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Research Article

Oscillation Criteria for Delay and Advanced Differential Equations with Nonmonotone Arguments

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We study the oscillatory behavior of differential equations with nonmonotone deviating arguments and nonnegative coefficients. New oscillation criteria, involving lim sup and lim inf, are obtained based on an iterative method. Examples, numerically solved in MATLAB, are given to illustrate the applicability and strength of the obtained conditions over known ones.

1. Introduction

In mathematics, delay differential equations (DDEs) are that type of differential equations where the derivative of the unknown function, at a certain time, is given in terms of the values of the function, at previous times. DDEs are also referred in the literature as time-delay systems, systems with aftereffect or dead-time, hereditary systems, or equations with delay arguments.

Mathematical modelling involving DDEs is widely used for analysis and predictions in various areas of the life sciences, for example, population dynamics, epidemiology, immunology, physiology, neural networks. See, for example, [1–10] and the references cited therein. The time delays add to these models memory effects, taking into account the dependence of the model's present state on its past history [9]. The delay can be related to the duration of certain hidden processes, like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, and so on.

In analogy, advanced differential equations (ADEs) are used in many applied problems where the evolution rate depends not only on the present, but also on the future. While delays in DDEs represent the retrospective memory of the past, advances in ADEs represent the prospective memory of the future, accounting for the influence on the system of potential future actions, which are available, at the present time. For instance, population dynamics, economics problems, or mechanical control engineering are typical fields where such phenomena are thought to occur (see [11, 12] for details).

The earliest delay model in mathematical biology is Hutchinson's equation, in 1948 [6]. Hutchinson modified the classical logistic equation, with a delay term to incorporate hatching and maturation periods into the model and account for oscillations, in the population of Daphnia,

$$y'(t) = ry(t)\left(1 - \frac{y(t-\tau)}{K}\right), \qquad (1)$$

where $y(t)$ denotes the size of the population, in the present time t, $y'(t)$ describes the change of this size, at time t, $y(t-\tau)$ is the size, in some past time $t-\tau$, $\tau > 0$ is the delay, representing the time for new eggs to hatch, and r is the reproduction rate of the population, while K is the carrying capacity, for the population.

Many physiological processes, including the concentration of red blood cells, the concentration of CO_2 in the blood, causing the observed periodic oscillations in the breathing frequency, and the production of new blood cells, in the bone marrow, exhibit oscillations and several DDE models have been proposed to model these processes.

Below, we present two applications indicating the relevance of the DDEs we study in this paper to real world problems. The two examples are taken from the areas of physiology and population dynamics.

Application 1 (blood cells production [9]). The production of red and white blood cells, in the bone marrow, is regulated by the level of oxygen, in the blood. A reduction in the number of cells in the blood, as a result of the loss of cells, causes the level of oxygen in the blood to decrease. When the level of oxygen in the blood decreases, a substance is released that in turn leads to the release of blood elements, from the bone marrow. Thus, the concentration $c(t)$ of cells in the blood stream, at any time t, changes according to the loss of cells and the release of new cells, from the bone marrow. But the bone marrow responds to a reduction in the number of blood cells and the decrease in the level of oxygen, with a delay that is in the order of 6 days. That means the release of new cells, into the blood stream, at time t, depends on the cell concentration, at an earlier time, namely, $t-\tau$, where τ is the delay with which the bone marrow responds to a reduced level of oxygen in the blood. The simplest model of the concentration of the cells in the blood stream can be described by the DDE

$$c'(t) = \lambda c(t-\tau) - \gamma c(t), \tag{2}$$

where λ represents the flux of cells into the blood stream, γ is the death rate, and τ is the delay. All of them are positive constants. The solutions of the above equation exhibit similar oscillations to the actual oscillatory pattern observed in the concentration of cells in the blood stream.

Application 2. Imagine a biological population composed of adult and juvenile individuals. Let $N(t)$ denote the density of adults at time t. Assume that the length of the juvenile period is exactly h units of time for each individual. Assume that adults produce offspring at a per capita rate α and that their probability per unit of time of dying is μ. Assume that a newborn survives the juvenile period with probability ρ and put $t=\alpha\rho$. Then the dynamics of N can be described by the differential equation

$$N'(t) = -\mu N(t) + rN(t-h) \tag{3}$$

which involves a nonlocal term, $rN(t-h)$ meaning that newborns become adults with some delay. So the time variation of the population density N involves the current as well as the past values of N.

The use of DDEs, from the initial application, in population dynamics, has spread to every area of the life sciences: immunology, physiology, epidemiology, and cell growth. The original delay logistic equation has led to several new DDE forms, like Volterra's integrodifferential equations and neutral DDEs [9], and several new models, from the delayed Hopfield model, in neural networks to the SIR model, in epidemiology [7]. More recently, the idea of state dependent delays has been introduced, involving "a delay that itself is governed by a differential equation that represents adaptation to the system's state" [9].

From the above review of DDEs, in the biological sciences, it is apparent that if DDEs are so extensively used in this area, this is because the dynamics of those equations, namely, the stability and oscillatory properties of the solutions of those equations, replicate the stability and oscillatory patterns, we actually observe in processes, in those areas. Thus, the study of the stability and oscillatory behavior of the solutions of DDEs has become the principal subject of the research on those equations. For more advanced treatises on oscillation theory, the reader is referred to [13–33].

In the paper, we consider a differential equation with delay argument of the form

$$x'(t) + p(t)\,x(\tau(t)) = 0, \quad t \ge t_0, \tag{E}$$

where p is a function of nonnegative real numbers and τ is a function of positive real numbers such that

$$\begin{gathered} \tau(t) < t, \quad t \ge t_0, \\ \lim_{t\to\infty} \tau(t) = \infty. \end{gathered} \tag{4}$$

By a *solution* of (E) we understand a continuously differentiable function defined on $[\tau(T_0),\infty)$ for some $T_0 \ge t_0$ and such that (E) is satisfied for $t \ge T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

A parallel problem to that of establishing oscillation criteria for the solutions of equation (E) is the one concerning the solutions of the advanced differential equation (ADE)

$$x'(t) - q(t)\,x(\sigma(t)) = 0, \quad t \ge t_0, \tag{E$'$}$$

where q is a function of nonnegative real numbers and σ is a function of positive real numbers such that

$$\sigma(t) > t, \quad t \ge t_0. \tag{5}$$

The objective of this paper is to consider the oscillatory dynamics of both delay and advanced differential equations, from the perspective of the qualitative analysis of those equations. In that framework, (i) we formulate new iterative oscillation conditions, for testing whether all solutions of a DDE of the form of (E) or an ADE of the form of (E$'$) are oscillatory, (ii) we show that these tests significantly improve on all the previous, iterative, and noniterative oscillation criteria which, briefly, are reviewed in the Historical and Chronological Review, in Section 2, requiring fewer iterations to determine whether an equation of the considered form is oscillatory, and (iii) these criteria apply to a more general class of equations, having nonmonotone arguments $\tau(t)$ or $\sigma(t)$, in contrast to the large majority of the other studies where the criteria apply to equations with nondecreasing arguments.

From this point onward, we will use the notation

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds,$$
$$\beta := \liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s)\,ds,$$
$$D(\omega) := \begin{cases} 0, & \text{if } \omega > \frac{1}{e}, \\ \dfrac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in \left[0, \frac{1}{e}\right], \end{cases} \quad (6)$$
$$\text{LD} := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds, \quad \text{where } \tau(t) \text{ is nondecreasing},$$
$$\text{LA} := \limsup_{t\to\infty} \int_{t}^{\sigma(t)} q(s)\,ds, \quad \text{where } \sigma(t) \text{ is nondecreasing}.$$

2. Historical and Chronological Review

2.1. DDEs. The first systematic study for the oscillation of all solutions of equation (E) was made by Myškis in 1950 [31], when he proved that every solution of (E) oscillates, if

$$\limsup_{t\to\infty} [t-\tau(t)] < \infty,$$
$$\liminf_{t\to\infty} [t-\tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}. \quad (7)$$

In 1972, Ladas et al. [27] proved that if

$$\text{LD} > 1, \quad (8)$$

then all solutions of (E) are oscillatory.

In 1982, Koplatadze and Chanturiya [24] improved (7) to

$$\alpha > \frac{1}{e}. \quad (9)$$

Regarding the constant $1/e$ in (9), it should be remarked that if the inequality

$$\int_{\tau(t)}^{t} p(s)\,ds \le \frac{1}{e} \quad (10)$$

holds eventually, then, according to [24], (E) has a nonoscillatory solution.

It is apparent that there is a gap between conditions (8) and (9), when

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds \quad (11)$$

does not exist. How to fill this gap is an interesting problem which has been investigated by several authors. For example, in 2000, Jaroš and Stavroulakis [23] proved that if λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$ and

$$\text{LD} > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \quad (12)$$

then all solutions of (E) oscillate.

Now we come to the general case where the argument $\tau(t)$ is nonmonotone. Set

$$h(t) := \sup_{s\le t} \tau(s), \quad t \ge t_0. \quad (13)$$

Clearly, the function $h(t)$ is nondecreasing and $\tau(t) \le h(t) < t$, for all $t \ge t_0$.

In 1994, Koplatadze and Kvinikadze [25] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{h(s)}^{h(t)} p(u)\,\psi_j(u)\,du\right) ds > 1 - D(\alpha), \quad (14)$$

where

$$\psi_1(t) = 0,$$
$$\psi_j(t) = \exp\left(\int_{\tau(t)}^{t} p(u)\,\psi_{j-1}(u)\,du\right), \quad j \ge 2, \quad (15)$$

then all solutions of (E) oscillate.

In 2011, Braverman and Karpuz [14] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u)\,du\right) ds > 1, \quad (16)$$

then all solutions of (E) oscillate, while in 2014, Stavroulakis [32] improved (16) to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p(u)\,du\right) ds > 1 - D(\alpha). \quad (17)$$

In 2016, El-Morshedy and Attia [30] proved that if

$$\limsup_{t\to\infty} \left[\int_{g(t)}^{t} p_n(s)\,ds + D(\alpha) \exp\left(\int_{g(t)}^{t} \sum_{j=0}^{n-1} p_j(s)\,ds\right)\right] > 1, \quad (18)$$

where $p_0(t) = p(t)$ and

$$p_n(t) = p_{n-1}(t) \int_{g(t)}^{t} p_{n-1}(s) \exp\left(\int_{g(s)}^{t} p_{n-1}(u)\,du\right) ds, \quad n \ge 1, \quad (19)$$

then all solutions of (E) are oscillatory. Here, $g(t)$ is a nondecreasing continuous function such that $\tau(t) \leq g(t) \leq t$, $t \geq t_1$, for some $t_1 \geq t_0$. Clearly, $g(t)$ is more general than $h(t)$ defined by (13).

Recently, Chatzarakis [15, 16] proved that if, for some $j \in \mathbb{N}$,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_j(u)\,du\right) ds > 1 \tag{20}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_j(u)\,du\right) ds > 1 - D(\alpha), \tag{21}$$

where

$$p_j(t) = p(t)\left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(t)} p_{j-1}(u)\,du\right) ds\right], \tag{22}$$

with $p_0(t) = p(t)$, then all solutions of (E) are oscillatory.

Lately, Chatzarakis [17] studied a more general form of (E); namely,

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0, \quad t \geq t_0, \tag{23}$$

and established sufficient oscillation conditions. Those conditions can lead to (20) and (21) when $m = 1$.

2.2. ADEs. By Theorem 2.4.3 [29], if

$$\text{LA} > 1, \tag{24}$$

then all solutions of (E$'$) are oscillatory.

In 1984, Fukagai and Kusano [21] proved that if

$$\beta > \frac{1}{e}, \tag{25}$$

then all solutions of (E$'$) are oscillatory, while if

$$\int_{t}^{\sigma(t)} q(s)\,ds \leq \frac{1}{e} \quad \text{for all sufficiently large } t, \tag{26}$$

then (E$'$) has a nonoscillatory solution.

Assume that the argument $\sigma(t)$ is not necessarily monotone. Set

$$\rho(t) = \inf_{s\geq t} \sigma(s), \quad t \geq t_0. \tag{27}$$

Clearly, the function $\rho(t)$ is nondecreasing and $\sigma(t) \geq \rho(t) > t$, for all $t \geq t_0$.

In 2015, Chatzarakis and Öcalan [18] proved that if

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)\,du\right) ds > 1, \tag{28}$$

or

$$\liminf_{t\to\infty} \int_{t}^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u)\,du\right) ds > \frac{1}{e}, \tag{29}$$

then all solutions of (E$'$) are oscillatory.

Recently, Chatzarakis [15, 16] proved that if, for some $j \in \mathbb{N}$,

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)\,du\right) ds > 1, \tag{30}$$

or

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_j(u)\,du\right) ds > 1 - D(\beta), \tag{31}$$

where

$$q_j(t) = q(t)\left[1 + \int_{t}^{\sigma(t)} q(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_{j-1}(u)\,du\right) ds\right], \quad j \geq 1 \tag{32}$$

with $q_0(t) = q(t)$, then all solutions of (E$'$) oscillate.

Lately, Chatzarakis [17] studied a more general form of (E$'$), namely,

$$x'(t) - \sum_{i=1}^{m} q_i(t)\, x(\sigma_i(t)) = 0, \quad t \geq t_0, \tag{33}$$

and established sufficient oscillation conditions. Those conditions can lead to (30) and (31) when $m = 1$.

3. Main Results

3.1. DDEs. In our main results, we state theorems, establishing new sufficient oscillation conditions. For the proofs of those theorems, we use the following lemmas.

Lemma 3 (see [19, Lemma 2.1.1]). *Assume that $h(t)$ is defined by (13). Then*

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = \liminf_{t\to\infty} \int_{h(t)}^{t} p(s)\,ds. \tag{34}$$

Lemma 4 (see [19, Lemma 2.1.3]). *Assume that $h(t)$ is defined by (13), $\alpha \in (0, 1/e]$, and $x(t)$ is an eventually positive solution of (E). Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq D(\alpha). \tag{35}$$

Lemma 5 (see [26]). *Assume that $h(t)$ is defined by (13), $\alpha \in (0, 1/e]$, and $x(t)$ is an eventually positive solution of (E). Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geq \lambda_0, \tag{36}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$.

Theorem 6. *Let $h(t)$ be defined by (13) and for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u)\exp\left(\int_{\tau(u)}^{u} P_j(\xi)\,d\xi\right)du\right)ds > 1, \quad (37)$$

where

$$P_j(t) = p(t)\left[1+\int_{\tau(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{j-1}(\xi)\,d\xi\right)du\right)ds\right] \quad (38)$$

with $P_0(t) = \lambda_0 p(t)$, and let λ_0 be the smaller root of the equation $\lambda = e^{\alpha\lambda}$. Then all solutions of (E) oscillate.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists a real number $t_1 > t_0$ such that $x(t), x(\tau(t)) > 0$ for all $t \geq t_1$. Thus, from (E) we have

$$x'(t) = -p(t)\,x(\tau(t)) \leq 0 \quad \forall t \geq t_1, \quad (39)$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers. Taking into account the fact that $\tau(t) \leq h(t)$, (E) implies that

$$x'(t) + p(t)\,x(h(t)) \leq 0, \quad t \geq t_1. \quad (40)$$

Observe that (36) implies that, for each $\epsilon > 0$, there exists a real number t_ϵ such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \epsilon \quad \forall t \geq t_\epsilon \geq t_1. \quad (41)$$

Combining inequalities (40) and (41), we obtain

$$x'(t) + p(t)(\lambda_0 - \epsilon)\,x(t) \leq 0, \quad t \geq t_\epsilon, \quad (42)$$

or

$$x'(t) + P_0(t,\epsilon)\,x(t) \leq 0, \quad t \geq t_\epsilon, \quad (43)$$

where

$$P_0(t,\epsilon) = p(t)(\lambda_0 - \epsilon). \quad (44)$$

Applying the Grönwall inequality in (43), we conclude that

$$x(s) \geq x(t)\exp\left(\int_s^t P_0(\xi,\epsilon)\,d\xi\right), \quad t \geq s \geq t_\epsilon. \quad (45)$$

Now we divide (E) by $x(t) > 0$ and integrate on $[s,t]$, so

$$-\int_s^t \frac{x'(u)}{x(u)}du = \int_s^t p(u)\frac{x(\tau(u))}{x(u)}du, \quad (46)$$

or

$$\ln\frac{x(s)}{x(t)} = \int_s^t p(u)\frac{x(\tau(u))}{x(u)}du, \quad t \geq s \geq t_\epsilon. \quad (47)$$

Since $\tau(u) < u$, equality (47) gives

$$\begin{aligned}\ln\frac{x(s)}{x(t)} &= \int_s^t p(u)\frac{x(\tau(u))}{x(u)}du \\ &\geq \int_s^t p(u)\frac{x(u)}{x(u)}\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du \\ &= \int_s^t p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du,\end{aligned} \quad (48)$$

or

$$x(s) \geq x(t)\exp\left(\int_s^t p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du\right). \quad (49)$$

Substituting $\tau(s)$ for s in (49), we get

$$x(\tau(s)) \geq x(t)\exp\left(\int_{\tau(s)}^t p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du\right). \quad (50)$$

Integrating (E) from $\tau(t)$ to t, we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)\,x(\tau(s))\,ds = 0. \quad (51)$$

Combining (50) and (51), we obtain

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du\right)ds \leq 0. \quad (52)$$

Multiplying inequality (52) by $p(t)$, we find

$$p(t)x(t) - p(t)x(\tau(t)) + p(t)x(t)\int_{\tau(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du\right)ds \leq 0, \quad (53)$$

which, in view of (E), becomes

$$x'(t) + p(t)x(t) + p(t)x(t)\int_{\tau(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_0(\xi,\epsilon)\,d\xi\right)du\right)ds \leq 0. \quad (54)$$

Hence, for sufficiently large t,

$$x'(t)+p(t)\left[1+\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_0(\xi,\epsilon)\,d\xi\right)du\right)ds\right]\cdot x(t)\le 0, \tag{55}$$

or

$$x'(t)+P_1(t,\epsilon)\,x(t)\le 0, \tag{56}$$

where

$$P_1(t,\epsilon)=p(t)\left[1+\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_0(\xi,\epsilon)\,d\xi\right)du\right)ds\right]. \tag{57}$$

Clearly (56) resembles (43), if we replace P_0 by P_1. Thus, integrating (56) on $[s,t]$ yields

$$x(s)\ge x(t)\exp\left(\int_{s}^{t}P_1(\xi,\epsilon)\,d\xi\right). \tag{58}$$

Repeating steps (45) through (50), we can see that x satisfies the inequality

$$x(\tau(s))\ge x(t)\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_1(\xi,\epsilon)\,d\xi\right)du\right). \tag{59}$$

Combining now (51) and (59), we obtain

$$x(t)-x(\tau(t))+x(t)\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_1(\xi,\epsilon)\,d\xi\right)du\right)ds\le 0. \tag{60}$$

Multiplying inequality (60) by $p(t)$, as before, we find

$$x'(t)+p(t)\left[1+\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_1(\xi,\epsilon)\,d\xi\right)du\right)ds\right]\cdot x(t)\le 0. \tag{61}$$

Therefore, for sufficiently large t, we have

$$x'(t)+P_2(t,\epsilon)\,x(t)\le 0, \tag{62}$$

where

$$P_2(t,\epsilon)=p(t)\left[1+\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_1(\xi,\epsilon)\,d\xi\right)du\right)ds\right]. \tag{63}$$

It becomes apparent, now, that, by repeating the above steps, we can build inequalities on $x'(t)$ with progressively higher indices $P_j(t,\epsilon)$, $j\in\mathbb{N}$. In general, for sufficiently large t, the positive solution $x(t)$ satisfies the inequality

$$x'(t)+P_j(t,\epsilon)\,x(t)\le 0,\quad j\in\mathbb{N}, \tag{64}$$

where

$$P_j(t,\epsilon)=p(t)\left[1+\int_{\tau(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_{j-1}(\xi,\epsilon)\,d\xi\right)du\right)ds\right]. \tag{65}$$

Proceeding to final step, we recall that $h(t)$, defined by (13), is a nondecreasing function. Since $\tau(s)\le h(s)\le h(t)$, we have

$$x(\tau(s))\ge x(h(t))\cdot\exp\left(\int_{\tau(s)}^{h(t)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right). \tag{66}$$

Hence

$$x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(t)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le 0, \tag{67}$$

or

$$x(h(t))\left[\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(t)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds-1\right]<0. \tag{68}$$

Thus

$$\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(t)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds-1<0. \tag{69}$$

Taking the limit as $t\to\infty$, we have

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(t)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le 1. \tag{70}$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (37).

This completes the proof of the theorem. □

Theorem 7. *Let $h(t)$ be defined by (13) and $\alpha \in (0, 1/e]$. If for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi)\, d\xi\right) du\right) ds > 1 - D(\alpha), \tag{71}$$

where P_j is defined by (38), then all solutions of (E) oscillate.

Proof. Assume x is an eventually positive solution of (E). Clearly, (67) is satisfied for sufficiently large t. Thus,

$$\int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds \le 1 - \frac{x(t)}{x(h(t))}, \tag{72}$$

which implies that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds \le 1 - \liminf_{t\to\infty} \frac{x(t)}{x(h(t))}. \tag{73}$$

Using Lemmas 3 and 4, it is evident that inequality (35) is satisfied. Thus, (73) leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds \le 1 - D(\alpha). \tag{74}$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (71).

This completes the proof of the theorem. □

Theorem 8. *Let $h(t)$ be defined by (13) and $\alpha \in (0, 1/e]$. If for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi)\, d\xi\right) du\right) ds > \frac{1}{D(\alpha)}, \tag{75}$$

where P_j is defined by (38), then all solutions of (E) oscillate.

Proof. Assume x is an eventually positive solution of (E). Then, as in the proof of Theorem 6, for sufficiently large t, we conclude that

$$x(\tau(s)) \ge x(t) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right). \tag{76}$$

Integrating (E) from $h(t)$ to t and using (76), we obtain

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s) x(t) \cdot \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds \le 0, \tag{77}$$

or

$$-x(h(t)) + \int_{h(t)}^{t} p(s) x(t) \cdot \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds < 0. \tag{78}$$

Hence

$$x(h(t)) \left[\frac{x(t)}{x(h(t))} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds - 1\right] < 0, \tag{79}$$

which yields, for all sufficiently large t,

$$\int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi,\epsilon)\, d\xi\right) du\right) ds < \frac{x(h(t))}{x(t)} \tag{80}$$

and consequently

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le\limsup_{t\to\infty}\frac{x(h(t))}{x(t)}. \quad (81)$$

Taking into account the fact that (35) is satisfied, inequality (81) leads to

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le\frac{1}{D(\alpha)}, \quad (82)$$

which contradicts (75), when $\epsilon\to 0$.

This completes the proof of the theorem. □

Theorem 9. *Let $h(t)$ be defined by (13) and $\alpha\in(0,1/e]$. If for some $j\in\mathbb{N}$*

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(s)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi)\,d\xi\right)du\right)ds>\frac{1+\ln\lambda_0}{\lambda_0}-D(\alpha), \quad (83)$$

where P_j is defined by (38) and λ_0 is the smaller root of the equation $\lambda=e^{\alpha\lambda}$, then all solutions of (E) oscillate.

Proof. Let x be an eventually positive solution of (E). As in the proof of Theorem 8, we can show that (76) holds; namely,

$$x(\tau(s))\ge x(t)\exp\left(\int_{\tau(s)}^{t}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right). \quad (84)$$

Since $\tau(s)\le h(s)$, inequality (84) gives

$$x(\tau(s))\ge x(h(s))\cdot\exp\left(\int_{\tau(s)}^{h(s)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right). \quad (85)$$

By Lemma 5, for each $\epsilon>0$, there exists a real number t_ϵ such that

$$\frac{x(h(t))}{x(t)}>\lambda_0-\epsilon\quad\forall t\ge t_\epsilon\ge t_1. \quad (86)$$

Note that, by the nondecreasing nature of the function $x(h(t))/x(s)$ in s, it holds

$$1=\frac{x(h(t))}{x(h(t))}\le\frac{x(h(t))}{x(s)}\le\frac{x(h(t))}{x(t)},\quad t_\epsilon\le h(t)\le s\le t. \quad (87)$$

In particular, for $\epsilon\in(0,\lambda_0-1)$, by continuity, we conclude that there exists a real number $t^*\in(h(t),t]$ satisfying

$$1<\lambda_0-\epsilon=\frac{x(h(t))}{x(t^*)}. \quad (88)$$

Integrating (E) from t^* to t and using (85), we obtain

$$x(t)-x(t^*)+x(h(t))\int_{t^*}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(s)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le 0, \quad (89)$$

or

$$\int_{t^*}^{t}p(s)\exp\left(\int_{\tau(s)}^{h(s)}p(u)\cdot\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le\frac{x(t^*)}{x(h(t))}-\frac{x(t)}{x(h(t))}. \quad (90)$$

Using (88) and Lemma 4, we deduce that, for the ϵ considered, there exists a real number $t'_\epsilon\ge t_\epsilon$ such that

$$\int_{t^*}^{t}p(s)\cdot\exp\left(\int_{\tau(s)}^{h(s)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds<\frac{1}{\lambda_0-\epsilon}-D(\alpha)+\epsilon \quad (91)$$

for $t\ge t'_\epsilon$.

Dividing (E) by $x(t)$, integrating from $h(t)$ to t^*, and using (85), we deduce that

$$\int_{h(t)}^{t^*}p(s)\cdot\frac{x(h(s))}{x(s)}\exp\left(\int_{\tau(s)}^{h(s)}p(u)\exp\left(\int_{\tau(u)}^{u}P_j(\xi,\epsilon)\,d\xi\right)du\right)ds\le-\int_{h(t)}^{t^*}\frac{x'(s)}{x(s)}ds. \quad (92)$$

Clearly, by means of (36), $x(h(s))/x(s) > \lambda_0 - \epsilon$, for $s \geq h(t) \geq t'_\epsilon$. Hence, for all sufficiently large t, we conclude that

$$(\lambda_0 - \epsilon) \int_{h(t)}^{t^*} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds < -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds \tag{93}$$

or

$$\int_{h(t)}^{t^*} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds < -\frac{1}{\lambda_0 - \epsilon} \int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds = \frac{1}{\lambda_0 - \epsilon} \ln \frac{x(h(t))}{x(t^*)} = \frac{\ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon}; \tag{94}$$

that is,

$$\int_{h(t)}^{t^*} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds < \frac{\ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon}. \tag{95}$$

Using (91) along with (95), we get

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds \leq \frac{1 + \ln(\lambda_0 - \epsilon)}{\lambda_0 - \epsilon} - D(\alpha) + \epsilon, \tag{96}$$

which contradicts (83), when $\epsilon \to 0$.

This completes the proof of the theorem. □

Theorem 10. *Let $h(t)$ be defined by (13). If for some $j \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi)\, d\xi\right) du\right) ds > \frac{1}{e}, \tag{97}$$

where P_j is defined by (38), then all solutions of (E) oscillate.

Proof. For the sake of contradiction, let x be a nonincreasing eventually positive solution and $t_1 > t_0$ be such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \geq t_1$. We note that we may obtain (85) as in the proof of Theorem 9.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t, we have

$$\ln\left(\frac{x(h(t))}{x(t)}\right) = \int_{h(t)}^{t} p(s) \frac{x(\tau(s))}{x(s)} ds \quad \forall t \geq t_2 \geq t_1, \tag{98}$$

from which, in view of $\tau(s) \leq h(s)$ and (85), we get

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s) \cdot \frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds. \tag{99}$$

Since x is nonincreasing and $h(s) < s$, inequality (99) becomes

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geq \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds. \tag{100}$$

From (97), it is clear that there exists a constant $c > 0$ such that

$$\int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi)\, d\xi\right) du\right) ds \geq c > \frac{1}{e}. \tag{101}$$

Choose c' such that $c > c' > 1/e$. For every $\epsilon > 0$, such that $c - \epsilon > c'$, we have

$$\int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds > c - \epsilon > c' > \frac{1}{e}. \tag{102}$$

Combining inequalities (100) and (102), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) > c', \tag{103}$$

or

$$\frac{x(h(t))}{x(t)} > e^{c'} > ec' > 1, \tag{104}$$

which yields

$$x(h(t)) > (ec')\, x(t). \tag{105}$$

Following the above steps, we can inductively show that, for any positive integer k,

$$\frac{x(h(t))}{x(t)} > (ec')^k \quad \text{for sufficiently large } t. \tag{106}$$

Since $ec' > 1$, there is a natural number $k \in \mathbb{N}$, satisfying $k > 2[\ln 2 - \ln c']/(1 + \ln c')$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} > (ec')^k > \left(\frac{2}{c'}\right)^2. \tag{107}$$

Further (cf. [13, 24]), for sufficiently large t, there exists a real number $t_m \in (h(t), t)$, such that

$$\begin{aligned}&\int_{h(t)}^{t_m} p(s) \\ &\quad \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds \\ &\quad > \frac{c'}{2}, \\ &\int_{t_m}^{t} p(s) \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds \\ &\quad > \frac{c'}{2}.\end{aligned} \tag{108}$$

Integrating (E) from $h(t)$ to t_m, using (85) and the fact that $x(t) > 0$, we obtain

$$\begin{aligned}x(h(t)) &> x(h(t_m)) \int_{h(t)}^{t_m} p(s) \\ &\quad \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds,\end{aligned} \tag{109}$$

which, in view of the first inequality in (108), implies that

$$x(h(t)) > \frac{c'}{2} x(h(t_m)). \tag{110}$$

Similarly, integrating (E) from t_m to t, using (85) and the fact that $x(t) > 0$, we have

$$\begin{aligned}x(t_m) &> x(h(t)) \int_{t_m}^{t} p(s) \\ &\quad \cdot \exp\left(\int_{\tau(s)}^{h(s)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi, \epsilon)\, d\xi\right) du\right) ds,\end{aligned} \tag{111}$$

which, in view of the second inequality in (108), yields

$$x(t_m) > \frac{c'}{2} x(h(t)). \tag{112}$$

Combining inequalities (110) and (112), we deduce that

$$x(h(t_m)) < \frac{2}{c'} x(h(t)) < \left(\frac{2}{c'}\right)^2 x(t_m), \tag{113}$$

which contradicts (107).

The proof of the theorem is complete. □

3.2. ADEs. Analogous oscillation conditions to those obtained for the delay equation (E) can be derived for the (dual) advanced differential equation (E′) by following similar arguments with the ones employed for obtaining Theorems 6–10.

Theorem 11. *Let $\rho(t)$ be defined by (27) and for some $j \in \mathbb{N}$*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \\ &\quad \cdot \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u) \exp\left(\int_{u}^{\sigma(u)} Q_j(\xi)\, d\xi\right) du\right) ds \\ &\quad > 1,\end{aligned} \tag{114}$$

where

$$\begin{aligned}Q_j(t) &= q(t) \left[1 + \int_{t}^{\sigma(t)} q(s)\right. \\ &\quad \left. \cdot \exp\left(\int_{t}^{\sigma(s)} q(u) \exp\left(\int_{u}^{\sigma(u)} Q_{j-1}(\xi)\, d\xi\right) du\right) ds\right]\end{aligned} \tag{115}$$

with $Q_0(t) = \lambda_0 q(t)$, and let λ_0 be the smaller root of the equation $\lambda = e^{\beta\lambda}$. Then all solutions of (E′) oscillate.

Theorem 12. *Let $\rho(t)$ be defined by (27) and $\beta \in (0, 1/e]$. If for some $j \in \mathbb{N}$*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \\ &\quad \cdot \exp\left(\int_{\rho(t)}^{\sigma(s)} q(u) \exp\left(\int_{u}^{\sigma(u)} Q_j(\xi)\, d\xi\right) du\right) ds \\ &\quad > 1 - D(\beta),\end{aligned} \tag{116}$$

where Q_j is defined by (115), then all solutions of (E′) oscillate.

Theorem 13. *Let $\rho(t)$ be defined by (27) and $\beta \in (0, 1/e]$. If for some $j \in \mathbb{N}$*

$$\begin{aligned}&\limsup_{t\to\infty} \int_{t}^{\rho(t)} q(s) \\ &\quad \cdot \exp\left(\int_{t}^{\sigma(s)} q(u) \exp\left(\int_{u}^{\sigma(u)} Q_j(\xi)\, d\xi\right) du\right) ds \\ &\quad > \frac{1}{D(\beta)},\end{aligned} \tag{117}$$

where Q_j is defined by (115), then all solutions of (E′) oscillate.

Theorem 14. *Let $\rho(t)$ be defined by (27) and $\beta \in (0, 1/e]$. If for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} q(s) \cdot \exp\left(\int_{\rho(s)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} Q_j(\xi)\, d\xi\right) du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\beta), \tag{118}$$

where Q_j is defined by (115) and λ_0 is the smaller root of the equation $\lambda = e^{\beta\lambda}$, then all solutions of (E') oscillate.

Theorem 15. *Let $\rho(t)$ be defined by (27). If for some $j \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_t^{\rho(t)} q(s) \cdot \exp\left(\int_{\rho(s)}^{\sigma(s)} q(u) \exp\left(\int_u^{\sigma(u)} Q_j(\xi)\, d\xi\right) du\right) ds > \frac{1}{e}, \tag{119}$$

where Q_j is defined by (115), then all solutions of (E') oscillate.

3.3. Differential Inequalities. A slight modification in the proofs of Theorems 6–15 leads to the following results about differential inequalities.

Theorem 16. *Assume that all the conditions of Theorem 6 [11], 7 [12], 8 [13], 9 [14], or 10 [15] hold. Then*

(*i*) *the delay [advanced] differential inequality*

$$x'(t) + p(t)\, x(\tau(t)) \le 0 \quad \left[x'(t) - q(t)\, x(\sigma(t)) \ge 0\right], \quad t \ge t_0 \tag{120}$$

has no eventually positive solutions;

(*ii*) *the delay [advanced] differential inequality*

$$x'(t) + p(t)\, x(\tau(t)) \ge 0 \quad \left[x'(t) - q(t)\, x(\sigma(t)) \le 0\right], \quad t \ge t_0 \tag{121}$$

has no eventually negative solutions.

Remark 17. The oscillation criteria established in this paper all depend on λ_0 (see, e.g., (37) and (71)) in contrast to the conditions obtained in [15, 16] and in [17, for $m = 1$]. In fact, the left-hand side of conditions (37) and (71) depends on λ_0, which is not the case with the left-hand side of conditions (20) and (21). Since $\lambda_0 > 1$ when $\alpha \in (0, 1/e]$, it is obvious that

$$P_0(t) = \lambda_0 p(t) > p(t) = p_0(t). \tag{122}$$

Consequently, the left-hand side of conditions (37) and (71) is greater than the corresponding parts of (20) and (21), respectively. This is the reason why the conditions in this paper improve on all known conditions mentioned in Section 2.

4. Examples and Comments

The oscillation tests we have proposed and established, in the main results, involve an iterative procedure. We iteratively compute limsup and liminf on the terms $P_j(t)$ and $Q_j(t)$, $j \in \mathbb{N}$ of a recurrent relation defined on the coefficients and the deviating argument of an equation of the form (E) or (E′) to determine whether that equation is oscillatory. But this computation cannot be performed on paper, but by means of a program, numerically computing limsup and liminf. The examples below illustrate the significance of our results and indicate the high level of improvement in the oscillation criteria. The calculations were performed using MATLAB code.

Example 1. Consider the delay differential equation

$$x'(t) + \frac{3}{25} x(\tau(t)) = 0, \quad t \ge 0, \tag{123}$$

with (see Figure 1(a))

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ -4t+40k+9, & \text{if } t \in [8k+2, 8k+3] \\ 5t-32k-18, & \text{if } t \in [8k+3, 8k+4] \\ -4t+40k+18, & \text{if } t \in [8k+4, 8k+5] \\ 5t-32k-27, & \text{if } t \in [8k+5, 8k+6] \\ -2t+24k+15, & \text{if } t \in [8k+6, 8k+7] \\ 6t-40k-41, & \text{if } t \in [8k+7, 8k+8], \end{cases} \tag{124}$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (13), we see (Figure 1(b)) that

$$h(t) = \begin{cases} t-1, & \text{if } t \in [8k, 8k+2] \\ 8k+1, & \text{if } t \in \left[8k+2, 8k+\frac{19}{5}\right] \\ 5t-32k-18, & \text{if } t \in \left[8k+\frac{19}{5}, 8k+4\right] \\ 8k+2, & \text{if } t \in \left[8k+4, 8k+\frac{29}{5}\right] \\ 5t-32k-27, & \text{if } t \in \left[8k+\frac{29}{5}, 8k+6\right] \\ 8k+3, & \text{if } t \in \left[8k+6, 8k+\frac{44}{6}\right] \\ 6t-40k-41, & \text{if } t \in \left[8k+\frac{44}{6}, 8k+8\right]. \end{cases} \tag{125}$$

It is obvious that

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds = \liminf_{t\to\infty} \int_{8k+1}^{8k+2} \frac{3}{25} ds = 0.12 \tag{126}$$

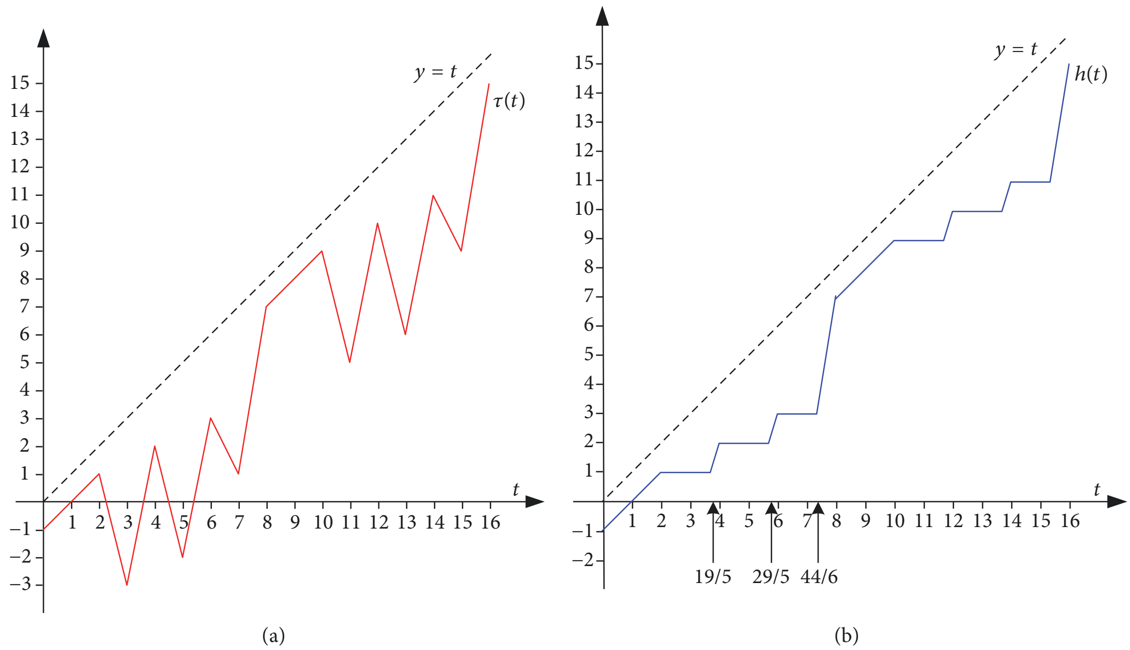

Figure 1: The graphs of $\tau(t)$ and $h(t)$.

and therefore, the smaller root of $e^{0.12\lambda} = \lambda$ is $\lambda_0 = 1.14765$.

Observe that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_{h(t)}^{t} p(s) \cdot \exp\left(\int_{\tau(s)}^{h(t)} p(u) \exp\left(\int_{\tau(u)}^{u} P_j(\xi)\, d\xi\right) du\right) ds \tag{127}$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. Specifically,

$$F_1\left(t = 8k + \frac{44}{6}\right) = \int_{8k+3}^{8k+44/6} p(s) \cdot \exp\left(\int_{\tau(s)}^{8k+3} p(u) \exp\left(\int_{\tau(u)}^{u} P_1(\xi)\, d\xi\right) du\right) ds \tag{128}$$

with

$$P_1(\xi) = p(\xi)\left[1 + \int_{\tau(\xi)}^{\xi} p(v) \cdot \exp\left(\int_{\tau(v)}^{\xi} p(w) \exp\left(\int_{\tau(w)}^{w} \lambda_0 p(z)\, dz\right) dw\right) dv\right]. \tag{129}$$

Using MATLAB, we obtain

$$F_1\left(t = 8k + \frac{44}{6}\right) \simeq 1.0417 \tag{130}$$

and therefore

$$\limsup_{t\to\infty} F_1(t) \simeq 1.0417 > 1. \tag{131}$$

Hence, condition (37) of Theorem 6 is satisfied, for $j = 1$. Consequently, all solutions of (123) are oscillatory.

Observe, however, that

$$\begin{aligned} \text{LD} &= \limsup_{k\to\infty} \int_{8k+3}^{8k+44/6} \frac{3}{35} ds = 0.52 < 1, \\ \alpha &= 0.12 < \frac{1}{e}, \\ 0.52 &< \frac{1 + \ln\lambda_0}{\lambda_0} - D(\alpha) \simeq 0.9831. \end{aligned} \tag{132}$$

Note that the function Φ_j defined by

$$\Phi_j(t) = \int_{h(t)}^{t} p(s) \exp\left(\int_{h(s)}^{h(t)} p(u)\, \psi_j(u)\, du\right) ds, \quad j \geq 2, \tag{133}$$

attains its maximum at $t = 8k + 44/6$, $k \in \mathbb{N}_0$, for every $j \geq 2$. Specifically,

$$\begin{aligned} &\Phi_2\left(8k + \frac{44}{6}\right) \\ &= \int_{8k+3}^{8k+44/6} p(s) \exp\left(\int_{h(s)}^{8k+3} p(s)\, \psi_2(u)\, du\right) ds \\ &= \int_{8k+3}^{8k+44/6} \frac{3}{25} \exp\left(\int_{h(s)}^{8k+3} \frac{3}{25} \exp\left(\int_{\tau(u)}^{u} \frac{3}{25} \cdot 0\, dw\right) du\right) ds \\ &= \int_{8k+3}^{8k+44/6} \frac{3}{25} \exp\left(\int_{h(s)}^{8k+3} \frac{3}{25} \cdot 1\, du\right) ds \\ &= \frac{3}{25} \cdot \left[\int_{8k+3}^{8k+19/5} \exp\left(\frac{3}{25} \int_{8k+1}^{8k+3} du\right) ds\right. \end{aligned}$$

$$+\int_{8k+19/5}^{8k+4}\exp\left(\frac{3}{25}\int_{5s-32k-18}^{8k+3}du\right)ds$$
$$+\int_{8k+4}^{8k+29/5}\exp\left(\frac{3}{25}\int_{8k+2}^{8k+3}du\right)ds$$
$$+\int_{8k+29/5}^{8k+6}\exp\left(\frac{3}{25}\int_{5s-32k-27}^{8k+3}du\right)ds$$
$$+\int_{8k+6}^{8k+44/6}\exp\left(\frac{3}{25}\int_{8k+3}^{8k+3}du\right)ds\Bigg]\simeq 0.57983. \quad (134)$$

Thus

$$\limsup_{t\to\infty}\Phi_2(t)\simeq 0.57983<1-D(\alpha)\simeq 0.99174. \quad (135)$$

Also

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{h(t)}p(u)\,du\right)ds$$
$$=\limsup_{k\to\infty}\int_{8k+3}^{8k+44/6}\frac{3}{25}\exp\left(\int_{\tau(s)}^{8k+3}\frac{3}{25}du\right)ds$$
$$=\frac{3}{25}\cdot\limsup_{t\to\infty}\Bigg[\int_{8k+3}^{8k+4}\exp\left(\frac{3}{25}\int_{5s-32k-18}^{8k+3}du\right)ds$$
$$+\int_{8k+4}^{8k+5}\exp\left(\frac{3}{25}\int_{-4s+40k+18}^{8k+3}du\right)ds$$
$$+\int_{8k+5}^{8k+6}\exp\left(\frac{3}{25}\int_{5s-32k-27}^{8k+3}du\right)ds \quad (136)$$
$$+\int_{8k+6}^{8k+7}\exp\left(\frac{3}{25}\int_{-2s+24k+15}^{8k+3}du\right)ds$$
$$+\int_{8k+7}^{8k+44/6}\exp\left(\frac{3}{25}\int_{6s-40k-41}^{8k+3}du\right)ds\Bigg]\simeq 0.7043$$
$$<1,$$
$$0.7043<1-D(\alpha)\simeq 0.99174.$$

In addition,

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{h(t)}p_1(u)\,du\right)ds\simeq 0.8052$$
$$<1, \quad (137)$$
$$0.8052<1-D(\alpha)\simeq 0.99174.$$

That is, none of conditions (8), (9), (12), (14) (for $j=2$), (16), (17), (20) (for $j=1$), and (21) (for $j=1$) is satisfied.

Comment. The improvement of condition (37) over the corresponding condition (8) is significant, approximately 100.33%. We get this measure by comparing the values, in the left-hand side of those conditions. Also, the improvement over conditions (14), (16), and (20) is very satisfactory, around 79.66%, 47.9%, and 29.37%, respectively. In addition, condition (37) is satisfied from the first iteration, while conditions (14), (20), and (21) need more than one iteration.

Example 2 (taken and adapted from [17]). Consider the advanced differential equation

$$x'(t)-\frac{333}{2500}x(\sigma(t))=0,\quad t\geq 0, \quad (138)$$

with (see Figure 2(a))

$$\sigma(t)=\begin{cases}5k+3, & \text{if } t\in[5k,5k+1]\\ 4t-15k-1, & \text{if } t\in[5k+1,5k+2]\\ -3t+20k+13, & \text{if } t\in[5k+2,5k+3]\\ 5t-20k-11, & \text{if } t\in[5k+3,5k+4]\\ -t+10k+13, & \text{if } t\in[5k+4,5k+5],\end{cases} \quad (139)$$

where $k\in\mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (27), we see (Figure 2(b)) that

$$\rho(t)=\begin{cases}5k+3, & \text{if } t\in[5k,5k+1]\\ 4t-15k-1, & \text{if } t\in[5k+1,5k+1.25]\\ 5k+4, & \text{if } t\in[5k+1.25,5k+3]\\ 5t-20k-11, & \text{if } t\in[5k+3,5k+3.8]\\ 5k+8, & \text{if } t\in[5k+3.8,5k+5].\end{cases} \quad (140)$$

It is obvious that

$$\beta=\liminf_{t\to\infty}\int_{5k+3}^{5k+4}\frac{333}{2500}ds=0.1332 \quad (141)$$

and therefore, the smaller root of $e^{0.1332\lambda}=\lambda$ is $\lambda_0=1.16839$.

Observe, that the function $G_j:\mathbb{R}_0\to\mathbb{R}_+$ defined as

$$G_j(t)=\int_{t}^{\rho(t)}q(s)\cdot\exp\left(\int_{\rho(t)}^{\sigma(s)}q(u)\exp\left(\int_{u}^{\sigma(u)}Q_j(\xi)\,d\xi\right)du\right)ds \quad (142)$$

attains its maximum at $t=5k+3.8$, $k\in\mathbb{N}_0$, for every $j\in\mathbb{N}$. Specifically,

$$G_1(t=5k+3.8)=\int_{5k+3.8}^{5k+8}q(s)\cdot\exp\left(\int_{5k+8}^{\sigma(s)}q(u)\exp\left(\int_{u}^{\sigma(u)}Q_1(\xi)\,d\xi\right)du\right)ds \quad (143)$$

with

$$Q_1(\xi)=q(\xi)\left[1+\int_{\xi}^{\sigma(\xi)}q(v)\cdot\exp\left(\int_{\xi}^{\sigma(v)}q(w)\exp\left(\int_{w}^{\sigma(w)}\lambda_0q(z)\,dz\right)dw\right)dv\right]. \quad (144)$$

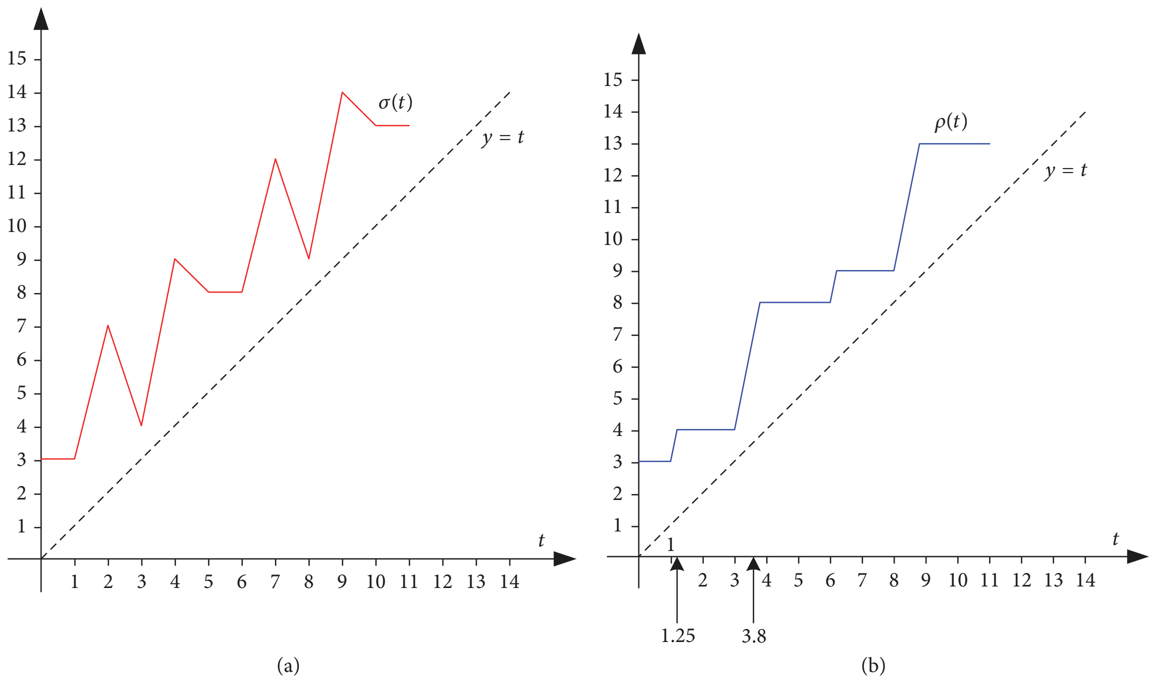

FIGURE 2: The graphs of $\sigma(t)$ and $\rho(t)$.

Using MATLAB, we obtain

$$G_1\left(t = 5k + 3.8\right) \simeq 0.9915. \tag{145}$$

Therefore

$$\limsup_{t\to\infty} G_1\left(t\right) \simeq 0.9915 > 1 - D\left(\beta\right) \simeq 0.9896. \tag{146}$$

Hence, condition (116) of Theorem 12 is satisfied, for $j = 1$. Consequently, all solutions of (138) oscillate.

Observe, however, that

$$\begin{aligned}
LA &= \limsup_{k\to\infty} \int_{5k+3.8}^{5k+8} \frac{333}{2500} ds = 0.55944 < 1, \\
\beta &= 0.1332 < \frac{1}{e}, \\
&\limsup_{t\to\infty} \int_t^{\rho(t)} q\left(s\right) \exp\left(\int_{\rho(t)}^{\sigma(s)} q\left(u\right) du\right) ds \\
&= \limsup_{k\to\infty} \int_{5k+3.8}^{5k+8} q\left(s\right) \exp\left(\int_{5k+8}^{\sigma(s)} q\left(u\right) du\right) ds \\
&= \limsup_{k\to\infty} \left[\int_{5k+3.8}^{5k+4} q\left(s\right)\right. \\
&\cdot \exp\left(\int_{5k+8}^{5s-20k-11} q\left(u\right) du\right) ds + \int_{5k+4}^{5k+5} q\left(s\right) \\
&\cdot \exp\left(\int_{5k+8}^{-s+10k+13} q\left(u\right) du\right) ds + \int_{5k+5}^{5k+6} q\left(s\right) \\
&\cdot \exp\left(\int_{5k+8}^{5k+8} q\left(u\right) du\right) ds + \int_{5k+6}^{5k+7} q\left(s\right) \\
&\cdot \exp\left(\int_{5k+8}^{4s-15k-16} q\left(u\right) du\right) ds + \int_{5k+7}^{5k+8} q\left(s\right) \\
&\cdot \left.\exp\left(\int_{5k+8}^{-3s+20k+33} q\left(u\right) du\right) ds\right] \simeq 0.6672 < 1, \\
&\liminf_{t\to\infty} \int_t^{\rho(t)} q\left(s\right) \exp\left(\int_{\rho(t)}^{\sigma(s)} q\left(u\right) du\right) ds \\
&= \liminf_{t\to\infty} \int_{5k+3}^{5k+4} q\left(s\right) \exp\left(\int_{5k+4}^{\sigma(s)} q\left(u\right) du\right) ds \\
&\simeq 0.1893 < \frac{1}{e}, \\
&\limsup_{t\to\infty} \int_t^{\rho(t)} q\left(s\right) \exp\left(\int_{\rho(t)}^{\sigma(s)} q_1\left(u\right) du\right) ds \simeq 0.7196 \\
&< 1, \\
&0.7196 < 1 - D\left(\beta\right) \simeq 0.9896.
\end{aligned} \tag{147}$$

That is, none of conditions (24), (25), (28), (29), (30) (for $j = 1$), and (31) (for $j = 1$) is satisfied.

Comment. The improvement of condition (116) over the corresponding condition (24) is significant, approximately 77.23%. We get this measure by comparing the values, in the left-hand side of those conditions. Also, the improvement over conditions (28) and (30) is very satisfactory, around 48.61% and 37.78%, respectively. In addition, condition (116) is satisfied from the first iteration, while conditions (30) and (31) need more than one iteration.

Remark 3. Similarly, one can provide examples, illustrating the other main results.

5. Concluding Remarks

In the present paper, we have considered the oscillatory dynamics of differential equations, having nonmonotone deviating arguments and nonnegative coefficients. New sufficient conditions have been established, for the oscillation of all solutions of (E) and (E′). These conditions include (37), (71), (75), (83), and (97) and (114), (116), (117), (118), and (119), for (E) and (E′), respectively. Applying these conditions involves a procedure that checks for oscillations by iteratively computing limsup and liminf, on terms recursively defined on the equation's coefficients and deviating argument.

The main advantage of these conditions is that they achieve a major improvement over all the related oscillation conditions for (E) [(E′)], in the literature. For example, condition (37) [(114)] improves upon the noniterative conditions that are reviewed in the introduction, namely, conditions (8) [(24)], (12), (16)≡(20) (for $j = 1$) [(28)≡(30) (for $j = 1$)], and (17)≡(21) (for $j = 1$) [(31) (for $j = 1$)]. That immediately becomes evident by inspecting the left-hand side of (37) [(114)] and the left-hand side of each of the above conditions.

The improvement of (37) [(114)] over the other iterative conditions, namely, (14) (for j > 2), (20) (for $j > 1$) [(30) (for $j > 1$)], and (21) (for $j > 1$) [(31) (for $j > 1$)], is that it requires far fewer iterations to establish oscillation than the other conditions.

This advantage, easily, can be verified computationally, by running the MATLAB programs (see Appendix), for computing limsup and liminf and comparing the number of iterations required by each condition to establish oscillation. Then we see that we achieve a significant improvement over all known oscillation criteria.

Another advantage and a significant departure from the large majority of the other studies is that the criteria in this paper apply to a more general class of equations, having nonmonotone arguments $\tau(t)$ or $\sigma(t)$, in contrast to most of the other oscillation criteria that apply to equations with nondecreasing arguments.

Appendix

In this appendix, for completeness, we give the algorithm on MATLAB software used in Example 1 for calculation of $\limsup_{t\to\infty} F_1(t) \simeq 1.0417$. For Example 2, the algorithm is omitted since it is similar to the one in Example 1.

Algorithm for Example 1

```
clear; clc;
c = .12;
n = 50;
λ0 = 1.14765;
a5 = 19;
b5 = 70/3;
h5 = (b5 − a5)/n;
for i5 = 1 : 1 : n + 1;
x5 = a5 + (i5 − 1) ∗ h5;
a4 = TFunction(x5);
b4 = 19;
h4 = (b4 − a4)/n;
for i4 = 1 : 1 : n + 1;
x4 = a4 + (i4 − 1) ∗ h4;
a3 = TFunction(x4);
b3 = x4;
h3 = (b3 − a3)/n;
for i3 = 1 : 1 : n + 1;
x3 = a3 + (i3 − 1) ∗ h3;
a2 = TFunction(x3);
b2 = x3;
h2 = (b2 − a2)/n;
for i2 = 1 : 1 : n + 1;
x2 = a2 + (i2 − 1) ∗ h2;
a1 = TFunction(x2);
b1 = x3;
h1 = (b1 − a1)/n;
for i1 = 1 : 1 : n + 1;
x1 = a1 + (i1 − 1) ∗ h1;
f1(i1) = c ∗ exp(λ0 ∗ c ∗ (x1 − TFunction(x1)));
end
I1 = f1(1) + f1(n + 1);
for i1 = 2 : 2 : n;
I1 = I1 + f1(i1) ∗ 4;
end
for i1 = 3 : 2 : n − 1;
I1 = I1 + f1(i1) ∗ 2;
end
I1 = I1 ∗ h1/3;
f2(i2) = c ∗ exp(I1);
end
I2 = f2(1) + f2(n + 1);
for i2 = 2 : 2 : n;
I2 = I2 + f2(i2) ∗ 4;
end
for i2 = 3 : 2 : n − 1;
I2 = I2 + f2(i2) ∗ 2;
end
I2 = I2 ∗ h2/3;
f3(i3) = c ∗ (1 + I2);
end
```

```
I3 = f3(1) + f3(n + 1);
for i3 = 2 : 2 : n;
I3 = I3 + f3(i3) * 4;
end
for i3 = 3 : 2 : n − 1;
I3 = I3 + f3(i3) * 2;
end
I3 = I3 * h3/3;
f4(i4) = c * exp(I3);
end
I4 = f4(1) + f4(n + 1);
for i4 = 2 : 2 : n;
I4 = I4 + f4(i4) * 4;
end
for i4 = 3 : 2 : n − 1;
I4 = I4 + f4(i4) * 2;
end
I4 = I4 * h4/3;
f5(i5) = c * exp(I4);
end
I5 = f5(1) + f5(n + 1);
for i5 = 2 : 2 : n;
I5 = I5 + f5(i5) * 4;
    end
    for i5 = 3 : 2 : n − 1;
    I5 = I5 + f5(i5) * 2;
    end
    I5 = I5 * h5/3
```

Algorithms for functions $\tau(t)$ and $h(t)$

```
function[a] = TFunction(x)
r = mod(x, 8);
k = floor(x/8);
if (r >= 0) && (r < 2)
a = x − 1;
end
if (r >= 2) && (r < 3)
a = −4 * x + 40 * k + 9;
end
if (r >= 3) && (r < 4)
a = 5 * x − 32 * k − 18;
end
if (r >= 4) && (r < 5)
a = −4 * x + 40 * k + 18;
end
if (r >= 5) && (r < 6)
a = 5 * x − 32 * k − 27;
end
if (r >= 6) && (r < 7)
a = −2 * x + 24 * k + 15;
end
if (r >= 7) && (r < 8)
a = 6 * x − 40 * k − 41;
end
end
function[a] = HFunction(x)
r = mod(x, 8);
k = floor(x/8);
if (r >= 0) && (r < 2)
a = x − 1;
end
if (r >= 2) && (r < 19/5)
a = 8 * k + 1;
end
if (r >= 19/5) && (r < 4)
a = 5 * x − 32 * k − 18;
end
if (r >= 4) && (r < 29/5)
a = 8 * k + 2;
end
if (r >= 29/5) && (r < 6)
a = 5 * x − 32 * k − 27;
end
if (r >= 6) && (r < 44/6)
a = 8 * k + 3;
end
if (r >= 44/6) && (r < 8)
a = 6 * x − 40 * k − 41;
end
end
```

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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