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Interval-valued Possibility Quadripartitioned Single Valued Neutrosophic Soft Sets and some uncertainty based measures on them

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Abstract: The theory of quadripartitioned single valued neutrosophic sets was proposed very recently as an extension to the existing theory of single valued neutrosophic sets. In this paper the notion of possibility fuzzy soft sets has been generalized into a new concept viz. interval-valued possibility quadripartitioned single valued neutrosophic soft sets. Some basic set-theoretic operations have been defined on them. Some distance, similarity, entropy and inclusion measures for possibility quadripartitioned single valued neutrosophic sets have been proposed. An application in a decision making problem has been shown.

Keywords: Neutrosophic set, entropy measure, inclusion measure, distance measure, similarity measure.

1 Introduction

The theory of soft sets (introduced by D. Molodstov, in 1999) ([10],[15]) provided a unique approach of dealing with uncertainty with the implementation of an adequate parameterization technique. In a very basic sense, given a crisp universe, a soft set is a parameterized representation or parameter-wise classification of the subsets of that universe of discourse with respect to a given set of parameters. It was further shown that fuzzy sets could be represented as a particular class of soft sets when the set of parameters was considered to be $[0, 1]$. Since soft sets could be implemented without the rigorous process of defining a suitable membership function, the theory of soft sets, which seemed much easier to deal with, underwent rapid developments in fields pertaining to analysis as well as applications (as can be seen from the works of [1],[6],[7],[12],[14],[16],[17] etc.)

On the otherhand, hybridized structures, often designed and obtained as a result of combining two or more existing structures, have most of the inherent properties of the combined structures and thus provide for a stronger tool in handling application oriented problems. Likewise, the potential of the theory of soft sets was enhanced to a greater extent with the introduction of hybridized structures like those of the fuzzy soft sets [8], intuitionistic fuzzy soft sets [9], generalized fuzzy soft sets [13], neutrosophic soft sets [11], possibility fuzzy soft sets [2], possibility intuitionistic fuzzy soft sets [3] etc. to name a few.

While in case of generalized fuzzy soft sets, corresponding to each parameter a degree of possibility is assigned to the corresponding fuzzy subset of the universe; possibility fuzzy sets, a further modification of the generalized fuzzy soft sets, characterize each element of the universe with a possible degree of belongingness along with a degree of membership. Based on Belnap's four-valued logic [4] and Smarandache's n-valued refined neutrosophic set [18], the theory of quadripartitioned single valued neutrosophic sets [5] was proposed as a generalization of the existing theory of single valued neutrosophic sets [19]. In this paper the concept of interval valued possibility quadripartitioned single valued neutrosophic soft sets (IPQSVNSS, in short) has been proposed. In the existing literature studies pertaining to a possibility degree has been dealt with so far. Interval valued possibility assigns a closed sub-interval of $[0, 1]$ as the degree of chance or possibility instead of a number in $[0, 1]$ and thus it is a generalization of the existing concept of a possibility degree. The proposed structure can be viewed as a generalization of the existing theories of possibility fuzzy soft sets and possibility intuitionistic fuzzy soft sets.

The organization of the rest of the paper is as follows: a couple of preliminary results have been stated in Section 2, some basic set-theoretic operations on IPQSVNSS have been defined in Section 3, some uncertainty based measures viz. entropy, inclusion measure, distance measure and similarity measure, have been defined in Section 4 and their properties, applications and inter-relations have been studied. Section 5 concludes the paper.

2 Preliminaries

In this section some preliminary results have been outlined which would be useful for the smooth reading of the work that follows.

2.1 An outline on soft sets and possibility intuitionistic fuzzy soft sets

Definition 1 [15]. Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and $A \subset E$. A pair (F, A) is called a soft set iff F is a mapping of A into

$\mathcal{P}(X)$.
The following results are due to [3].

Definition 2 [3]. Let $U = \{x_1, x_2, ..., x_n\}$ be the universal sets of elements and let $E = \{e_1, e_1, ..., e_m\}$ be the universal set of parameters. The pair (U, E) will be called a soft universe. Let $F : E \to (I \times I)^U \times I^U$ where $(I \times I)^U$ is the collection of all intuitionistic fuzzy subsets of U and I^U is the collection of all fuzzy subsets of U. Let p be a mapping such that $p : E \to I^U$ and let $F_p : E \to (I \times I)^U \times I^U$ be a function defined as follows:
$F_p(e) = (F(e)(x), p(e)(x))$, where $F(e)(x) = (\mu_e(x), \nu_e(x)) \, \forall x \epsilon U$.
Then F_p is called a possibility intuitionistic fuzzy soft set (PIFSS in short) over the soft universe (U, E). For each parameter e_i, $F_p(e_i)$ can be represented as:
$F_p(e_i) = \left\{ \left(\frac{x_1}{F(e_i)(x_1)}, p(e_i)(x_1) \right), ..., \left(\frac{x_n}{F(e_i)(x_n)}, p(e_i)(x_n) \right) \right\}$

Definition 3 [3]. Let F_p and G_q be two PIFSS over (U, E). Then the following operations were defined over PIFSS as follows:
Containment: F_p is said to be a possibility intuitionistic fuzzy soft subset (PIFS subset) of G_q and one writes $F_p \subseteq G_q$ if
(i) $p(e)$ is a fuzzy subset of $q(e)$, for all $e\epsilon E$,
$(ii) F(e)$ is an intuitionistic fuzzy subset of $G(e)$, for all $e\epsilon E$.
Equality: F_p and G_q are said to be equal and one writes $F_p = G_q$ if F_p is a PIFS subset of G_q and G_q is a PIFS subset of F_p
Union: $F_p \tilde{\cup} G_q = H_r$, $H_r : E \to (I \times I)^U \times I^U$ is defined by $H_r(e) = (H(e)(x), r(e)(x))$, $\forall e\epsilon E$ such that $H(e) = \cup_{Atan}(F(e), G(e))$ and $r(e) = s(p(e), q(e))$, where $\cup_{Atan}$ is Atanassov union and s is a triangular conorm.
Intersection: $F_p \tilde{\cap} G_q = H_r$, $H_r : E \to (I \times I)^U \times I^U$ is defined by $H_r(e) = (H(e)(x), r(e)(x))$, $\forall e\epsilon E$ such that $H(e) = \cap_{Atan}(F(e), G(e))$ and $r(e) = t(p(e), q(e))$, where $\cap_{Atan}$ is Atanassov intersection and t is a triangular norm.

Definition 4 [3]. A PIFSS is said to be a possibility absolute intuitionistic fuzzy soft set, denoted by A_1, if $A_1 : E \to (I \times I)^U \times I^U$ is such that $A_1(e) = (F(e)(x), P(e)(x))$, $\forall e\epsilon E$ where $F(e) = (1, 0)$ and $P(e) = 1, \forall e\epsilon E$.

Definition 5 [3]. A PIFSS is said to be a possibility null intuitionistic fuzzy soft set, denoted by ϕ_0, if $\phi_0 : E \to (I \times I)^U \times I^U$ is such that $\phi_0 = (F(e)(x), p(e)(x))$, $\forall e\epsilon E$ where $F(e) = (0, 1)$ and $p(e) = 0, \forall e\epsilon E$.

2.2 An outline on quadripartitioned single valued neutrosophic sets

Definition 6 [5]. Let X be a non-empty set. A quadripartitioned neutrosophic set (QSVNS) A, over X characterizes each element x in X by a truth-membership function T_A, a contradiction-membership function C_A, an ignorance-membership function U_A and a falsity membership function F_A such that for each $x \,\epsilon\, X, T_A, C_A, U_A, F_A \,\epsilon\, [0, 1]$
When X is discrete, A is represented as,
$A = \sum_{i=1}^{n} \langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i) \rangle / x_i, \, x_i \epsilon X$.
However, when the universe of discourse is continuous, A is represented as,
$A = \langle T_A(x), C_A(x), U_A(x), F_A(x) \rangle / x, x\epsilon X$

Definition 7 [5]. A QSVNS is said to be an absolute QSVNS, denoted by $\mathcal{A}$, iff its membership values are respectively defined as $T_{\mathcal{A}}(x) = 1$, $C_{\mathcal{A}}(x) = 1$, $U_{\mathcal{A}}(x) = 0$ and $F_{\mathcal{A}}(x) = 0, \forall x\epsilon X$.

Definition 8 [5]. A QSVNS is said to be a null QSVNS, denoted by Θ, iff its membership values are respectively defined as $T_\Theta(x) = 0$, $C_\Theta(x) = 0$, $U_\Theta(x) = 1$ and $F_\Theta(x) = 1, \forall x\epsilon X$

Definition 9 [5]. Let A and B be two QSVNS over X. Then the following operations can be defined:
Containment: $A \subseteq B$ iff $T_A(x) \leq T_B(x)$, $C_A(x) \leq C_A(x)$, $U_A(x) \geq U_A(x)$ and $F_A(x) \geq F_A(x)$, $\forall x\epsilon X$.
Complement: $A^c = \sum_{i=1}^{n} \langle F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i) \rangle / x_i, x_i \,\epsilon\, X$
i.e. $T_{A^c}(x_i) = F_A(x_i)$, $C_{A^c}(x_i) = U_A(x_i)$, $U_{A^c}(x_i) = C_A(x_i)$ and $F_{A^c}(x_i) = T_A(x_i)$, $x_i \,\epsilon\, X$
Union: $A \cup B = \sum_{i=1}^{n} < (T_A(x_i) \vee T_B(x_i)), (C_A(x_i) \vee C_B(x_i)), (U_A(x_i) \wedge U_B(x_i)), (F_A(x) \wedge F_B(x)) > / x_i, x_i \,\epsilon\, X$
Intersection: $A \cap B = \sum_{i=1}^{n} < (T_A(x_i) \wedge T_B(x_i)), (C_A(x_i) \wedge C_B(x_i)), (U_A(x_i) \vee U_B(x_i)), (F_A(x_i) \vee F_B(x_i)) > / x_i, x_i \,\epsilon\, X$

Proposition 1[5]. Quadripartitioned single valued neutrosophic sets satisfy the following properties under the aforementioned set-theoretic operations:

1.(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$
2.(i) $A \cup (B \cup C) = (A \cup B) \cup C$
(ii) $A \cap (B \cap C) = (A \cap B) \cap C$
3.(i) $A \cup (A \cap B) = A$
(ii) $A \cap (A \cup B) = A$
4.(i) $(A^c)^c = A$
(ii) $\mathcal{A}^c = \Theta$
(iii) $\Theta^c = \mathcal{A}$
(iv) De-Morgan's laws hold viz. $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B$
5.(i) $A \cup \mathcal{A} = \mathcal{A}$
(ii) $A \cap \mathcal{A} = A$
(iii) $A \cup \Theta = A$
(iv) $A \cap \Theta = \Theta$

3 Interval-valued possibility quadripartitioned single valued neutrosophic soft sets and some of their properties

Definition 10. Let X be an initial crisp universe and E be a set of parameters. Let $I=[0,1]$, $QSVNS(X)$ represents the collection of all quadripartitioned single valued neutrosophic sets over X , $Int([0,1])$ denotes the set of all closed subintervals of $[0,1]$ and $(Int([0,1]))^X$ denotes the collection of interval valued fuzzy subsets over X . An interval-valued possibility quadripartitioned single valued neutrosophic soft set (IPQSVNSS, in short) is a mapping of the form $F_\rho: E \to QSVNS(X)\times(Int([0,1]))^X$ and is defined as $F_\rho(e)=(F_e,\rho_e)$, $e\epsilon E$, where, for each $x\epsilon X$, $F_e(x)$ is the quadruple which represents the truth membership, the contradiction-membership, the ignorance-membership and the falsity membership of each element x of the universe of discourse X viz. $F_e(x)=\langle t^e_F(x), c^e_F(x), u^e_F(x), f^e_F(x)\rangle$,$\forall x\epsilon X$ and $\rho_e(x)=[\rho^-_e(x), \rho^+_e(x)]\epsilon Int([0,1])$. If $X=\{x_1,x_2,...,x_n\}$ and $E=\{e_1,e_2,...,e_m\}$, an interval-valued possibility quadripartitioned single valued neutrosophic soft set over the soft universe (X,E) is represented as, $F_\rho(e_i)=\{\left(\frac{x_1}{F_{e_i}(x_1)}, \rho_{e_i}(x_1)\right), \left(\frac{x_2}{F_{e_i}(x_2)}, \rho_{e_i}(x_2)\right), ...,$ $\left(\frac{x_n}{F_{e_i}(x_n)}, \rho_{e_i}(x_n)\right)\}$ viz.

$$F_\rho(e_i)=\{\left(\frac{x_1}{\langle t^{e_i}_F(x_1), c^{e_i}_F(x_1), u^{e_i}_F(x_1), f^{e_i}_F(x_1)\rangle}, [\rho^-_{e_i}(x_1), \rho^+_{e_i}(x_1)]\right),$$
$$..., \left(\frac{x_n}{\langle t^{e_i}_F(x_n), c^{e_i}_F(x_n), u^{e_i}_F(x_n), f^{e_i}_F(x_n)\rangle}, [\rho^-_{e_i}(x_n), \rho^+_{e_i}(x_n)]\right)\}, e_i\epsilon E,$$
$i=1,2,...,m$.

Example 1. Let $X=\{x_1,x_2,x_3\}$ and $E=\{e_1,e_2\}$. Define an IPQSVNSS over the soft universe (X,E), $F_\rho: E\to QSVNS(X)\times(Int([0,1]))^X$ as,

$F_\rho(e_1)=\{\left(\frac{x_1}{\langle 0.3,0.1,0.4,0.5\rangle}, [0.5,0.6]\right),$
$\left(\frac{x_2}{\langle 0.6,0.2,0.1,0.01\rangle}, [0.25,0.3]\right), \left(\frac{x_3}{\langle 0.7,0.3,0.4,0.6\rangle}, [0.6,0.7]\right)\}$
$F_\rho(e_2)=\{\left(\frac{x_1}{\langle 0.7,0.3,0.5,0.2\rangle}, [0.1,0.2]\right),$
$\left(\frac{x_2}{\langle 0.1,0.2,0.6,0.7\rangle}, [0.45,0.6]\right), \left(\frac{x_3}{\langle 0.5,0.5,0.3,0.2\rangle}, [0.3,0.4]\right)\}$

Another IPQSVNSS G_μ can be defined over (X,E) as

$G_\mu(e_1)=\{\left(\frac{x_1}{\langle 0.8,0.6,0.3,0.4\rangle}, [0.8,0.85]\right),$
$\left(\frac{x_2}{\langle 0.2,0.1,0.1,0.6\rangle}, [0.4,0.5]\right), \left(\frac{x_3}{\langle 0.5,0.5,0.3,0.4\rangle}, [0.4,0.6]\right)\}$
$G_\mu(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.6,0.3,0.7\rangle}, [0.6,0.75]\right),$
$\left(\frac{x_2}{\langle 0.4,0.2,0.2,0.7\rangle}, [0.8,0.9]\right), \left(\frac{x_3}{\langle 0.9,0.7,0.1,0.6\rangle}, [0.35,0.5]\right)\}$

Definition 11. The absolute IPQSVNSS over (X,E) is denoted by $\tilde{A}_{\bar{1}}$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $\tilde{A}_e(x)=\langle 1,1,0,0\rangle$ and $\bar{1}_e(x)=[1,1]$

Definition 12. The null IPQSVNSS over (X,E) is denoted by $\tilde{\theta}_{\bar{0}}$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $\tilde{\theta}_e(x)=\langle 0,0,1,1\rangle$ and $\bar{0}_e(x)=[0,0]$

3.1 Operations over IPQSVNSS

Definition 13. Let F_ρ and G_μ be two IPQSVNSS over the common soft universe (X,E). Some elementary set-theoretic operations on IPQSVNSS are defined as,

(i) Union: $F_\rho\tilde{\cup}G_\mu=H_\eta$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $H_e(x)=\langle t^e_F(x)\vee t^e_G(x), c^e_F(x)\vee c^e_G(x), u^e_F(x)\wedge u^e_G(x), f^e_F(x)\wedge f^e_G(x)\rangle$ and $\eta_e(x)=[sup(\rho^-_e(x), \mu^-_e(x)), sup(\rho^+_e(x), \mu^+_e(x))]$.

(ii) Intersection: $F_\rho\tilde{\cap}G_\mu=H_\eta$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $H_e(x)=\langle t^e_F(x)\wedge t^e_G(x), c^e_F(x)\wedge c^e_G(x), u^e_F(x)\vee u^e_G(x), f^e_F(x)\vee f^e_G(x)\rangle$ and $\eta_e(x)=[inf(\rho^-_e(x), \mu^-_e(x)), inf(\rho^+_e(x), \mu^+_e(x))]$.

(iii) Complement: $(F_\rho)^c=F^c_\rho$ such that for each $e\epsilon E$ and $\forall x\epsilon X$, $F^c_e(x)=\langle f^e_F(x), u^e_F(x), c^e_F(x), t^e_F(x)\rangle$ and $\rho^c_e(x)=[1-\rho^+_e(x), 1-\rho^-_e(x)]$

(iv) Containment: $F_\rho\tilde{\subseteq}G_\mu$ if for each $e\epsilon E$ and $\forall x\epsilon X$, $t^e_F(x)\le t^e_G(x), c^e_F(x)\le c^e_G(x), u^e_F(x)\ge u^e_G(x), f^e_F(x)\ge f^e_G(x)$ and $\rho^-_e(x)\le\mu^-_e(x), \rho^+_e(x)\le\mu^+_e(x)$.

Example 2. Consider the IPQSNSS F_ρ and G_μ over the same soft universe (X,E) defined in example 1. Then, F^c_ρ is obtained as,

$F^c_\rho(e_1)=\{\left(\frac{x_1}{\langle 0.5,0.4,0.1,0.3\rangle}, [0.4,0.5]\right),$
$\left(\frac{x_2}{\langle 0.01,0.1,0.2,0.6\rangle}, [0.7,0.75]\right), \left(\frac{x_3}{\langle 0.6,0.4,0.3,0.7\rangle}, [0.3,0.4]\right)\}$
$F^c_\rho(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.5,0.3,0.7\rangle}, [0.8,0.9]\right),$
$\left(\frac{x_2}{\langle 0.7,0.6,0.2,0.1\rangle}, [0.4,0.55]\right), \left(\frac{x_3}{\langle 0.2,0.3,0.5,0.5\rangle}, [0.6,0.7]\right)\}$

$H_\eta=F_\rho\tilde{\cup}G_\mu$ is obtained as,

$H_\eta(e_1)=\{\left(\frac{x_1}{\langle 0.8,0.6,0.3,0.4\rangle}, [0.8,0.85]\right),$
$\left(\frac{x_2}{\langle 0.6,0.2,0.1,0.01\rangle}, [0.4,0.5]\right), \left(\frac{x_3}{\langle 0.7,0.5,0.3,0.4\rangle}, [0.6,0.7]\right)\}$
$H_\eta(e_2)=\{\left(\frac{x_1}{\langle 0.7,0.6,0.3,0.2\rangle}, [0.6,0.75]\right),$
$\left(\frac{x_2}{\langle 0.4,0.2,0.2,0.7\rangle}, [0.8,0.9]\right), \left(\frac{x_3}{\langle 0.9,0.7,0.1,0.2\rangle}, [0.35,0.5]\right)\}$

Also, the intersection $K_\delta=F_\rho\tilde{\cap}G_\mu$ is defined as,

$K_\delta(e_1)=\{\left(\frac{x_1}{\langle 0.3,0.1,0.4,0.5\rangle}, [0.5,0.6]\right),$
$\left(\frac{x_2}{\langle 0.2,0.1,0.1,0.6\rangle}, [0.25,0.3]\right), \left(\frac{x_3}{\langle 0.5,0.3,0.4,0.6\rangle}, [0.4,0.6]\right)\}$
$K_\delta(e_2)=\{\left(\frac{x_1}{\langle 0.2,0.3,0.5,0.7\rangle}, [0.1,0.2]\right),$
$\left(\frac{x_2}{\langle 0.1,0.2,0.6,0.7\rangle}, [0.45,0.6]\right), \left(\frac{x_3}{\langle 0.5,0.5,0.3,0.6\rangle}, [0.3,0.4]\right)\}$

Proposition 2. For any $F_\rho, G_\mu, H_\eta\epsilon IPQSVNSS(X,E)$, the following results hold:

1. (i) $F_\rho\tilde{\cup}G_\mu=G_\mu\tilde{\cup}F_\rho$
(ii) $F_\rho\tilde{\cap}G_\mu=G_\mu\tilde{\cap}F_\rho$
2. (i) $F_\rho\tilde{\cup}(G_\mu\tilde{\cup}H_\eta)=(F_\rho\tilde{\cup}G_\mu)\tilde{\cup}H_\eta$
(ii) $F_\rho\tilde{\cap}(G_\mu\tilde{\cap}H_\eta)=(F_\rho\tilde{\cap}G_\mu)\tilde{\cap}H_\eta$

3. (i) $F_\rho \tilde{\cup} \tilde{\theta}_{\bar{0}} = F_\rho$
(ii) $F_\rho \tilde{\cap} \tilde{\theta}_{\bar{0}} = \tilde{\theta}_{\bar{0}}$
(iii) $F_\rho \tilde{\cup} \tilde{A}_{\bar{1}} = \tilde{A}_{\bar{1}}$
(iv) $F_\rho \tilde{\cap} \tilde{A}_{\bar{1}} = F_\rho$
4. (i) $\left(F_\rho^c\right)^c = F_\rho$
(ii) $\tilde{A}_{\bar{1}}^c = \tilde{\theta}_{\bar{0}}$
(iii) $\left(\tilde{\theta}_{\bar{0}}\right)^c = \tilde{A}_{\bar{1}}$
5. (i) $\left(F_\rho \tilde{\cup} G_\mu\right)^c = \left(F_\rho\right)^c \tilde{\cap} \left(G_\mu\right)^c$
(ii) $\left(F_\rho \tilde{\cap} G_\mu\right)^c = \left(F_\rho\right)^c \tilde{\cup} \left(G_\mu\right)^c$

Proofs are straight-forward.

4 Some uncertainty-based measures on IPQSVNSS

4.1 Entropy measure

Definition 14. Let $IPQSVNSS(X,E)$ denotes the set of all IPQSVNSS over the soft universe (X,E). A mapping $\varepsilon : IPQSVNSS(X,E) \to [0,1]$ is said to be a measure of entropy if it satisfies the following properties:
$(e1)$ $\varepsilon\left(F_\rho^c\right) = \varepsilon\left(F_\rho\right)$
$(e2)\varepsilon\left(F_\rho\right) \leq \varepsilon\left(G_\mu\right)$ whenever $F_\rho \tilde{\subseteq} G_\mu$with $f_F^e(x) \geq f_G^e(x) \geq t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x) \geq c_G^e(x) \geq c_F^e(x)$ and $\rho_e^-(x) + \rho_e^+(x) \leq 1$.
$(e3)$ $\varepsilon\left(F_\rho\right) = 1$ iff $t_F^e(x) = f_F^e(x)$, $c_F^e(x) = u_F^e(x)$ and $\rho_e^-(x) + \rho_e^+(x) = 1$, $\forall x \epsilon X$ and $\forall e \epsilon E$.

Theorem 1. *The mapping* $e : IPQSVNSS(X,E) \to [0,1]$ *defined as,* $\varepsilon\left(F_\rho\right) = 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$ *is an entropy measure for IPQSVNSS.*

Proof:

(i) $\varepsilon\left(F_\rho^c\right) = 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|f_F^e(x) - t_F^e(x)|.|u_F^e(x) - c_F^e(x)|.|1 - \{(1-\rho_e^-(x)) + (1-\rho_e^+(x))\}|$
$= 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = \varepsilon\left(F_\rho\right)$.

(ii) Suppose that $F_\rho \tilde{\subseteq} G_\mu$ and $f_G^e(x) \geq t_G^e(x)$, $u_G^e(x) \geq c_G^e(x)$, $\rho_e^-(x) + \rho_e^+(x) \leq 1$. Automatically, $\mu_e^-(x) + \mu_e^+(x) \leq 1$. Thus, $f_F^e(x) \geq f_G^e(x), t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x), c_G^e(x) \geq c_F^e(x), \mu_e^-(x) \geq \rho_e^-(x)$,
$\mu_e^+(x) \geq \rho_e^+(x)$, and $f_G^e(x) \geq t_G^e(x)$, $u_G^e(x) \geq c_G^e(x)$, $\rho_e^-(x) + \rho_e^+(x) \leq 1$.
$\Rightarrow f_F^e(x) \geq f_G^e(x) \geq t_G^e(x) \geq t_F^e(x)$, $u_F^e(x) \geq u_G^e(x) \geq c_G^e(x) \geq c_F^e(x)$, $\mu_e^-(x) \geq \rho_e^-(x), \mu_e^+(x) \geq \rho_e^+(x)$ and $\rho_e^-(x) + \rho_e^+(x) \leq 1, \mu_e^-(x) + \mu_e^+(x) \leq 1$.
From the above relations it follows that $t_G^e(x) - f_G^e(x) \geq t_F^e(x) - f_F^e(x)$ but $t_G^e(x) - f_G^e(x) \leq 0, t_F^e(x) - f_F^e(x) \leq 0$
$\Rightarrow |t_G^e(x) - f_G^e(x)| \leq |t_F^e(x) - f_F^e(x)|$. Similarly, $|c_G^e(x) - u_G^e(x)| \leq |c_F^e(x) - u_F^e(x)|$ and $|1-\{\mu_e^+(x)+\mu_e^-(x)\}| \leq |1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$, $\forall x\epsilon X$, $\forall e \epsilon E$. Then,
$|t_G^e(x) - f_G^e(x)|.|c_G^e(x) - u_G^e(x)|.|1 - \{\mu_e^+(x) + \mu_e^-(x)\}|$
$\leq |t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$
$\Rightarrow 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}|$
$\leq 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_G^e(x) - f_G^e(x)|.|c_G^e(x) - u_G^e(x)|.|1 - \{\mu_e^+(x) + \mu_e^-(x)\}|$
$\Rightarrow \varepsilon\left(F_\rho\right) \leq \varepsilon\left(G_\mu\right)$

(iii) $\varepsilon\left(F_\rho\right) = 1$
$\Leftrightarrow 1 - \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 1$
$\Leftrightarrow \frac{1}{||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}|t_F^e(x) - f_F^e(x)|.|c_F^e(x) - u_F^e(x)|.|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 0$
$\Leftrightarrow |t_F^e(x) - f_F^e(x)| = 0$, $|c_F^e(x) - u_F^e(x)| = 0$, $|1 - \{\rho_e^+(x) + \rho_e^-(x)\}| = 0$, for each $x\epsilon X$ and each $e\epsilon E$.
$\Leftrightarrow t_F^e(x) = f_F^e(x)$, $c_G^e(x) = u_G^e(x)$, $\rho_e^+(x) + \rho_e^-(x) = 1$, for each $x\epsilon X$ and each $e\epsilon E$.

Remark 1. In particular, from *Theorem 1*, it follows that, $\varepsilon\left(\tilde{A}_{\bar{1}}\right) = 0$ and $\varepsilon\left(\tilde{\theta}_{\bar{0}}\right) = 0$.

Proof is straight-forward.

4.1.1 An application of entropy measure in decision making problem

The entropy measure not only provides an all over information about the amount of uncertainty ingrained in a particular structure, it can also be implemented as an efficient tool in decision making processes. Often while dealing with a selection process subject to a predefined set of requisitions, the procedure involves allocation of weights in order to signify the order of preference of the criteria under consideration. In what follows next, the entropy measure corresponding to an IPQSVNSS has been utilized in defining weights corresponding to each of the elements of the parameter set over which the IPQSVNSS has been defined.

The algorithm is defined as follows:

Step 1: Represent the data in hand in the form of an IPQSVNSS, say F_ρ.
Step 2: Calculate the entropy measure $\varepsilon\left(F_\rho\right)$, as defined in Theorem A.
Step 3: For each $\alpha\epsilon E$, assign weights $\omega_F(\alpha)$, given by the formula,
$\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$, where $\kappa_F(\alpha) = 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$.
Step 4: Corresponding to each option $x\epsilon X$, calculate the net score, defined as,
$score(x_i) = \sum_e \omega_F(\alpha).[t_F^\alpha(x_i) + c_F^\alpha(x_i) + \{1 - u_F^\alpha(x_i)\} + \{1 - f_F^\alpha(x_i)\}].\{\frac{\rho_\alpha^+(x_i)+\rho_\alpha^-(x_i)}{2}\}$.

Step 5: Arrange $score(x_i)$ in the decreasing order of values.
Step 6: Select $max_i\{score(x_i)\}$. If $max_i\{score(x_i)\} = score(x_m), x_m \epsilon X$, then x_m is the selected option.

Theorem 2. *Corresponding to each parameter $\alpha \epsilon E$, $\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$ is such that $0 \leq \omega_F(\alpha) \leq 1$.*

Proof:

From the definition of $\kappa_F(\alpha)$ and $\varepsilon(F_\rho)$, it is clear that $\omega_F(\alpha) \geq 0$.
Consider $|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$. It follows that,
$\sum_{\alpha\epsilon E}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}| \geq \sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$, whenever $||X|| \geq 1$.
$\Rightarrow 1 - \frac{1}{||X||.||E||}\sum_{\alpha\epsilon E}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}| \leq 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t_F^\alpha(x) - f_F^\alpha(x)|.|c_F^\alpha(x) - u_F^\alpha(x)|.|1 - \{\rho_\alpha^+(x) + \rho_\alpha^-(x)\}|$
$\Rightarrow \varepsilon(F_\rho) \leq \kappa_F(\alpha)$
$\Rightarrow \omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)} \leq 1$, for each $\alpha \epsilon E$.

Example 3. Suppose a person wishes to buy a phone and the judging parameters he has set are a: appearance, c: cost, b: battery performance, s: storage and l: longevity. Further suppose that he has to choose between 3 available models, say x_1, x_2, x_3 of the desired product. After a survey has been conducted by the buyer both by word of mouth from the current users and the salespersons, the resultant information is represented in the form of an IPQSVNSS, say F_ρ as follows, where it is assumed that corresponding to an available option, a higher degree of belongingness signifies a higher degree of agreement with the concerned parameter:

$F_\rho(a) = \{\left(\frac{x_1}{\langle 0.4,0.3,0.1,0.5\rangle}, [0.5, 0.6]\right),$
$\left(\frac{x_2}{\langle 0.8,0.1,0.0,0.01\rangle}, [0.6, 0.7]\right), \left(\frac{x_3}{\langle 0.6,0.3,0.2,0.5\rangle}, [0.45, 0.5]\right)\}$
$F_\rho(c) = \{\left(\frac{x_1}{\langle 0.8,0.1,0.1,0.2\rangle}, [0.7, 0.75]\right),$
$\left(\frac{x_2}{\langle 0.5,0.01,0.1,0.6\rangle}, [0.4, 0.55]\right), \left(\frac{x_3}{\langle 0.7,0.2,0.1,0.1\rangle}, [0.6, 0.65]\right)\}$
$F_\rho(b) = \{\left(\frac{x_1}{\langle 0.65,0.3,0.1,0.2\rangle}, [0.6, 0.65]\right),$
$\left(\frac{x_2}{\langle 0.8,0.2,0.1,0.0\rangle}, [0.75, 0.8]\right), \left(\frac{x_3}{\langle 0.4,0.5,0.3,0.6\rangle}, [0.7, 0.8]\right)\}$
$F_\rho(s) = \{\left(\frac{x_1}{\langle 0.5,0.4,0.3,0.6\rangle}, [0.7, 0.8]\right),$
$\left(\frac{x_2}{\langle 0.85,0.1,0.0,0.01\rangle}, [0.8, 0.85]\right), \left(\frac{x_3}{\langle 0.8,0.2,0.1,0.02\rangle}, [0.85, 0.9]\right)\}$
$F_\rho(l) = \{\left(\frac{x_1}{\langle 0.6,0.3,0.2,0.5\rangle}, [0.45, 0.55]\right),$
$\left(\frac{x_2}{\langle 0.75,0.3,0.3,0.2\rangle}, [0.67, 0.75]\right), \left(\frac{x_3}{\langle 0.75,0.3,0.2,0.2\rangle}, [0.7, 0.75]\right)\}$

Following steps 2-6, we have the following results:

(2) $\varepsilon(F_\rho) = 0.982$
(3) $\omega_F(a) = 0.984, \omega_F(c) = 0.983, \omega_F(b) = 0.988, \omega_F(s) = 0.99, \omega_F(l) = 0.984$
(4) $score(x_1) = 7.193, score(x_2) = 9.097, score(x_3) = 8.554$
(5) $score(x_2) > score(x_3) > score(x_1)$
(6) x_2 is the chosen model.

4.2 Inclusion measure

Definition 15. A mapping $I : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to [0,1]$ is said to be an inclusion measure for IPQSVNSS over the soft universe (X,E) if it satisfies the following properties:
$(I1)$ $I\left(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}\right) = 0$
$(I2)$ $I(F_\rho, G_\mu) = 1 \Leftrightarrow F_\rho \tilde{\subseteq} G_\mu$
$(I3)$ if $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$ then $I(H_\eta, F_\rho) \leq I(G_\mu, F_\rho)$ and $I(H_\eta, F_\rho) \leq I(H_\eta, G_\mu)$

Theorem 3. *The mapping $I : IPQSVNSS(X,E) \to [0,1]$ defined as,*
$I(F_\rho, G_\mu) = 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}[|t_F^e(x) - min\{t_F^e(x), t_G^e(x)\}| + |c_F^e(x) - min\{c_F^e(x), c_G^e(x)\}| + |max\{u_F^e(x), u_G^e(x)\} - u_F^e(x)| + |max\{f_F^e(x), f_G^e(x)\} - f_F^e(x)| + |\rho_e^-(x) - min\{\rho_e^-(x), \mu_e^-(x)\}| + |\rho_e^+(x) - min\{\rho_e^+(x), \mu_e^+(x)\}|]$, *is an inclusion measure for IPQSVNSS.*

Proof:

(i) Clearly, according to the definition of the proposed measure, $I\left(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}\right) = 0$

(ii) From the definition of the proposed measure, it follows that,
$I(F_\rho, G_\mu) = 1$,
$\Leftrightarrow \sum_{e\epsilon E}\sum_{x\epsilon X}[|t_F^e(x) - min\{t_F^e(x), t_G^e(x)\}| + |c_F^e(x) - min\{c_F^e(x), c_G^e(x)\}| + |max\{u_F^e(x), u_G^e(x)\} - u_F^e(x)| + |max\{f_F^e(x), f_G^e(x)\} - f_F^e(x)| + |\rho_e^-(x) - min\{\rho_e^-(x), \mu_e^-(x)\}| + |\rho_e^+(x) - min\{\rho_e^+(x), \mu_e^+(x)\}|] = 0, \forall x\epsilon X, \forall e\epsilon E$.
$\Leftrightarrow |t_F^e(x) - min\{t_F^e(x), t_G^e(x)\}| = 0$, $|c_F^e(x) - min\{c_F^e(x), c_G^e(x)\}| = 0$, $|max\{u_F^e(x), u_G^e(x)\} - u_F^e(x)| = 0$, $|max\{f_F^e(x), f_G^e(x)\} - f_F^e(x)| = 0$, $|\rho_e^-(x) - min\{\rho_e^-(x), \mu_e^-(x)\}| = 0$ and $|\rho_e^+(x) - min\{\rho_e^+(x), \mu_e^+(x)\}| = 0, \forall x\epsilon X, \forall e\epsilon E$.
Now, $|t_F^e(x) - min\{t_F^e(x), t_G^e(x)\}| = 0 \Leftrightarrow t_F^e(x) \leq t_G^e(x)$.
Similarly, it can be shown that, $c_F^e(x) \leq c_G^e(x), u_F^e(x) \geq u_G^e(x), f_F^e(x) \geq f_G^e(x), \rho_e^-(x) \leq \mu_e^-(x)$ and $\rho_e^+(x) \leq \mu_e^+(x), \forall x\epsilon X, \forall e\epsilon E$ which proves $F_\rho \tilde{\subseteq} G_\mu$.

(iii) Suppose, $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. Thus we have, $t_F^e(x) \leq t_G^e(x) \leq t_H^e(x)$, $c_F^e(x) \leq c_G^e(x) \leq c_H^e(x)$, $u_F^e(x) \geq u_G^e(x) \geq u_H^e(x)$, $f_F^e(x) \geq f_G^e(x) \geq f_H^e(x)$, $\rho_e^-(x) \leq \mu_e^-(x) \leq \eta_e^-(x)$ and $\rho_e^+(x) \leq \mu_e^+(x) \leq \eta_e^+(x)$ for all $x\epsilon X$ and $e\epsilon E$.
$\Rightarrow I(H_\eta, F_\rho) \leq I(G_\mu, F_\rho)$.
In an exactly analogous manner, it can be shown that, $I(H_\eta, F_\rho) \leq I(H_\eta, G_\mu)$. This completes the proof.

Example 4. Consider IPQSVNSS F_ρ, G_μ in *Example 1*, then $I(F_\rho, G_\mu) = 0.493$.

4.3 Distance measure

Definition 16. A mapping $d : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ is said to be a distance measure between IPQSVNSS if for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X,E)$ it satisfies the following properties:
$(d1)$ $d(F_\rho, G_\mu) = d(G_\mu, F_\rho)$
$(d2)$ $d(F_\rho, G_\mu) \geq 0$ and $d(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$
$(d3)$ $d(F_\rho, H_\eta) \leq d(F_\rho, G_\mu) + d(G_\mu, H_\eta)$
In addition to the above conditions, if the mapping d satisfies the condition
$(d4)$ $d(F_\rho, G_\mu) \leq 1, \forall F_\rho, G_\mu \epsilon IPQSVNSS(X,E)$
it is called a Normalized distance measure for IPQSVNSS.

Theorem 4. *The mapping* $d_h : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ *defined as,*
$d_h(F_\rho, G_\mu) = \sum_{e\epsilon E}\sum_{x\epsilon X}(|t_F^e(x) - t_G^e(x)| + |c_F^e(x) - c_G^e(x)| + |u_F^e(x) - u_G^e(x)| + |f_F^e(x) - f_G^e(x)| + |\rho_e^-(x) - \mu_e^-(x)| + |\rho_e^+(x) - \mu_e^+(x)|)$ *is a distance measure for IPQSVNSS. It is known as the Hamming Distance*.

Proofs are straight-forward.

Definition 17. The corresponding Normalized Hamming distance for IPQSVNSS is defined as $d_h^N(F_\rho, G_\mu) = \frac{1}{6||X||.||E||} d_h(F_\rho, G_\mu)$, where $||.||$ denotes the cardinality of a set.

Theorem 5. *The mapping* $d_E : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ *defined as,*
$d_E(F_\rho, G_\mu) = \sum_{e\epsilon E}\sum_{x\epsilon X}\{(t_F^e(x) - t_G^e(x))^2 + (c_F^e(x) - c_G^e(x))^2 + (u_F^e(x) - u_G^e(x))^2 + (f_F^e(x) - f_G^e(x))^2 + (\rho_e^-(x) - \mu_e^-(x))^2 + (\rho_e^+(x) - \mu_e^+(x))^2\}^{\frac{1}{2}}$ *is a distance measure for IPQSVNSS. It is known as the Euclidean Distance*.

Proofs are straight-forward.

Definition 18. The corresponding Normalized Hamming distance for IPQSVNSS is defined as $d_E^N(F_\rho, G_\mu) = \frac{1}{6||X||.||E||} d_E(F_\rho, G_\mu)$.

Proposition 3. $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$ iff
(i) $d_h(F_\rho, H_\eta) = d_h(F_\rho, G_\mu) + d_h(G_\mu, H_\eta)$
(ii) $d_h^N(F_\rho, H_\eta) = d_h^N(F_\rho, G_\mu) + d_h^N(G_\mu, H_\eta)$

Proofs are straight-forward.

Example 5. Consider the IPQSVNSS given in *Example 1*. The various distance measures between the sets are obtained as, $d_h(F_\rho, G_\mu) = 5.29$, $d_h^N(F_\rho, G_\mu) = 0.882$,$d_E(F_\rho, G_\mu) = 4.387$, $d_E^N(F_\rho, G_\mu) = 0.731$

4.4 Similarity measure

Definition 19. A mapping $s : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ is said to be a quasi-similarity measure between IPQSVNSS if for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X,E)$ it satisfies the following properties:
$(s1)$ $s(F_\rho, G_\mu) = s(G_\mu, F_\rho)$
$(s2)$ $0 \leq s(F_\rho, G_\mu) \leq 1$ and $s(F_\rho, G_\mu) = 1 \Leftrightarrow F_\rho = G_\mu$
In addition, if it satisfies
$(s3)$ if $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$then $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu) \wedge s(G_\mu, H_\eta)$
then it is known as a similarity measure between IPQSVNSS.

Various similarity measures for quadripartitioned single valued neutrosophic sets were proposed in [5]. Undertaking a similar line of approach, as in our previous work [5] we propose a similarity measure for IPQSVNSS as follows:

Definition 20. Consider $F_\rho, G_\mu \epsilon IPQSVNSS(X,E)$. Define functions $\tau_{i,e}^{F,G} : X \to [0,1]$, $i = 1,2,..,5$ such that for each $x\epsilon X$, $e\epsilon E$
$\tau_{1,e}^{F,G}(x) = |t_G^e(x) - t_F^e(x)|$
$\tau_{2,e}^{F,G}(x) = |f_F^e(x) - f_G^e(x)|$
$\tau_{3,e}^{F,G}(x) = |c_G^e(x) - c_F^e(x)|$
$\tau_{4,e}^{F,G}(x) = |u_F^e(x) - u_G^e(x)|$
$\tau_{5,e}^{F,G}(x) = |\rho_e^-(x) - \mu_e^-(x)|$
$\tau_{6,e}^{F,G}(x) = |\rho_e^+(x) - \mu_e^+(x)|$
Finally, define a mapping $s : IPQSVNSS(X,E) \times IPQSVNSS(X,E) \to R^+$ as, $s(F_\rho, G_\mu) = 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6}\tau_{i,e}^{F,G}(x)$

Theorem 6. *The mapping* s $(F_\rho,$ $G_\mu)$ *defined above is a similarity measure*.

Proof:

(i) It is easy to prove that $s(F_\rho, G_\mu) = s(G_\mu, F_\rho)$.

(ii) We have, $t_F^e(x), c_F^e(x), u_F^e(x), f_F^e(x)\epsilon[0,1]$ and $\rho_e(x), \mu_e(x)\epsilon Int([0,1])$ for each $x\epsilon X, e\epsilon E$. Thus, $\tau_{1,e}^{F,G}(x)$ attains its maximum value if either one of $t_F^e(x)$ or $t_G^e(x)$ is equal to 1 while the other is 0 and in that case the maximum value is 1. Similarly, it attains a minimum value 0 if $t_F^e(x) = t_G^e(x)$. So, it follows that $0 \leq \tau_{1,e}^{F,G}(x) \leq 1$, for each $x\epsilon X$. Similarly it can be shown that $\tau_{i,e}^{F,G}(x)$, $i = 2,...,6$ lies within $[0,1]$ for each $x\epsilon X$. So,
$0 \leq \sum_{i=1}^{6}\tau_{i,e}^{F,G}(x) \leq 6$
$\Rightarrow 0 \leq \sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n}\tau_{i,e}^{F,G}(x) \leq 6||X||.||E||$
which implies $0 \leq s(F_\rho, G_\mu) \leq 1$.
Now $s(F_\rho, G_\mu) = 1$ iff $\sum_{i=1}^{n}\tau_{i,e}(x) = 0$ for each $x\epsilon X, e\epsilon E$ $\Leftrightarrow$ $t_F^e(x) = t_G^e(x)$, $c_F^e(x) = c_G^e(x)$, $u_F^e(x) = u_G^e(x)$,

$f^e_F(x) = f^e_G(x)$ and $\rho^-_e(x) = \mu^-_e(x)$, $\rho^+_e(x) = \mu^+_e(x)$, for all $x\epsilon X, e\epsilon E$ i.e.. iff F_ρ, G_μ.

(iii) Suppose $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. then, we have, $t^e_F(x) \le t^e_G(x) \le t^e_H(x)$, $c^e_F(x) \le c^e_G(x) \le c^e_H(x)$, $u^e_F(x) \ge u^e_G(x) \ge u^e_H(x)$, $f^e_F(x) \ge f^e_G(x) \ge f^e_H(x)$, $\rho^-_e(x) \le \mu^-_e(x) \le \eta^-_e(x)$ and $\rho^+_e(x) \le \mu^+_e(x) \le \eta^+_e(x)$ for all $x\epsilon X$ and $e\epsilon E$. Consider $\tau^{F,G}_{1,e}(x)$ and $\tau^{F,G}_{2,e}(x)$. Since $t^e_F(x) \le t^e_G(x)$ holds, it follows that, $|t^e_G(x) - t^e_F(x)| \le |t^e_H(x) - t^e_F(x)| \Rightarrow \tau^{F,G}_{1,e}(x) \le \tau^{F,H}_{1,e}(x)$. Similarly it can be shown that $\tau^{F,G}_{i,e}(x) \le \tau^{F,H}_{i,e}(x)$, for $i = 3, 5, 6$ and all $x\epsilon X$. Next, consider $\tau^{F,G}_{2,e}(x)$.
Since, $f^e_F(x) \ge f^e_G(x) \ge f^e_H(x)$, it follows that $f^e_F(x) - f^e_G(x) \le f^e_F(x) - f^e_H(x)$ where $f^e_F(x) - f^e_G(x) \ge 0$, $f^e_F(x) - f^e_H(x) \ge 0$. Thus, $|f^e_F(x) - f^e_G(x)| \le |f^e_F(x) - f^e_H(x)| \Rightarrow \tau^{F,G}_{3,e}(x) \le \tau^{F,H}_{3,e}(x)$.
Also, it can be shown that $\tau^{F,G}_{4,e}(x) \le \tau^{F,H}_{4,e}(x)$ respectively for each $x\epsilon X$.
Thus, we have, $\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n} \tau^{F,G}_{i,e}(x) \le \sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n} \tau^{F,H}_{i,e}(x)$
$\Rightarrow 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n} \tau^{F,H}_{i,e}(x) \le 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{n} \tau^{F,G}_{i,e}(x)$
$\Rightarrow s(F_\rho, H_\eta) \le s(F_\rho, G_\mu)$
In an analogous manner, it can be shown that $s(F_\rho, H_\eta) \le s(G_\mu, H_\eta)$. Thus, we have, $s(F_\rho, H_\eta) \le s(F_\rho, G_\mu) \wedge s(G_\mu, H_\eta)$

Remark 2. $s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 0$.

Proof :

For each $x\epsilon X$ and $e\epsilon E$,
$\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_1(x) = |t^e_{\tilde{\theta}_{\bar{0}}}(x) - t^e_{\tilde{A}_{\bar{1}}}(x)| = 1$, $\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_2(x) = |f^e_{\tilde{A}_{\bar{1}}}(x) - f^e_{\tilde{\theta}_{\bar{0}}}(x)| = 1$
$\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_3(x) = |c^e_{\tilde{\theta}_{\bar{0}}}(x) - c^e_{\tilde{A}_{\bar{1}}}(x)| = 1$, $\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_4(x) = |u^e_{\tilde{A}_{\bar{1}}}(x) - u^e_{\tilde{\theta}_{\bar{0}}}(x)| = 1$
$\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_5(x) = |\rho^-_e(x) - \mu^-_e(x)| = 1$, $\tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_6(x) = |\rho^+_e(x) - \mu^+_e(x)| = 1$
which yields $\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6} \tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_i(x) = 6||X||.||E||$
$\Rightarrow s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 1 - \frac{1}{6||X||.||E||}\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6} \tau^{\tilde{A}_{\bar{1}},\tilde{\theta}_{\bar{0}}}_i(x) = 0$.

Definition 21. Suppose $F_\rho, G_\mu \epsilon IPQSVNSS(X, E)$. Consider functions $\tau^{F,G}_{i,e} : X \to [0, 1]$, $i = 1, 2, .., 5$ as in *Definition 1*. Define a mapping $s_\omega : IPQSVNSS(X, E) \times IPQSVNSS(X, E) \to R^+$ as, $s_\omega(F_\rho, G_\mu) = 1 - \frac{\sum_{e\epsilon E}\sum_{x\epsilon X}\sum_{i=1}^{6} \omega(e)\tau^{F,G}_{i,e}(x)}{6||X||.||E||\sum_{e\epsilon E}\omega(e)}$, where $\omega(e)$ is the weight allocated to the parameter $e\epsilon E$ and $\omega(e)\epsilon[0, 1]$, for each $e\epsilon E$.

Theorem 7. *$s_\omega(F_\rho, G_\mu)$ is a similarity measure.*

Proof is similar to that of *Theorem 6*.

Remark 3. $s_\omega(F_\rho, G_\mu)$ is the weighted similarity measure between any two IPQSVNSS F_ρ and G_μ.

4.4.1 Allocation of entropy-based weights in calculating weighted similarity

It was shown in *Section 4.1.1* how entropy measure could be implemented to allocate specific weights to the elements of the parameter set. In this section, it is shown how the entropy-based weights can be implemented in calculating weighted similarity. Consider an IPQSVNSS F_ρ defined over the soft universe (X, E). Let $\omega_F(e)\epsilon[0, 1]$ be the weight allocated to an element $e\epsilon E$, w.r.t. the IPQSVNSS F_ρ.
Define $\omega_F(\alpha)$ as before, viz.
$\omega_F(\alpha) = \frac{\varepsilon(F_\rho)}{\kappa_F(\alpha)}$, where $\kappa_F(\alpha) = 1 - \frac{1}{||X||.||E||}\sum_{x\epsilon X}|t^\alpha_F(x) - f^\alpha_F(x)|.|c^\alpha_F(x) - u^\alpha_F(x)|.|1 - \{\rho^+_\alpha(x) + \rho^-_\alpha(x)\}|$
Consider any two IPQSVNSS $F_\rho, G_\mu \epsilon IPQSVNSS(X)$. Following Definition C, the weighted similarity measure between these two sets can be defined as
$s_\omega(F_\rho, G_\mu) = 1 - \frac{\sum_{e\epsilon E}\omega(\alpha)\{\sum_{x\epsilon X}\sum_{i=1}^{6}\tau^{F,G}_i(x)\}}{6||X||.||E||\sum_{e\epsilon E}\omega(\alpha)}$, where $\omega(\alpha) = \frac{\omega_F(\alpha)+\omega_G(\alpha)}{2}$, and $\omega_G(\alpha) = \frac{\varepsilon(G_\mu)}{\kappa_G(\alpha)}$ is the weight allocated to the parameter $\alpha\epsilon E$ w.r.t. the IPQSVNSS G_μ.
From previous results clearly, $\omega_F(\alpha), \omega_G(\alpha)\epsilon[0, 1] \Rightarrow \omega(\alpha)\epsilon[0, 1]$.

Example 6. Consider $F_\rho, G_\mu \epsilon IPQSVNSS(X)$ as defined in Example 1. Then $s(F_\rho, G_\mu) = 0.738$. Also, $\omega_F(e_1) = 0.983, \omega_G(e_1) = 0.987, \omega_F(e_2) = 0.993, \omega_G(e_2) = 0.988$, which gives, $\omega(e_1) = 0.985, \omega(e_2) = 0.991$ which finally yields $s_\omega(F_\rho, G_\mu) = 0.869$.

5 Relation between the various uncertainty based measures

Theorem 8. *$s^1_d(F_\rho, G_\mu) = 1 - d^N_h(F_\rho, G_\mu)$ is a similarity measure.*

Proof:

(i) $d^N_h(F_\rho, G_\mu) = d^N_h(G_\mu, F_\rho) \Rightarrow s^1_d(F_\rho, G_\mu) = s^1_d(G_\mu, F_\rho)$
(ii) $0 \le d^N_h(F_\rho, G_\mu) \le 1 \Rightarrow 0 \le s^1_d(F_\rho, G_\mu) \le 1$
Also, $s^1_d(F_\rho, G_\mu) = 1 \Leftrightarrow d^N_h(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$.
(iii) Whenever $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$, $d^N_h(F_\rho, H_\eta) = d^N_h(F_\rho, G_\mu) + d^N_h(G_\mu, H_\eta)$. Thus,
$s^1_d(F_\rho, G_\mu) - s^1_d(F_\rho, H_\eta) = 1 - d^N_h(F_\rho, G_\mu) - 1 + d^N_h(F_\rho, H_\eta) = d^N_h(F_\rho, H_\eta) - d^N_h(F_\rho, G_\mu) = d^N_h(G_\mu, H_\eta) \ge 0$, from property of distance measure.
$\Rightarrow s^1_d(F_\rho, H_\eta) \le s^1_d(F_\rho, G_\mu)$.
Similarly, it can be shown that, $s^1_d(F_\rho, H_\eta) \le s^1_d(G_\mu, H_\eta)$.

Hence, $s_d^1(F_\rho, H_\eta) \leq s_d^1(F_\rho, G_\mu) \wedge s_d^1(G_\mu, H_\eta)$.

Remark 4. For any similarity measure$s(F_\rho, G_\mu)$, $1-s(F_\rho, G_\mu)$ may not be a distance measure.

Theorem 9.$s_d^2(F_\rho, G_\mu) = \frac{1}{1+d_h(F_\rho,G_\mu)}$ *is a similarity measure.*

Proof:

(i) $d_h(F_\rho, G_\mu) = d_h(G_\mu, F_\rho) \Rightarrow s_d^2(F_\rho, G_\mu) = s_d^2(G_\mu, F_\rho)$
(ii) $d_h(F_\rho, G_\mu) \geq 0 \Rightarrow 0 \leq s_d^2(F_\rho, G_\mu) \leq 1$. Also, $s_d^2(F_\rho, G_\mu) = 1 \Leftrightarrow d_h(F_\rho, G_\mu) = 0 \Leftrightarrow F_\rho = G_\mu$.
(iii) $d_h(F_\rho, H_\eta) = d_h(F_\rho, G_\mu) + d_h(G_\mu, H_\eta)$ whenever $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$.
$\Rightarrow d_h(F_\rho, H_\eta) \geq d_h(F_\rho, G_\mu)$ and $d_h(F_\rho, H_\eta) \geq d_h(G_\mu, H_\eta)$.
$\Rightarrow \frac{1}{1+d_h(F_\rho,H_\eta)} \leq \frac{1}{1+d_h(F_\rho,G_\mu)} \Rightarrow s_d^2(G_\mu, F_\rho) \leq s_d^2(F_\rho, G_\mu)$.
Similarly, it can be shown that, $s_d^2(G_\mu, F_\rho) \leq s_d^2(G_\mu, H_\eta)$.

Corollary 1. $s_d^3(F_\rho, G_\mu) = \frac{1}{1+d_h^N(F_\rho,G_\mu)}$ *is a similarity measure*.

Proofs follow in the exactly same way as the previous theorem.

Remark 5. For any similarity measure $s(F_\rho, G_\mu)$,$\frac{1}{s(F_\rho,G_\mu)} - 1$ may not be a distance measure.

Theorem 10 *Consider the similarity measure* $s(F_\rho, G_\mu)$. $s(F_\rho, F_\rho \tilde{\cap} G_\mu)$*is an inclusion measure*.

Proof:

(i) Choose $F_\rho = \tilde{A}_{\bar{1}}$ and $G_\mu = \tilde{\theta}_{\bar{0}}$. Then, $s(F_\rho, F_\rho \tilde{\cap} G_\mu) = s(\tilde{A}_{\bar{1}}, \tilde{\theta}_{\bar{0}}) = 0$, from previous result.
(ii) $s(F_\rho, F_\rho \tilde{\cap} G_\mu) = 1 \Leftrightarrow F_\rho = F_\rho \tilde{\cap} G_\mu \Leftrightarrow F_\rho \tilde{\subseteq} G_\mu$.
(iii) Let $F_\rho \tilde{\subseteq} G_\mu \tilde{\subseteq} H_\eta$. Then, $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu)$ and $s(F_\rho, H_\eta) \leq s(G_\mu, H_\eta)$ hold. Consider $s(F_\rho, H_\eta) \leq s(F_\rho, G_\mu)$. From commutative property of similarity measure, it follows that, $s(H_\eta, F_\rho) \leq s(G_\mu, F_\rho) \Rightarrow s(H_\eta, H_\eta \tilde{\cap} F_\rho) \leq s(G_\mu, G_\mu \tilde{\cap} F_\rho)$. Similarly, $s(H_\eta, H_\eta \tilde{\cap} F_\rho) \leq s(F_\rho, F_\rho \tilde{\cap} G_\mu)$.

Theorem 11.$1 - d_h(F_\rho, F_\rho \tilde{\cap} G_\mu)$ *is an inclusion measure*.

Proof follows from the results of *Theorem 8* and *Theorem 10*.

Theorem 12. $\frac{1}{1+d_h(F_\rho,F_\rho \tilde{\cap} G_\mu)}$ *and* $\frac{1}{1+d_h^N(F_\rho,F_\rho \tilde{\cap} G_\mu)}$ *are inclusion measures*.

Proofs follow from *Theorem 9*,*Corollary 1* and *Theorem 10*.

Theorem 13. *Let* $e : IPQSVNSS(X,E) \to [0,1]$ *be a measure of entropy such that* $\varepsilon(F_\rho) \leq \varepsilon(G_\mu) \Rightarrow F_\rho \tilde{\subseteq} G_\mu$. *Then* $|\varepsilon(F_\rho) - \varepsilon(G_\mu)|$ *is a distance measure*.

Proof:

(i) $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| = |\varepsilon(G_\mu) - \varepsilon(F_\rho)|$
(ii) $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| \geq 0$ and in particular, $|\varepsilon(F_\rho) - \varepsilon(G_\mu)| = 0 \Leftrightarrow \varepsilon(F_\rho) = \varepsilon(G_\mu) \Leftrightarrow \varepsilon(F_\rho) \leq \varepsilon(G_\mu)$ and $\varepsilon(F_\rho) \geq \varepsilon(G_\mu) \Leftrightarrow F_\rho = G_\mu$
(iii) Triangle inequality follows from the fact that, $|\varepsilon(F_\rho) - \varepsilon(H_\eta)| \leq |\varepsilon(F_\rho) - \varepsilon(G_\mu)| + |\varepsilon(G_\mu) - \varepsilon(H_\eta)|$ for any $F_\rho, G_\mu, H_\eta \epsilon IPQSVNSS(X,E)$.

6 Conclusions and Discussions

In this paper, the concept of interval possibility quadripartitioned single valued neutrosophic sets has been proposed. In the present set-theoretic structure an interval valued gradation of possibility viz. the chance of occurrence of an element with respect to a certain criteria is assigned and depending on that possibility of occurrence the degree of belongingness, non-belongingness, contradiction and ignorance are assigned thereafter. Thus, this structure comes as a generalization of the existing structures involving the theory of possibility namely, possibility fuzzy soft sets and possibility intuitionistic fuzzy soft sets. In the present work, the relationship between the various uncertainty based measures have been established. Applications have been shown where the entropy measure has been utilized to assign weights to the elements of the parameter set which were later implemented in a decision making problem and also in calculating a weighted similarity measure. The proposed theory is expected to have wide applications in processes where parameter-based selection is involved.

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