Theory Dec.
DOI 10.1007/s11238-013-9394-z

Implementing equal division with an ultimatum threat

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Abstract We modify the payment rule of the standard *divide the dollar* (*DD*) game by introducing a second stage and thereby resolve the multiplicity problem and implement equal division of the dollar in equilibrium. In the standard *DD* game, if the sum of players' demands is less than or equal to a dollar, each player receives what he demanded; if the sum of demands is greater than a dollar, all players receive zero. We modify this second part, which involves a harsh *punishment*. In the modified game (DD'), if the demands are incompatible, then players have one more chance. In particular, they play an ultimatum game to avoid the *excess*. In the two-player version of this game, there is a unique subgame perfect Nash equilibrium in which players demand (and receive) an equal share of the dollar. We also provide an n-player extension of our mechanism. Finally, the mechanism we propose eliminates not only all pure strategy equilibria involving unequal divisions of the dollar, but also all equilibria where players mix over different demands in the first stage.

Keywords Arbitration · Divide the dollar game · Equal division · Implementation · Subgame perfect Nash equilibrium · Ultimatum game

"Practically speaking, why threaten such punishment if the threat is likely to be empty and there are, moreover, 'softer' ways of inducing reasonable behavior?"

(Brams and Taylor 1996, p. 173)

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Published online: 08 September 2013

1 Introduction

The quote above refers to the punishment threat and egalitarian behavior in the celebrated Divide the Dollar (*DD*) game. The divide-the-dollar game is a special case of Nash's (1953) demand game, where the bargaining frontier is linear. In the *DD* game, two players simultaneously demand shares of a dollar. If the sum of their demands is less than or equal to a dollar, they receive their corresponding demands; if the sum of their demands is greater than a dollar, each player receives zero. Since this simple game encapsulates important characteristics of most bargaining problems (e.g., parties can secure a surplus if they can agree on the way it is to be shared; otherwise they cannot), it received great interest (Binmore 1998; Kilgour 2003). However, despite its appealing features the *DD* game suffers from an essential drawback: the Nash equilibrium does not have much predictive power since any division of the dollar is a Nash equilibrium. Therefore, in its perfectly divisible dollar version, there are infinitely many Nash equilibria in this game. One may think that 50–50 division is the most likely outcome, possibly due to its prominence, symmetry, or presumed fairness. In fact, this is what is suggested theoretically (Schelling 1960; Sugden 1986; Young 1993; Huyck et al. 1995; Skyrms 1996; Bolton 1997)[1] and observed empirically (Siegel and Fouraker 1960; Nydegger and Owen 1975; Roth and Malouf 1979). Nevertheless, from a standard game theoretical point of view the 50–50 division is just one of the infinitely many Nash equilibria. This makes the research program on mechanisms that resolve the multiplicity problem and induce *reasonable* behavior an auspicious one.

To the best of our knowledge, Brams and Taylor (1994) is the first study that aims to resolve the multiplicity problem in the *DD* game in favor of the equal division. They offer three modifications of the standard *DD* game, in which they change the payment rule (*DD1*), add a second stage (*DD2*), or do both (*DD3*). All of these mechanisms implement equal division in equilibrium. Brams and Taylor (1994) also list five characteristics that any *reasonable* mechanism should satisfy. These are (i) equal demands are treated equally; (ii) no player receives more than his demand; (iii) if the estate to be divided is sufficient to respect all players' demands, then everyone should receive his demand; (iv) the estate should be completely distributed in any case; and (v) if each player's demand is greater than E/n (E is the value of the estate to be divided and n is the number of players), then the player with the highest demand should not receive more than what the player with the lowest demand receives. They show that no *reasonable* mechanism can implement the equal division if the equilibrium concept used is the iterated elimination of weakly (or strongly) dominated strategies.

In a similar fashion, Anbarcı (2001) modifies the payment rule in case the sum of players' demands exceeds a dollar. The author interprets incompatible demands (i.e., $d_1 + d_2 > 1$) as each player asking for a compromise from the other. Anbarcı (2001) mechanism requires each player to make a compromise identical to the one he asks from the other player. He shows that the iterated elimination of strictly dominated strategies attains a unique equilibrium that involves equal demands. Intuitively speak-

[1] Schelling's (1960) argument is based on the *prominence* of 50–50 division. Arguments in Sugden (1986), Young (1993), Skyrms (1996), and Bolton (1997) are based on evolutionary accounts.

ing, this mechanism requires each player to put himself in the other player's shoes. Thus, it can be thought as a mechanism that implements equal division by forcing players to empathize.

Ashlagi et al. (2012) also modify the payment rule in case the sum of players' demands is greater than the estate. Instead of each player receiving zero in this case, the estate is completely distributed by using an allocation rule. The authors resort to bankruptcy (or conflicting claims) problems to find such allocation rules. They show that equal division prevails in all Nash equilibria of the modified game, if the allocation rule satisfies *equal treatment of equals, efficiency*, and *order preservation of awards* (Theorem 1).[2] Alternatively, in case the rule satisfies *equal treatment of equals*, *efficiency*, *claims monotonicity*, and *non-bossiness*, the same result holds (Theorem 3).[3] Thus, Ashlagi et al. (2012) show that it is possible to implement equal division in the Nash equilibria of the modified *DD* game even after avoiding the punishment (i.e., everyone receiving zero in the case of incompatible demands) and using allocation rules that satisfy certain properties (e.g., *claims monotonicity* and *order preservation of awards*) that encourage greedy behavior. The class of rules they consider is large and contains—but is not restricted to—most of the prominent bankruptcy rules (e.g., proportional rule, constrained equal awards rule, constrained equal losses rule, and Talmud rule).

A result by Trockel (2000) in a different line of research is also of interest. Trockel (2000) implements the Nash solution based on its Walrasian characterizations. He considers convex bargaining sets (i.e., including ones with non-linear frontiers). Although his focus is somewhat different than the papers mentioned above, if one limits attention to *DD* the methods Trockel (2000) employs can be seen as other ways to implement the equal division, because the Nash bargaining solution in *DD* is the 50–50 division.[4] Walrasian mechanisms that he employs and Brams and Taylor (1994) and Anbarcı (2001) offer share some common features: all of these mechanisms (i) alter the payment rule in case of incompatible demands and (ii) discriminate between players and favor the player with a more modest demand in this case.

Our modification (DD') involves a second stage with a *softer* (in comparison to *DD*) punishment rule in the case of incompatible demands. In such a case, players have one more chance to divide the dollar. Specifically, they can avoid the *excess* they generated in the first stage with an ultimatum game in the second stage. In the two-player version, the player who demanded less (more) in the first stage is the proposer (responder). In case of a tie, a player will be chosen randomly. The proposer makes an offer, which suggests the part of the excess that will be deducted from his demand and the responder's demand. If the proposal is accepted, then both

[2] *Equal treatment of equals:* any two players with equal demands should receive the same amount. *Efficiency:* if the sum of demands exceeds the estate, the estate should be completely distributed. *Order preservation of awards:* a player with a higher demand should not receive a lower amount than a player with a lower demand.

[3] *Claims monotonicity:* a player should not receive a lower amount after increasing his demand. *Non-bossiness:* a change in a player's demand should not be able to change other players' payoffs, if this change does not influence his own payoff.

[4] Harsanyi (1977), Howard (1992), and Miyagawa (2002) are some other studies that implement the Nash bargaining solution in subgame perfect equilibrium with sequential game forms.

players receive their corresponding (post-deduction) demands. In case of a rejection, the excess cannot be avoided and players do not stand another chance: they both receive zero. If one interprets the modifications to DD as guidelines of different types of arbitrators, the arbitrator type that corresponds to DD' would be one that (i) does not immediately enforce harsh punishments, (ii) leaves the floor to bargaining parties as much as possible, and (iii) encourages agreeable behavior. Moreover, in the two-player version, DD' attains a unique subgame perfect Nash equilibrium in pure strategies, in which players demand (and receive) an equal share of the dollar in the first stage (Theorem 1). In the $n > 2$ version, there are payoff-equivalent subgame perfect Nash equilibria with the same characteristics.

In all three studies on the DD game mentioned above, the authors focus on pure-strategy equilibria. However, if one also considers mixed-strategy equilibria, the problem of multiplicity is even more severe, as hinted by Myerson (1991, p. 112) and illustrated in detail by Malueg (2010). In particular, Malueg (2010) shows that practically any set of *balanced* demands can support a mixed-strategy Nash equilibrium. As we show below, the mechanism we propose not only eliminates all pure-strategy equilibria involving unequal division of the dollar but also eliminates all equilibria where players mix over different demands in the first stage (Theorem 2).[5]

The organization of the paper is as follows: In Sect. 2, we introduce the modified DD game (DD') and prove our results for *two* players. Section 3 concludes. Extensions of our results to $n > 2$ players are provided in the appendix.

2 Modified DD

Let $i = 1, 2$ denote players and $d_i \in [0, 1]$, player i's demand in the first stage. Let π_i denote the amount player i receives at the end of the game. If $d_1 + d_2 \leq 1$, then the game ends in the first stage and for all i, $\pi_i = d_i$. If $d_1 + d_2 > 1$, then the player with the lowest (highest) demand is chosen to be the proposer (responder) in the ultimatum game in the second stage. If $d_1 + d_2 > 1$ and $d_1 = d_2$, then each player is chosen to be the proposer with probability $1/2$. We assume that players are *risk-neutral*. We denote the *excess* by $x \equiv d_1 + d_2 - 1$. If player j is the proposer (i.e., either $d_j < d_k$ or $d_j = d_k$ and player j is chosen as a result of a random draw), to avoid x he makes a proposal to player $k \neq j$, (x_j, x_k) such that $x_j + x_k = x$. x_j is the part of the excess to be deducted from d_j and x_k is the part of the excess to be deducted from d_k. Player k can either *accept* or *reject* this proposal. Thus, the proposer's action set in the second stage is $x_k \in [0, x]$ and the responder's action set in the second stage is $\{accept, reject\}$. If player k accepts the proposal, then $\pi_j = d_j - x_j$ and $\pi_k = d_k - x_k$. If player k rejects the proposal, then $\pi_j = \pi_k = 0$.[6]

The following theorem shows that when only pure-strategy equilibria are considered, there exists a unique subgame perfect Nash equilibrium of DD' and demanding an amount different than $1/2$ is not part of this equilibrium,

[5] We thank an anonymous reviewer for bringing Malueg (2010) to our attention and suggesting an alternative proof, which also considers mixed strategies (Theorem 2).

[6] To keep the paper simple, reader-friendly and comparable to Brams and Taylor (1994) and Anbarcı (2001), we intentionally used *actions* as the working horse instead of *strategies*.

Theorem 1 *In the unique subgame perfect (pure strategy) Nash equilibrium of* DD', $(d_1, d_2) = (1/2, 1/2)$.

Proof First, we analyze the equilibrium behavior in the second stage. Player j is the proposer and player k is the responder. Two cases need to be analyzed.

Case 1 $d_k - x > 0$. In this case, player j proposes $x_k = x$ and player k accepts the proposal since rejection brings $\pi_k = 0$, whereas acceptance brings $\pi_k = d_k - x > 0$.

Case 2 $d_k - x = 0$. In this case, player j proposes $x_k = x - \epsilon$ where ϵ is an arbitrarily small, positive real number and player k accepts the proposal since rejection brings $\pi_k = 0$, whereas acceptance brings $\pi_k = d_k - x + \epsilon = \epsilon > 0$. Note that when $n = 2$, this case occurs only if $d_1 = d_2 = 1$.

Also note that $d_k - x < 0$ is not possible since for all $i = 1, 2$, $d_i \in [0, 1]$. Therefore, given the first-stage demands, the subgame perfect Nash equilibrium behavior in the second stage is uniquely determined.

Now, knowing what will happen in the second stage if the game reaches there, we analyze the equilibrium behavior in the first stage. First, we show that $(d_1, d_2) = (1/2, 1/2)$ is part of the subgame perfect Nash equilibrium. Without loss of generality, we use player *1* in our arguments. Clearly, any unilateral deviation $d_1' < 1/2$ is not profitable for player *1*, since $\pi_1' = d_1{}' < 1/2 = \pi_1$. Any unilateral deviation $d_1' > 1/2$ is not profitable for player *1* either, since in that case $d_1' + d_2 > 1$, player *2* will be the proposer in the second stage, and $\pi_1' = d_1' - (d_1' + d_2 - 1) = 1/2 = \pi_1$. Thus, $(d_1, d_2) = (1/2, 1/2)$ is a Nash equilibrium in the first stage of DD' and part of the subgame perfect Nash equilibrium.

Now, we show that $(d_1, d_2) = (1/2, 1/2)$ is the *only* Nash equilibrium (in the first stage of DD') that is part of the subgame perfect Nash equilibrium. The following exhaustive set of cases needs to be analyzed. In each case, we show that at least one profitable unilateral deviation exists.

Case 1 $d_1 = 1/2$ and $d_2 < 1/2$. Clearly, a unilateral deviation $d_2' = 1/2$ brings $\pi_2' = 1/2 > d_2 = \pi_2$. Note that a similar argument is valid for $d_1 < 1/2$ and $d_2 = 1/2$.

Case 2 $d_1 < 1/2$ and $d_2 < 1/2$. Without loss of generality, take player *1*. A unilateral deviation $d_1' = 1 - d_2$ brings $\pi_1' = 1 - d_2 > 1/2 > d_1 = \pi_1$.

Case 3 $d_1 = 1/2$ and $d_2 > 1/2$. A unilateral deviation $d_1' = d_2 - \epsilon$ brings $\pi_1' = d_2 - \epsilon > 1/2 = \pi_1$. A similar argument is valid for $d_1 > 1/2$ and $d_2 = 1/2$.

Case 4 $d_1 = d_2 > 1/2$. Without loss of generality, take player *1*. A unilateral deviation $d_1' = d_2 - \epsilon$ brings $\pi_1' = d_2 - \epsilon > (0.5)(d_1) + (0.5)(d_1 - (d_1 + d_2 - 1)) = 1/2 = \pi_1$.

Case 5 $d_1 < 1/2$ and $d_2 > 1/2$. There are two subcases here.

Case 5.1 $d_1 < 1/2, d_2 > 1/2$, and $d_1 + d_2 < 1$. A unilateral deviation $d_1' = 1 - d_2$ brings $\pi_1' = 1 - d_2 > d_1 = \pi_1$.

Case 5.2 $d_1 < 1/2, d_2 > 1/2$, and $d_1 + d_2 \geq 1$. A unilateral deviation $d_1' = d_2 - \epsilon$ brings $\pi_1' = d_2 - \epsilon > 1/2 > d_1 = \pi_1$. Note that similar arguments are valid for $d_1 > 1/2$ and $d_2 < 1/2$.

Case 6 $1/2 < d_1 < d_2$. A unilateral deviation $d_2' = d_1 - \epsilon$ brings $\pi_2' = d_1 - \epsilon > d_2' - (d_1 + d_2' - 1) = 1 - d_1 = \pi_2$. Clearly, a similar argument is valid for $1/2 < d_2 < d_1$.

Hence, in the unique subgame perfect Nash equilibrium of DD', $(d_1, d_2) = (1/2, 1/2)$. □

Note that $x = 0$ in equilibrium and the game ends in the first stage. Hence, it is the presence of the ultimatum threat rather than its execution, which induces players to demand equal shares in the first stage. One can also see that the presence of the ultimatum threat and the way the proposer role is assigned make the behavior of players in DD' similar to firm behavior in *Bertrand* competition: as equilibrium prices in Bertrand competition with two firms (with equal and constant marginal costs) are situated at the common marginal cost, the equilibrium demands of the players in DD' are situated at equal division.

Theorem 1 shows that the mechanism we propose eliminates all (pure strategy) equilibria involving unequal divisions of the dollar. However, as Malueg (2010) points out there are many mixed-strategy equilibria of *DD*. For instance, both players mixing over demands, 1/3 and 2/3 with equal probabilities (i.e., 1/2 and 1/2) constitute a mixed-strategy Nash equilibirum in *DD*. A natural question is whether such mixing behavior can be observed in the subgame perfect Nash equilibria of DD' or not. The following theorem shows that it cannot.

Theorem 2 *There is no subgame perfect Nash equilibrium of DD', which involves mixing in the first stage.*

Proof We will prove this result in four steps.[7]

Step 1 Claiming $d_i < 1/2$ is strictly dominated by claiming $d_i = 1/2$, for every player $i = 1, 2$.

Without loss of generality, we focus on player *1*. It is sufficient to investigate the following two cases: $x \leq 0$ and $x > 0$.

Case 1 $x \leq 0$. Let us analyze a deviation from $d_1 < 1/2$ to $d_1' = 1/2$. This is always profitable for player *1*. Note that even if this deviation leads to $x > 0$ (i.e., an excess emerges), it brings $\pi_1' = d_1' > \pi_1$. The reason is that player *1* will be the proposer at the second stage, set $x_2 = x$ and player *2* will accept this offer.[8]

Case 2 $x > 0$. In this case, since $x > 0$ and d_1 is less than 1/2, d_2 must be greater than 1/2. Thus, a deviation from $d_1 < 1/2$ to $d_1' = 1/2$ brings $\pi_1' = d_1' > \pi_1$.

Therefore, $d_1 = 1/2$ strictly dominates any $d_1 < 1/2$. Since player *1* is chosen without loss of generality, the same result holds for player *2*, as well.

Step 2 In any subgame perfect Nash equilibrium, $E(\pi_i) \geq 1/2$ for every player $i = 1, 2$, where E stands for the expectation operator.

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[7] The following arguments—explicitly or implicity—use the fact that sequential rationality implies acceptance of any offer in the second stage.

[8] The subcase, $d_2 < 1/2$, is not analyzed here since it is trivial and analyzed in Theorem 1.

From Step 1, we know that setting $d_i = 1/2$ guarantees $\pi_i = 1/2$, independent of what the other player does. Therefore, in any subgame perfect Nash equilibrium of DD', the expected payoff of any player should be greater than or equal to $1/2$.

Step 3 Subgame perfection implies $\pi_1 + \pi_2 = 1$.

First, note that $\pi_1 + \pi_2 < 1$ cannot hold in any subgame perfect Nash equilibrium since it implies that at least one of the players has a demand less than $1/2$ and claiming less than $1/2$ in the first stage is a strictly dominated action. Moreover, note that if the game reached the second stage, then $d_1 + d_2 > 1$ must hold. There are three subcases to be analyzed here:

Case 1 $d_1 < d_2$. Then, $\pi_1 = d_1$ and $\pi_2 = 1 - d_1$.
Case 2 $d_2 < d_1$. Then, $\pi_1 = 1 - d_2$ and $\pi_2 = d_2$.
Case 3 $d_1 = d_2$. Then, $E(\pi_1) = E(\pi_2) = 1/2$.

Therefore, $\pi_1 + \pi_2 = 1$ holds in any subgame perfect Nash equilibrium of DD'.

Next, combining Steps 1, 2, and 3 it follows that in any subgame perfect Nash equilibrium, $E(\pi_1) = E(\pi_2) = 1/2$. Note that this equal division outcome can, in principle, be reached in different ways. For instance, both players may mix uniformly over $[1/2, 1]$, or one player may (purely) demand $1/2$ and the other player mixes over $[1/2, 1]$.[9] In Step 4 we show that this cannot be the case in equilibrium.

Step 4 In any subgame perfect Nash equilibrium, it is impossible to have players demanding a share above $1/2$.

Suppose without loss generality that player *2* mixes over $[1/2, 1]$ using a probability distribution F.[10] First, we show that F cannot have atoms above $1/2$. Suppose for a contradiction that it has an atom above $1/2$. Hence, we can pick an arbitrary point a such that $a > 1/2$ and $P(a) > 0$. Now, suppose player *1* mixes using a distribution G, which has a support $[1/2, a)$ and is first-order stochastically dominated by F and $G(1/2) \neq 1$. As a result, we have $E(\pi_1) > E(\pi_2)$ and hence $E(\pi_1) > 1/2$ (a contradiction to $E(\pi_1) = E(\pi_2) = 1/2$), where the second inequality follows from subgame perfection. This shows that F cannot have atoms above $1/2$. In other words, we should not worry about ties in demands.

Therefore, playing $d_1 \geq 1/2$ brings an expected payoff, $\pi_1(d_1) = [1 - F(d_1)]d_1 + \int_{1/2}^{d_1}(1-d_2) f(d_2)\mathbf{d}(d_2)$ to player *1* implying $\pi_1'(d_1) = 1 - F(d_1) + (1-2d_1) f(d_1)$.[11] Obviously, if $F(1/2) < 1$ then $\pi_1'(1/2) > 0$. This implies that player *1* can receive a share higher than $1/2$ just by marginally increasing his demand. But this means player *2* must receive a share less than $1/2$ in that case, which is in contradiction with

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[9] We assume that players mix over only the Borel measurable subsets of $[0, 1]$. Also note that since players utility functions are linear they are Borel measurable.

[10] Note that it is sufficient to check the best response of a player against a mixing opponent.

[11] For simplicity, we assume that F is an absolutely continuous function, which implies that there exists a Lebesgue-integrable function f equal to the derivative of F almost everywhere. Moreover, this f is called the density function. Note that, alternatively F can be assumed to be continuous and have a derivative almost everywhere, which would then imply that there exists a Henstock–Kurzweil integrable f (Bartle 2001, Theorem 4.7).

findings from Steps 1, 2, and 3. Hence, a positive probability cannot be assigned to any demand above 1/2, in equilibrium.

Therefore, the result follows from Steps 1, 2, 3, and 4. □

The intuition for this result is simple. First, playing $d_i < 1/2$ can be an equilibrium strategy in *DD*, whereas it is strictly dominated in DD'. Second, playing $d_i > 1/2$ can pay off in some equilibria of *DD*, whereas the competition for the proposer role in DD' kills players' incentives for playing $d_i > 1/2$. Hence, no $d_i \neq 1/2$ can be assigned a positive probability in any equilibrium of DD′, which eliminates mixing behavior in the first-stage.

3 Conclusion

Similar to Brams and Taylor (1994), Anbarcı (2001), and Ashlagi et al. (2012), we provide a simple mechanism, which resolves the multiplicity problem and implements equal division in the divide-the-dollar game. We introduce a second stage, in case the sum of players' demands exceeds a dollar. This replaces the harsh punishment of the original *DD* game, where all players receive zero if their demands are incompatible. In this second stage, the players have a chance to avoid the excess and divide the dollar in an *efficient* manner. The player with the lowest (highest) demand in the first stage is assigned the proposer (responder) role in an ultimatum game. The proposer makes an offer, which suggests the amounts of excess to be deducted from his own and the responder's first-stage demands. If the responder accepts this proposal, players succeed in avoiding the excess and receive their corresponding shares. Otherwise, they both receive zero.

Our mechanism can be interpreted as a guideline of an arbitrator, who (i) refrains from intervening the negotiation process as much as possible; (ii) in case of a disagreement, gives one more chance to parties to resolve their conflict instead of imposing a harsh punishment right away; and (iii) encourages players to be agreeable and modest, by favoring such a player in the case of a conflict. All of these are qualities of a good arbitrator (see Bloom 1981 and references therein). We show that for $n = 2$, DD' attains a unique subgame perfect Nash equilibrium (in pure strategies), which induces equal demands and equal division, in the first stage (Theorem 1). Moreover, the mechanism we propose also eliminates all equilibria where players mix over different demands in the first stage (Theorem 2). Finally, for $n > 2$, there exists payoff-equivalent subgame perfect Nash equilibria that induce equal demands and equal division, in the first stage (Theorems 3 and 4).

Our modified divide-the-dollar game, DD', shares some similarities with *DD3* in Brams and Taylor (1994) in that it has a second stage where the excess can be avoided. However, there are important differences between the two mechanisms. First, we use an ultimatum game in this stage whereas Brams and Taylor (1994) use a sort of *sequential dictatorship* game. Therefore, our mechanism still gives all players (that face an effective ultimatum) a veto power, whereas *DD3* may not (i.e., the dollar may be divided before some players take their *dictatorship* turn). Second, DD' assigns the proposer role based on the players' first-stage demands, whereas *DD3* assigns sequential dictatorships based on the players' second stage decisions (e.g., confirming with own first-stage demand or usurping others' demands). In addition, from an ex-ante point of view, this difference implies that in DD' the excess *may* be avoided in the second stage, whereas in *DD3* it is *surely* avoided, which makes DD' more interesting from a strategic point of view. Finally, for $n > 2$ our game attains payoff-equivalent multiple equilibria in all of which players' demand equal shares of the dollar in the first stage, whereas *DD3* attains a unique equilibrium. Which of these mechanisms would be more effective in implementing equal division in reality is an interesting empirical question that could be addressed in future work with experimental techniques.

Another interesting question future studies may focus on is the characterization of mechanism classes, which implement equal division (or any particular division) in modified *DD* games. Properties of mechanisms in this class—and whether they are consonant with good arbitration practices—would be interesting to know for arbitration practitioners as well as bargaining, fair division, and conflict resolution researchers. Finally, future work studying similar issues in divide-the-dollar games where the pie is not exogenously given but rather endogenously and jointly produced by bargainers will also be of great interest [see Andreozzi (2010) for theoretical work and Karagözoğlu (2012) for experimental work on such problems].

Acknowledgments We would like to thank two anonymous reviewers for their comments on an earlier version. We would also like to thank Arno Riedl for helpful comments. All remaining errors are ours.

4 Appendix

In this appendix, we provide an extension of our mechanism to $n > 2$ players. First, we introduce some new notation. $i = 1, ..., n$ still denotes players. If $\sum_{i=1}^{n} d_i \leq 1$, then the game ends in the first stage and for all i, $\pi_i = d_i$. If $\sum_{i=1}^{n} d_i > 1$, then the game reaches the second stage and the player with the minimal first-stage demand is chosen to be the proposer in the ultimatum game. If $\sum_{i=1}^{n} d_i > 1$ and there are multiple players with the common minimal first-stage demand, then each such player is chosen to be the proposer with equal probability. We assume that players are *risk-neutral.* We denote the *excess* by $x \equiv \sum_{i=1}^{n} d_i - 1$. If player j is the proposer (i.e., either d_j is the unique minimal demand or player j is chosen among players with the minimal demand as a result of a random draw), to avoid x he makes a proposal $(x_1, ..., x_n)$ to other players such that $\sum_{i=1}^{n} x_i = x$. Each other player can either *accept* or *reject* this proposal. The proposer's action set in the second stage is for every i, $x_i \in [0, d_i]$ and the responders' action sets in the second stage are $\{accept, reject\}$. [12] If each responder accepts the proposal, then for all i, $\pi_i = d_i - x_i$. If at least one responder rejects the proposal, then $\pi_1 = ... = \pi_n = 0$. Therefore, the mechanism requires a unanimous agreement. For clarification, below we provide some examples on how our mechanism works.

Example 1 Let $n = 5$ and $(d_1, d_2, d_3, d_4, d_5) = (0.2, 0.3, 0.4, 0.8, 0.9)$. In this case, $\sum_{i=1}^{n} d_i > 1$, $x = 1.6$ and the minimal demand is unique. Player *1* proposes a split of x. A possible allocation would be $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0.7, 0.9)$. This

[12] Note that when $n = 2$, it is not necessary to assume $x_i \in [0, d_i]$ since $x_i > d_i$ is not possible.

proposal would be accepted by all responding players and thus, for all $i = 1, 2, 3$, $\pi_i = d_i$; $\pi_4 = 0.1$, and $\pi_5 = 0$.

Example 2 Let $n = 4$ and $(d_1, d_2, d_3, d_4) = (0.3, 0.3, 0.6, 0.7)$. In this case, $\sum_{i=1}^{n} d_i > 1$, $x = 0.9$ and the minimal demand is not unique. A random draw determines the proposer. Without loss of generality, assume that player 2 is chosen to be the proposer. A possible allocation would be $(x_1, x_2, x_3, x_4) = (0, 0, 0.4, 0.5)$. This proposal would be accepted by all responding players and thus, for all $i = 1, 2$, $\pi_i = d_i$; $\pi_3 = 0.2$, and $\pi_4 = 0.2$.

Note that the proposals in these examples are just given for clarification purposes and hence may not be optimal in the subgame perfect sense. Characteristics of *sequentially rational* ultimatum proposals will be provided in the following proof.

Theorem 3 *There exists payoff-equivalent multiple subgame perfect Nash equilibria of DD' and $(d_1, ..., d_n) = (1/n, ..., 1/n)$ in all equilibria.*

Proof We first show that the subgame perfect Nash equilibrium play induces $x = 0$ and there are multiple ways to avoid the excess in the second stage if it is reached (Claim 1). Then, we show that in subgame perfect Nash equilibrium, the minimal demand is equal to the maximal demand in the first stage (Claim 2). The result directly follows from these arguments.

Claim 1 Subgame perfect Nash equilibrium play induces $x = 0$.
The following exhaustive set of cases needs to be analyzed. In each case, we show that at least one profitable unilateral deviation exists.

Case 1 $x < 0$. Without loss of generality take player j. Clearly, a unilateral deviation $d_j' = 1 - \sum_{i \neq j} d_i$ brings $\pi' = 1 - \sum_{i \neq j} d_i > d_j = \pi_j$.

Case 2 $x > 0$. There are two subcases here. Before analyzing these subcases let us first take a look at the second-stage behavior. If and when the second stage of the game is reached, without loss of generality assume that player j is the proposer. As in Brams and Taylor (1994), we employ iterated elimination of weakly dominated strategies here.

For any responding player k, playing *accept* brings $\pi_k \in [0, d_k]$ and hence it weakly dominates (also there exist cases in which strict dominance occurs) playing *reject*. Thus, responding players choose to accept any proposal. This establishes the second-stage behavior of responding players.

Now, let us denote the set of players with the common maximal first-stage demand as H and the common value of the maximal demand as $h \equiv \max\{d_1, ..., d_n\}$. Given that any responding player chooses to accept any proposal, there are two cases to be studied: if $x \geqslant \sum_{i \in H} d_i$, then "offering $x_i = d_i$ to all players in H and distributing the remaining amount $x - \sum_{i \in H} d_i$ to $n - 1 - |H|$ players in such a way that for all remaining players, $\pi_i \geq 0$" weakly dominates all other strategies. Note that there are (infinitely) many ways of distributing $x - \sum_{i \in H} d_i$ to $n - 1 - |H|$ players, which is why there exists multiple equilibria. On the other hand, if $x < \sum_{i \in H} d_i$ then player j distributes x among the players in H. In these two cases, weak domination follows from the fact that playing such, player j can always gurantee $\pi_j = d_j$. This establishes the proposers' second-stage behavior.

Case 2.1 Minimal demand is unique. Without loss of generality, assume that $d_j = \min\{d_1, ..., d_n\}$. Denote the set of the remaining demands as A, where $A \equiv \{d_i : d_i \neq d_j\}$. Clearly, for player j, a unilateral deviation $d'_j = \min(A) - \epsilon$ brings $\pi'_j = \min(A) - \epsilon > d_j = \pi_j$.

Case 2.2 Minimal demand is not unique. This means that there exists a set of players with the common minimal demand. Denote this set as M and the common value of the minimal demand as $m \equiv \min\{d_1, ..., d_n\}$. There are two subcases.

Case 2.2.1 $\sum_{i \in M} d_i < 1$. Then there are two subcases.

Case 2.2.1.1 $\sum_{i \in H} d_i \leqslant x$. Without loss of generality, take player $j \in H$. Then for player j, a unilateral deviation, $d'_j = m - \epsilon$ brings $\pi'_j = d'_j = m - \epsilon > \pi_j = \epsilon$.

Case 2.2.1.2 $\sum_{i \in H} d_i > x$. Without loss of generality, take player $j \in M$. Let d''_j be the demand that satisfies $\sum_{i \in H} d_i = x$. Then, for player j a unilateral deviation, $d'_j = \min(h - \epsilon, d''_j)$ brings $\pi'_j = d'_j > \pi_j = d_j$.

Case 2.2.2 $\sum_{i \in M} d_i \geqslant 1$. This implies $x \geqslant \sum_{i \in H} d_i$. Without loss of generality, assume that player j is the one with the maximal demand. Thus, a unilateral deviation, $d'_j = m - \epsilon$ brings $\pi'_j = d'_j = m - \epsilon > \pi_j = \epsilon$.

Arguments above imply that in the subgame perfect Nash Equilibrium $x = 0$.

Claim 2 In the subgame perfect Nash equilbrium, $x = 0$ induces $h = m$.

Case 1 $h < m$. This case is impossible by definition.

Case 2 $h > m$. Without loss generality player j is the one with the minimal first-stage demand. Therefore, a unilateral deviation to $d'_j = h - \epsilon$ brings $\pi'_j = h - \epsilon > \pi_j = d_j$. □

The following theorem is a generalization of Theorem 2.

Theorem 4 *There is no subgame perfect Nash equilibrium of* DD' *(with* $n > 2$*), which involves mixing in the first stage.*

Proof We will prove this result in steps.

Step 1 Claiming $d_i < 1/n$ is strictly dominated by claiming $d_i = 1/n$, for every player $i = 1, 2, ...n$. Without loss of generality, let us focus on player *1*. It is sufficient to investigate the following two cases $x \leq 0$ and $x > 0$.

Case 1 $x \leq 0$. Let us analyze a deviation from $d_1 < 1/n$ to $d'_1 = 1/n$. Note here, we assume that (as in Theorem 3) in case the second stage is reached, the proposer will distribute x as follows: (i) he starts from h and if possible deducts the whole x from h, if not possibe deducts as much as possible from h; (ii) then he moves to the agent with the second highest demand follows the procedure in (i); and (iii) continues in the same fashion until x vanishes. Under this assumption, a deviation from $d_1 < 1/n$ to

$d'_1 = 1/n$ is always profitable for player *1*. Note that, even if this deviation creates an excess, $x > 0$, it brings $\pi'_1 = d'_1 > \pi_1$. To see why, observe that $h > d'_1$ and $x < h$.

Case 2 $x > 0$. In this case, since $x > 0$ there must exist a player j with $d_j > 1/n$. Now, consider a deviation from $d_1 < 1/n$ to $d'_1 = 1/n$. Clearly, $\sum_{i \in b} d_i < 1$ where $b \equiv \{i : d_i \leq 1/n\}$. Therefore, under the assumption made in *Case 1, x* will vanish before reaching player *1*. This implies that $\pi'_1 = d'_1 > \pi_1$. Since player *1* was chosen without loss of generality, the result is valid for any player.

Step 2 In any subgame perfect equilibrium $E(\pi_i) \geq 1/n$, for every player $i = 1, 2, ..., n$.

From Step 1, we know that for any i setting $d_i = 1/n$ guarantees $\pi_i = 1/n$ independent of what other players do. Therefore, in any subgame perfect Nash Equilibrium of DD', the expected payoff of any player should be greater or equal to $1/n$.

Step 3 Subgame perfection implies $\sum_{i=1}^{n} \pi_i = 1$.

Suppose that, $\sum_{i=1}^{n} d_i > 1$ and without loss of generality player *1* is the proposer in the second stage. Observe that there is always a way to divide x such that player *1* is not punished: by definition $x = \sum_{i=1}^{n} d_i - 1$ and $d_1 \leq 1$; therefore, $x \leq \sum_{j \neq 1} d_j$. This implies, player *1* have uncountably many ways to divide the x to other players. Then, in any subgame perfect equilibrium $\sum_{i=1}^{n} \pi_i = 1$.

Therefore, combining Steps 1, 2, and 3 implies that in any subgame perfect equilibrium, $E[\pi_1] = ... = E[\pi_n] = 1/n$.

Note that this equal division outcome, in principle, can be reached in different ways. For instance, all players may mix uniformly over $[1/n, 1]$ [13], or one player may (purely) demand $1/n$ and the others mix between $(1/n, 1]$, or yet some of the players (purely) demand $1/n$ and others mix over different demands. Nevertheless, Step 4 shows that this cannot be the case in equilibrium.

Step 4 Suppose without loss generality that players *2*, ..., N mix over $[1/n, 1]$ using a probability distribution F. First, we show that F cannot have atoms above $1/n$. Suppose for a contradiction that it has an atom above $1/n$. Hence, we can pick an arbitrary point a such that $a > 1/n$ and $P(a) > 0$. Now, suppose player *1* mixes using a distribution G, which has a support $[1/n, a)$, which is first-order stochastically dominated by F and $G(1/n) \neq 1$. As a result, we have $E(\pi_1) > E(\pi_2) = ... = E(\pi_n)$ and hence $E(\pi_1) > 1/n$ (a contradiction to $E(\pi_1) = E(\pi_2) = ... = E(\pi_n) = 1/n$), where the second inequality follows from subgame perfection. This shows that F cannot have atoms above $1/2$. In other words, we should not worry about ties in demands.

Suppose $k < n$ players mix over demands within the support, $[1/n, 1]$. For simplicity, assume that they use the same distribution, F, which has a density above $1/n$. $\mathcal{F}$ denotes the joint distribution of demands for k players. Therefore, for player i playing

[13] As in Theorem 2, we assume that players mix over only the Borel measurable subsets of [0, 1].

$d_i > 1/n$ will bring an expected payoff , $\pi_i(d_i)$,

$$\pi_i(d_i) = (1 - F(d_i))^k d_i + \sum_{h=1}^{k} \binom{k}{h} (1 - F(d_i))^h \int_{1/n}^{d_i} \cdots \int [\mathbf{I}_{\{\varphi > x\}} d_i$$
$$+((1 - \mathbf{I}_{\{\varphi > x\}}) \max(0, d_i - x + \varphi))] f(d_1)...f(d_h)\mathbf{d}(d_1)...\mathbf{d}(d_h)$$
$$+ \max\left(0, \int_{1/n}^{d_i} \cdots \int (d_i - x) f(d_1)...f(d_n)\mathbf{d}(d_1)....\mathbf{d}(d_n)\right),$$

where $\varphi \equiv \sum_{j \in L} d_j$, $L \equiv \{j : d_j > d_i\}$, is the set of agents with demands higher than d_i and $\mathbf{I}_{\{\varphi > x\}}$ is the indicator function, which takes the value 1 if the sum of demands higher than d_i is less than x and 0 otherwise.[14]

Observe that if we take the derivative of $\pi_i(d_i)$ with respect to d_i, we get $\pi_i'(d_i) = (1 - F(d_i))^{k-1} - f(d_i)\Phi + (1 - F(d_i))\Psi + \sum_{i=1}^{k} \gamma_i f(d_i)(1 - F(d_i))^k$, where Φ,Ψ, and γ are some positive constants. Obviously, if $F(d_i) < 1$, this derivative is greater than zero. This implies that player i can receive a share higher than $1/n$ just by marginally increasing his demand. This means that there must exist some player j with $\pi_j < 1/n$, which is in contradiction with findings from Steps 1, 2, and 3. Hence, a positive probability cannot be assigned to any demand above $1/n$ in equilibrium.

Therefore, the result follows from Steps 1, 2, 3, and 4. □

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[14] As in Theorem 2, for simplicity, we assume that F is an absolutely continuous function.

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