Brownian Motion, Dynamical Friction, and Stellar Dynamics

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I. INTRODUCTION

AS is well known, one of the triumphs of classical physics was the unraveling of the phenomenon of Brownian motion by Einstein and von Smoluchowski. Out of these early investigations of Einstein and von Smoluchowski an extensive literature on the theory of probability and random variables has grown. In this paper we shall not attempt to summarize these varied developments; we shall limit ourselves, rather, to an analysis of the physical (as distinct from the mathematical) foundations of the theory and to an illustration from stellar dynamics how Einstein's ideas have found fruitful applications in a very different field.

II. THE BASIC ASSUMPTIONS OF THE PHYSICAL THEORY OF BROWNIAN MOTION

The theory of Brownian motion is concerned with the irregular, perpetual motions of colloidal particles in suspension in a liquid. It is known that these motions have their origin in the collisions which the colloidal particles suffer with the molecules of the surrounding fluid. Under normal conditions, in a liquid, a Brownian particle will suffer about 10^{21} collisions per second. Since each of these collisions can be thought of as producing a kink in the path of the particle, it is evident that we cannot hope to follow the path of a particle in any very great detail: to our senses, the details are impossibly fine.

And, reduced to its essentials, the theory of Brownian motion as initiated by Einstein derives from the following set of assumptions:

The motion of a *free particle* (i.e., one in the absence of an external field of force) is assumed to be governed by an equation of the form

$$d\mathbf{u}/dt = -\eta\mathbf{u} + \mathbf{A}(t), \tag{1}$$

where $\mathbf{u}$ denotes the instantaneous velocity of the particle. In writing this equation, the assumption has been made that the influence of the surrounding medium can be split up into two parts: a *systematic part*, $-\eta\mathbf{u}$, which represents the operation of *dynamical friction*, and a *fluctuating part*, $\mathbf{A}(t)$, which is characteristic of Brownian motion.

Regarding the frictional term, $-\eta\mathbf{u}$, it is *assumed* that it is governed by Stokes' law according to which the frictional force decelerating a spherical particle of radius a and mass m is given by $6\pi a\nu\mathbf{u}/m$, where ν denotes the coefficient of viscosity of the surrounding liquid. In other words,

$$\eta = 6\pi a\nu/m. \tag{2}$$

As for the part $\mathbf{A}(t)$ the following principal assumptions are made: (i) $\mathbf{A}(t)$ is independent of $\mathbf{u}$,[1] and (ii) $\mathbf{A}(t)$ varies extremely rapidly compared with $\mathbf{u}$. The second of these assumptions implies that time intervals Δt exist such that during Δt the changes in $\mathbf{u}$ to be expected are very small, while during the same interval $\mathbf{A}(t)$ may undergo a very large number of fluctuations. Alternatively, we may express this assumption by the statement that though $\mathbf{u}(t)$ and $\mathbf{u}(t+\Delta t)$ are expected to differ by a negligible amount, no correlation between $\mathbf{A}(t)$ and $\mathbf{A}(t+\Delta t)$ is expected. Considering then the net increment in velocity,

$$\mathbf{B}(\Delta t) = \int_t^{t+\Delta t} \mathbf{A}(\xi)d\xi, \tag{3}$$

which a particle experiences (due to random fluctuations) during an interval Δt, we assert (i) that the increments between the successive intervals $(t, t+\Delta t_1)$ and $(t+\Delta t_1, t+\Delta t_1+\Delta t_2)$ have no correlation, and (ii) that the probability of occurrence of different net increments during an interval Δt is given by

$$w[\mathbf{B}(\Delta t)] = \frac{1}{(4\pi q\Delta t)^{\frac{3}{2}}} \exp[-|\mathbf{B}(\Delta t)|^2/4q\Delta t], \tag{4}$$

where q is a certain *diffusion coefficient* (in velocity space) related to the frictional coefficient, η, by

$$q = \eta kT/m, \tag{5}$$

where k is the Boltzmann constant and T is the absolute temperature.

III. A DISCUSSION OF THE BASIC ASSUMPTIONS OF THE THEORY OF BROWNIAN MOTION

The basic assumptions of the theory of Brownian motion which we have set out, barely in Section II, emphasize the drastic nature of these assumptions. The intuitive character of the assumptions is apparent, already, in the separation of a systematic from a fluctuating part in the acceleration in Eq. (1) which, by implication, supposes that we can divide the phenomenon into two parts: a part in which the discontinuity of the events taking place is essential, and a part in which it is inessential and can be ignored. Granting this separation, we next inquire into the meaning and justification of the assumptions underlying Eqs. (4) and (5).

[1] We shall indicate later the generalizations required when $\mathbf{A}(t)$ depends on $\mathbf{u}$ (see Eq. (13)).

The justification for the form of the distribution function (4) is derived from the theory of random flights.

In the problem of random flights, a particle suffers a sequence of displacements, $\mathbf{r}_i$ $(i=1, 2, \cdots)$, the magnitude and direction of each of the displacements being governed by a probability distribution. After N such displacements, the position of the particle will be given by

$$\mathbf{R}=\sum_{i=1}^{N}\mathbf{r}_i. \tag{6}$$

We ask for the probability distribution of $\mathbf{R}$. While it is not difficult to write down the formal solution of the problem,[2] the case of greatest interest is when N is large and the different displacements, $\mathbf{r}_i$, are governed by the same spherically symmetric probability distribution, $\tau(\mathbf{r})$. In that case, the distribution of $\mathbf{R}$ is given by[3]

$$w(\mathbf{R})=\frac{1}{(2\pi N\langle r^2\rangle_{\text{Av}}/3)^{\frac{3}{2}}}\exp(-3|\mathbf{R}|^2/2N\langle r^2\rangle_{\text{Av}}), \tag{7}$$

where

$$\langle r^2\rangle_{\text{Av}}=\int_{-\infty}^{+\infty}\tau(\mathbf{r})r^2d\mathbf{r} \tag{8}$$

is the mean square displacement to be expected on any particular occasion.

If we suppose that the particle experiences n displacements per unit time, the net displacement, $\mathbf{R}$, after an interval Δt during which a very large number of displacements take place, is given by

$$w(\mathbf{R})=\frac{1}{(4\pi q\Delta t)^{\frac{3}{2}}}\exp(-|\mathbf{R}|^2/4q\Delta t), \tag{9}$$

where we have written

$$q=\tfrac{1}{6}n\langle r^2\rangle_{\text{Av}}. \tag{10}$$

Returning to the problem of Brownian motion, we recall that intervals of time Δt exist during which a particle, though it suffers a very large number of collisions with the molecules of the surrounding fluid, experiences only "infinitesimal" increments in the velocity. During such an interval of time, the net random increment in velocity, $\mathbf{B}(\Delta t)$, which the particle will experience, is the resultant of the effects of a very large number of collisions, each of which causes a certain "minute" acceleration $\delta\mathbf{v}$. The appropriateness of the problem of random flights to determine the probability distribution of $\mathbf{B}(\Delta t)$ is apparent. Indeed, we can write this directly from (9) if we interpret q by

$$q=\tfrac{1}{6}n\langle|\delta\mathbf{v}|^2\rangle_{\text{Av}}, \tag{11}$$

where n is the number of collisions per unit time between a Brownian particle and the molecules of the surrounding fluid, and $\langle|\delta\mathbf{v}|^2\rangle_{\text{Av}}$ is the mean square increment in velocity of a particle, per collision. In this fashion, we recover the form of the distribution function (4).

Turning next to the relation (5) between q and η, we introduce considerations of the following type:

Let the distribution of velocities among the Brownian particles at a particular instant of time t be given by $W(\mathbf{u}, t)$. After a time Δt the distribution function will have changed as during such an interval a particle with a velocity $\mathbf{u}$, for example, would have suffered an increment of velocity

$$\Delta\mathbf{u}=-\eta\mathbf{u}\Delta t+\mathbf{B}(\Delta t), \tag{12}$$

and the probability of such an increment will be governed by (see Eq. (4))

$$\psi(\mathbf{u};\Delta\mathbf{u})=\frac{1}{(4\pi q\Delta t)^{\frac{3}{2}}}\exp\{-|\Delta\mathbf{u}+\eta\mathbf{u}\Delta t-\text{grad}_{\mathbf{u}}q\Delta t|^2\}/4q\Delta t, \tag{13}$$

where we have slightly generalized (4) to allow for a dependence of q on the velocity. We therefore expect that the distribution function $W(\mathbf{u}, t+\Delta t)$ at time $t+\Delta t$ will be given by

$$W(\mathbf{u}, t+\Delta t)=\int_{-\infty}^{+\infty}W(\mathbf{u}-\Delta\mathbf{u};t)\psi(\mathbf{u}-\Delta\mathbf{u};\Delta\mathbf{u})d(\Delta\mathbf{u}). \tag{14}$$

We may parenthetically remark that in expecting the integral Eq. (14) between $W(\mathbf{u}, t+\Delta t)$ and $W(\mathbf{u}, t)$ to be valid, we are actually supposing that the course which a Brownian particle will take depends only on the instantaneous values of the physical parameters and is entirely independent of its whole previous history. In probability theory, a stochastic process, having this property, namely, that what happens at a given instant depends only on the state of the system at that instant, is said to be a *Markoff process*. We may describe a Markoff process by the statement that it represents "a gradual unfolding of a transition probability" in exactly the same sense as the development of a conservative dynamical system can be described as "the gradual unfolding of a contact transformation" (Whittaker). That we should be able to picture Brownian motion as a Markoff process is reasonable: its "reasonableness" arising principally from the circumstance that in Eq. (14) we can consider intervals Δt during which a very large number of collisions take place and which, nevertheless, change the distribution of velocities among the particles only insensibly. With this understanding, we can expand $W(\mathbf{u}, t+\Delta t)$, $W(\mathbf{u}-\Delta\mathbf{u}, t)$, and $\psi(\mathbf{u}-\Delta\mathbf{u};\Delta\mathbf{u})$ in Eq. (14) by Taylor

[2] See e.g., S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943) (see particularly Chapter I, Section 4 of this paper).

[3] Reference 2, Eq. (93).

series and obtain

$$W(\mathbf{u},t)+(\partial W/\partial t)\Delta t+O(\Delta t^2)$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Big\{W(\mathbf{u},t)-\sum_i\frac{\partial W}{\partial u_i}\Delta u_i$$
$$+\tfrac{1}{2}\sum_i\frac{\partial^2 W}{\partial u_i^2}\Delta u_i^2+\sum_{i<j}\frac{\partial^2 W}{\partial u_i\partial u_j}\Delta u_i\Delta u_j+\cdots\Big\}$$
$$\times\Big\{\psi(\mathbf{u};\Delta\mathbf{u})-\sum_i\frac{\partial\psi}{\partial u_i}\Delta u_i+\tfrac{1}{2}\sum_i\frac{\partial^2\psi}{\partial u_i^2}\Delta u_i^2$$
$$+\sum_{i<j}\frac{\partial^2\psi}{\partial u_i\partial u_j}\Delta u_i\Delta u_j\Big\}d(\Delta u_1)d(\Delta u_2)d(\Delta u_3),\quad(15)$$

or writing

$$\langle\Delta u_i\rangle_{\text{Av}}=\int_{-\infty}^{+\infty}\Delta u_i\psi(\mathbf{u};\Delta\mathbf{u})d(\Delta\mathbf{u})\quad\text{etc.},\qquad(16)$$

we have

$$(\partial W/\partial t)\Delta t+O(\Delta t^2)$$
$$=-\sum_i W\frac{\partial}{\partial u_i}\langle\Delta u_i\rangle_{\text{Av}}+\tfrac{1}{2}\sum_i W\frac{\partial^2}{\partial u_i^2}\langle\Delta u_i^2\rangle_{\text{Av}}$$
$$+\sum_{i<j}W\frac{\partial^2}{\partial u_i\partial u_j}\langle\Delta u_i\Delta u_j\rangle_{\text{Av}}-\sum_i\langle\Delta u_i\rangle_{\text{Av}}\frac{\partial W}{\partial u_i}$$
$$+\sum_i\frac{\partial W}{\partial u_i}\frac{\partial}{\partial u_i}\langle\Delta u_i^2\rangle_{\text{Av}}+\sum_{i<j}\frac{\partial W}{\partial u_i}\frac{\partial}{\partial u_j}\langle\Delta u_i\Delta u_j\rangle_{\text{Av}}$$
$$+\tfrac{1}{2}\sum_i\frac{\partial^2 W}{\partial u_i^2}\langle\Delta u_i^2\rangle_{\text{Av}}+\sum_{i<j}\frac{\partial^2 W}{\partial u_i\partial u_j}\langle\Delta u_i\Delta u_j\rangle_{\text{Av}}$$
$$+O(\langle\Delta u_i\Delta u_j\Delta u_k\rangle_{\text{Av}}),\quad(17)$$

where the remainder term involves the averages of the quantities Δu_i^2, $\Delta u_i^2\Delta u_j$, and $\Delta u_i\Delta u_j\Delta u_k$ $(i,j,k=1,2,3)$, and similar larger combinations. Equation (17) can be written more conveniently in the form

$$\frac{\partial W}{\partial t}\Delta t+O(\Delta t^2)$$
$$=-\sum_i\frac{\partial}{\partial u_i}(W\langle\Delta u_i\rangle_{\text{Av}})+\tfrac{1}{2}\sum_i\frac{\partial^2}{\partial u_i^2}(W\langle\Delta u_i^2\rangle_{\text{Av}})$$
$$+\sum_{i<j}\frac{\partial^2}{\partial u_i\partial u_j}(W\langle\Delta u_i\Delta u_j\rangle_{\text{Av}})+O(\langle\Delta u_i\Delta u_j\Delta u_k\rangle_{\text{Av}}).\quad(18)$$

This is the *Fokker-Planck equation* in its most general form.

For the transition probability (13),

$$\langle\Delta\mathbf{u}\rangle_{\text{Av}}=-(\eta\mathbf{u}-\text{grad}_{\mathbf{u}}q)\Delta t,$$
$$\langle\Delta u_i^2\rangle_{\text{Av}}=2q\Delta t\quad(i=1,2,3)\qquad(19)$$
$$\langle\Delta u_i\Delta u_j\rangle_{\text{Av}}=O(\Delta t^2)\quad\text{and}\quad\langle\Delta u_i\Delta u_j\Delta u_k\rangle_{\text{Av}}=O(\Delta t^3).$$

Substituting these in Eq. (18), we obtain

$$(\partial W/\partial t)\Delta t+O(\Delta t^2)=\textstyle\sum_i(\partial/\partial u_i)(\eta u_i W-W(\partial q/\partial u_i))$$
$$+\textstyle\sum_i(\partial^2/\partial u_i^2)(qW)+O(\Delta t^2).\quad(20)$$

Now, passing to the limit $\Delta t=0$, we have

$$\partial W/\partial t=\textstyle\sum_i(\partial/\partial u_i)[\eta u_i W-W(\partial q/\partial u_i)$$
$$+(\partial/\partial u_i)Wq]\quad(21)$$

or

$$\partial W/\partial t=\text{div}_{\mathbf{u}}(\eta\mathbf{u}W+q\,\text{grad}_{\mathbf{u}}W).\qquad(22)$$

According to this equation, we may visualize the motions of the representative points in the velocity space as a process of diffusion in which the rate of flow across an element of surface $d\sigma$ is given by

$$-(q\,\text{grad}_{\mathbf{u}}W+\eta W\mathbf{u})\cdot\mathbf{1}_{d\sigma}d\sigma,\qquad(23)$$

where $\mathbf{1}_{d\sigma}$ is a unit vector normal to the element of surface considered. It should, however, be understood that this visualization ceases to be valid when time intervals less than Δt are considered.

So far we have not restricted q and η in any manner. We now assert that *a Maxwellian distribution of velocities must be invariant to the underlying stochastic process and that any arbitrary initial distribution of velocities must eventually become Maxwellian.* In other words, we require that

$$W(\mathbf{u})=(m/2\pi kT)^{\frac{3}{2}}\exp(-m|\mathbf{u}|^2/2kT),\qquad(24)$$

satisfies Eq. (22) identically; this condition, as may be readily verified, is equivalent to imposing the relation (5) between q and η.

In some ways it is remarkable that we can obtain as complete a specification, as we have, of the stochastic process characteristic of Brownian motion, without, at any point, having been required to analyze the mechanics of the collision process itself; but it emphasizes Einstein's extraordinary perception into the physical character of the problem.

IV. STELLAR ENCOUNTERS AS AN EXAMPLE OF BROWNIAN MOTION

The discussion of the physical foundations of the theory of Brownian motion in the preceding sections has disclosed certain inherent limitations in the theory. The limitations are nowhere more serious than in the circumstance that the coefficients q and η are not derived from a microscopic analysis of the individual encounters. It is therefore of interest that stellar dynamics provides a case of Brownian motion in which all phases of the problem can be explicitly analyzed.

In stellar dynamics, one of the fundamental problems

is to incorporate in the framework of a general theory the effect of encounters between stars; and stellar encounters under Newtonian inverse square attractions influence the motions of stars in the manner of Brownian motion. The analogy with Brownian motion arises from the peculiar character of inverse square forces: Encounters with small values of the impact paraments (which produce appreciable deflections, for example) are very rare, and encounters with large values of the impact parameter, which are frequent, are very ineffective. Thus, as in Brownian motion, it is only the cumulative effect of a large number of encounters which will produce sensible changes in the directions and the magnitudes of the motions. There is, however, one inessential difference: In the stellar case, stars influence one another, while in the Brownian motion of colloidal particles, the particles are primarily influenced by the molecules of the surrounding fluid. But, physically, the close analogy that exists between the motion of a star in the gravitational field of its neighbors and the motion of a colloidal particle describing Brownian motion results from the following circumstance: Even as collisions with single molecules of the surrounding fluid hardly affect the motion of a colloidal particle, so also does an average encounter with another star hardly affect the motion of a star; and in both cases what is of importance is the cumulative effect of a large number of separate events each of which has only a very minute effect. Moreover, in both problems, during a time interval, Δt, necessary for the velocity of a particle (star) to change sensibly, a very large number of collisions (encounters) take place. In the stellar case, this time interval is of the order of 10^6 years: during such an interval of time an average star will have experienced about 100 encounters since the time required for an average star to traverse a distance equal to the average distance between the stars is of the order of 10^4 years.

V. DYNAMICAL FRICTION

If our analogy of the effect of stellar encounters with Brownian motion is correct, then we should expect to establish by a direct analysis of stellar encounters the operation of dynamical friction superposed on random fluctuations. It is remarkable that such a separation of the effects of stellar encounters can be accomplished without appealing to any heuristic concepts.

Turning then to an analysis of stellar encounters, we recall that during an encounter each star will describe a hyperbola relative to the other. As a result of the encounter, a star will suffer certain increments $\Delta u_{\parallel}$ and $\Delta u_{\perp}$ in its velocity in directions parallel, respectively, perpendicular to the initial direction of motion. The exact amounts of these increments will depend on the parameters which are necessary to specify an encounter. Considering an encounter of a star of mass m and velocity $\mathbf{u}$ with another "field star" of mass m_1 and velocity $\mathbf{v}_1$, we find from a straightforward analysis based on the classical two body problem that[4]

$$\Delta u_{\parallel}=-[2m_1/(m_1+m)][(u-v_1\cos\theta)\cos\psi + v_1\sin\theta\cos\Theta\sin\psi]\cos\psi, \quad (25)$$

and

$$\Delta u_{\perp}=\pm[2m_1/(m_1+m)][u^2+v_1^2-2uv_1\cos\theta - \{(u-v_1\cos\theta)\cos\psi+v_1\sin\theta\cos\Theta\sin\psi\}^2]^{\frac{1}{2}}\cos\psi, \quad (26)$$

where θ denotes the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}_1$, Θ the inclination of the orbital plane to the plane containing $\mathbf{u}$ and $\mathbf{v}_1$,

$$\cos\psi=\{1+D^2(u^2+v_1^2-2uv_1\cos\theta)^2/G^2(m_1+m)^2\}^{-\frac{1}{2}}, \quad (27)$$

D the impact parameter, and G the constant of gravitation.

Consider an interval of time Δt ($\sim 10^6$ years) long compared with the time ($\sim 10^4$ years) required for two stars to separate by a distance equal to the average distance between the stars, but short compared to the time intervals during which the velocity of a star may be expected to change appreciably. During such an interval of time the net increments $\sum\Delta u_{\parallel}$ and $\sum\Delta u_{\perp}$, which a star with an initial velocity $\mathbf{u}$ may be expected to suffer, can be obtained by simply averaging the expressions (25) and (26) for $\Delta u_{\parallel}$ and $\Delta u_{\perp}$.

According to Eq. (26) and as can indeed be expected on general symmetry grounds, $\Delta u_{\perp}$ when summed over a large number of encounters vanishes. But this is not the case with $\Delta u_{\parallel}$; it is given by

$$\sum\Delta u_{\parallel}=\Delta t\int_0^{\infty}dv_1\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\int_0^{D_0}dD\int_0^{2\pi}\frac{d\Theta}{2\pi} \times\{2\pi N(v_1,\theta,\varphi)\}VD\Delta u_{\parallel}, \quad (28)$$

where V denotes the relative velocity between the two stars, $N(v_1,\theta,\varphi)dv_1d\theta d\varphi$ is the number of field stars with the specified parameters in the indicated ranges, and D_0 is the average distance between the stars. Further, in Eq. (28) the various integrations are with respect to the different parameters defining a single encounter. Carrying out the various integrations, except the last, we find[5]

$$\sum\Delta u_{\parallel}=-\pi m_1(m+m_1)\frac{G^2}{u^2}\Delta t\int_0^{\infty}N(v_1)Q(v_1)dv_1, \quad (29)$$

where

$$Q(v_1)=\begin{cases}\log[(1+q^2[u+v_1]^4)(1+q^2[u-v_1]^4)] & (v_1<u)\\ \log[(1+16q^2v_1^4)]-4 & (v_1=u)\\ \log\dfrac{\{1+q^2(v_1+u)^4\}}{\{1+q^2(v_1-u)^4\}}-\dfrac{8u}{v_1} & (v_1>u)\end{cases} \quad (30)$$

[4] S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942), p. 229, Eq. (5.721).

[5] For the details of the derivation see S. Chandrasekhar, Astrophys. J. **97**, 255 (1943) (pp. 258–260).

and

$$q=\frac{D_0}{G(m_1+m_2)}$$

$$=2.33\times10^4\frac{D_0/\text{parsec.}}{[(m_1+m_2)/\odot][10\text{ km/sec.}]^2}. \quad (31)$$

Under normal conditions, $q^2(v_1+u)^4$ and $q^2(v_1-u)^4$ are very large compared to unity, and we can simplify Eq. (30) to

$$Q=\begin{cases}4\log q(u^2-v_1^2) & (v_1<u)\\ 2\log 4qv_1^2-4 & (v_1=u)\\ 4\log[(v_1+u)/(v_1-u)]-8u/v_1 & (v_1>u).\end{cases} \quad (32)$$

From Eq. (32) the remarkable result emerges that *to a sufficient accuracy only stars with velocities less than the one under consideration contribute to* $\sum\Delta u_{11}$. It is precisely on this account that dynamical friction appears on our present analysis.

With the further approximation

$$Q=\begin{cases}4\log q\langle|\mathbf{u}|^2\rangle_{Av} & (v_1<u)\\ 0 & (v_1>u),\end{cases} \quad (33)$$

where $\langle|\mathbf{u}|^2\rangle_{Av}$ is the mean square velocity of the field stars, Eq. (29) becomes

$$\sum\Delta u_{11}=-4\pi(m+m_1)m_1NG^2\frac{1}{|\mathbf{u}|^2}\times\left(\log\left[\frac{D_0\langle|\mathbf{u}|^2\rangle_{Av}}{G(m_1+m)}\right]\right)\Delta t\int_0^{|\mathbf{u}|}f(v_1)dv_1, \quad (34)$$

where N denotes the number of field stars per unit volume and $f(v_1)$ is the distribution function governing the probability of occurrence of a star with velocity $|\mathbf{v}_1|=v_1$. According to Eq. (34) the star experiences dynamical friction with a coefficient of dynamical friction η given by

$$\eta=4\pi Nm_1(m+m_1)\frac{G^2}{|\mathbf{u}|^3}\times\left(\log\left[\frac{D_0\langle|\mathbf{u}|^2\rangle_{Av}}{G(m_1+m)}\right]\right)\int_0^{|\mathbf{u}|}f(v_1)dv_1. \quad (35)$$

Again from Eq. (26) we similarly find, after averaging over the various parameters of the encounter, that

$$\sum\Delta u_{11}^2=\frac{8}{3}\pi Nm_1^2\frac{G^2}{|\mathbf{u}|^3}\left(\log\left[\frac{D_0\langle|\mathbf{u}|^2\rangle_{Av}}{G(m_1+m)}\right]\right)\langle|\mathbf{u}|^2\rangle_{Av}\Delta t\times\int_0^{|\mathbf{u}|}f(v_1)dv_1, \quad (36)$$

which represents, in analogy with Eq. (19), a diffusion in velocity space. The completeness of the analogy of our present problem with Brownian motion is seen even more clearly when we note that, according to Eqs. (35) and (36),

$$\frac{\sum\Delta u_{11}^2}{\eta\Delta t}=\frac{2}{3}\frac{m_1}{m+m_1}\langle|\mathbf{u}|^2\rangle_{Av} \quad (37)$$

which, in our present context, is the equivalent of Eq. (5).

In some ways the emergence of dynamical friction from a straightforward analysis of stellar encounters is surprising. Indeed, it is contrary to what one might have expected on the following arguments which sound "plausible" enough.

(a) Suppose we consider a star of velocity $|\mathbf{u}|$ appreciably less than the root-mean-square velocity $\langle|\mathbf{u}|^2\rangle_{Av}^{\frac{1}{2}}$. We should then expect that it encounters oftener stars, with velocities greater than its own, than stars with velocities less than its own. Consequently, we might be led to believe that stars with velocities less than the average would be systematically accelerated and similarly, that stars with velocities greater than the average would be systematically decelerated.

(b) We might go even farther and argue that the conclusions reached in (a) are "reasonable," for, it might be supposed that systematically different effects on stars with relatively large, respectively, small velocities are required for the statistical maintenance of the average (i.e., normal) conditions.

In view of the great importance of dynamical friction for statistical dynamics, it is important to see the fallacy in these arguments:

The fallacy in (a) is simply that for inverse square encounters, the effect on the velocity in the direction of motion of a given star, by stars with velocities greater than that of the given one, nearly cancels out on the average; and it is only stars with velocities less than that of the given one which predominantly affect the velocity in the direction of motion.

The fallacy in (b) is due to a misunderstanding. There is nothing really obvious in the requirement that for the statistical maintenance of the average conditions stars differing from the average conditions should be affected differently according to the *sense* of their departure from the normal state. Indeed, the requirement that the normal conditions are self-perpetuating is to state in a different way one of two things: Either, that starting from any arbitrary initial state we always approach the normal state (i.e., the Maxwellian distribution of velocities) as $t\rightarrow\infty$; or, that once the normal state has been attained it continues to be maintained. It is now apparent that these conditions can be met only if a given star behaves at later times in a manner less and less dependent on an initial state as time goes on; or expressing the same thing somewhat differently, we should much rather expect that a star gradually

loses all trace of its initial state as time progresses. Such a gradual loss of "memory" can be achieved only by the operation of a dissipative force like dynamical friction which will gradually damp out any given initial velocity. Thus, if we assume for the sake of simplicity, that η is independent of $|\mathbf{u}|$, then the average velocity at later times will tend to zero like

$$\bar{\mathbf{u}}=\mathbf{u}_0 e^{-\eta t}; \tag{38}$$

but this is not to imply that the mean square velocity also tends to zero. Indeed, the restoration of a Maxwellian distribution of velocities from an arbitrary initial state requires that

$$\bar{\mathbf{u}}\to 0 \text{ while } \langle|\mathbf{u}|^2\rangle_{Av}\to \text{a constant as } t\to\infty. \tag{39}$$

To achieve the first of these conditions we need dynamical friction and to achieve the second we need random fluctuations as expressed by a diffusion coefficient. The recognition of these facts is, of course, Einstein's achievement.

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A Special Method for Solving the Dirac Equations

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Exact solutions to the Dirac equations for an electron in an external field of an arbitrary plane wave are obtained by transforming plane wave solutions for a free electron by variable Lorentz matrices. These matrices are those which occur in the discussion of the classical orbits. An example shows that this method, which applies classically, may fail for the Dirac equations when the external field is a constant one.

1. INTRODUCTION

IN a previous paper[1] it was shown that solutions of the classical relativistic equations of motion for a charged particle in the external field of a plane electromagnetic wave may be obtained in terms of Lorentz matrices determined by the antisymmetric tensor describing the external field. In this paper it is shown that when plane wave solutions of the Dirac equations for a free particle are transformed by these Lorentz matrices then the exact solutions to the Dirac equations for an electron in the external field described above are obtained. These solutions have been discussed by Volkow[2] and Singupta.[3]

We first obtain the necessary and sufficient conditions that a variable set of Lorentz matrices must satisfy in order that they be able to transform plane wave solutions of the Dirac equation for a free particle into solutions of these equations when an external field is present. It is then shown that these conditions can be satisfied in case the external field is that of a plane wave.

In the binary spinor formalism[4] the Dirac equations are

$$\begin{aligned} g^\sigma\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\sigma}-\frac{e}{c}\Phi_\sigma\right)\psi&=-imc\bar{\varphi},\\ g^\sigma\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\sigma}+\frac{e}{c}\Phi_\sigma\right)\varphi&=-imc\bar{\psi}, \end{aligned} \tag{1.1}$$

where m and e are the mass and charge of the particle, $\hbar$ is Planck's constant divided by 2π, c is the velocity of light, Φ_σ is the four-vector potential describing the external field and ψ, φ and g^σ are spinors. The first two are single index spinors and the g^σ are two index ones satisfying the matrix equation

$$\tfrac{1}{2}(\bar{g}^\sigma g^\tau+\bar{g}^\tau g^\sigma)=-g^{\sigma\tau}\cdot 1, \tag{1.2}$$

where 1 is the 2×2 identity matrix and in a galilean frame,

$$g_{\sigma\tau}=\begin{vmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 \end{vmatrix}=g^{\sigma\tau}. \tag{1.3}$$

An explicit set of matrices satisfying (1.2) are given in T.E., p. 938.

Equations (1.1) are numerically invariant under a proper Lorentz transformation of the independent variables x^σ, namely,

$$x^\sigma\to x^{\sigma *}=L_\tau{}^\sigma x^\tau,$$

where the $L_\tau{}^\sigma$ are constants, provided the spinors ψ and φ have the transformation law

$$\begin{aligned} \psi\to\psi^*(x)&=\Gamma\psi(L^{-1}x),\\ \varphi\to\varphi^*(x)&=\Gamma\varphi(L^{-1}x), \end{aligned} \tag{1.4}$$

where Γ is the spin-image of the Lorentz matrix L. Thus Γ is determined in terms of L by the equations:

$$\bar{\Gamma}^{-1}g^\sigma\Gamma=L_\tau{}^\sigma g^\tau, \tag{1.5}$$

and satisfied the condition

$$\det\Gamma=1. \tag{1.6}$$

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