in fact, when $i < m$, the accessible worlds are exactly the same as in $\mathfrak{A}$, by b), so that also the values of the formulas of LS_i are the same as in $\mathfrak{A}$.

So, using (o), starting from W_m' (by oo), in m steps (by ooo) we prove the thesis.

2) Now we determine a suitable restriction of the accessibility relation R' that respects the values of some selected modal subformulas of A in every world; we show every world has only a finite number of accessible worlds with respect to this restricted relation and every intermediate relation between this restriction and R' also respects the value of some selected modal subformulas of A in every world.

As to notation, given two relations X and Y, X is a restriction of Y means that X is defined on a subset of the domain of Y, and that X and Y are the same on such set; on the contrary, X is a reduction of Y means only that X is a subrelation of Y. That distinction will be really meaningful when considering the properties of the accessibility for the other $GNLs$: in fact, restrictions remain reflexive, symmetric, transitive, euclidean, while reductions in general do not.

Given $w \in W'$ (say $w \in W_i'$), we select $\bigcup\{M_j : j \in \rho(w)\}$ as the set of formulas whose values are to be respected in w: so let

$$
\begin{aligned}
I(w) &= \bigcup\{\, I_j : j \in \rho(w)\,\} \\
M^+(w) &= \Big\{\, C \in \bigcup\{\, M_j : j \in \rho(w)\,\} : V(w,C) = 1 \,\Big\} \\
M^-(w) &= \bigcup\{\, M_j : j \in \rho(w)\,\} \setminus M^+(w) \\
R'(w) &= \{\, w' \in W' : wR'w'\,\} \subseteq W_{i+1}'.
\end{aligned}
$$

To respect the value of the formulas of $\bigcup\{M_j : j \in \rho(w)\}$ suffices restricting the accessibility relation in a way such that:

d) the values of the formulas of $I(w)$ are respected in all of the w–accessible worlds;

e) for every $w' \in M^+(w)$ there are *more than* n w–accessible worlds where B is true;

f) for every $w' \in M^-(w)$ there are *not more than* n w–accessible worlds where B is true.

Recalling usual filtrations, we partition $R'(w)$ according to the values of the formulas in $I(w)$, i.e. two worlds are equivalent ($\equiv$) iff they agree on the values of all of the formulas of $I(w)$. Since $I(w)$ is really a subset of the set of subformulas of A (by definition), a finite number of classes $\mathcal{C}_o, \ldots, \mathcal{C}_h$ is induced by $\equiv$.

Let us consider one $\mathcal{C} \in \{\mathcal{C}_o, \ldots, \mathcal{C}_h\}$; it may happen $\mathcal{C}$ is infinite, so that we need to reduce $\mathcal{C}$ in a reasonable way.

To respect the values of the formulas of $M^-(w)$ we must only avoid adding any new accessible world to $\mathcal{C}$.

As to formulas of $M^+(w)$, a reduction of $\mathcal{C}$ could affect only the values in w of formulas $\Diamond_n B$ of $M^+(w)$ such that B is true in (every world of) $\mathcal{C}$; in any case, we are sure to respect the value of $\Diamond_n B$ by making accessible at least $n+1$ worlds of $\mathcal{C}$. So let

$$
\begin{aligned}
g(w,\mathcal{C}) &= max\{\, n+1 : \Diamond_n B \in M^+(w), \\
&\qquad\qquad V'(w',B) = 1 \text{ for } w' \in \mathcal{C}\,\} \quad (\text{when } M^+(w) \neq \emptyset) \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (\text{when } M^+(w) = \emptyset)
\end{aligned}
$$

that is the number of worlds of $\mathcal{C}$ that are enough to respect in any case the values in w of all of the formulas of $M^+(w)$.

So, to respect both $M^-(w)$ and $M^+(w)$, we need

$$r(w,\mathcal{C}) = min\{g(w,\mathcal{C}), card(\mathcal{C})\}$$

worlds of $\mathcal{C}$. In fact, since we have not added any new accessible world we respect $M^-(w)$; furthermore, let $\Diamond_n B \in M^+(w)$, let $\mathcal{C}_{i_0}, \ldots, \mathcal{C}_{i_m}$ be the classes of $R(w)/\equiv$ where B is true; since $\mathfrak{A}'$ satisfies $\Diamond_n B$ in w (by hypothesis), then $n < card(\mathcal{C}_{i_0}) + \cdots + card(\mathcal{C}_{i_m})$. Two cases can occur: (a) $card(\mathcal{C}_{i_h}) \leq g(w,\mathcal{C}_{i_h})$, which implies $r(w,\mathcal{C}_{i_h}) = card(\mathcal{C}_{i_h})$, for every $h \leq m$, so that $n < card(\mathcal{C}_{i_0}) + \cdots + card(\mathcal{C}_{i_m}) = r(w,\mathcal{C}_{i_0}) + \cdots + r(w,\mathcal{C}_{i_m})$; (b) $g(w,\mathcal{C}_{i_k}) \leq card(\mathcal{C}_{i_k})$, which implies $r(w,\mathcal{C}_{i_k}) = g(w,\mathcal{C}_{i_k})$, for some $k \leq m$, so that (by the definition of $g(w,\mathcal{C}_{i_k})$) $n < g(w,\mathcal{C}_{i_k}) = r(w,\mathcal{C}_{i_k}) \leq r(w,\mathcal{C}_{i_0}) + \cdots + r(w,\mathcal{C}_{i_m})$. In any case, we have enough accessible worlds to respect $\Diamond_n B \in M^+(w)$.

This proof works also when $R'(w) = \emptyset$ and when $I(w) = \emptyset$: in these cases $r(w,\mathcal{C})=0$.

So we reduce the accessibility relation R', stating that the number of the w–accessible worlds is $r(w,\mathcal{C})$ for each class $\mathcal{C}$; a simple way to do this is to well–order each class $\mathcal{C}$ and to make accessible the first $r(w,\mathcal{C})$ worlds; in any case, by changing the ordering we change the relation, so that, really, we have a family of restrictions. Namely, given a well–order for each class $\mathcal{C}$, let S be the reduction of R' defined as:

$$wSx_s \text{ iff } s < r(w,\mathcal{C}), \text{ for each } w \in W', \mathcal{C} \in R'(w)/\equiv, \text{ and } x_s \in \mathcal{C}.$$

Since we have a finite number of equivalence classes and we allow only a finite number of worlds in each class to be accessible, as a result only a finite number of worlds are accessible (by S) from any w.

So let $\mathfrak{B} = \langle Wd, Re|_{Wd}, V|_{Wd} \rangle$ be a model such that

i) $\{w' \in W' : wSw', w \in Wd\} \subseteq Wd \subseteq W'$

ii) $S \subseteq Re \subseteq R'$

iii) $v \in Wd$.

Every $w \in Wd$ satisfies in $\mathfrak{B}$ the same formulas of LS_j (and P_j, M_j) as it does in $\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$; in fact:

(o) in every world of Wd the values of the formulas of P_j are the same as in $\mathfrak{A}'$ (since the evaluation is still V);

(oo) since $LS_m = P_m$, in every world of $W'_m \cap Wd$ the values of the formulas of LS_m are the same as in $\mathfrak{A}'$ (by o);

(ooo) for each $i < m$, if for every world of $W'_{i+1} \cap Wd$ the values of the formulas of $LS_{i+1} (\supseteq I_i)$ are the same as in $\mathfrak{A}'$ then for every world of $W'_i \cap Wd$ the values of the formulas of M_i are the same as in $\mathfrak{A}'$ (in fact, such a world w satisfies the property d — since $I_i \subseteq LS_{i+1}$, by the inductive hypothesis — the property e — since there are enough accessible worlds, because Wd contains all of the worlds accessible by S, by iii, and $Re \supseteq S$, by ii — and the property f, too — since we added no new accessible world, because $Re \subseteq R'$, by ii), so that also the values of the formulas of LS_i are the same as in $\mathfrak{A}'$.

So, using (o), starting from $W'_m \cap Wd$ (by oo), in m steps (by ooo) we prove the thesis.

3) Now we set:

$$
\begin{aligned}
W^*_0 &= \{v\}, \\
W^*_1 &= \{w \in W' : vSw\}, \\
&\cdots \\
W^*_{n+1} &= \{w \in W' : \text{ there exists } w' \in W^*_n \text{ such that } w'Sw\}, \\
&\cdots \\
W^*_m &= \{w \in W' : \text{ there exists } w' \in W^*_{m-1} \text{ such that } w'Sw\},
\end{aligned}
$$

and

$$
W^* = \bigcup\{W^*_i : i \leq m\}.
$$

W^* is finite: in fact, each world has only a finite number of S–accessible worlds (by definition of S) and we construct W^* starting from only one world and considering only S–accessible worlds at each step; since we stop the construction after a finite number of steps, W^* is finite.

Let $\mathfrak{A}^* = \langle W^*, S|_{W^*}, V|_{W^*} \rangle$: by construction, $\mathfrak{A}^*$ respects all of the conditions i, ii and iii, so that every $w \in W^*$ satisfies in $\mathfrak{A}^*$ the same formulas of LS_j (and P_j, M_j) as it does in $\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A (\in LS_0)$. So $\mathfrak{A}^*$ is a finite model that satisfies A in at least a world; the *f.m.p.* for K° is proved.

Now we prove the *f.m.p.* for all of the $GNLs$ using the proof used for K° with suitable modifications for each system; in fact we must take care the model we construct is also a model of the system.

When constructing the finite model $\mathfrak{A}^*$ for K°, we have reduced the relation R to R' and R' to S: but R' is really a restriction of R, and so

it maintains all the usual properties of R (but, possibly, seriality: this fact will oblige to do specific considerations for those systems in which seriality is involved), while S is a reduction of R', so that it can lose some of those properties.

When constructing the finite model $\mathfrak{A}^*$ for any system S among KT°, $K4^\circ$, $KT4^\circ$, $KT5^\circ$, KB°, KBT°, $KB4^\circ$, $K5^\circ$, $K45^\circ$, the simplest way to avoid those problems is to consider the suitable (reflexive, symmetric, transitive, euclidean) closure of the relation S, $\overline{S}$, with respect to the properties of the accessibility of S–models, and to set $\mathfrak{A}^* = \left\langle W^*, \overline{S}|_{W^*}, V|_{W^*} \right\rangle$: clearly $\mathfrak{A}^*$ respects the conditions i and iii; furthermore, $S \subseteq R'$ so that $\overline{S} \subseteq \overline{R}'$; since R' (as R) satisfies the properties of the accessibility of S–models, it coincides with its closure: thus $\overline{S} \subseteq \overline{R}' = R'$; moreover $S \subseteq \overline{S}$, so that $\mathfrak{A}^*$ respects also the condition ii: $\mathfrak{A}^*$ is a finite model of the system that satisfies A in at least one world; the *f.m.p.* for KT°, $K4^\circ$, $KT4^\circ$ $(S4^\circ)$, $KT5^\circ$ $(S5^\circ)$, KB°, KBT°, $KB4^\circ$, $K5^\circ$, $K45^\circ$ is proved.

As to systems in which the axiom schema D° is involved, both R' and S are, in general, not serial. One could try to extend S to a serial relation, by making accessible from any world w at least one world accessible by R'; but this attempt fails for any w with $R'(w) = \emptyset$; since R is serial, such a w must belong only to the last level of W' (whose accessible worlds were cut), i.e. $w \in W' \setminus \bigcup\{W'_i : i < m\}$. Furthermore, the attempt to obtain a serial accessibility by making any world accessible from itself fails for every world w that must respect the values of the formulas in $M^-(w)$; in any case, for every $w \in W' \setminus \bigcup\{W'_i : i < m\}$, $M^-(w) = \emptyset$. In conclusion, those two ideas fail — but on different worlds, so that we can apply one where the other fails, and obtain a finite serial model.

First we prove the *f.m.p.* for the minimal serial system KD°: for any $w \in W'$ such that $R'(w) \neq \emptyset$ let x_w be a world of $R'(w)$; so let:

$$\begin{aligned} T &= S \cup \{(w, x_w) : w \in W' \text{ and } R'(w) \neq \emptyset\}, \\ W''_0 &= \{v\}, \\ &\cdots \\ W''_{n+1} &= \{w \in W' : \text{ there exists } w' \in W''_n \text{ such that } w'Tw\} \\ &\cdots \end{aligned}$$

and

$$W'' = \bigcup\{W''_i : i \leq m\}.$$

W'' is finite: in fact, each world has only a finite number of T–accessible worlds (the S–accessible worlds plus one, by definition) and we construct W'' starting from only one world and considering only T–accessible worlds

each step; since we stop the construction after a finite number of steps, W''' is finite.

Let $\mathfrak{A}'' = \langle W'', T|_{W''}, V|_{W''} \rangle$: $\mathfrak{A}''$ clearly respects the conditions i and iii; furthermore, by the definition of x_w, $\mathfrak{A}''$ respects also the condition ii, so that every $w \in W''$ satisfies in $\mathfrak{A}''$ the same formulas of LS_j (and P_j, M_j) as it does in $\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$. $\mathfrak{A}''$ is serial except for those $w \in W''$ such that $R'(w) = \emptyset$. So let:

$$Q = T|_{W''} \cup \{ (w,w) : w \in W'' \text{ and } R'(w) = \emptyset \}$$

and

$$\mathfrak{A}^* = \langle W'', Q, V|_{W''} \rangle ;$$

Q is serial; furthermore, $R'(w)$ is empty only for any $w \in W'' \cap [W' \setminus \bigcup \{ W'_i : i < m \}]$, i.e. worlds affecting only the values of formulas in P_m; so every $w \in W''$ satisfies in $\mathfrak{A}^*$ the same formulas of LS_j (and P_j, M_j) as it does in $\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$. Thus $\mathfrak{A}^*$ is a finite serial model that satisfies A in at least one world; the *f.m.p.* for **KD°** is proved.

As to the other serial systems, **KD4°**, **KD5°**, **KD45°**, **KBD°**, we suitably modify the construction of both $\mathfrak{A}''$ and $\mathfrak{A}^*$ in a way similar to that done for non-serial systems. Namely, for any system **S** among **KD4°**, **KD5°**, **KD45°**, **KBD°**, we consider the suitable (symmetric, transitive, euclidean) closure of T, $\overline{T}$, with respect to the properties of the accessibility of **S**–models other than seriality, and set $\mathfrak{A}'' = \langle W'', \overline{T}|_{W''}, V|_{W''} \rangle$: $\mathfrak{A}''$ clearly respects the conditions i and iii; furthermore, reasoning as for non-serial systems, $\mathfrak{A}''$ respects also the condition ii: thus every $w \in W''$ satisfies in $\mathfrak{A}''$ the same formulas of LS_j (and P_j, M_j) as it does in $\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$. As for **KD°**, $\mathfrak{A}''$ is serial except for those $w \in W''$ such that $R'(w) = \emptyset$. So let:

$$Q = \overline{T}|_{W''} \cup \{ (w,w) : w \in W'' \text{ and } R'(w) = \emptyset \}$$

and

$$\mathfrak{A}^* = \langle W'', Q, V|_{W''} \rangle ;$$

Q is serial and also symmetric, transitive, euclidean as T is (depending on the system **S**); furthermore, $R'(w)$ is empty only for any $w \in W'' \cap [W' \setminus \bigcup \{ W'_i : i < m \}]$, i.e. worlds affecting only the values of formulas in P_m; so every $w \in W''$ satisfies in $\mathfrak{A}^*$ the same formulas of LS_j (and P_j, M_j) as it does in

$\mathfrak{A}$, for each $j \in \rho(w)$, and in particular v satisfies $A(\in LS_0)$. Thus $\mathfrak{A}^*$ is a finite $\boldsymbol{S}$–model that satisfies A in at least one world; the f.m.p. for $\boldsymbol{KD4^\circ}$, $\boldsymbol{KD5^\circ}$, $\boldsymbol{KD45^\circ}$, $\boldsymbol{KBD^\circ}$ is proved.

Finally we remark that we have always supposed every GNL $\boldsymbol{S}$ complete with respect to the class of $\boldsymbol{S}$–models: in fact completeness was proved, by general canonical models, in [2] — but the proof of the existence of the general canonical model for $\boldsymbol{KD4}$ (and so the proof of completeness) was unintentionally omitted by the author; that canonical model is easily seen to be definable along the same lines as for $\boldsymbol{K}$ or for $\boldsymbol{KD}$ or for $\boldsymbol{K4}$, and does not give rise to any technical trouble; so completeness holds, and general filtrations give rise to the *f.m.p.*

4. The decidability of $GNLs$

In general, the finite model property is not sufficient to assure we have found a recursively enumerable (*r.e.*) class of models: in fact, in [15] was exhibited a *r.e.*–axiomatized modal system (that is a recursively axiomatized one, by [4]) whose class of finite models is not *r.e.* However, when the system has also only a finite number of axiom schemata we can establish in a finite number of steps if a finite model is a system model, and so the class of finite models of the system is *r.e.*

In our case, as shown in [1], since $GNLs$ are only a syntactical extension of the usual normal logics (NLs), the models of $GNLs$ are the same of the corresponding NLs: e.g. to establish a model is a $\boldsymbol{K4^\circ}$–model we must only prove it is a transitive model, that is a $\boldsymbol{K4}$–model; so we do not need to test all the schemata $\Box_n A \to \Box_0 \Box_n A$ but only the usual $\Box_0 A \to \Box_0 \Box_0 A$. Since the NLs corresponding to the main $GNLs$ have a finite number of axiom schemata ([3]) the classes of finite $GNLs$–models are *r.e.*

Furthermore, $GNLs$ are recursively axiomatized ([7], [5], [6], [2]), so that we have both a *r.e.* set of theorems and a *r.e.* set of non–theorems: $GNLs$ are decidable.

5. Conclusions and acknowledgements

We have proved the decidability of the main $GNLs$ between $\boldsymbol{K^\circ}$ and $\boldsymbol{S5^\circ}$ by general filtrations. The use of filtrations for graded modalities was suggested in [9] and developed in [1], [10], [11]. Unfortunately symmetric systems inhibit any attempt to globally control models (see [2]), so that old canonical models do not exist and the usual filtrations fail. In [2] the author introduced general canonical models and proved completeness for the $GNLs$; now we

have introduced a general filtration method for $GNLs$ that works well in every case, and we have proved both the finite model property and decidability for the $GNLs$.

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