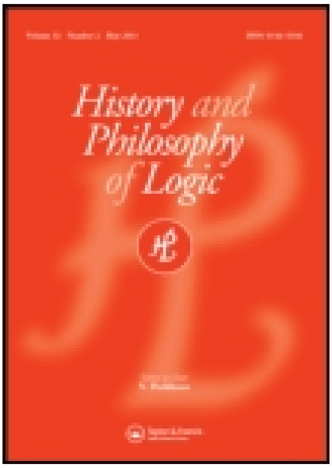

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The Cube, the Square and the Problem of Existential Import

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We re-examine the problem of existential import by using classical predicate logic. Our problem is: How to distribute the existential import among the quantified propositions in order for all the relations of the logical square to be valid? After defining existential import and scrutinizing the available solutions, we distinguish between three possible cases: explicit import, implicit non-import, explicit negative import and formalize the propositions accordingly. Then, we examine the 16 combinations between the 8 propositions having the first two kinds of import, the third one being trivial and rule out the squares where at least one relation does not hold. This leads to the following results: (1) three squares are valid when the domain is non-empty; (2) one of them is valid even in the empty domain: the square can thus be saved in arbitrary domains and (3) the aforementioned eight propositions give rise to a cube, which contains two more (non-classical) valid squares and several hexagons. A classical solution to the problem of existential import is thus possible, without resorting to deviant systems and merely relying upon the symbolism of First-order Logic (FOL). Aristotle's system appears then as a fragment of a broader system which can be developed by using FOL.

1. Introduction

The problem of existential import might be seen as a challenge to the theory of oppositions expressed by the traditional square of oppositions. For it seems that the different oppositions of the square, which originates from Aristotle and is explicitly defended by Apuleius and Boethius, do not go together harmoniously when the categorical propositions are formalized in the modern way; this invalidates the square as a whole. In modern logic, **A** and **E** are formalized as conditionals (true when Sx is false) and **I** and **O** as conjunctions; but formalizing the propositions in these ways makes modern logicians such as Russell[1] and Quine,[2] for instance, to cite the most famous ones, assume that **A** and **E** do not have an import while **I** and **O** do. This assumption and the formalizations that go with it invalidate most of the relations of the square, and validate only the contradiction between the pairs **A–O** and **E–I**. However, some logicians like Blanché defend the square despite these invalidations, being convinced that contradiction does not exhaust the sense of the word 'opposition'. These logicians think that a solution to the problem of existential import is possible, but neither of these solutions has been entirely convincing until now. The problem is: What are the distributions of the import that validate the square? Which propositions have an existential import? We provide a new solution to this problem in the present paper, which is entirely classical and uses the tools of modern predicate logic by demonstrating the validity of the relations of the square without any exception.

[1] The case of Frege is more complex; in his *Begriffsschrift,* he presents a square he assumes to be valid; but he formalizes the propositions in exactly the same way as Russell, and these formalizations *do invalidate* the square.

[2] See for instance *Quine 1950*, Chapter 15, French translation (p. 96) and *Russell 1959*, French translation, Chapter VI (p. 83) where both authors assume that the universals do not have an import, when translated by conditionals in the modern symbolism.

We will first consider the different 'solutions' available in the literature about the square and show their limits, before presenting our own one: the latter validates all the relations of the square, as can be shown by means of the well known truth-table method. Then, we will consider what ensues from this way of resolving the problem by thinking about the consequences that emerge

2. The problem of existential import

The theory of opposition illustrated by the so-called 'Aristotelian square' has been greatly improved and amended during the last 50 years by many logicians concerned with the notion of opposition. Among these logicians, let us mention the French philosopher Robert Blanché. Blanché supplemented the square with two new vertices **Y** and **U** (in *Blanché 1953, 1966*) by inventing new structures applicable to both quantified and modal propositions as well as some other kinds of logical relations. Thus, he showed that the **O** vertex should be completed by a more complex proposition including an affirmative side: 'Some Ss are Ps and some are not'; the contradictory of this new vertex is the following disjunction: 'No S is P or every S is P'. The same could be said about the modal propositions and applied to other expressions. This has increasingly developed the theory of opposition, for other more complex structures have been discovered since Blanché's seminal contribution. There is another crucial problem related to the theory of oppositions, however, in that it has to do with the central notion of logical consequence. It has not been treated with the same efficiency and lacks until now a really satisfactory solution, namely: the problem of existential import. The latter is related more specifically to the quantified propositions and stems from the very modern formalizations of these propositions. This problem makes the relations of the square incompatible with each other, thereby invalidating the square as a whole unless adequate formalizations are given to the different kinds of propositions. To characterize it more precisely, we will first define the notion of existential import itself. This notion is traditionally defined as follows:

DEFINITION 1 *A proposition has existential import if and only if it cannot be true unless its subject refers to some existing object(s).*

However, in modern predicate logic, the above definition merely applies to *singular* propositions which do contain a subject (see *Russell 1959*, French translation, p. 83). When the proposition is quantified the subject turns into a predicate, as shown by *Russell 1959*; for instance, 'Every Greek is mortal' is expressed as 'For all possible values of x, if x is Greek, x is mortal' (*Russell 1959*). The definition above has to be modified accordingly in order to fit with modern predicate logic.[3]

DEFINITION 2 *A quantified proposition has existential import if and only if Sx is true of some (real) value of the variable x.*

DEFINITION 3 *If Sx is true of some value of the variable x, then the proposition implies $(\exists x)Sx$.*

According to these definitions, the universal affirmative (**A**) has existential import in traditional logic since it entails the particular affirmative (**I**) and, consequently, $(\exists x)Sx$; but

[3] We thank one anonymous referee for pointing out the fact that modern mathematical logic rejects the notion of subject. This remark holds only for quantified propositions, however: Russell does talk about the subject when he analyses the singular propositions. This is why we provide the second definition above.

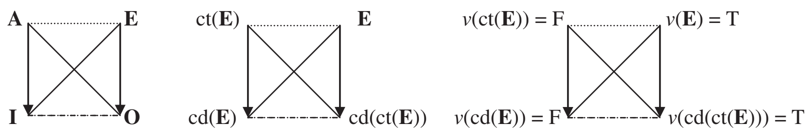
it does not have existential import in modern logic: it does not entail **I** or (∃x)Sx, because it is translated by a conditional which is true even when Sx is false. Indeed, **A** is formalized in the following way by modern and most contemporary logicians: (x)(Sx ⊃ Px), while **I** is formalized as (∃x)(Sx ∧ Px). However, with these modern formalizations of **A** and **I**, the square is either *inappropriate* or *invalid* because the universal propositions do *not* have import.

In the first case, when S is an empty term, one can be led to inappropriate squares. To illustrate this, let S stand for the term 'griffin' and P for 'evil'. Then:

A: Every S is P = 'Every griffin is evil' = (x)(Sx ⊃ Px)
E: No S is P = 'No griffin is evil' = (x)(Sx ⊃ ∼Px)
I: Some S is P = 'Some griffin is evil' = (∃x)(Sx ∧ Px)
O: Some S is not P = 'Some griffin is not evil' = (∃x)(Sx ∧ ∼Px)

Assuming that there is no griffin, then according to the logic of oppositions:

- that there is no griffin is true, so that $v(\mathbf{E}) = \mathrm{T}$
- its contrary must be false, so that $v(\mathrm{ct}(\mathbf{E})) = v(\mathbf{A}) = \mathrm{F}$
- the contradictory of its contrary must be true, so that $v(\mathrm{cd}(\mathrm{ct}(\mathbf{E}))) = v(\mathrm{cd}(\mathbf{A})) = v(\mathbf{O}) = \mathrm{T}$
- according to the square, the truth of **E** entails that **O** is also true (sb(T) = T)[4]



This seems to be an incorrect square, for given that **O** is formalized in modern logic by the formula (∃x)(Sx ∧ ∼Px) which is an existential proposition, it could not be true if Sx is false, as the case at stake assumes. Consequently, the truth-value $v(\mathbf{O}) = \mathrm{T}$ seems inappropriate in this particular case.

In the second case, assuming that $v(\mathrm{Sx}) = \mathrm{F}$ and that universal propositions do not have import, then:

- if $v(\mathrm{Sx}) = \mathrm{F}$, then **E** is true, i.e. $v((\mathrm{x})(\mathrm{Sx} \supset \sim\mathrm{Px})) = \mathrm{T}$
- if $v(\mathrm{Sx}) = \mathrm{F}$, then **A** is true, i.e. $v((\mathrm{x})(\mathrm{Sx} \supset \mathrm{Px})) = \mathrm{T}$
- in this same case, **E** being true, then by subalternation **O** must also be true.

But then, we have the following truth-values:

$$v(\mathbf{A}) = \mathrm{T};\quad v(\mathbf{E}) = \mathrm{T};\quad v(\mathbf{I}) = \mathrm{F}; v(\mathbf{O}) = \mathrm{T}$$

which lead to an invalid square, since $v(\mathbf{A}) = v(\mathbf{O}) = \mathrm{T}$.

This shows that, when subalternation is considered as valid, the contradictory relation is not valid; this invalidates the whole square, since its relations are not equally admissible. However, given the way the propositions are formalized, it is not subalternation that is admitted by modern logicians but rather contradiction for, in modern logic, subalternation is *not* valid because a conditional cannot imply a conjunction, whether the propositions are affirmative or negative, for **E** does not entail **O** either.

[4] Here is a list of opposite-forming operators to express oppositions between propositions: cd = contradictory; ct = contrary, sb = subaltern and sct = subcontrary.

As to the other relations of the square, contrariety is definable as the validity of the negated conjunction between **A** and **E**, that is as $\models \sim(\mathbf{A} \wedge \mathbf{E})$, and subcontrariety is expressed by a valid inclusive disjunction, that is by $\models \mathbf{I} \vee \mathbf{O}$.[5] These relations are not valid if the propositions are formalized in the modern way, since there are some cases of falsity in the lines of the corresponding tables.

Consequently, there is a difficulty related to existential import: if we deny this import for the universal propositions, then most of the relations of the square become unacceptable because they are no more valid. Does this mean that the square is incompatible with the modern formalizations of the propositions? The fact is that what invalidates most of the relations of the square is the assumption that universals do not have an import, while particulars do. Such an assumption can be questioned, however, since there are many possible interpretations of the quantified propositions. The problem consists in determining which interpretations validate the square. Two questions can thus be raised:

(1) How to solve the problems encountered whenever there is no S?
(2) How to account for the meaning of {**A**,**E**,**I**,**O**} in natural language and formal logic?

In other words, what are the combinations that make all the relations valid, whether the propositions have an import or not? This seems to require the independence of logic from extra-logical assumptions, in order for the logical square to be a reliable logical tool in the applied contexts.

We will show in this paper, that the assumption of modern logicians is false, while the modern symbolism is still adequate and can be used to reconcile modern logic with the square, by considering that all the propositions *may have an import or not* and formalizing them accordingly (i.e. in different ways in each case).

But before presenting our own view, let us see what the different available 'solutions' are in the literature. These 'solutions' are given by *Horn* 2001 (p. 24), in the following quotation:

There are, as it happens, at least four distinct ways of answering such questions:

(i) Existential import is determined by the *quality* of the proposition: affirmative (**A** and **I**) entail existence, while negative ones (**E** and **O**) do not.
(ii) Existential import is determined by the *quantity* of the proposition: universals (**A** and **E**) have no existential import, while particulars (**I** and **O**) do.
(iii) Existential import corresponds to a *presupposition* associated with **A**, **E**, **I** and **O** propositions.
(iv) The question of existential import is entirely absent from the square of opposition.

Let us examine these different 'solutions' and see if they really save the relations of the square.

3. The proposed 'solutions' to the problem

The aforementioned quotation clearly summarizes the main answers to the question, for it considers different ways of distributing the existential import among the propositions. Do the alleged 'solutions' save the square or not?

3.1. Import and quality

The first one states that only affirmative propositions have existential import, the negative ones being free of it; Terence Parsons endorsed this position in *Parsons 2006*, 2008. In the

[5] Contrariey, subcontrariety, subalternation and contradiction are defined as *valid* relations, not only as disjunctions, or negations etc., as has been rightly noticed by one anonymous referee whom we thank for stressing that point.

vein of some medieval logicians, Terence Parsons' argument is the following: if the subject does not exist, then **I** is false (which is perfectly rational); therefore, **E** will be true as the contradictory of **I**, which makes consequently **O** true by subalternation and **A** false by contradiction. Therefore, **A** is false when the subject does not exist, which means that it must have existential import to be true. And since **O** and **E** are true in this case, they do not have existential import. This argument leads to the view that '*affirmatives* have existential import, and *negatives* do not'.[6] But although it seems convincing at first sight, it has some defaults when we examine it more scrupulously. On the one hand, it presupposes the validity of the relations of the square; this cannot be accepted unless we do have evidence for this validity, thus making the argument somewhat circular by presupposing what has to be demonstrated. On the other hand, Terence Parsons does not say how one should formalize 'Not all S are P'; if we formalize it in the usual way[7] and construct truth-tables corresponding to the different relations as they are described in *Parsons 2006*, they show that the relations considered as valid by him are not. To prove this, let us see what ensues from these formalizations. If we formalize **A** by $(\exists x)Sx \wedge (x)(Sx \supset Px)$ (since it has existential import),[8] **O** by $\sim(x)(Sx \supset Px)$, and construct a truth-table by considering that there are only two existing entities x_1 and x_2 in the universe, we obtain a table where the following line (1 for 'true', 0 for 'false'):

$$((Sx_1 \vee Sx_2) \wedge ((Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)))\underline{\vee}\sim((Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2))$$

$$0 \quad 0\ 0 \quad 0 \quad 0 \quad 1\ 0 \quad 1\ 0 \quad 1 \quad 0 \quad \mathbf{0}\ 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0$$

shows that there is no contradiction: there *is* a case of falsity under $\underline{\vee}$ entailing that the exclusive disjunction is not valid. Since $\underline{\vee}$ is a good test for contradiction, this result shows that the relation of contradiction does not hold in the case considered by Parsons. This also means that the translation adopted for **O** is not the right one: translated in that way, it does have existential import as its equivalent $(\exists x)(Sx \wedge \sim Px)$ which cannot have but an explicit one. Therefore, 'Not all S are P' should be rendered in another way. But which one? Parsons does not give any answer to that question in *Parsons 2006*, and this is why his solution to the problem ought to be completed. We have to conclude that this first interpretation evoked by Horn would be satisfactory only in case one gives the right formalization of 'Not all S are P'. The latter needs to be formalized and completed in order for the demonstration of the validity of the relations to be convincing.[9] Moreover, we will show in the sequel that the combination he gives is not the only one that makes the relations of the square valid.

Returning to Horn's text, we find that he gives other names to illustrate this solution; these are for instance Buridan and other logicians (Apuleius, Boethius and Abelard) who, according to (*Horn 2001*) 'generalize' to quantified propositions what Aristotle said about singular propositions, i.e. that the negative ones have no import while affirmatives do. Buridan has even given a clear view about the way in which a universal proposition could be rendered false. He thought that the right contradictory of **A** should be equivalent to the following disjunction (formalized in a modern way in (*Horn 2001*)): $\sim(\exists x)Fx \vee (\exists x)(Fx \wedge \sim Gx)$. This makes it possible to find the contradictory of a sentence like 'All unicorns

[6] See *Parsons 2006*, Section 2.2.

[7] That is: $\sim(x)(Sx \supset Px)$, which seems to follow faithfully the way this proposition is expressed in natural language since 'Not' = $\sim$, 'All' = (x), 'S are P' = $S \supset P$, and is by the way the translation of **O** in *Frege 1879*.

[8] *Kleene 1967* expresses the existential import of **A** in this way.

[9] However, *Parsons 2008* gives the equivalent of the right formalization of **O**, by saying that **O** means: 'either nothing is A or something is A that is not B' (6, we have corrected an error in the original text). This corresponds to Buridan's solution, which will be considered later.

are equine', given as an example by Horn and which can be falsified either by the non-existence of unicorns or by the existence of unicorns that would not be equines. This solution has also been defended by some modern authors such as Brentano, Peirce and Thompson, according to Horn. Our own solution retains it partly, relying on a formula which is equivalent (by De Morgan's law) to that of Buridan while using conjunction instead of disjunction. Thus when **A** has existential import, 'not **A**' is expressed in the following way: $\sim[(\exists x)Fx \wedge (x)(Fx \supset Gx)]$, where the negation is external since it puts on the whole formula. However, this first solution has the inconvenience of constraining **A** to have existential import, which could be seen as an arbitrary restriction by modern logicians since **A** could be true of empty terms. One must look for a more general solution accordingly. Furthermore, it makes the negative particular free of existential import, which is not very satisfactory since **O** has existential import in most cases. Some logicians, namely the modern logicians, go on saying that particulars always have existential import since they are translated as existential propositions.

3.2. Import and quantity

The second interpretation is expressed as follows: '**A** and **E** have no existential import while **I** and **O** do'. This is the way Boole interprets among other ones the existential import of the propositions. It is also the Russellian position, because in modern logic universal propositions have no existential import while particulars do. It has already been shown that, when formalized in the modern way, the universals are true if Sx is false, because the conditionals are true in that case, while the particulars are false, because the conjunctions are false if Sx is false. Consequently, both subalternations do not hold. Moreover, the two conditionals expressing the universals are respectively equivalent to $\sim(\exists x)(Fx \wedge \sim Gx)$ for **A** and $\sim(\exists x)(Fx \wedge Gx)$ for **E,** which explicitly say that there is no object satisfying both predicates hence explicitly assume the non-existence of these objects (i.e. those satisfying both Fx and Gx).

These formulas do *not* say that Fx is never satisfied, however; rather, they say that the formula can be true, irrespective of whether Fx is satisfied or not. Hence, we can say that the modern formulas corresponding to the universal propositions are more neutral than explicit about the import: they do not say that Sx is not satisfied; rather, what they say is that the proposition can be true *even* when Sx is not satisfied. They do not have an explicit import, nor do they deny the import explicitly; they merely stand in the middle, because they do not argue for or against the import. The reason why the universal propositions are interpreted in this way might be that the import is not obvious in mathematics and in other scientific fields, so that it is better to avoid assuming it explicitly.

This interpretation does not save the square, however, as we have seen above. It is, then, not quite satisfying if not entirely unsatisfying since it does not preserve most of the relations of the square. Their unsatisfying character is also related to the fact, noticed by Horn, that the modern symbolization makes the proposition always true whenever its subject is empty, and furthermore true *because* the subject is empty, which is not acceptable if we consider that some sentences talking of non-existent objects might be false.[10]

3.3. Import and presupposition

As to the third interpretation, that is, the one which states that existential import is 'associated' with 'existential *presuppositions*' related to all kinds of propositions, Horn imputes it to Peter Strawson and Herbert Hart who explicitly talk about *presuppositions*. According to Strawson, the notion of presupposition is different from implication because a proposition could presuppose the existence of the subject without implying it. For instance,

[10] See *Horn 2001*.

the Russellian sentence 'The present king of France is wise' does not imply 'There is a king of France' but presupposes it. Therefore, if the presupposition is false (since there is no king of France nowadays) then the sentence 'The king of France is wise' is not false: it simply has no truth-value; whereas in Russell's argument, if 'There is a king of France' is false then 'The present king of France is wise' will be false. The Russellian sentence which has challenged so many people does not lead to a falsity but, rather, to a proposition which is neither true nor false. This makes presupposition different from implication as it is usually understood. This view can be used to save the square since subalternation, for instance, could be validated without having to add the clause (∃x)Sx to the modern formula of **A**. If the existence of the subject's referent is presupposed, then subalternation holds; but if it is not, then the propositions are neither true nor false, which is compatible with the validity of subalternation since this relation holds only when propositions have a truth-value and does not apply to those which have neither value.

But this view is not quite satisfying, for two main reasons.

First, as noticed in *Linsky 1967* (p. 134), the notion of presupposition 'contains'[11] that of implication:

> Suppose that S presupposes S′,[12] this means that from the premise saying that S has a truth-value, it follows that S′ is true. But if S is true it follows that S has a truth-value. Therefore if S is true, it follow that S′ is true. Now S is true if and only if the king of France is wise and S′ is true if and only if one and only one person is a king of France. Therefore the statement 'the king of France is wise' implies the statement 'there is one and only one king of France'.

Linsky deduces from his argument that Strawson's theory is not incompatible with Russell's theory and, above all, that 'if a statement presupposes another one, it implies it too' (see *Linsky 1967*, p. 139). This conclusion is especially important for us: it shows that the notion of presupposition is not very different from implication and can be handled in classical terms; this makes Strawson's theory criticizable.

Second, Strawson relies on a non-classical logic because he overtly mentions propositions which are neither true nor false; this has been criticized by Smiley, for instance, who showed that conversion does not hold in this case. Smiley's argument is the following: if we consider a universal negative like 'No A is B' and if A exists but B does not, then since each proposition has a truth-value only when its subject's referent exists but does not have a truth-value when its subject's referent does not exist, the **E**-conversion which leads from 'No A is B' to 'No B is A' would lead from a true proposition to a proposition that is neither true nor false. A parallel argument is given for other rules like inversion. This result is not acceptable, because it invalidates some central rules in traditional logic. Moreover, Strawson's view admits two definitions of implication and three definitions of contradiction and inconsistency; this is neither very satisfactory nor convincing, according to Smiley, since it uselessly complexifies the theory.

Strawson and Hart's solution leads to a restriction of the domain of application of the square, *Horn 2001* (p. 29) noticed for:

> We can no longer say with Aristotle that of any two contradictories (**A/O**, **E/I**) one must be true simpliciter, but only that if either is true the other is false (and vice versa).

This is not compatible with the traditional square and its underlying logic, since Aristotle's logic is classical (i.e. bivalent). Strawson and Hart's interpretation goes then beyond the

[11] *Linsky 1967* (p. 134).
[12] S = 'The present king of France is wise' and S′ = 'There is one and only one king of France'.

Aristotelian theory and is not in accordance with it. Saving the square is not what has been made by Strawson; rather, his theory is a personal interpretation of how the square should be viewed and this interpretation is not Aristotelian at all.

The notion of presupposition, as understood by Strawson, does not therefore solve all the problems and seems to create further ones while requiring a non-classical logic. Since we want to use a classical framework preferably, because any solution available in such a framework would be much better and simpler, we will try to avoid this notion or to express it explicitly in a classical framework. Following Linsky's objection, we find that presuppositions can be expressed explicitly by means of existential premises. This is shown very clearly by Kleene, for instance, or even *Quine 1950*. Therefore, we do not need to construct an alternative logic to express them: these are perfectly expressible in our classical framework, as we have seen earlier. The problem arises when we consider that *all* propositions have existential import. For if we consider for instance that **E** has existential import, i.e. is expressed thus: $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$, then the relation of contradiction **I**–**E** does not hold (see Appendix A, Table 2). In a nutshell when all the propositions of the square have an import, the relations are not all validated and the square as a whole is not valid. This refutes also a common interpretation according to which Aristotle's logic does not admit empty terms.[13] Thus *Łukasiewicz 1951*,[14] for instance, says 'Aristotle does not introduce into his logic singular or empty terms or quantifiers [. . .] Singular, empty, and also negative terms are excluded as values' (cited in *Read 2012*, p. 1).

As to Geach's solution, which is given in his paper 'Subject and Predicate' (1950), it also seems to rely on a non-classical logic: Geach assumes that whenever the subject's referent is non-existent, the square is not invalid but rather inadequate because in this case the propositions are neither true nor false. He says: 'If 'S' is a pseudo-name like 'dragon' or 'round square', and names nothing, *none of the above forms has a truth-value*, and so the 'square of opposition' becomes not invalid but *inapplicable*' (*Geach 1950*, p. 480, emphasis added). But this admission of truth-value gaps is not in accordance with either Aristotelian logic or traditional logic, which gives determinate values to all categorical propositions: they are either true or false, even when the subject's referent is non-existent. For instance, Aristotle says that a singular proposition whose subject's referent is non-existent is true when it is negative and false when it is affirmative. The same could be said about quantified propositions, as many commentators assume it. In addition, as T. Smiley has rightly noticed, this admission of truth-value gaps invalidates some main rules of traditional logic. For this reason, it is no more admissible than Strawson's solution.

However, in *Logic Matters* (*Geach 1972*, §2.1, p. 64), Geach gives the following interpretations to the quantified propositions:

'S a P' is read as '$S = \wedge . P = \wedge . \upsilon . S = \vee . P = \vee . \upsilon . S \neq \wedge . P \neq \vee . S \subset P$'

'S e P' is read as '$S = \wedge . P = \vee . \upsilon . S = \vee . P = \wedge . \upsilon . S \neq \wedge . P \neq \wedge . S \cap P = \wedge$'

'S i P' is read as '$S = \wedge . P \neq \vee . \upsilon . S \neq \vee . P = \wedge . \upsilon . S \cap P \neq \wedge$'

'S o P' is read as '$S = \wedge . P \neq \wedge . \upsilon . S \neq \vee . P = \vee . \upsilon . \sim(S \subset P)$'

These interpretations make use of the logic of classes and assume that **A** is true whenever S is included in P, or both are the null class or both are the universal class; that **E** is true when there is no common element between S and P, or S is the null class and P the universal

[13] This interpretation has been defended by many people, such as *Kneale and Kneale 1962*, *Patzig 1968* and some others. Almost all of them are cited in *Read, 2012* (pp. 1–2)

[14] We thank Professor Read for reading a previous version of this paper and asking about the place of Łukasiewicz's theory in this classification.

class, or vice versa; that **I** is true whenever there is a common element between S and P, or S is the null class while P is not the universal class, or else S is different from the universal class while P is the null class; and finally, that **O** is true whenever S is not included into P or S is the null class while P is not the null class or S is not the universal class and P is the universal class. In these interpretations (*Geach 1972*, p. 64) claims, all the relations of the square are valid, for he says what follows:

> This interpretation preserves the square of opposition, and all and only those syllogisms which are traditionally so regarded.

However plausible this solution, it does not tell exactly in which cases **A** implies **I**, or **A** is contrary to **E,** and the like, because Geach does not give the precise combinations that validate all the relations. In which case does SaP imply SiP, for instance? Should we say that the case where SaP is read as $S = \wedge$ and $P = \wedge$ and SiP is read as $S \neq \vee$ and $P = \wedge$ is the one that makes SaP imply SiP? Or is it another case?

In addition, Geach states hereby the cases where the propositions do not have import while he considered in his article that these propositions are inappropriate, so that the square is 'inapplicable' to them. Does this mean that his opinion changed meanwhile? Or is it just a different way to express the same opinion? In any case, he does not give a precise analysis of the relations of the square and the cases where they are valid. Geach's theory is also criticized by *Seuren 2012b* (p. 134), who says the following:

> Geach proposed the theory of *restricted quantification*, where variables rotate not over the whole of ENT but only over [[F]], the extension of the restrictor predicate. This, however, landed Geach and his followers in the predicament that in cases where $[[F]] = \emptyset$ no substitution is possible for the matrix predicate G(x), so that the truth-value of propositions like $\exists x : F(x) \mid [G(x)]$ or $\forall x : F(x) \mid [G(x)]$ must remain undecided when $[[F]] = \emptyset$. Geach never solved this problem (...)

Restricting the quantification to the extension of the predicate Fx is thus not only arbitrary, but highly unsatisfactory, because it makes the theory not applicable to empty terms. As rightly noticed by *Read 2012* (pp. 8–9), who shares Seuren's opinion in this particular point,

> Aristotle's system would be 'non-valid', that is, useless as a logical system, if it applied only to non-empty terms, for there is no logical guarantee that a term is non-empty.

3.4. No import in the square

We have finally to consider the fourth and last position, to the effect that 'the question of existential import is entirely absent from the square of opposition'. This position is defended by Nelson, for instance, who argues in *Horn 2001* (p. 30) that

> Universals cannot be claimed to have or to lack existential import: they are simply neutral. Thus any **A**-Form proposition like *All ogres are wicked* – will entail the corresponding **I**-proposition (*some ogres are wicked*) since the question of import (are there ogres?) need never be broached: the question of existential import is entirely absent from the square of opposition.

This is not quite satisfying either, even if it is true that universals are neutral (in the modern sense for instance), it is not true that the question of existential import does not influence the relations of the square and that these relations remain valid whatever import the propositions may have. As we have seen, questions of existence cannot be excluded from the square because, if they were, we would have a Strawsonian theory which is not compatible with Aristotelian logic; and if we express sentences in the modern way, their very formalization includes questions of import since the particular propositions are not free of it as they are expressed by modern logic. Questions of existence are then crucial for the validity of the relations of the square, in the sense that this validity is related to them if we stick to a classical framework. Therefore, if one wants to make all the relations of the square go together consistently, one has to solve the problem of existential import by showing which propositions have import and which ones do not. We could add that the relation of opposition is much richer than contradiction, which is the only one accepted by modern logicians as being expressible in several ways, and this is why many philosophers and logicians actually give interest to the square. But this interest makes it even more crucial to solve the problem of existential import, since it creates some kind of discomfort to all those who study the square and think that it has many things to teach us about oppositions.

It follows from this that none of the four main positions is quite satisfying. If we consider that Horn's summary evokes the state-of-the-art results in the contemporary literature, then the problem of existential import is still quite vivid and has not yet been solved. This is why we want to return to this problem in order to solve it in preference with classical tools, since the modern symbolism is largely able to translate existential presuppositions as well as all kinds of interpretation of the quantified propositions. But since we wish to examine all the possible readings of all the propositions, we will start by scrutinizing these different propositions together with their negations in order to identify the contradictories of each kind of proposition. This will require an examination of the role of negation and its place in a formula.

4. Where should negation be?

First of all, we assume that the solution can and should be found in classical predicate logic with its usual tools and symbolism (i.e. the Russellian symbolism), because this symbolism can express all kinds of propositions and takes into account the non-existent objects as well; the logic of FOL is also adequate for our purpose because it is bivalent (such as traditional logic), unlike the systems which use the notion of presupposition. This is an advantage over the other alternative symbolisms, such as Geach's *restricted quantification*, which do not solve the problem adequately because they leave the question of non-existent objects outside the frame. The solution we will give owes to Russell many things apart from the symbolism, such as the different scopes of the negation which are involved in what contemporary logicians call 'external negation' and 'internal negation'. If Russell himself did not give it, it might be because the problem did not interest him that much or because his aim was not primarily to save Aristotelian logic since, as is well known, he criticized this logic and is one prominent champion of the modern position.

But before turning to this solution, let us first examine all the possibilities that could be used to translate the quantified propositions. For we think that an exhaustive review of all the possible cases is the only way to find a really convincing solution to the problem.

To begin with, let us first see what are the possibilities regarding the existential import itself. There seems to be exactly three options:

(a) There is an explicit import.

This means that some S exists, or to use a modern language, some existing object satisfies the predicate Sx. We have to add $(\exists x)Sx$ to the proposition. In this case, the proposition implies $(\exists x)Sx$. We will translate it by 'imp!' and define it as follows:

DEFINITION 4 *For any proposition* **X** *in* {**A**,**E**,**I**,**O**}:

$$\mathbf{X}_{\text{imp!}} =_{df} (\exists x)Sx \wedge \mathbf{X}$$

(b) *There is no explicit import.*

This case is neutral, in the sense that such a proposition does not imply either $(\exists x)Sx$ or $\sim(\exists x)Sx$. It corresponds to what we generally call a case of non-import and amounts to the modern position with respect to the universal propositions. We will translate it by 'imp?' and define it as follows:

DEFINITION 5 *For any proposition* **X** *in* {**A**,**E**,**I**,**O**}*:*

$$\mathbf{X}_{\text{imp?}} =_{df} \sim[(\exists x)Sx \wedge \sim\mathbf{X}]^{15}$$

(c) *There is an explicit non-import.*

This means that no S exists (quite explicitly). We have to add therefore '$\sim(\exists x)Sx$' to the proposition. It is a kind of negative import, since we assume explicitly the inexistence of objects satisfying Sx. We will translate this by '$\sim$imp!' and define it as follows:

DEFINITION 6 *For any proposition* X *in* {**A**,**E**,**I**,**O**}*:*

$$X_{\sim\text{imp!}} =_{df} \sim(\exists x)Sx \wedge \mathbf{X}$$

In this case, the proposition implies $\sim(\exists x)Sx$.
Accordingly,

(1) $\mathbf{A}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset Px)$ differs both from $\mathbf{A}_{\text{imp?}} : \sim[(\exists x)Sx \wedge (\exists x)(Sx \wedge \sim Px)]$ (which by simplification becomes: $\sim(\exists x)(Sx \wedge \sim Px) \equiv (x)(Sx \supset Px)$) and from $\mathbf{A}_{\sim\text{imp!}} : \sim(\exists x)Sx \wedge (x)(Sx \supset Px)$.
(2) $\mathbf{E}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset \sim Px)$ differs from $\mathbf{E}_{\text{imp?}} : (x)(Sx \supset \sim Px)$ (obtained by the same simplification as above) and also from $\mathbf{E}_{\sim\text{imp!}} : \sim(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$.
(3) $\mathbf{I}_{\text{imp!}} : (\exists x)Sx \wedge (\exists x)(Sx \wedge Px)$ (i.e. $(\exists x)(Sx \wedge Px)$, by simplification) differs from $\mathbf{I}_{\text{imp?}} : \sim[(\exists x)Sx \wedge (Sx \supset \sim Px)]$ and also from $\mathbf{I}_{\sim\text{imp!}} : \sim(\exists x)Sx \wedge (\exists x)(Sx \wedge Px)$.
(4) $\mathbf{O}_{\text{imp!}} : (\exists x)Sx \wedge (\exists x)(Sx \wedge \sim Px)$ (i.e. $(\exists x)(Sx \wedge \sim Px)$, by simplification) differs from both $\mathbf{O}_{\text{imp?}} : \sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$[16] and $\mathbf{O}_{\sim\text{imp!}} : \sim(\exists x)Sx \wedge (\exists x)(Sx \wedge \sim Px)$.

Furthermore,

$$\mathbf{A}_{\sim\text{imp!}} \equiv \mathbf{E}_{\sim\text{imp!}} \equiv \sim(\exists x)Sx;$$

$\mathbf{I}_{\sim\text{imp!}}$ and $\mathbf{O}_{\sim\text{imp!}}$ are self-contradictory.

To summarize: there are three possibilities for **A** and **E**, but in the third possibility (i.e. when the existence of the subject is explicitly denied) they are both equivalent to $\sim(\exists x)Sx$.

[15] Note that the negation in front of **X** may be explained as follows: As shown below, $\mathbf{A}_{\text{imp?}}$, for instance, which is generally formalized as $(x)(Sx \supset Px)$, that is $\sim(\exists x)(Sx \wedge \sim Px)$, is obtained by simplification from the following formula: $\sim[(\exists x)Sx \wedge (\exists x)(Sx \wedge \sim Px)]$ (or, equivalently, $\sim[(\exists x)Sx \wedge \sim(x)(Sx \supset Px)]$); this formula is a strict application of our definition. The same simplification applies to $\mathbf{E}_{\text{imp?}}$. As to $\mathbf{O}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp?}}$, no simplification is possible, so that they are stated as follows: $\mathbf{O}_{\text{imp?}} : \sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$ and $\mathbf{I}_{\text{imp?}} : \sim[(\exists x)Sx \wedge (Sx \supset \sim Px)]$ as will appear below. These formulas are also strict applications of definition 5.

[16] Some authors like Allan Bäck have mentioned this formalization of **O** (see e.g. *Bäck 2000*, p. 242). It is equivalent to the medieval formula (given above) which preferably uses disjunction.

There are three possibilities for **I** and **O**, but in the third possibility (i.e. when the existence of the subject is explicitly denied) **O** is equivalent to **I** because both are self-contradictory in that case. $\mathbf{O}_{imp?}$ and $\mathbf{I}_{imp?}$ contain an external negation (putting on the whole formula), while $\mathbf{O}_{imp!}$, for instance, contains an internal negation (putting on the predicate). The difference between these propositions has to do with the scope of negation. Actually, our reasoning is similar to that of Leibniz who said that, since the universal propositions are the contradictories of the particular propositions, they ought to be translated simply as their negations.[17] In the same way, we say that since $\mathbf{I}_{imp?}$ and $\mathbf{O}_{imp?}$ are the contradictories of $\mathbf{E}_{imp!}$ and $\mathbf{A}_{imp!}$, they ought to be translated as their negations. The translations we have adopted neutralize the import of the usual particular propositions in that they do not imply $(\exists x)Sx$. This is why they give a really new insight and make it possible to solve the problem of existential import adequately. With such particulars, one does not reject the import explicitly nor does one admit it explicitly; rather, one sees it exactly as the modern logicians see the import for the universals: the particulars become as neutral as the modern universals.

Assuming that $\mathbf{A}_{\sim imp!}$ and $\mathbf{E}_{\sim imp!}$ are both equivalent to $\sim(\exists x)Sx$ and that $\mathbf{I}_{\sim imp!}$ and $\mathbf{O}_{\sim imp!}$ are both equivalent and self-contradictory, we have to calculate the number of cases for the first and the second import; the third is very special and even trivial, since squares that contain one or more contradictory propositions as vertices are not likely to have valid relations; we will then leave this case aside, because of its triviality. If we consider only 'imp!' and 'imp?', we have the following $2^4 = 16$ possibilities:

(I)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp!}$
(II)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp?}$
(III)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp!}$
(IV)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp?}$
(V)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp!}$
(VI)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp?}$
(VII)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp!}$
(VIII)	$\mathbf{A}_{imp!}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp?}$
(IX)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp!}$
(X)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp?}$
(XI)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp!}$
(XII)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp!}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp?}$
(XIII)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp!}$
(XIV)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp!}$	$\mathbf{O}_{imp?}$
(XV)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp!}$
(XVI)	$\mathbf{A}_{imp?}$	$\mathbf{E}_{imp?}$	$\mathbf{I}_{imp?}$	$\mathbf{O}_{imp?}$

Let us now check these possibilities in order to see which ones, if any, make *all* the relations of the square valid. For this purpose, we will proceed as follows: since we look for squares in which all the relations are valid, it suffices that only one of these relations be not valid to rule out the square in consideration. In order to find out which squares are admissible, we have to check the relations one by one and to consider how the import and non-import influence these relations.

[17] See *Couturat 1901* (p. 350).

We will define the relations of the square as follows[18]:

$$\begin{array}{lll} \alpha \text{ and } \beta \text{ are contrary (CT)} & \text{iff} & \models \sim(\alpha \wedge \beta) \\ \alpha \text{ and } \beta \text{ are contradictory (CD)} & \text{iff} & \models \alpha \veebar \beta \\ \alpha \text{ and } \beta \text{ are subcontrary (SCT)} & \text{iff} & \models \alpha \vee \beta \\ \alpha \text{ and } \beta \text{ are subaltern (SB)} & \text{iff} & \models \alpha \supset \beta \end{array}$$

Let us start with the contradictory relation, given that it involves all the propositions and will help to rule out the greatest number of possibilities. If we combine all kinds of propositions, we have several 'contradictions' which are the following: (i) $\mathbf{E}_{\text{imp!}}$ and $\mathbf{I}_{\text{imp!}}$, (ii) $\mathbf{E}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp?}}$, (iii) $\mathbf{A}_{\text{imp!}}$ and $\mathbf{O}_{\text{imp!}}$, (iv) $\mathbf{A}_{\text{imp?}}$ and $\mathbf{O}_{\text{imp?}}$, (v) $\mathbf{E}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp!}}$, (vi) $\mathbf{E}_{\text{imp!}}$ and $\mathbf{I}_{\text{imp?}}$, (vii) $\mathbf{A}_{\text{imp!}}$ and $\mathbf{O}_{\text{imp?}}$, (viii) $\mathbf{A}_{\text{imp?}}$ and $\mathbf{O}_{\text{imp!}}$.

The formalizations of the eight propositions yield the following:

$$\mathbf{A}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset Px)$$

$$\mathbf{E}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset \sim Px)$$

$$\mathbf{I}_{\text{imp!}} : (\exists x)(Sx \wedge Px)$$

$$\mathbf{O}_{\text{imp!}} : \sim(x)(Sx \supset Px)[or(\exists x)(Sx \wedge \sim Px)]$$

$$\mathbf{A}_{\text{imp?}} : (x)(Sx \supset Px)$$

$$\mathbf{E}_{\text{imp?}} : (x)(Sx \supset \sim Px)$$

$$\mathbf{I}_{\text{imp?}} : \sim\{(\exists x)Sx \wedge (x)(Sx \supset \sim Px)\}$$

$$\mathbf{O}_{\text{imp?}} : \sim\{(\exists x)Sx \wedge (x)(Sx \supset Px)\}$$

Here, is the first case:

$$(\#1)\ \ \mathbf{E}_{\text{imp!}} \veebar \mathbf{I}_{\text{imp!}}$$

That is: $[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)] \veebar (\exists x)(Sx \wedge Px)$
Assuming that there are two elements $\{x_1, x_2\}$ in the universe,[19]

$$\mathbf{E}_{\text{imp!}} : \{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$$

$$\mathbf{I}_{\text{imp!}} : [(Sx_1 \wedge Px_1) \vee (Sx_2 \wedge Px_2)]$$

The whole formula is shown to be invalid (see Appendix A, Table 2), since the exclusive disjunction *can* be false. This rules out the squares nos. **I**, **II**, **IX** and **X**.

As to the second case, that is:

$$(\#2)\ \ \mathbf{E}_{\text{imp?}} \veebar \mathbf{I}_{\text{imp?}}$$

it is shown not to be valid (see Appendix A, Table 3). This rules out the squares nos. **VII**, **VIII**, **XV** and **XVI**.

[18] These definitions correspond to the traditional definitions in terms of truth and falsity, that is CT: the propositions are never true together but possibly false together; CD: the propositions are never true nor false together; SCT: the propositions are never false together but possibly true together and SB: either both propositions are false or both are true, and β is always true when α is true.

[19] We consider *two* elements (and not only *one*) because what is valid in a universe with *one* element may not be valid in universes containing more elements, whereas what is valid with *two* elements is valid for more than two elements. It appears then that a universe with two elements is the simplest way to validate the relations in *all* non empty domains.

In the same way, we can show that the third case, i.e. the following:

$$(\#3)\ \ \mathbf{A}_{\text{imp!}} \veebar \mathbf{O}_{\text{imp!}}$$

is not valid (see Appendix A, Table 4), which rules out two more squares, i.e. the squares nos. **III** and **V**(**I** and **VII** being already ruled out).

The invalidity of

$$(\#4)\ \ \mathbf{A}_{\text{imp?}} \veebar \mathbf{O}_{\text{imp?}}$$

(see Appendix A, Table 5) rules out two more squares which are nos. **XII** and **XIV** (**X** and **XVI** being already ruled out).

We can see therefore that the contradictory relation *rules out* the following squares: **I**, **II**, **III**, **V**, **VII**, **VIII**, **IX**, **X**, **XII**, **XIV**, **XV** and **XVI**. These squares cannot be interesting for us, since they contain at least one invalid relation and sometimes two (as in **I**, **VII**, **X** and **XVI**).

As to the other squares, that is, the squares nos. **IV**, **VI**, **XI** and **XIII**, they may be checked by the same method. The square no. **XIII** is the modern one and, as we know, its contradictory relations, which are $\mathbf{A}_{\text{imp?}} \veebar \mathbf{O}_{\text{imp!}}$ and $\mathbf{E}_{\text{imp?}} \veebar \mathbf{I}_{\text{imp!}}$ are valid.

However, in the square no.**XI** the contradictory relations are expressed by $\mathbf{E}_{\text{imp!}} \veebar \mathbf{I}_{\text{imp?}}$, and in the squares nos. **IV** and **VI** we also have $\mathbf{A}_{\text{imp!}} \veebar \mathbf{O}_{\text{imp?}}$

The propositions involved are the following:

$$\mathbf{E}_{\text{imp!}} :\ (\exists x)Sx\ \wedge (x)(Sx \supset\ \sim Px)$$

$$\mathbf{I}_{\text{imp?}} :\ \sim[(\exists x)Sx\ \wedge (x)(Sx \supset\ \sim Px)]$$

and

$$\mathbf{A}_{\text{imp!}} : (\exists x)Sx\ \wedge (x)(Sx \supset Px)$$

$$\mathbf{O}_{\text{imp?}} :\ \sim[(\exists x)Sx\ \wedge (x)(Sx \supset Px)]$$

The contradictions are valid, both of them having the compound form $(\ldots \veebar \sim \ldots)$.

Let us consider the following contrariety relations, i.e. the following:

In the square no. **IV**, we have: $(1) \sim(\mathbf{A}_{\text{imp!}} \wedge \mathbf{E}_{\text{imp!}})$

In the square no. **VI,** we have: $(2) \sim(\mathbf{A}_{\text{imp!}} \wedge \mathbf{E}_{\text{imp?}})$

In the square no. **XI**, we have: $(3) \sim(\mathbf{A}_{\text{imp?}} \wedge \mathbf{E}_{\text{imp!}})$

In the square no. **XIII**, we have: $(4) \sim(\mathbf{A}_{\text{imp?}} \wedge \mathbf{E}_{\text{imp?}})$

In the square no. **XIII,** which is the modern one, the contrariety (4) is invalid (see Appendix A, Table 1), so that the square no. **XIII** has to be ruled out.

But the three other contrarieties are all valid (see Appendix A, Tables 6–8). Hence contrariety holds in the squares nos. **IV, VI** and **XI.**

We need to check both subcontrariety and subalternation, however. As to subcontrariety, it is defined as a valid inclusive disjunction which relates the two particulars. Thus we have three formulas, namely: $\mathbf{I}_{\text{imp!}} \vee \mathbf{O}_{\text{imp?}}$ (square no. **VI**), $\mathbf{I}_{\text{imp?}} \vee \mathbf{O}_{\text{imp?}}$ (square no. **IV**) and $\mathbf{I}_{\text{imp?}} \vee \mathbf{O}_{\text{imp!}}$ (square no. **XI**). All these disjunctions are valid (see Appendix A, Tables 9–11).

As to the subalternations, we have $\mathbf{A}_{\text{imp!}} \supset \mathbf{I}_{\text{imp!}}$ in the square no. **VI**, which is valid without any doubt since both **A** and **I** have an import; then the two following: $\mathbf{A}_{\text{imp!}} \supset \mathbf{I}_{\text{imp?}}$

(square no. **IV**) and $\mathbf{A}_{\text{imp?}} \supset \mathbf{I}_{\text{imp?}}$ (square no. **XI**). Both are shown to be valid (see Appendix A, Tables 12 and 13).

The subalternations involving **E** and **O** are the following: $\mathbf{E}_{\text{imp?}} \supset \mathbf{O}_{\text{imp?}}$ (square no. **VI**), $\mathbf{E}_{\text{imp!}} \supset \mathbf{O}_{\text{imp?}}$ (square no. **IV**) and $\mathbf{E}_{\text{imp!}} \supset \mathbf{O}_{\text{imp!}}$ (square no. **XI**). All of them are shown to be valid (see Appendix A, Tables 14–16).

We can conclude that *all* the relations are valid *only* in the squares nos **IV**, **VI** and **XI**. Consequently, *all* these squares are valid in all non-empty domains. These squares are the following:

Square no. **IV**: $\mathbf{A}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{E}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset {\sim}Px)$
$\mathbf{I}_{\text{imp?}} : {\sim}[(\exists x)Sx \wedge (x)(Sx \supset {\sim}Px)]$
$\mathbf{O}_{\text{imp?}} : {\sim}[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

Square no. **VI**: $\mathbf{A}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{E}_{\text{imp?}} : (x)(Sx \supset {\sim}Px)$
$\mathbf{I}_{\text{imp!}} : (\exists x)(Sx \wedge Px)$
$\mathbf{O}_{\text{imp?}} : {\sim}[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

Square no. **XI**: $\mathbf{A}_{\text{imp?}} : (x)(Sx \supset Px)$
$\mathbf{E}_{\text{imp!}} : (\exists x)Sx \wedge (x)(Sx \supset {\sim}Px)$
$\mathbf{I}_{\text{imp?}} : {\sim}[(\exists x)Sx \wedge (x)(Sx \supset {\sim}Px)]$
$\mathbf{O}_{\text{imp!}} : (\exists x)(Sx \wedge {\sim}Px)$ [or the equivalent: ${\sim}(x)(Sx \supset Px)$]

They are drawn in the Appendix B.

Furthermore, the square no. **IV** is also valid in the empty domain since all the relations are valid in that square when there are no elements in the domain. Contradiction **A**–**O** holds, because $\mathbf{A}_{\text{imp!}}$ is false in the empty domain while $\mathbf{O}_{\text{imp?}}$ is true; **E**–**I** equally holds, because $\mathbf{E}_{\text{imp!}}$ is false while $\mathbf{O}_{\text{imp?}}$ is true in the empty domain; contrariety **A**–**E** holds, because both $\mathbf{A}_{\text{imp!}}$ and $\mathbf{E}_{\text{imp!}}$ are false in the empty domain; subcontrariety holds, because both $\mathbf{I}_{\text{imp?}}$ and $\mathbf{O}_{\text{imp?}}$ are true in the empty domain; and finally, the two subalternations hold because $\mathbf{A}_{\text{imp!}}$ and $\mathbf{E}_{\text{imp!}}$ are both false while $\mathbf{I}_{\text{imp?}}$ and $\mathbf{O}_{\text{imp?}}$ are both true, which makes $\mathbf{A} \supset \mathbf{I}$ and $\mathbf{E} \supset \mathbf{O}$ true in the empty domain.

These results show that the square is valid in empty as well as all non-empty domains. They also show that the existential import influences the different relations and their validity. This influence may be stated by the following theorems:

THEOREM 1 *Contradiction is valid only if both propositions have a different import.*

THEOREM 2 *Contrariety is valid only if both propositions have existential import or a different import.*

THEOREM 3 *Subcontrariety is valid only if both propositions have no existential import or a different import.*

THEOREM 4 *Subalternation is valid only if both propositions have the same import or the universals have existential import while the particulars do not.*

What about the laws of duality, that is, $\models \mathbf{A} \equiv {\sim}\mathbf{O}$ and $\models \mathbf{E} \equiv {\sim}\mathbf{I}$? Are they preserved by the new formulations of the propositions? We can show that they are preserved indeed, for consider the formulations of

$\mathbf{A}_{\text{imp!}}$ and $\mathbf{E}_{\text{imp!}}$, that is:

$$\mathbf{A}_{\text{imp!}} = (\exists x)Sx \wedge (x)(Sx \supset Px)$$
$$\mathbf{E}_{\text{imp!}} = (\exists x)Sx \wedge (x)(Sx \supset \sim Px)$$

These propositions are equivalent to $\sim\mathbf{O}_{\text{imp?}}$ and to $\sim\mathbf{I}_{\text{imp?}}$ respectively, for

$$\begin{aligned}\mathbf{A}_{\text{imp!}} &= \sim\mathbf{O}_{\text{imp?}} \text{ since } \sim\mathbf{O}_{\text{imp?}} = \sim\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]\\ &= (\exists x)Sx \wedge (x)(Sx \supset Px)\\ \mathbf{E}_{\text{imp!}} &= \sim\mathbf{I}_{\text{imp?}} \text{ since } \sim\mathbf{I}_{\text{imp?}} = \sim\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]\\ &= [(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]\end{aligned}$$

and

$$\mathbf{I}_{\text{imp?}} = \sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)] = \sim\mathbf{E}_{\text{imp!}}$$
$$\mathbf{O}_{\text{imp?}} = \sim[(\exists x)Sx \wedge (x)(Sx \supset Px)] = \sim\mathbf{A}_{\text{imp!}}$$

This refutes Seuren's opinion, according to which the duality laws are no more valid when one expresses **A**, **E**, **I** and **O** in these ways. The equivalences we have exhibited show clearly that Seuren is wrong when he says that the system containing the eight propositions above (i.e. **A**, **E**, **I** and **O** with and without import), which were known by the Medievals and especially Abelard, does not admit the duality laws but only 'one-way entailments' (*Seuren 2012*b, p. 133).

We have to note here that these eight propositions are given by *Seuren 2012a*, who expresses both **O** and **I** without import, in what he calls the system AAPC (an abbreviation for 'Aristotelian Abelardian Predicate Calculus') by using disjunctions, following the Medieval classical formulations. As already noted, these disjunctions are exactly equivalent by De Morgan's laws to our own formulas which use negations of conjunctions instead. They are also given in a less formal way by *Read 2012*, who finds in Aristotle's text itself eight propositions (and not only four) because each proposition may contain either an affirmative or a negative predicate. These negative predicates (called 'indefinite' by Aristotle) do not transform the proposition that contains them into a negative one: it remains affirmative.[20] This results in eight propositions, because each proposition among the four traditional ones can include an affirmative or a negative predicate. We will return to this analysis in the next section. But we will show in the following that, despite many similarities between our treatment and those of these two authors, there are some differences which are sometimes crucial.

As to $\mathbf{A}_{\text{imp?}}$ and $\mathbf{E}_{\text{imp?}}$, they are respectively equivalent to $\sim\mathbf{O}_{\text{imp!}}$ and to $\sim\mathbf{I}_{\text{imp!}}$, as is well known and witnessed by the following equivalences:

$$\mathbf{A}_{\text{imp?}} : (x)(Sx \supset Px) = \sim(\exists x)(Sx \wedge \sim Px) = \sim\mathbf{O}_{\text{imp!}}$$
$$\mathbf{I}_{\text{imp!}} : (\exists x)(Sx \wedge Px) = \sim(x)(Sx \supset \sim Px) = \sim \mathrm{E}_{\text{imp?}}$$
$$\mathbf{E}_{\text{imp?}} : (x)(Sx \supset \sim Px) = \sim(\exists x)(Sx \wedge Px) = \sim\mathbf{I}_{\text{imp!}}$$
$$\mathbf{O}_{\text{imp!}} : (\exists x)(Sx \wedge \sim Px) = \sim(x)(Sx \supset Px) = \sim\mathbf{A}_{\text{imp?}}$$

However, in the traditional doctrine there are other relations that are considered and used in the inferences: these relations are the so-called *conversions*. Conversions are of two kinds:

[20] *Englebretsen 1981* calls these propositions 'counteraffirmatives', by contradistinction to the 'denials' which are genuine negative propositions.

simple conversion, that is **E**-conversion, which leads from; 'No S is P' to 'No P is S', and **I**-conversion, which leads from 'Some Ss are Ps' to 'Some Ps are Ss'; and *partial* conversion (also called *per accidens*), which leads from 'All Ss are Ps' (**A**-propositions) to 'Some Ps are Ss' (**I**-propositions).

Simple conversions are taken into consideration by *Parsons 2006* and said to contribute to the doctrine that he calls '[SQUARE]', combining between the relations of the square and this kind of conversion. He calls SQUARE the doctrine that admits only the relations of the square. SQUARE is valid in all non-empty domains and in the empty domain too for the square no. **IV**; but what about [SQUARE]? Is it also valid for all of the three squares? This will be examined in the following section.

5. Conversion in the three squares

If we consider the square no. **VI**, we find that it contains $\mathbf{E}_{imp?}$ and $\mathbf{I}_{imp!}$. So **E**-conversion is expressed in this way: $(x)(Sx \supset \sim Px) \supset (x)(Px \supset \sim Sx)$. This formula is valid without any doubt by the principle of contraposition, (i.e. $(p \supset q) \equiv (\sim q \supset \sim p)$, which leads to $(p \supset \sim q) \equiv (q \supset \sim p)$).

As to **I**-conversion, which leads from $\mathbf{I}_{imp!}$ to $\mathbf{I}_{imp!}$, i.e. from $(\exists x)(Sx \wedge Px)$ to $(\exists x)(Px \wedge Sx)$, it is even easier to validate since conjunction is commutative and turns the first formula into the second (and conversely). Therefore, the square no. **VI** validates the doctrine called [SQUARE] by Terence Parsons.

What about the other ones? Let us consider simple conversions in these squares. In the squares nos. **IV** and **XI**, **E** has import while **I** has no import. If we express **E**-conversion in this way: $[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)] \supset [(\exists x)Px \wedge (x)(Px \supset \sim Sx)]$ and assume that both S and P exist, **E**-conversion does not hold in this case as shown by (Appendix A, Table 17). This proves that when **E** has an import, the simple conversion does not hold. $\mathbf{E}_{imp!}$ is then really different from $\mathbf{E}_{imp?}$, which admits conversion without any doubt. However, could we express the simple conversion in this way? If we take an ordinary sentence of the form $\mathbf{E}_{imp!}$ or, as Read expresses it 'Every S is not-P'[21] such as the following: 'Every savant is not-stupid', what would be its (simple) converse? Would it be 'Every stupid is not-savant'? Or rather 'Every not-stupid (thing) is savant', i.e. 'Everything that is not stupid is savant'? It seems more reasonable and faithful to the linguistic conventions to choose the second sentence as the converse of our initial example. However, even in this case the simple conversion does not hold: from 'Every S is not-P' we cannot deduce 'Every not-P is S' (see Appendix A, Table 18).

Given that simple conversion does not hold, could we talk about partial conversion for this kind of propositions? We raise the question and think it legitimate because, as we will show below, there are correspondences between our eight propositions and those mentioned by Read and claimed by Aristotle himself. To show this, let us first present the different propositions as they occur in Read's paper. These propositions and their names are the following:

$$\mathbf{A} = \text{Every S is P}$$
$$\mathbf{A}^* = \text{Every S is not-P}$$
$$\mathbf{I} = \text{Some S is P}$$
$$\mathbf{I}^* = \text{Some S is not-P}$$

[21] See *Read 2012* (p. 8).

$$\mathbf{E} = \text{No S is P}$$

$$\mathbf{E}^* = \text{No S is not-P}$$

$$\mathbf{O}^* = \text{Not every S is not-P}$$

$$\mathbf{O} = \text{Not every S is P}$$

Following Aristotle, Read considers that every proposition with an indefinite predicate (i.e. not-P) is affirmative while a negative proposition contains an external negation, which puts not only on the predicate but on the whole proposition. Thus, the first four propositions are affirmative while the last four propositions are negative, according to Read (and Aristotle). Although we do not fully share this claim about the affirmative character of the propositions containing 'not-P', we find a clear correspondence between our own propositions and Read's (Aristotle's) ones. For it seems clear that $\mathbf{A}^*$, for instance, is the ordinary way of expressing $\mathbf{E}_{\text{imp!}}$, while $\mathbf{E}^*$ is just the ordinary way of expressing $\mathbf{A}_{\text{imp?}}$ (which could be formalized by $\sim(\exists x)(Sx \wedge \sim Px)$). But unlike Read, we do not think that the difference between these kinds of propositions is related to the negation as such; rather, it is a difference of *import*. Nevertheless, as we will show below, the propositions *with import* behave *as if* they were affirmative and this is why partial conversions may be hold by this kind of propositions (as they are by the traditional **A** propositions).

The correspondences between our propositions and Read's ones are stated in the following table:

Natural language	FOL	Our symbolizations	Read's symbolizations
(1) Every S is P	$(\exists x)Sx \wedge (x)(Sx \supset Px)$	$\mathbf{A}_{\text{imp!}}$	$\mathbf{A}$
(2) Every S is not-P	$(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$	$\mathbf{E}_{\text{imp!}}$	$\mathbf{A}^*$
(3) Some S is P	$(\exists x)(Sx \wedge Px)$	$\mathbf{I}_{\text{imp!}}$	$\mathbf{I}$
(4) Some S is not-P	$(\exists x)(Sx \wedge \sim Px)$	$\mathbf{O}_{\text{imp!}}$	$\mathbf{I}^*$
(5) No S is not-P	$(x)(Sx \supset Px)$	$\mathbf{A}_{\text{imp?}}$	$\mathbf{E}^*$
(6) No S is P	$(x)(Sx \supset \sim Px)$	$\mathbf{E}_{\text{imp?}}$	$\mathbf{E}$
(7) Not every S is not-P	$\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$	$\mathbf{I}_{\text{imp?}}$	$\mathbf{O}^*$
(8) Not every S is P	$\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$	$\mathbf{O}_{\text{imp?}}$	$\mathbf{O}$

Let us then check partial conversion for $\mathbf{E}_{\text{imp!}}$. Partial conversion will be expressed by the following formula: $[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)] \supset (\exists x)(\sim Px \wedge Sx)$, that is, 'Every S is not-P' implies 'Some not-P are S'. Is this formula valid? It is (see Appendix A, Table 19) so that we can say that partial conversion holds for $\mathbf{E}_{\text{imp!}}$. This shows that $\mathbf{E}_{\text{imp!}}$ behaves like $\mathbf{A}_{\text{imp!}}$ with regard to conversion (and only in this respect). An ordinary example will help understand this relation. From 'Every table is not-round' (which is not equivalent to 'No table is round' because it implies that there are tables) one can deduce 'Some not-round (things) are tables', which seems to be correct even in ordinary language. But this does not mean that these propositions *are* fully affirmative because the negation is present and cannot be removed. Our opinion is that they are different from the usual negative propositions because of their import, but they behave *like* affirmative propositions *precisely because* of that *import*. This opinion is corroborated by the fact that the sentence may not include the copula 'is' at all, but another verb, as in the following: 'Some men do not walk': it is hard to distinguish between 'not walk' and 'not-walk' in that case (or perhaps 'is not walking' and 'is not-walking', following Aristotle's formulations, for this seems to be a very unnatural way to express things). Our view is thus different from both Read's view (who thinks that the propositions containing 'not-P' are affirmative) and from Seuren's view to the effect that the propositions containing 'not-P' and those containing 'not P' are not different at all since they are both negative and say the same thing. Talking about Seuren's opinion, *Read 2012* (p. 8) says the following:

> Seuren's mistake, we now see, is to equate 'Some S is not-P' with 'Some S is not P'. The former is an affirmative proposition, false if there is no S, whereas the latter is a negative proposition, true in those circumstances.

As to our own opinion, we agree with Seuren in that the propositions containing 'not-P' are negative in some sense, because the predicate is negated and one cannot equate them with a proposition containing an affirmative predicate. But unlike him, we hold that they are nevertheless different from those containing 'not P' because of their different import. Those with 'not-P' have an import, while those with 'not P' do not, and this is what explains their different behavior. The problem does not concern the negation as such: it concerns the import before the scope of the negation. We could add that the second sentence in Read's quotation is more precisely expressed by 'Not every S is P', because it is this reading that makes it without import: the reading given here, that is, 'Some S is not P' becomes '$(\exists x)(Sx \wedge \sim Px)$' when formalized and does have an import. This is why we believe that the modern formalizations clarify things by showing in which cases the different propositions have an import or not.

What about **I**-conversion? It is well known that $\mathbf{I}_{\mathrm{imp!}}$ converts simply, since $(\exists x)(Sx \wedge Px)$ is equivalent to $(\exists x)(Px \wedge Sx)$. At the same time, $\mathbf{I}_{\mathrm{imp?}}$ does not convert since it is just the negation of $\mathbf{E}_{\mathrm{imp!}}$; it has been shown (see Appendix A, Table 20) that simple conversion does not hold for it. **I**-conversion is therefore valid in the square no. **VI**, but not in the other two ones. This leads to the following theorems:

THEOREM 5 *If* **E** *has no import,* **E***-conversion holds simply.*

THEOREM 6 *If* **E** *has import,* **E***-conversion holds only partially.*

THEOREM 7 *If* **I** *has import,* **I***-conversion holds simply.*

THEOREM 8 *If* **I** *has no import,* **I***-conversion does not hold.*

As to **A**-conversions, $\mathbf{A}_{\mathrm{imp!}}$ converts partially since '$(\exists x)Sx \wedge (x)(Sx \supset Px)$' obviously implies '$(\exists x)(Px \wedge Sx)$', but what about $\mathbf{A}_{\mathrm{imp?}}$? If we consider that $\mathbf{A}_{\mathrm{imp?}}$ is expressed both by (1) $(x)(Sx \supset Px)$ and (2) $\sim(\exists x)(Sx \wedge \sim Px)$, i.e. 'No S is not-P' in Read's interpretation, we may say that it converts simply if we consider that its converse is 'No not-P is S', which will be formalized by (3) $\sim(\exists x)(\sim Px \wedge Sx)$. The latter is obviously equivalent to (2) by the commutativity of conjunction. If we express (3) by a conditional, we obtain (4) $(x)(\sim Px \supset \sim Sx)$, which is also equivalent to (1) by contraposition. So it seems that $\mathbf{A}_{\mathrm{imp?}}$ converts simply. An ordinary example will help understand that. If we say 'No savant is not-intelligent', we may deduce 'No not-intelligent (being) is a savant'.

This leads to the following theorems for **A**-conversion:

THEOREM 9 *If* **A** *has import,* **A***-conversion holds partially.*

THEOREM 10 *If* **A** *has no import,* **A***-conversion holds simply.*

We finally have to examine **O**-conversions. As is well known, **O** does not convert in the Aristotelian logic. But what is the exact kind of **O** that does not convert: Is it $\mathbf{O}_{\mathrm{imp?}}$, Or $\mathbf{O}_{\mathrm{imp!}}$? Consider first $\mathbf{O}_{\mathrm{imp!}}$, that is $(\exists x)(Sx \wedge \sim Px)$ or, as Read expresses it, 'Some S is not-P'.

We find that it converts simply if its converse is expressed as $(\exists x)(\sim Px \wedge Sx)$, just by the commutativity of conjunction. So it does convert in this reading. Is it, however, possible to express it in that other way: $(\exists x)(Px \wedge \sim Sx)$ = Some P are not-S? Not if we stick to Read's distinction between internal and external negations and consider that the negation in the first formula puts on the predicate and nothing else. This seems to be the right reading, for $\mathbf{O}_{\text{imp!}}$ is different from $\mathbf{O}_{\text{imp?}}$ in that the latter does not have an import. If so, $\mathbf{O}_{\text{imp?}}$ could not be formalized simply by $(\exists x)(Sx \wedge \sim Px)$ or equivalently by $\sim(x)(Sx \supset Px)$ because these formalizations make **O** have an import.

Consequently, $\mathbf{O}_{\text{imp?}}$ or 'Not every S is P' in usual words must be expressed in a complex way and formalized by $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$. But this formula does not convert (see Appendix A, Table 21). This leads to the following theorems:

THEOREM 11 *If* **O** *has import, it converts simply.*

THEOREM 12 *If* **O** *does not have import, it does not convert.*

6. Consequences of this approach

6.1. A cube as a solution

Given that we have eight propositions and not only four, the appropriate figure should not be a mere square, but a cube (or an octagon, as depicted in Seuren's theory). Our choice is to construct a cube and to consider all the relations between the eight propositions in an exhaustive way. This choice is also motivated by strictly geometrical reasons, for a cube is a two-dimensional figure, and it is easier to literally *see* what other figures it contains. By contrast, an octagon is a one-dimensional figure and one cannot see what it contains in a so easy way.

To calculate the number of relations among the eight propositions, we will use the following equation: The number of n-adic logical relations among m propositions $= m!/(n!(m-n!))$, when $m = 8$ and $n = 2$, we have 28 relations.

The logical relations are expressed as follows: CT (contrariety), CD (contradiction), SCT (subcontrariety), SB (subalternation) and NCD (non-contradiction). This last relation does not appear among the usual relations in the square of opposition: it means that the two propositions may be true together or false together; or both may not share the same truth-value. These are *independent* of each other, in the sense that one could not deduce the truth-value of either from the truth-value of the other.

Starting from the eight propositions:

(1) $\mathbf{A}_{\text{imp!}}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
(2) $\mathbf{E}_{\text{imp!}}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
(3) $\mathbf{I}_{\text{imp!}}$ $(\exists x)(Sx \wedge Px)$
(4) $\mathbf{O}_{\text{imp!}}$ $(\exists x)(Sx \wedge \sim Px)$
(5) $\mathbf{A}_{\text{imp?}}$ $(x)(Sx \supset Px)$
(6) $\mathbf{E}_{\text{imp?}}$ $(x)(Sx \supset \sim Px)$
(7) $\mathbf{I}_{\text{imp?}}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
(8) $\mathbf{O}_{\text{imp?}}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

We obtain the following list by using the equation above:

1/ (1)–(2) CT ($\mathbf{A}_{\text{imp!}}/\mathbf{E}_{\text{imp!}}$, square no. **IV**)
2/(1)–(3) SB ($\mathbf{A}_{\text{imp!}}/\mathbf{I}_{\text{imp!}}$, square no. **VI**)
3/ (1)–(4) CT ($\mathbf{A}_{\text{imp!}}/\mathbf{O}_{\text{imp!}}$)
4/ (1)–(5) SB ($\mathbf{A}_{\text{imp!}}/\mathbf{A}_{\text{imp?}}$)

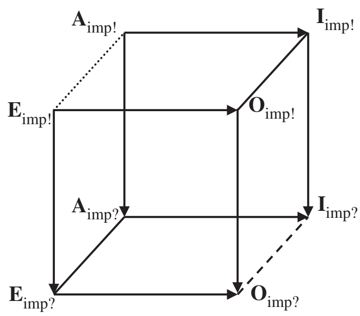
5/ (1)–(6) CT ($\mathbf{A}_{\text{imp!}}/\mathbf{E}_{\text{imp?}}$, square no. **VI**)
6/ (1)–(7) SB ($\mathbf{A}_{\text{imp!}}/\mathbf{I}_{\text{imp?}}$, square no. **IV**)
7/ (1)–(8) CD ($\mathbf{A}_{\text{imp!}}/\mathbf{O}_{\text{imp?}}$, square no. **IV**)
8/ (2)–(3) CT ($\mathbf{E}_{\text{imp!}}/\mathbf{I}_{\text{imp!}}$)
9/ (2)–(4) SB ($\mathbf{E}_{\text{imp!}}/\mathbf{O}_{\text{imp!}}$, square no. **XI**)
10/ (2)–(5) CT ($\mathbf{E}_{\text{imp!}}/\mathbf{A}_{\text{imp?}}$, square no. **XI**)
11/ (2)–(6) SB ($\mathbf{E}_{\text{imp!}}/\mathbf{E}_{\text{imp?}}$)
12/ (2)–(7) CD ($\mathbf{E}_{\text{imp!}}/\mathbf{I}_{\text{imp?}}$, square no. **IV**)
13/ (2)–(8) SB ($\mathbf{E}_{\text{imp!}}/\mathbf{O}_{\text{imp?}}$, square no. **IV**)
14/ (3)–(4) NCD
15/ (3)–(5) NCD
16/ (3)–(6) CD ($\mathbf{I}_{\text{imp!}}/\mathbf{E}_{\text{imp?}}$, square no. **VI**)
17/ (3)–(7) SB ($\mathbf{I}_{\text{imp!}}/\mathbf{I}_{\text{imp?}}$)
18/ (3)–(8) SCT ($\mathbf{I}_{\text{imp!}}/\mathbf{O}_{\text{imp?}}$, square no. **VI**)
19/ (4)–(5) CD ($\mathbf{O}_{\text{imp!}}/\mathbf{A}_{\text{imp?}}$, square no. **XI**)
20/ (4)–(6) NCD
21/ (4)–(7) SCT ($\mathbf{O}_{\text{imp!}}/\mathbf{I}_{\text{imp?}}$, square no. **XI**)
22/ (4)–(8) SB ($\mathbf{O}_{\text{imp!}}/\mathbf{O}_{\text{imp?}}$)
23/ (5)–(6) NCD
24/ (5)–(7) SB ($\mathbf{A}_{\text{imp?}}/\mathbf{I}_{\text{imp?}}$, square no. **XI**)
25/ (5)–(8) SCT ($\mathbf{A}_{\text{imp?}}/\mathbf{O}_{\text{imp?}}$)
26/ (6)–(7) SCT ($\mathbf{E}_{\text{imp?}}/\mathbf{I}_{\text{imp?}}$)
27/ (6)–(8) SB ($\mathbf{E}_{\text{imp?}}/\mathbf{O}_{\text{imp?}}$, square no. **VI**)
28/ (7)–(8) SCT ($\mathbf{I}_{\text{imp?}}/\mathbf{O}_{\text{imp?}}$, square no. **IV**)

Among these relations, 16 are already parts of the three squares above, as appears in our classification. Others can be easily shown as valid, among which the subalternations: $\mathbf{A}_{\text{imp!}} \supset \mathbf{A}_{\text{imp?}}$, $\mathbf{E}_{\text{imp!}} \supset \mathbf{E}_{\text{imp?}}$, $\mathbf{O}_{\text{imp!}} \supset \mathbf{O}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp!}} \supset \mathbf{I}_{\text{imp?}}$. For if we consider the formulas expressing them, we find that the two first ones have the compound structure '$(\alpha \wedge \beta) \supset \beta$' which is obviously valid, while the last two ones have the compound structure '$\alpha \supset (\sim\beta \vee \alpha)$', given that $\mathbf{O}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp?}}$ are equivalent to the following disjunctions: $\mathbf{O}_{\text{imp?}} = \sim(\exists x)Sx \vee (\exists x)(Sx \wedge \sim Px)$, $\mathbf{I}_{\text{imp?}} = \sim(\exists x)Sx \vee (\exists x)(Sx \wedge Px)$. The first disjunction is implied by $(\exists x)(Sx \wedge \sim Px)$ and the second is implied by $(\exists x)(Sx \wedge Px)$. If we consider in addition that $\mathbf{A}_{\text{imp!}}$ and $\mathbf{O}_{\text{imp?}}$ are contradictories as well as $\mathbf{E}_{\text{imp!}}$ and $\mathbf{I}_{\text{imp?}}$, we obtain two more squares which are the following:

Square 4:	$\mathbf{A}_{\text{imp!}}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
	$\mathbf{O}_{\text{imp!}}$ $(\exists x)(Sx \wedge \sim Px)$
	$\mathbf{A}_{\text{imp?}}$ $(x)(Sx \supset Px)$
	$\mathbf{O}_{\text{imp?}}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$
Square 5:	$\mathbf{E}_{\text{imp!}}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
	$\mathbf{I}_{\text{imp!}}$ $(\exists x)(Sx \wedge Px)$
	$\mathbf{E}_{\text{imp?}}$ $(x)(Sx \supset \sim Px)$
	$\mathbf{I}_{\text{imp?}}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$

If these squares are valid, $\mathbf{A}_{\text{imp!}}$ and $\mathbf{O}_{\text{imp!}}$ should be contrary, whereas $\mathbf{A}_{\text{imp?}}$ and $\mathbf{O}_{\text{imp?}}$ should be subcontrary and $\mathbf{E}_{\text{imp!}}$ and $\mathbf{I}_{\text{imp!}}$ should be contrary whereas $\mathbf{E}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp?}}$ should be subcontrary.

As a matter of fact, these contrarieties and subcontrarieties do hold (see Appendix A, Tables 22–25). Although these two squares are 'non-standard', in the sense that their vertices are not the standard **AEIO**-sequence of propositions, they are nevertheless valid because all their relations are valid. Furthermore, we could even let them close to the standard squares if we use Read's symbolizations, for the square 4 contains $\{\mathbf{A}, \mathbf{I}^*, \mathbf{E}^*, \mathbf{O}\}$ and the square 5 contains $\{\mathbf{A}^*, \mathbf{I}, \mathbf{E}, \mathbf{O}^*\}$. Consequently, we find six more valid relations, the remaining four relations being non-contradictions or independence, because they do not correspond to any of the four classical relations of opposition. The cube containing the eight propositions as vertices also contains these 24 valid relations plus the four non-contradictions.



In addition to the usual squares, the cube contains six hexagons that can be constructed starting from the different squares. These hexagons are the following:

From **Square IV**:

Hexagon 1:
$\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{I}_{imp!}$ $(\exists x)(Sx \wedge Px)$
$\mathbf{E}_{imp?}$ $(x)(Sx \supset \sim Px)$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

Hexagon 2:
$\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{A}_{imp?}$ $(x)(Sx \supset Px)$
$\mathbf{O}_{imp!}$ $(\exists x)(Sx \wedge \sim Px)$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

From **Square VI**:

Hexagon 3:
$\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{E}_{imp?}$ $(x)(Sx \supset \sim Px)$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{I}_{imp!}$ $(\exists x)(Sx \wedge Px)$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

From **Square XI**:

Hexagon 4:
$\mathbf{A}_{imp?}$ $(x)(Sx \supset Px)$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{O}_{imp!}$ $(\exists x)(Sx \wedge \sim Px)$

From **Square 4**:

Hexagon 5: $\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{O}_{imp!}$ $(\exists x)(Sx \wedge \sim Px)$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{A}_{imp?}$ $(x)(Sx \supset Px)$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$

From **Square 5**:

Hexagon 6: $\mathbf{I}_{imp!}$ $(\exists x)(Sx \wedge Px)$
$\mathbf{E}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset \sim Px)$
$\mathbf{A}_{imp!}$ $(\exists x)Sx \wedge (x)(Sx \supset Px)$
$\mathbf{O}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset Px)]$
$\mathbf{I}_{imp?}$ $\sim[(\exists x)Sx \wedge (x)(Sx \supset \sim Px)]$
$\mathbf{E}_{imp?}$ $(x)(Sx \supset \sim Px)$

The hexagons 1 and 2 are isomorphic with the hexagon displayed in *Czeżowski 1955*, while the four last hexagons, although similar to Czeżowski's hexagon, are non-standard in the sense that the arrows go from the bottom to the top and the places of the lines of contrariety and of subcontrariety are not the same as in Czeżowski's hexagon. These hexagons are all drawn in Appendix B.

The above cube is comparable to Read's cube presented in *Read 2012* (p. 8) but not equivalent to it, for the following reasons. First, the vertices in Read's cube are not arranged in the same way as ours, for he places all universals in the top and the particulars in the bottom, while we place all propositions with import in the top and those without import in the bottom. Second, and more importantly, is the following difference: Read admits only 18 relations between the propositions in his cube while we admit 24 (classical) relations in ours and 4 non-contradictions, i.e. relations of independence between the propositions.

As to Seuren, he admits as we do the 24 valid relations available in the cube in what he calls the system AAPC; he also talks of one relation of independence (between $\mathbf{A}_{imp?}$ and $\mathbf{E}_{imp?}$ called in his frame $\sim$I* and $\sim$I) but he uses a different way to validate these relations.

However, he claims in several writings that Aristotle's logic does not admit what he calls the 'Conversions' i.e. the duality rules, for he says, for instance, in (*Seuren 2012a*) the following:

> Careful reading of [Aristotle's] texts (mainly his *On Interpretatione*) shows that he stopped short of positing the Conversions (All F is G ≡ No F is not-G and Some F is G ≡ Not all F is not-G). All he did was allow for an entailment from All F is G to No F is not-G but not vice versa, which breaks the chain of existential import. (pp. 236–237)

But this means that he talks of the duality rules and says that **A** is not equivalent to $\sim$**O** and **I** is not equivalent to $\sim$**E**. Rather **A** implies $\sim$**O**, but not the other way round and **I** implies $\sim$**E** but not the other way round. Unfortunately, this could not be true because Aristotle admits the contradictions between **A** and **O** and between **E** and **I**. So $\mathbf{A} \veebar \mathbf{O}$ and $\mathbf{E} \veebar \mathbf{I}$ are valid in his frame, but if so, by the usual logical rules, **A** must be equivalent to $\sim$**O** and **I** must be equivalent to $\sim$**E**. (if $\alpha \veebar \beta$, then $\alpha \equiv \sim\beta$, by simple propositional logic, as anyone can easily see). Therefore, his opinion about one-way entailments cannot be correct and the equivalences must hold. As a matter of fact, they hold indeed as we have already shown.

We have to add, though, that Seuren does not take AAPC as the adequate solution to the problem of existential import. Rather, he constructs his own system based on linguistic natural intuitions, which is different from AAPC. But he adds in this system some very unusual classical concepts such as the concept of 'radical falsity' (in *Seuren 2012b*, p. 137) which does not have any equivalent in Aristotle's frame. This makes his theory really far from Aristotle's initial one.

6.2. The punch line

A number of relevant additional consequences are in order.

First, it seems that the particular is no more identifiable with an existential proposition, as has become usual in modern logic, since it is possible to neutralize its import. We argue in the same vein that the universals and particulars are not different with respect to existential import: both could have existential import or not, depending on the way we formalize them.

Second, we find that the traditional square may be valid also in the empty domain, so that its validity is not strictly related to non-empty domains. This is really a new insight, since all those who have studied the square have found difficulties with the empty domain. Our results show that this is not a problem any longer, since one of the interpretations of the square (the no **IV**) makes it valid in all domains without exception.

Third, our theorems about the import and the conversion rules clarify Aristotle's system as well: for given the rules that Aristotle admits, e.g. simple and partial conversions, **A** and **I** are supposed to have an import, while **E** and **O** do not, since **E** converts simply in his system, while **O** does not convert at all. He seems then to endorse square no. **VI**, as Terence Parsons and others have already said, although he does admit the eight kinds of propositions (albeit in an informal way). But we have given here some more arguments to confirm that thesis. We can add that Aristotle's system may be seen just as a *fragment* of the whole system that we have presented. Indeed, Aristotle admits the square no. **VI** and some rules that are consistent with it, along with some other elements of our whole system, such as the eight propositions; but he does not fully develop the relations between all these propositions, as we have done previously. His theory is thus incomplete but clearly compatible with our own one.

The fourth consequence concerns the relations of the square themselves. These relations hold under the conditions stated by our theorems. And this is important for the theory of oppositions in general, since existential import is also considered with more complex figures such as the hexagon or the cube. The conditions under which these relations are valid are equally applicable to these more complex figures.

Fifth and finally, the cube shows that there are two additional valid squares and several valid hexagons, which demonstrates the richness of the theory.

Our analysis is then fruitful in that it exhibits more structures than the ones usually admitted while remaining close to the initial Aristotelian system, although it uses the language of FOL, which is far more powerful than ordinary language and helps clarify the propositions and their import much more precisely.

Besides that, we think that the translations we have given to $\mathbf{O}_{\text{imp?}}$ and especially $\mathbf{I}_{\text{imp?}}$ are not as counterintuitive as they might appear, since we could express the first by the following sentence: 'It is not the case that there are Ss and that all of them are Ps', and the second by a parallel one which would say something like 'It is not the case that there are Ss and that none of them is P'.

They are even useful, since they are the real contradictions of sentences like 'All of my friends are intelligent', for instance, which clearly presupposes that I have friends and talks about existent persons. Universal propositions generally have existential import in daily life,[22] especially when they are affirmative, and it is always useful to know what is exactly the contradictory of such a proposition even if this contradictory seems not very natural at first sight. This is why $\mathbf{O}_{\text{imp?}}$ and $\mathbf{I}_{\text{imp?}}$, despite their non-intuitive (and perhaps unnatural) character, are actually useful because they make it possible to avoid fallacies in the argumentation.

7. Conclusion

The preceding shows that the problem of existential import can be solved in a way that makes all the relations of the square valid, by using only the tools of contemporary quantificational logic and without calling for any kind of non-classical logic. This should be considered as an advantage, since monadic predicate logic is decidable, complete and consistent, and a solution requiring only its tools and symbolism is always welcome. The solution to this problem also shows that the expressive power of classical quantificational logic should not be neglected or under evaluated, since its symbolism can express many kinds of propositions and arguments. Although some formalizations that we have used seem to be somewhat derived and indirect, they are nevertheless useful in that they make it possible to express adequately the oppositions between propositions in a quite explicit way. Our solution has many advantages. First, it validates the square in all domains, even the empty one; second, it reconciles traditional logic with modern logic since the way we have formalized the quantified propositions is merely a complexification of their form that does not contradict the usual contemporary definitions; third, it enriches and goes into all the details of the theory by showing that the entire set of propositions amounts to eight propositions and not only four. This leads to a cube which may be considered as a more adequate figure to represent all the relations between these eight propositions. However, the cube itself contains five valid squares and six valid hexagons; one of these squares seems to be admitted by Aristotle himself. As a consequence, we may say that our frame is broader than Aristotle's one, but still close to it, since it contains it as a fragment.

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References

Aristotle. 1991. 'Prior Analytics', in J. Barnes, *The Complete Works of Aristotle*, the Revised Oxford Translation, Princeton University Press, Princeton/Bollingen Series LXXI* 2, volume 1.

Bäck, A. 2000. *Aristotle's Theory of Predication, Philosophia Antiquia*, Leiden: Brill.

Blanché, R. 1953. 'Sur l'opposition des concepts', *Theoria,* **19**, 89–130.

Blanché, R. 1966, *Structures Intellectuelles (Essai Sur l'organisation Systématique des Concepts)*, Paris: Vrin.

Blanché, R. 1970. *La Logique et Son Histoire d'Aristote à Russell*, Paris: Armand Colin.

Czeżowski, T, 1955, 'On certain peculiarities of singular propositions', *Mind*, **64**, 287–308.

Couturat, L. 1901. *La Logique de Leibniz*, Hildesheim: Georg Olms Verlagsbuchhandlung [1969].

Englebretsen, G. 1981. *Logical Negation*, Assen, The Netherlands: Van Gorcum, Ltd.

Frege, G. 1879. 'Begriffsschrift', in J. Van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic*, Cambridge, MA: Harvard University Press [1981].

Geach, P. T. 1950. 'Subject and predicate', *Mind*, **59**, 461–482.

Geach, P. T. 1980. *Logic Matters*, Berkeley and Los Angeles: University of California Press.

[22] Blanché says in *Blanché 1970* (p. 258) : 'If I said to someone "All my children are musicians" and if he discovers after a while that I have no children, he would certainly blame me for misleading him' (our translation).

Hart, H. L. A. 1951. 'A logician's fairy tale', *Philosophical Review*, **60**, 98–212.
Horn, L. 2001. *A Natural History of Negation*, The David Hume Series, CSLI Stanford, University of Chicago Press.
Kleene, S.C. 1967. *Mathematical Logic*, French translation *Logique mathématique*, Paris: Armand Colin [1971].
Linsky, L. 1967. *Referring*, French translation, *Le Problème de La Référence*, Paris: Edition du Seuil [1974].
Parsons, T. 2006. 'The traditional square of opposition', in E. N. Zalta, *Stanford Encyclopedia of Philosophy,* New York, N.Y. http://plato.stanford.edu/entries/square/index.html
Parsons, T. 2008. 'Things that are right with the traditional square of opposition', *Logica Universalis*, **2**, 3–11.
Quine, W. V. O. 1950. *Methods of Logic*, French translation: *Méthodes de logique*, Paris: Armand Colin, [1973].
Read, S. 2012. 'Aristotle and Lukasiewicz on existential import', www.st-andrews.ac.uk/~slr/Existential_import.pdf
Russell, B. 1959. *My philosophical development*, London: George Allen & Unwin. French translation: *Histoire de mes idées philosophiques*, TEL Gallimard [1961].
Russell, B. and Whitehead, A. N. 1962. *Principia Mathematica*, Paperback edition to *56, New York, NY: Cambridge University Press [1973].
Seuren, P. 2012a. 'From logical intuitions to natural logic', in J.-Y. Béziau and G. Payette, *The Square of Opposition, A General Framework for Cognition,* Bern: Peter Lang.
Seuren, P. 2012b. 'Does a leaking O-Corner save the square?', in J.-Y. Béziau and D. Jacquette, *Around and Beyond the Square of Opposition*, Birkhäuser: Springer Basel.
Smiley, T. 1967. 'Mr Strawson on the traditional logic', *Mind* (New Series), **76**, 118–120.
Strawson, P. 1971. 'On referring', in J. F. Rosenberg and T. Travis, *Readings in the Philosophy of Language*, Englewood Cliffs, NJ: Prentice Hall.

Appendix A: Proofs of the validity or the invalidity of the logical relations

We will proceed from the main operator in each case, supposing that it is false (in boldface); then we will consider the operator(s) that come next, assigning values to the other operators and elementary propositions in order to see if the case of falsity is possible or not. If yes, this means that the relation is not valid. If no, this means that we always arrive to a contradiction (marked by underlined values) and, consequently, that the relation is never false hence always true, i.e. valid. We borrow the *reductio method* from Hugues and Cresswell (*An Introduction to modal logic*, p. 13).

1. $\nvDash \sim(\mathbf{A}_{imp?} \wedge \mathbf{E}_{imp?})$

Table 1

$\sim\{[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)] \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

0 0 1 1 1 0 1 1 1 0 1 0 1 1 0 1 0 1

2. $\nvDash (\mathbf{I}_{imp!} \veebar \mathbf{E}_{imp!})$

Table 2

$[(Sx_1 \wedge Px_1) \vee (Sx_2 \wedge Px_2)] \veebar \{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

0 0 1 0 0 0 1 **0** 0 0 0 0 0 1 0 1 1 0 1 0 1

3. $\nvDash \mathbf{E}_{imp?} \veebar \mathbf{I}_{imp?}$

Table 3

$[(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)] \veebar \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

0 1 1 0 1 0 1 1 0 **0** 1 0 0 0 0 0 1 1 0 1 0 1 1 0

4. $\nvDash \mathbf{A}_{imp!} \veebar \mathbf{O}_{imp!}$

Table 4

$\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\} \veebar \sim[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\}$

0 0 0 0 0 1 1 1 0 1 1 **0** 0 0 1 1 1 0 1 1

5. $\nvDash \mathbf{A}_{imp?} \veebar \mathbf{O}_{imp?}$

Table 5

$[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)] \veebar \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\}$

0 1 1 1 0 1 1 **0** 1 0 0 0 0 0 1 1 1 0 1 1

6. $\vDash \sim(\mathbf{A}_{imp!} \wedge \mathbf{E}_{imp!})$

Table 6

$\sim\{\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\} \wedge \{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 1 1 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 0 1

7. $\models \sim(\mathbf{A}_{imp!} \wedge \mathbf{E}_{imp?})$

Table 7

$\sim\{\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\} \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 0 1 1 1 1 $\underline{0}$ 1 0 1 1 0

8. $\models \sim(\mathbf{A}_{imp?} \wedge \mathbf{E}_{imp!})$

Table 8

$\sim\{[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)] \wedge \{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}\}$

0 1 1 $\underline{1}$ 1 0 1 1 1 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 0 1

9. $\models \mathbf{I}_{imp!} \vee \mathbf{O}_{imp?}$

Table 9

$[(Sx_1 \wedge Px_1) \vee (Sx_2 \wedge Px_2)] \vee\sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\}$

1 0 $\underline{0}$ 0 0 0 0 **0** 0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 0

10. $\models \mathbf{I}_{imp?} \vee \mathbf{O}_{imp?}$

Table 10

$\sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\} \vee\sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\}$

0 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 0 1 **0** 0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 1

11. $\models \mathbf{I}_{imp?} \vee \mathbf{O}_{imp!}$

Table 11

$\sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\} \vee\sim[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]$

0 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 0 1 **0** 0 1 1 $\underline{1}$ 1 0 1 1

12. $\models \mathbf{A}_{imp!} \supset \mathbf{I}_{imp?}$

Table 12

$\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\} \supset \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\}$

1 1 0 1 1 1 $\underline{1}$ 1 0 1 0 **0** 0 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 1 0

13. $\models \mathbf{A}_{imp?} \supset \mathbf{I}_{imp?}$

Table13

$[(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)] \supset \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]$

1 1 $\underline{1}$ 1 0 1 1 **0** 0 1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 0 1

14. $\models \mathbf{E}_{imp?} \supset \mathbf{O}_{imp?}$

Table 14

$[(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)] \supset \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]\}$

1 1 1 $\underline{0}$ 1 0 1 1 0 **0** 0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 0

15. $\models \mathbf{E}_{imp!} \supset \mathbf{O}_{imp?}$

Table 15

$\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset \sim Px_1) \wedge (Sx_2 \supset \sim Px_2)]\} \supset \sim\{(Sx_1 \vee Sx_2) \wedge [(Sx_1 \supset Px_1) \wedge (Sx_2 \supset Px_2)]$

1 1 0 1 1 1 1 $\underline{0}$ 1 0 1 1 0 **0** 0 1 1 0 1 1 1 $\underline{1}$ 1 0 1 0

16. $\models \mathbf{E}_{imp!} \supset \mathbf{O}_{imp!}$

Table 16

{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]} ⊃ ∼[(Sx$_1$ ⊃ Px$_1$) ∧ (Sx$_2$ ⊃ Px$_2$)]

1 1 0 1 1 1 1 **0** 1 0 1 1 0 **0** 0 1 1 **1** 1 0 1 0

17. $\not\models$ Simple conversion $\mathbf{E}_{imp!}$ (1) (every P is not-S)

Table 17

{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]} ⊃ {(Px$_1$ ∨ Px$_2$) ∧ [(Px$_1$ ⊃ ∼Sx$_1$) ∧ (Px$_2$ ⊃ ∼Sx$_2$)]}

1 1 1 1 1 1 1 0 1 1 1 1 0 **0** 0 0 0 0 0 1 0 1 1 0 1 0 1

18. $\not\models$ Simple conversion $\mathbf{E}_{imp!}$ (2) (Every not-P is S)

Table 18

{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]} ⊃ {(∼Px$_1$ ∨ ∼Px$_2$) ∧ [(∼Px$_1$ ⊃ Sx$_1$) ∧ (∼Px$_2$ ⊃ Sx$_2$)]}

1 1 0 1 1 1 1 0 1 0 1 0 1 **0** 0 1 0 0 1 0 0 1 1 1 1 0 1 1 0

19. $\models$ Partial conversion $\mathbf{E}_{imp!}$

{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]} ⊃ [(∼Px$_1$ ∧ Sx$_1$) ∨ (∼Px$_2$ ∧ Sx$_2$)]

1 1 0 1 1 1 1 0 1 0 1 0 1 **0** 1 0 0 **0** 0 0 1 0 0

20. $\not\models$ Conversion $\mathbf{I}_{imp?}$ (with (∃x)Sx and (∃x)Px)

Table 20

∼{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]}⊃ ∼{(Px$_1$ ∨ Px$_2$) ∧ [(Px$_1$ ⊃ ∼Sx$_1$) ∧ (Px$_2$ ⊃ ∼Sx$_2$)]

1 0 0 0 0 0 1 0 1 1 0 1 0 1 **0** 0 1 1 1 1 1 1 1 0 1 1 1 1 0

21. $\not\models$ Conversion $\mathbf{O}_{imp?}$

Table 21

∼{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ Px$_1$) ∧ (Sx$_2$ ⊃ Px$_2$)]}⊃ ∼{(Px$_1$ ∨ Px$_2$) ∧ [(Px$_1$ ⊃ Sx$_1$) ∧ (Px$_2$ ⊃ Sx$_2$)]

1 1 1 1 0 1 1 1 0 1 0 0 **0** 0 1 1 0 1 1 1 1 1 0 1 1

22. $\models \sim(\mathbf{A}_{imp!} \wedge \mathbf{O}_{imp!})$

Table 22

∼{{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ Px$_1$) ∧[(Sx$_2$ ⊃ Px$_2$)]}∧∼[(Sx$_1$ ⊃ Px$_1$) ∧ (Sx$_2$ ⊃ Px$_2$)]}

0 1 1 1 1 1 1 1 0 1 0 1 1 1 0 0 0 0 1 0

23. $\models \sim(\mathbf{E}_{imp!} \wedge \mathbf{I}_{imp!})$

Table 23

∼{{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]}∧ [(Sx$_1$ ∧ Px$_1$) ∨ (Sx$_2$ ∧ Px$_2$)]}

0 1 1 0 1 1 1 1 0 1 0 1 1 0 1 1 1 1 1 0 0 0

24. $\models \mathbf{A}_{imp?} \vee \mathbf{O}_{imp?}$

Table 24

[(Sx$_1$ ⊃ Px$_1$) ∧ (Sx$_2$ ⊃ Px$_2$)] ∨∼{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ Px$_1$) ∧ (Sx$_2$ ⊃ Px$_2$)]

1 0 0 0 1 1 1 **0** 0 1 1 1 1 1 1 1 1 1 1 1

25. $\models \mathbf{E}_{imp?} \vee \mathbf{I}_{imp?}$

Table 25

[(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)] ∨∼{(Sx$_1$ ∨ Sx$_2$) ∧ [(Sx$_1$ ⊃ ∼Px$_1$) ∧ (Sx$_2$ ⊃ ∼Px$_2$)]

1 0 0 1 0 0 1 0 1 **0** 0 1 1 0 1 1 1 1 0 1 0 1 0 1

Appendix B: Valid oppositions

We want to determine how many logical squares and hexagons can be constructed with a given set of formulas, under the proviso that the order of the vertices does not matter and no single formula can occur more than once in the same polygon.

This can be characterized according to the following combinatorial formula: $C^m_n = m!/((m-n)!n!)$, m standing for the number of formulas and n for the number of vertices that correspond to a formula. Accordingly, our squares of quantified oppositions are to be found within a list of polygons such that $m = 8$ and $n = 4$. Let us symbolize each of the formulas by an integer:

$$1 = \mathbf{A}_{\text{imp!}}; 2 = \mathbf{E}_{\text{imp!}}; 3 = \mathbf{I}_{\text{imp!}}; 4 = \mathbf{O}_{\text{imp!}}; 5 = \mathbf{A}_{\text{imp?}}; 6 = \mathbf{E}_{\text{imp?}}; 7 = \mathbf{I}_{\text{imp?}}; 8 = \mathbf{O}_{\text{imp!}}.$$

The exhaustive number of logical squares amounts to $C^8_4 = 70$. Let us symbolize each of these 70 squares by the following bit strings of length 4 among 8 available elements, where the three standard squares and the two non-standard ones are marked in boldface.

(1) 1234	**(15) 1278**	(29) 1467	**(43) 2367**	(57) 3457
(2) 1235	(16) 1345	(30) 1468	(44) 2368	(58) 3458
(3) 1236	(17) 1346	(31) 1478	(45) 2378	(59) 3467
(4) 1237	(18) 1347	(32) 1567	(46) 2456	(60) 3468
(5) 1238	(19) 1348	(33) 1568	**(47) 2457**	(61) 3478
(6) 1245	(20) 1356	(34) 1578	(48) 2458	(62) 3567
(7) 1246	(21) 1357	(35) 1678	(49) 2467	(63) 3568
(8) 1247	(22) 1358	(36) 2345	(50) 2468	(64) 3578
(9) 1248	(23) 1367	(37) 2346	(51) 2478	(65) 3678
(10) 1256	**(24) 1368**	(38) 2347	(52) 2567	(66) 4567
(11) 1257	(25) 1378	(39) 2348	(53) 2568	(67) 4568
(12) 1258	(26) 1456	(40) 2356	(54) 2578	(68) 4578
(13) 1267	(27) 1457	(41) 2357	(55) 2678	(69) 4678
(14) 1268	**(28) 1458**	(42) 2358	(56) 3456	(70) 5678

It is taken to be granted that a good deal of such squares could not be entertained in our preceding investigation: a lot of these includes the same sort of formula more than once, whereas the traditional Aristotelian square rules out any bit string other than a **AEIO**-sequence.

The exhaustive number of logical hexagons amounts to $C^8_6 = 28$. It results in a set of 28 possible hexagons, as witnessed by the following bit strings of length 6 among 8 available elements.

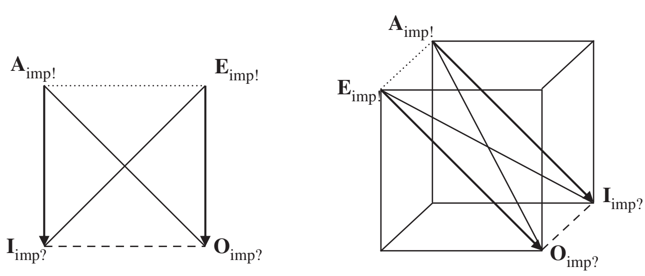
(1) 123456	(8) 123568	(15) 125678	(22) 234567
(2) 123457	(9) 123578	(16) 134567	(23) 234568
(3) 123458	**(10) 123678**	(17) 134568	(24) 234578
(4) 123467	(11) 124567	(18) 134578	(25) 234678
(5) 123468	(12) 124568	(19) 135678	(26) 235678
(6) 123478	**(13) 124578**	(20) 135678	(27) 245678
(7) 123567	(14) 124678	(21) 145678	(28) 345678

The valid hexagons are isomorphic to those displayed by *Czeżowski 1955*, rather than *Blanché 1953*. They are said to be non-standard, insofar as each of their vertices are not distinct from each other. The same does for the two additional non-standard squares which are not of the form **AEIO**.

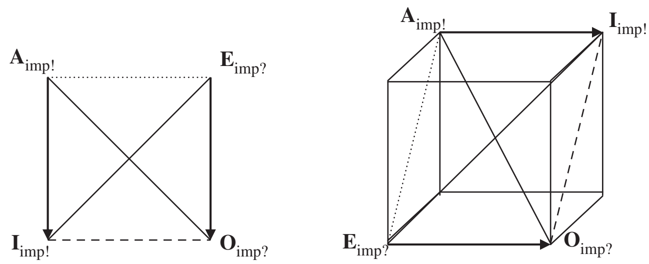
B.1. Valid standard squares (AEIO-sequences)

Here are the squares of the form **AEIO**, as they have been described in Section 5. The two additional squares are 'non-standard' cases in which one and the same vertex occurs twice.

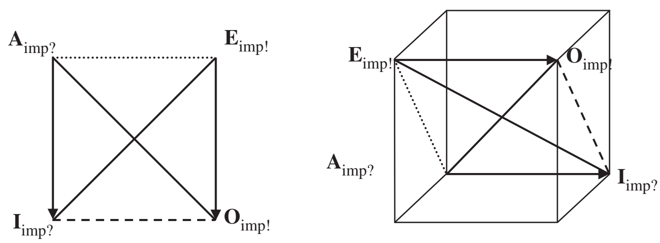
Square n°IV
1278



Square n°VI
1368

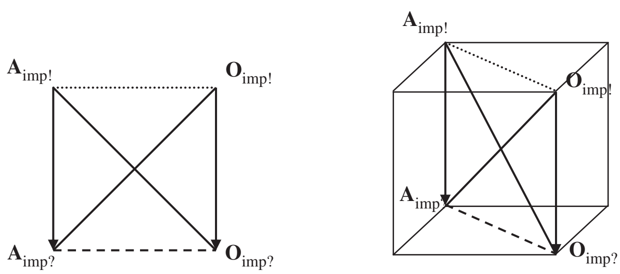


Square n˚XI
2457

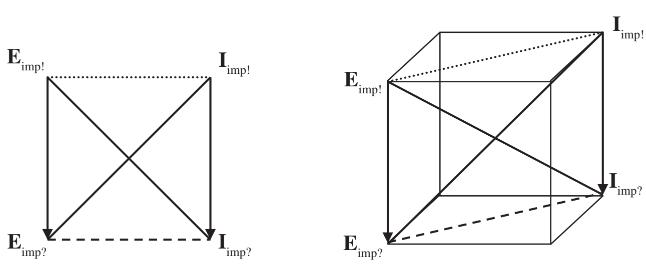


B.2. Valid non-standard squares (non-AEIO-sequences)

Square n°4
1458

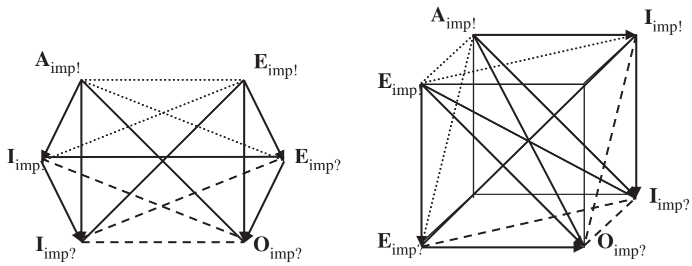


Square n°5
2367

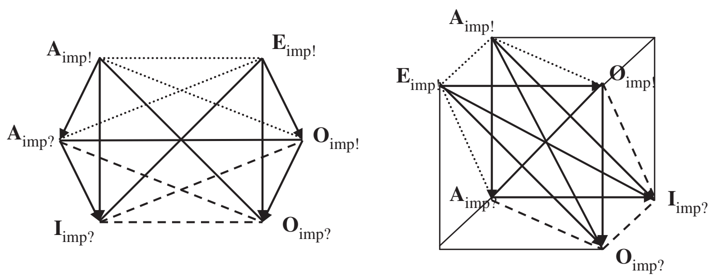


B.3. Valid non-standard hexagons (non-AEIO-sequences)

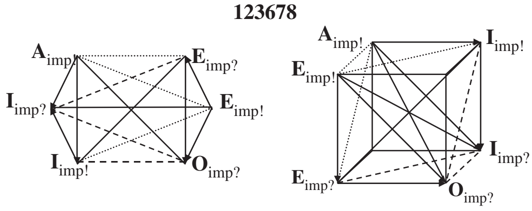
Hexagon n° 1
123678



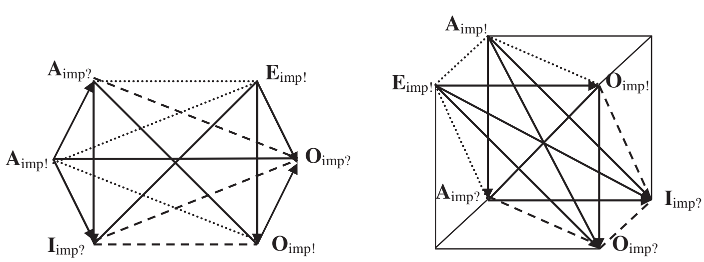
Hexagon n° 2
124578



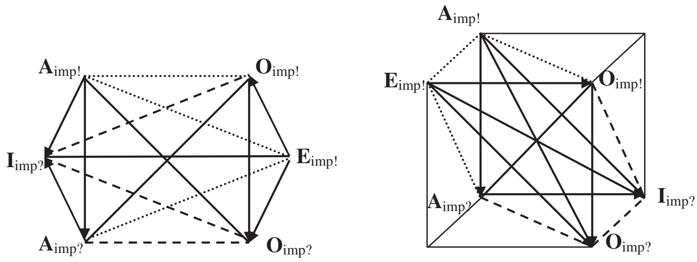
Hexagon n˚3
123678



Hexagon n° 4
124578



Hexagon n° 5
124578



Hexagon n°6
123678