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General Canonical Models for Graded Normal Logics (Graded Modalities IV)

Abstract. We prove the canonical models introduced in [D] do not exist for some graded normal logics with symmetric models, namely ***KB***°, ***KBD***°, ***KBT***°, so that we define a new kind of canonical models, the general ones, and show they exist and work well in every case.

1. Introduction

Graded modalities investigate combinatorial properties of Kripke models: graded possibility operators $\Diamond_i (i \in N)$ and their dual ones, graded necessity operators $\Box_i = \neg \Diamond_i \neg (i \in N)$, were introduced in [FD], using natural numbers as grades so to capture models' features expressible with reference to finite cardinalities.

Using notation from [FD], let $T_w(\alpha) = \{w' \in W\colon wRw' \text{ and } V(w', \alpha) = 1\}$, $\langle W, R, V \rangle$ being a Kripke model and α a well formed formula, so that the "meaning" of the graded possibility operators are explicated by

$$V(w, \Diamond_i \alpha) = 1 \quad \text{iff} \quad |T_w(\alpha)| > i \quad (i \in N).$$

Thus, as in [FD], we can

a) easily obtain, by duality, suitable "meanings" for the graded necessity operators $\Box_i$;

b) introduce new useful operators $\Diamond !_i (i \in N)$, whose "meaning" is

$$V(w, \Diamond !_i \alpha) = 1 \quad \text{iff} \quad |T_w(\alpha)| = i \quad (i \in N);$$

c) observe that graded modalities extend standard ones, i.e. $\Diamond_0$ and $\Box_0$ have the same "meanings" of $\Diamond$ and $\Box$ respectively.

In fact the study of Kripke models involving the graded modalities has been developed in a way closely resembling the classical one, and the research has been so far focused on complete axiomatizations of the graded versions of normal logics (*GNLs*): in [FD], [D], [FC] ***K***°, ***K4***°, ***KT***°, ***KT4***°, ***KT5***° were introduced and shown complete with respect to the relevant class of Kripke models (we denote by ***S***° the graded version of a normal modal logic ***S***).

In this paper we show that the canonical models, introduced in [D] as tools to derive more easily completeness theorems for *GNLs*, cannot make this job for systems like ***KB***°, ***KBD***°, ***KBT***°, because they do not exist at all.

So we introduce a new kind of canonical model for a *GNL*, that exists and works well in every case and allows to complete the program to prove completeness for all the relevant *GNLs*.

2. A critical model of $KB°$, $KBD°$, $KBT°$

Now we introduce a model $\mathfrak{A}$ of $KB°$, $KBD°$, $KBT°$ which shall play a central role to introduce the new kind of canonical models.

We build $\mathfrak{A}$ by dividing its worlds into three disjoint classes, called a-, b-, c-class respectively, each being the set of worlds in which only the propositional variable A, B, C are respectively true.

Furthermore, each world of $\mathfrak{A}$ has only a finite number of accessible worlds, and we shall prove that, assuming the "old" strong canonical models of $KB°$, $KBD°$, $KBT°$ exist, they must have a finite submodel equivalent to $\mathfrak{A}$, i.e. such to validate the same set of sentences which $\mathfrak{A}$ validates. On the other hand we shall prove also that $\mathfrak{A}$ cannot be equivalent (in the above sense) to any finite model of $KB°$, $KBD°$, $KBT°$, so getting a contradiction and concluding those strong canonical models do not exist.

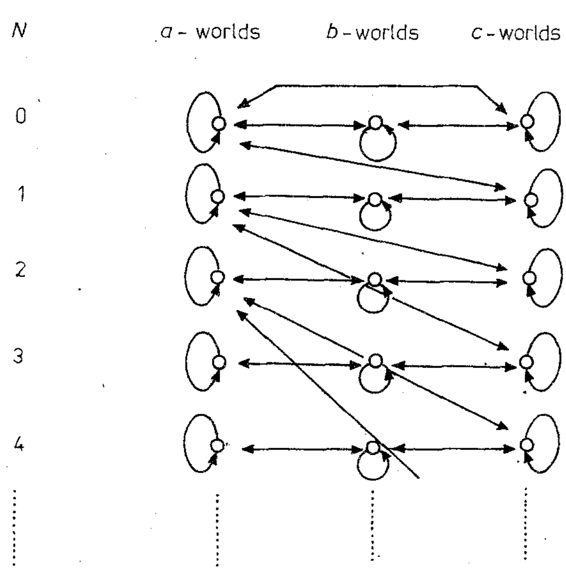
As to notation, we use capital latin letter for propositional variables, small ones for Kripke worlds and small greek letters for wff (= well formed formula(s)).

Let $\mathfrak{A} = \langle W, R, V \rangle$ where

- $W = \{a_i\colon i \in N\} \cup \{b_i\colon i \in N\} \cup \{c_i\colon i \in N\}$
- $V(a_i, A \wedge \neg B \wedge \neg C) = V(b_i, \neg A \wedge B \wedge \neg C) =$
  $= V(c_i, \neg A \wedge \neg B \wedge C) = 1 \quad (i \in N)$
  $V(w, P) = 0 \quad (w \in W, P \neq A, B, C)$
- R is the least reflexive and symmetric binary relation that satisfies

$$a_i R b_i R c_i,\ a_i R c_{2i},\ a_i R c_{2i+1} \qquad (i \in N).$$

So one can easily realize that $\mathfrak{A}$ looks like the following picture:



By definition, one can see also that each wff has always the same truth value in all of the a-worlds, and the same thing happens in b- and c-worlds.

Furthermore, $\Diamond !_1 B$ and $\Diamond !_2 C$ are true in a-worlds, $\Diamond !_1 A$ and $\Diamond !_1 C$ are true in b-worlds, and $\Diamond !_1 A$ and $\Diamond !_1 B$ are true in c-worlds.

Finally, let us observe that R is symmetric, serial and reflexive so that $\mathfrak{A}$ is a $\boldsymbol{KB}^\circ$-, $\boldsymbol{KBD}^\circ$- and a $\boldsymbol{KBT}^\circ$-model.

Now we are ready to prove that $\mathfrak{A}$ cannot be equivalent to any finite model.

THEOREM 1. *The following sentences*

(1) $A \to \Diamond !_1 B \wedge \Diamond !_2 C$

(2) $B \to \Diamond !_1 A \wedge \Diamond !_1 C$

(3) $C \to \Diamond !_1 A \wedge \Diamond !_1 B$

(4) $A \vee B \vee C$

are valid in $\mathfrak{A}$; *on the other hand, each finite symmetric model that validates* (1), (2), (3), *validates* $\neg (A \vee B \vee C)$ *too.*

PROOF. The first statement obviously holds: (1), (2), (3) and (4) are clearly true in every world of $\mathfrak{A}$, by definition.

Let us prove $\neg (A \vee B \vee C)$ is valid in every finite symmetric model that validates (1), (2), (3). Let $\langle W, R, V \rangle$ be such a model and define

$$\mathscr{A} = \{w \in W\colon\ V(w, A) = 1\}$$
$$\mathscr{B} = \{w \in W\colon\ V(w, B) = 1\}$$
$$\mathscr{C} = \{w \in W\colon\ V(w, C) = 1\}.$$

We shall show that

$$|\mathscr{A}| = |\mathscr{B}| = |\mathscr{C}|$$
$$|\mathscr{C}| = |\mathscr{A}| \cdot 2.$$

From these relations the claim is easily proved, because they imply $|\mathscr{A}| = |\mathscr{A}| \cdot 2$, and this implies in turn that either $|\mathscr{A}| = 0$ or $|\mathscr{A}| \geqslant \omega$, but W is finite so $|\mathscr{A}| = 0$ and then $|\mathscr{B}| = |\mathscr{C}| = 0$ too. As a conclusion $\neg (A \vee B \vee C)$ holds in every world, so is valid in $\langle W, R, V \rangle$.

We have to show the above relations among $|\mathscr{A}|$, $|\mathscr{B}|$, $|\mathscr{C}|$ hold. Actually R defines a bijection between $\mathscr{A}$ and $\mathscr{B}$: if $w \in \mathscr{A}$ then $V(w, A) = 1$ and, by (1), there exists exactly one $w' \in \mathscr{B}$ such that wRw', i.e. R is a function from $\mathscr{A}$ to $\mathscr{B}$; on the other hand $w' \in \mathscr{B}$ implies $V(w', B) = 1$, so that, by (2), there exists exactly one $w \in \mathscr{A}$ such that $w'Rw$, namely, by symmetry, such that wRw': so R is both surjective and injective and $|\mathscr{A}| = |\mathscr{B}|$. The same argument applied to $\mathscr{B}$ and $\mathscr{C}$, instead of $\mathscr{A}$ and $\mathscr{B}$, gives $|\mathscr{B}| = |\mathscr{C}|$, and the first relation is proved.

Taking into account (1) and (3), one can see, arguing as above, that R is a surjective function from $\mathscr{C}$ to $\mathscr{A}$, and in fact that $|R^{-1}(w)| = 2$, for every $w \in \mathscr{A}$, i.e. the second relation holds. ■

COROLLARY 1. $\mathfrak{A}$ *is not equivalent to any finite symmetric model.*

3. General canonical models

In this section we shall find axiomatizations for $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$: after a brief syntactical introduction we shall show that the "old" and strong canonical models, introduced in [D], for those systems do not exist. This fact will force the introduction of a new (in fact, weaker) kind of canonical models, which we shall call general.

Let us define the axiom schemata B°, D° as the natural translations in the graded environment of the classical ones:

$$B^\circ = \alpha \to \Box_0 \Diamond_0 \alpha$$
$$D^\circ = \Box_0 \alpha \to \Diamond_0 \alpha$$

and write again the characteristic axiom of $\boldsymbol{KT}^\circ$, i.e. T°, using $\Box_0$ instead of L_0 as in [FD], [D]:

$$T^\circ = \Box_0 \alpha \to \alpha.$$

Now we can define the syntactic bases for our systems:

Axioms of $\boldsymbol{KB}^\circ$ = the axioms of $\boldsymbol{K}^\circ + B^\circ$
Axioms of $\boldsymbol{KBD}^\circ$ = the axioms of $\boldsymbol{K}^\circ + B^\circ + D^\circ$
Axioms of $\boldsymbol{KBT}^\circ$ = the axioms of $\boldsymbol{K}^\circ + B^\circ + T^\circ$

and the rules of inference are the usual ones: modus ponens and (graded) necessitation (see [FD], [D]).

It is obvious that the above defined systems are sound with respect to the relevant classes of models: the symmetric, the symmetric and serial, the symmetric and reflexive Kripke models.

Now we recall the main features of the "old" canonical models and we point out their limits: they are defined in [D], assuming as worlds all the *mc*-sets (= maximal consistent sets) of formulas and repeating each one a convenient number of times; this repetition is ruled by the two functions (Φ denotes the class of all the *mc*-sets of formulas, Γ and Γ' are *mc*-sets)

$$m\colon\ \Phi \times \Phi \to \omega + 1 = \omega \cup \{\omega\}$$
$$m'\colon\ \Phi \to \omega + 1$$

(we write m' instead of $m(-)$, as in [D], to avoid confusing notations), such that $m(\Gamma, \Gamma')$ represents the number of Γ'-copies which every Γ-copy needs to access to, and $m'(\Gamma)$ represents the global number of Γ-copies which are needed in the model.

Furthermore, an accessibility relation R is suitably defined (taking into account the properties induced by the various axiom schemata), and finally an evaluation V is defined in such a way that, roughly speaking, "$\models$ iff $\in$" holds for all formulas ("$\models$ iff $\in$" means that, for a formula, truth in such a world, namely in a *mc*-set, is the same thing as belonging to it).

From this, completeness and compactness are easily attained.

Now, if we want to preserve this conceptual schema, we cannot touch the "$\models$ iff $\in$" feature, of course; moreover we may note the purpose of the function m is to explicate power links between any two *mc*-sets, and in fact it is too closely related to the "meaning" of the $\Box_i$, $\Diamond_i$ operators to allow modifications (see [D]):

$$m(\Gamma, \Gamma') = \begin{cases} glb\ \{n \in N:\ \Diamond !_n \alpha \in \Gamma \text{ for some } \alpha \in \Gamma'\} & \text{if this set is not empty} \\ \omega & \text{otherwise.} \end{cases}$$

On the other hand the function m' cannot be strongly affected by R, that is only required to be symmetric; in fact m' is defined in [D] as

$$m'(\Gamma) = lub\ \{m(\Gamma', \Gamma):\ \Gamma' \in \Phi\}$$

and its purpose is to bound the number of replicas of Γ.

However, from our point of view graded canonical models are like puzzles, where one does not care for the number of pieces but needs to put every piece in the right place.

The following theorem strengthens this opinion.

THEOREM 2. *Strong canonical models for* $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$ *do not exist.*

To prove the above theorem we need a technical lemma.

LEMMA 1. *Let Λ be a GNL with axiom B^0 and Γ, Γ' mc-sets of Λ; then*

$$m(\Gamma, \Gamma') \neq 0 \quad \textit{iff} \quad m(\Gamma', \Gamma) \neq 0.$$

PROOF. We show only the left-to-right implication, the other one being proved similarly.

Assume $m(\Gamma, \Gamma') \neq 0$: for every formula β, if $\beta \in \Gamma$ then $\Box_0 \Diamond_0 \beta \in \Gamma$, so by Lemma 3-iii) of [D], we have $\Diamond_0 \beta \in \Gamma'$; therefore $m(\Gamma', \Gamma) \neq 0$, by Lemma 3-ii) of [D]. ■

Now we can prove theorem 2.

PROOF OF THEOREM 2. Let us consider the model $\mathfrak{A}$ and call Γ_a, Γ_b, Γ_c the sets of valid formulas in *a*-, *b*-, *c*-worlds respectively; Γ_a, Γ_b, Γ_c are *mc*-sets of $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$ because all of the *a*-worlds are copies of any fixed one of them (in the sense they coincide for formulas' evaluations), and the same holds for *b*- and *c*-worlds.

Using indexes $i, j \in \{a, b, c\}$, we easily have $m(\Gamma_i, \Gamma_j) = 1$ for $(i, j) \neq (a, c)$, and $m(\Gamma_a, \Gamma_c) = 2$.

Furthermore $m(\Gamma, \Gamma_i) = 0$, for every other *mc*-set Γ and $i \in \{a, b, c\}$: in fact, being $\Gamma \neq \Gamma_i$, there exists a wff $\beta_i \in \Gamma - \Gamma_i$ so that $\beta = \beta_a \wedge \beta_b \wedge \beta_c \in \Gamma$ and is false in all of the worlds of $\mathfrak{A}$; thus $\Diamond !_0 \beta \in \Gamma_i$ and this implies $m(\Gamma_i, \Gamma) = 0$, that is $m(\Gamma, \Gamma_i) = 0$, by Lemma 1.

So, by the definition of m', we have $m'(\Gamma_a) = m'(\Gamma_b) = 1$, $m'(\Gamma_c) = 2$: this means that strong canonical models of $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$ have one copy of

Γ_a and Γ_b, two copies of Γ_c. This situation implies "$\models$ iff $\in$" does not hold and then the claim follows.

In fact, suppose "$\models$ iff $\in$" does hold: then for every $\Gamma \neq \Gamma_i$, $i \in \{a, b, c\}$, if β is the related formula, as above, $\Diamond !_0 \beta \in \Gamma_i$ implies $\Diamond !_0 \beta$ holds in Γ_i, so that every copy of Γ is not accessible from any copy of $\Gamma_a, \Gamma_b, \Gamma_c$; thus the restriction of such a canonical model to the copies of $\Gamma_a, \Gamma_b, \Gamma_c$ saves the truth values of the formulas of these worlds, and, by the definition of Γ_i, the valid formulas of the restricted model are exactly the $\mathfrak{A}$-valid formulas. But this restricted model is a symmetric and finite one (it has four worlds) and by Corollary 1 it cannot be equivalent to $\mathfrak{A}$: contradiction. ∎

The preceding proof shows not only that the usual control on the power of the set of needed copies of *mc*-sets (i.e. m') fails its goal for some symmetric systems, but also that every control on the power of such sets which requires a finite number of copies of $\Gamma_a, \Gamma_b, \Gamma_c$ fails too. In fact in that case we could repeat the above proof, using Corollary 1 and getting again a contradiction.

On the other hand the only information we have about $\Gamma_a, \Gamma_b, \Gamma_c$ (namely, the function m, that controls the power of *mc*-sets' pairs, has always finite values) is not sufficient to deduce that at least $\aleph_0$ copies of $\Gamma_a, \Gamma_b, \Gamma_c$ are needed in a "good" canonical model.

This seems to condemn to failure any attempt to deduce a global control on the power of the set of copies of a *mc*-set by a world-to-world power bound, and a highly problematic task to try to find any "algorithmic" control of power in strong canonical models (i.e. by modifying the definition of m').

The main modifications we do now to introduce general canonical models are, firstly, to repeat each *mc*-set in a countable number of copies and, secondly, to associate with each of these copies various types of orderings, related to the particular *mc*-set and the particular system under examination.

Note that in what follows $m(\Gamma, \Gamma')$, Γ and Γ' being *mc*-sets, will be considered sometimes as an ordinal number and sometimes as a cardinal one (the context will reveal its nature, anyway) and we shall use $\otimes$ as the ordinal product, while $\times$ will indicate the usual cartesian product.

Formally, for each *mc*-set Γ we shall insert in a general canonical model a set $cp(\Gamma)$ of $\aleph_0$ copies of Γ; so we can define a *general canonical model* $\mathfrak{M} = \langle W, R, V \rangle$ as follows:

$W = \bigcup \{cp(\Gamma)\colon \Gamma \in \Phi\}$

R is defined with the help of various orderings on $\mathrm{cp}(\Gamma)$

V respects "$\models$ iff $\in$" for all the formulas.

Before proving the canonical model existence theorem we need a technical lemma:

LEMMA 2. *Let Λ be a GNL with the axiom D°: for any mc-set Γ of Λ there exists another mc-set Γ' of Λ such that $m(\Gamma, \Gamma') \neq 0$.*

PROOF. By a standard argument (see e.g. [C], theor. 5.13) we get that $\{\beta$: β is a wff and $\Box_0 \beta \in \Gamma\}$ is a Λ-consistent set, so there exists a *mc*-extension of it, say Γ'. We have

(0) for every wff $\beta \in \Gamma'$: $\Diamond !_0 \beta \notin \Gamma$;

in fact, otherwise, there exists a wff $\beta \in \Gamma'$ such that $\Box_0 \neg \beta \in \Gamma$, so that $\neg \beta \in \Gamma'$, by definition of Γ', and Γ' is inconsistent: contradiction.

From the definition of m and (0) we get $m(\Gamma, \Gamma') \neq 0$. ■

THEOREM 3. *General canonical models for* $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$ *exist.*

PROOF. Let us consider the model $\mathfrak{M} = \langle W, R, V\rangle$, where $W = \bigcup\{cp(\Gamma)$: $\Gamma \in \Phi\}$. We set the accessibility relation R as follows: for each $\Gamma, \Gamma' \in \Phi$

a) if $m(\Gamma, \Gamma') \neq 0$ (and so also $m(\Gamma', \Gamma) \neq 0$, by Lemma 1) we define on $cp(\Gamma)$ and $cp(\Gamma')$ orderings of type $\omega \otimes m(\Gamma', \Gamma)$ and $\omega \otimes m(\Gamma, \Gamma')$, respectively, and put

$$\langle \Gamma, n, i\rangle R \langle \Gamma', n, j\rangle \text{ for each } n \in \omega,\ i \in m(\Gamma', \Gamma),\ j \in m(\Gamma, \Gamma');$$

b) if $m(\Gamma, \Gamma') = 0$ (and so also $m(\Gamma', \Gamma) = 0$, by Lemma 1) we state any copy of Γ' is inaccessible from any copy of Γ, and viceversa; i.e. we put

$$R \cap [cp(\Gamma) \times cp(\Gamma')] = R \cap [cp(\Gamma') \times cp(\Gamma)] = \emptyset.$$

Finally we define V so that "$\models$ iff $\in$" holds for propositional symbols.

We show the accessibility relation has the required properties in the various cases: in every case R is clearly symmetric; moreover, in the case of $\boldsymbol{KBD}^\circ$ we have, by Lemma 2, that for any $\Gamma \in \Phi$ there exists $\Gamma' \in \Phi$ such that $m(\Gamma', \Gamma) \neq 0$, so R is serial by the above definition; in the case of $\boldsymbol{KBT}^\circ$, by Lemma 7 of [D], we have $m(\Gamma, \Gamma) \neq 0$ for any $\Gamma \in \Phi$, so that, by the use of the indexes in the definition, R is reflexive.

Now we prove "$\models$ iff $\in$" holds for every formula: by the definition of R we have clearly that for any copy of Γ there are exactly $m(\Gamma, \Gamma')$ copies of Γ' accessible from Γ, so that the Theorem 1 of [D] and the above choice of V imply the statement. ■

Theorem 3 yields completeness and compactness theorems for $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$ in the usual way.

4. Existence of general canonical models for *GNLs*

In this section we shall show that general canonical models exist for all the graded versions of the fifteen main normal graded systems between $\boldsymbol{K}^\circ$ and $\boldsymbol{S5}^\circ$ (see [C], fig. 4.1 and fig. 5.1), i.e. those graded systems whose models are the Kripke models with an accessibility relation that has any combination of the main properties (seriality, reflexivity, symmetry, transitivity, euclideaness). Of course, they allow to prove general satisfiability theorems (so completeness and compactness) for those systems.

The preceding section displays such a result for $\boldsymbol{KB}^\circ$, $\boldsymbol{KBD}^\circ$, $\boldsymbol{KBT}^\circ$.

Similar results for strong canonical models of other systems have been obtained already (see [FD], [D], [FC], [B]): so, as a first example, we shall adapt those proofs to show the existence of general canonical models of those systems.

Let us recall the axioms' schemata 4 and 5 and their graded versions 4° and 5°:

$$\begin{aligned} 4 &= \Box_0 \alpha \to \Box_0 \Box_0 \alpha \\ 4^\circ &= \Box_n \alpha \to \Box_0 \Box_n \alpha \qquad (n \in N) \\ 5 &= \Diamond_0 \Box_0 \alpha \to \Box_0 \alpha \\ 5^\circ &= \Diamond_0 \Box_n \alpha \to \Box_n \alpha \qquad (n \in N). \end{aligned}$$

THEOREM 5. *General canonical models exist for* $\boldsymbol{K}^\circ$, $\boldsymbol{KD}^\circ$, $\boldsymbol{KT}^\circ$, $\boldsymbol{K4}^\circ$, $\boldsymbol{KT4}^\circ$, $\boldsymbol{KT5}^\circ$. *So those systems are complete and compact.*

PROOF. Given any one of the quoted systems, we need to show that its general canonical model exist, that is it satisfies "$\models$ iff $\in$" for any formula.

To this purpose we can use the existing proofs (see the above quotation), redefining the accessibility relation R in such a way to make it work always on a countable set of copies of *mc*-sets: the passage from "$\models$ iff $\in$" for propositional symbols to "$\models$ iff $\in$" for all the formulas goes in the same way as in the old proofs.

We shall use the schema

system's name: – axioms
– models
– definition of R in the canonical model

to compare systems with each other and to stress their main technical features.

However we strongly point out that canonical models have different sets of worlds, depending on the system itself, and that the required properties of the accessibility relation are expressed by the function m in specific lemmas.

In the following schemata the relation R will be apparently the same for different systems, because of the same technical definition, but it links really different sets in different ways (by different lemmas), so having in fact different properties.

$\boldsymbol{K}^\circ$
– axioms as in [D]
– all Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$\langle \Gamma, i \rangle R \langle \Gamma', j \rangle$ for each $i \in \omega$, $j \in m(\Gamma, \Gamma')$;

$\boldsymbol{KD}^\circ$:
– $\boldsymbol{K}^\circ + D^\circ$
– all serial Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$\langle \Gamma, i \rangle R \langle \Gamma', j \rangle$ for each $i \in \omega$, $j \in m(\Gamma, \Gamma')$;

seriality follows from Lemma 2 (completeness and compactness theorems for this system were proved for the first time in [B], by strong canonical models);

KT° – $K^{\circ}+T^{\circ}$
– all reflexive Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$$\langle\Gamma, i\rangle R\langle\Gamma', j\rangle \text{ if } \Gamma\neq\Gamma' \text{ or } i\in m(\Gamma,\Gamma)$$
$$\langle\Gamma, i\rangle R\langle\Gamma, h\rangle \text{ otherwise}$$

for each $i\in\omega$, $j\in m(\Gamma,\Gamma')$, $h\in[m(\Gamma,\Gamma)-\{0\}]\cup\{i\}$;

K4° – $K^{\circ}+4^{\circ}$
– all transitive Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$$\langle\Gamma, i\rangle R\langle\Gamma', j\rangle \text{ for each } i\in\omega,\ j\in m(\Gamma,\Gamma');$$

KT4°: – axioms as in [FC]
(= ***S4***°) – all reflexive and transitive Kripke models
– R is defined as in [FC], but avoiding bounds on the number of blocks $b([\Gamma])$, which is put $=\omega$; so we associate with every $cp(\Gamma)$ an ordering of type $\omega\otimes m(\Gamma,\Gamma)$ and (recalling from [FC] the relation ϱ and the number $b(\Gamma,[\Gamma'])$, considered as an ordinal one) we define

$$\langle\Gamma, n, i\rangle R\langle\Gamma', k, j\rangle \text{ if not-}\Gamma\varrho\Gamma'$$
$$\langle\Gamma, n, i\rangle R\langle\Gamma', n, j\rangle \text{ otherwise}$$

for each $n\in\omega$, $i\in m(\Gamma,\Gamma)$, $j\in m(\Gamma',\Gamma')$, $k\in b(\Gamma,[\Gamma'])$;

KT5°: – $K^{\circ}+T^{\circ}+5^{\circ}$
(= ***S5***°) – all reflexive and euclidean (i.e. reflexive, symmetric and transitive) Kripke models
– as to R, we have $m(\Gamma,\Gamma)\neq 0$ for each *mc*-set Γ (see [D]), so we consider an ordering of type $\omega\otimes m(\Gamma,\Gamma)$ on $cp(\Gamma)$ and define
a) if $m(\Gamma,\Gamma')\neq 0$ then

$$\langle\Gamma, n, i\rangle R\langle\Gamma', n, j\rangle \text{ for each } n\in\omega,\ i\in m(\Gamma,\Gamma),\ j\in m(\Gamma',\Gamma')$$

b) if $m(\Gamma,\Gamma')=0$ then
no copy of Γ' is accessible from any copy of Γ.

This completes the proof of the theorem. ■

Now we can prove the completeness and compactness theorems for the last symmetric system ***KB4***°, whose models are the symmetric and transitive Kripke models, and which has $K^{\circ}+B^{\circ}+4^{\circ}$ as a complete axiomatization:

THEOREM 6. *The general canonical model of* ***KB4***°, *axiomatized as above, exist; so it is complete and compact.*

PROOF. We can use the proof we used for $KT5^o$, with slight modifications: first of all, we have to pay attention to keep copies of any *mc*-set Γ such that $m(\Gamma, \Gamma) = 0$ totally inaccessible from any other world; furthermore, following the completeness proof for $KT5^o$ ($= S5^o$) in [D], we have to replace the use of axiom 5^o in Lemma 8 of [D] by our Lemma 1. ∎

To complete the treatment of euclidean systems (i.e. those having 5^o among their axioms) we need the following lemma:

LEMMA 3. *For every system having* 5^o *among its axioms one has*

$$m(\Gamma'', \Gamma') \geqslant m(\Gamma, \Gamma'),$$

for any $\Gamma, \Gamma', \Gamma'' \in \Phi$ *such that* $m(\Gamma, \Gamma'') \neq 0$.

PROOF. By the definition of m we need only to prove that $\Diamond_n \alpha \in \Gamma''$ when $\Diamond_n \alpha \in \Gamma$ and $\alpha \in \Gamma'$.

In fact, for every wff α, if $\Diamond_n \alpha \in \Gamma$ then $\Box_n \Diamond_n \alpha \in \Gamma$ (by axiom 5^o), so $\Diamond_n \alpha \in \Gamma''$ (by the hypothesis $m(\Gamma, \Gamma'') \neq 0$ and Lemma 3 of [D]), i.e. the thesis. ∎

Now we can prove our last theorem, stressing that the proof of the critical passage from "$\models$ iff $\in$" for propositional symbols to "$\models$ iff $\in$" for all the formulas goes in the same way as in K^o, using Theorem 1 of [D].

THEOREM 7. *General canonical models of* $K5^o$, $KD5^o$, $K45^o$, $KD45^o$ *exist. So those systems are complete and compact.*

PROOF. Following the same style of Theorem 5, we set

$K5^o$:
- $K^o + 5^o$
- euclidean Kripke models
- R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

  $\langle \Gamma, i\rangle R \langle \Gamma', j\rangle$ for each $i \in \omega$, $j \in m(\Gamma, \Gamma')$;

  by Lemma 3, R is euclidean, that is if

  a) $\langle \Gamma, i\rangle R \langle \Gamma', h\rangle$ and b) $\langle \Gamma, i\rangle R \langle \Gamma'', k\rangle$

  then $\langle \Gamma'', k\rangle R \langle \Gamma', h\rangle$: in fact, by the definition of R, a) yields $h < m(\Gamma, \Gamma')$ and b) implies $m(\Gamma, \Gamma'') \neq 0$, that implies in turn (Lemma 3) $m(\Gamma, \Gamma') \leqslant m(\Gamma'', \Gamma')$, so $h < m(\Gamma'', \Gamma')$ and the claim follows;

$KD5^o$:
- $K^o + D^o + 5^o$
- all euclidean and serial Kripke models
- R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

  $\langle \Gamma, i\rangle R \langle \Gamma', j\rangle$ for each $i \in \omega$, $j \in m(\Gamma, \Gamma')$;

  seriality follows from Lemma 2 and euclideanness from Lemma 3, as above;

***K*45°**: – $K^{\circ}+4^{\circ}+5^{\circ}$
– all transitive and euclidean Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$$\langle \Gamma, i\rangle R \langle \Gamma', j\rangle \quad \text{for each } i\in\omega, j\in m(\Gamma, \Gamma');$$

transitivity follows from Lemma 6 of [D] and euclideanness from Lemma 3, as above;

***KD*45°**: – $K^{\circ}+D^{\circ}+4^{\circ}+5^{\circ}$
– all serial, transitive and euclidean Kripke models
– R as in [D], i.e. we associate with every $cp(\Gamma)$ an ordering of type ω and define

$$\langle \Gamma, i\rangle R \langle \Gamma', j\rangle \quad \text{for each } i\in\omega, j\in m(\Gamma, \Gamma');$$

Lemma 2 yields seriality, Lemma 6 of [D] transitivity and Lemma 3 euclideanness.

This completes the proof of the theorem. ■

5. Conclusions and acknowledgements

In this paper we have proved that the "old" and strong canonical models do not exist for some symmetric *GNLs*.

In fact they include a global control on the power of the set of needed replicas of *mc*-sets, based on the number of worlds that are one-step accessible.

So strong canonical models exist and work well both for systems whose accessibility relation does not care of distant worlds (as e.g. ***K***°, ***KD***°, ***KT***°) and for systems whose accessibility relations make many-steps distance equivalent to one-step distance (i.e. the transitive *GNLs*).

But for systems whose accessibility relation has an intermediate strength (as the symmetric *GNLs* ***KB***°, ***KBD***°, ***KBT***°) strong canonical models do not exist at all. Actually we have exhibited a semantical reason of this failure, that is the critical model $\mathfrak{A}$.

So we have introduced a weaker, but more useful, notion of canonical model of a *GNL*, that exists and works well in every case, so providing completeness and compactness theorems for all the relevant *GNLs*.

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