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PURE-STRATEGY EQUILIBRIA WITH NON-EXPECTED UTILITY PLAYERS

ABSTRACT. A pure-strategy equilibrium existence theorem is extended to include games with non-expected utility players. It is shown that to guarantee the existence of a Nash equilibrium in pure strategies, the linearity of preferences in the probabilities can be replaced by the weaker requirement of quasiconvexity in the probabilities.

KEY WORDS: Game theory, Pure strategies, Risk, Non-expected utility

1. INTRODUCTION

A great deal of attention has been paid to non-expected utility behavior, primarily to experimental evidence and models which can accommodate the experimental evidence (see, for example, Machina, 1987; Harless and Camerer, 1994). There have also been efforts to apply expected utility in realistic choice situations, such as in auctions (Chew, 1989; Karni and Safra, 1989; Neilson, 1994). Bidding behavior in standard (i.e. expected utility) auction theory is typically determined by a pure strategy equilibrium, and, indeed, several authors have presented conditions which guarantee the existence of a pure strategy equilibrium in general games with expected utility players (for example, Debreu, 1952; Glicksberg, 1952; Fan, 1952; Dasgupta and Maskin, 1986). As yet, however, and in spite of the existence of the auction papers, there is no corresponding result for non-expected utility players. This paper presents sets of conditions guaranteeing the existence of pure strategy equilibria with non-expected utility players. The existence of pure strategy equilibria is important not only for auction theory, but for other situations in which players have continuous strategy spaces, such as mechanism design, Cournot oligopoly, and public good contribution games.

Other researchers have addressed the issue of the existence of Nash equilibria, possibly in mixed strategies, in games with finite

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action spaces when players violate the expected utility hypothesis. In contrast, our goal is to determine conditions under which there exist pure-strategy equilibria in games with infinite, convex action spaces.[1] Crawford (1990) finds that in the former type of game, if preferences are quasiconcave in the probabilities then a Nash equilibrium exists, possibly in mixed strategies, but if preferences are quasiconvex in the probabilities, a Nash equilibrium may not exist. To remedy this problem, he introduces the notion of an equilibrium in beliefs. Cheng and Zhu (1995) analyze properties of mixed strategy equilibria in a non-expected utility model which assumes strict quasiconcavity in the probabilities when payoffs are gains. Ritzberger (1996) examines the existence of equilibria when player's preferences satisfy the assumptions of one particular model, expected utility with rank-dependent preferences. He demonstrates that if players are risk averse, any Nash equilibrium must be a pure-strategy equilibrium, because in the rank-dependent model risk aversion and quasiconvexity in the probabilities are linked (see Chew, Karni, and Safra, 1987).

We find that when the expected utility assumption is dropped from the usual set of assumptions used to guarantee the existence of pure- strategy equilibria, the game may no longer possess a Nash equilibrium in pure strategies. The additional requirement which restores the pure- strategy equilibrium existence result is that the players' preferences must be quasiconvex in the probabilities, which is a weaker requirement than expected utility maximization. For a narrow set of games, in which payoffs are deterministic monetary amounts whenever all players use pure strategies, a completely different restriction on preferences suffices: continuity and first-order stochastic dominance preference can replace expected utility maximization in the standard existence result.

In order to analyze a game in which players are not expected utility maximizers, a more complicated description of the game is needed. Specifically, the game must first be specified with monetary payoffs, and the players' preferences over probability distributions must be specified separately. This is because when players are not expected utility maximizers, the preference value from playing a mixed strategy is not necessarily a linear combination of the preference values from playing the component pure strategies. In Section

2 we construct a monetary game, in which payoffs are probability distributions over monetary prizes, and in Section 3 we complete the game by considering preferences. It is also shown in Section 3 that if preferences are continuous and quasiconvex in the probabilities, best-response correspondences contain pure strategies. This fact is used in Section 4 to prove a pure-strategy equilibrium existence result. The result we extend is the Debreu–Glicksberg–Fan result.[2] Section 5 considers the special case in which the monetary payoffs are nonstochastic whenever players use pure strategies. The paper concludes in Section 6 with a comparison with other theoretical restrictions on the quasiconcavity or quasiconvexity of non-expected utility preferences.

2. THE MONETARY GAME

In this section we introduce the game to be analyzed. Unlike in standard game theory, the payoffs are monetary payoffs instead of utility payoffs. While utility payoffs make sense when players are expected utility maximizers, they do not make much sense when expected utility is violated. In particular, if players are expected utility maximizers, the expected utility of a mixed strategy can be calculated from the utility values of the pure strategies. When players are not expected utility maximizers, the preference value of a mixed strategy cannot be calculated from the preference values of pure strategies, in general. Since the utility values of pure strategies do not contain enough information to fully characterize the game, we proceed in two steps. In this section we describe a game with monetary payoffs, called a monetary game.[3] In the next section preferences are added, forming a game in the standard sense.

There are n players, and let A_i denote agent i's (possibly finite) pure strategy set, with typical element $a_i \in A_i$. Let $A = \times_{i=1,\ldots,n} A_i$ denote the set of all possible pure strategy profiles. For any element a of A, we have $a = (a_i)_{i=1,\ldots,n}$. Also, define $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, and let A_{-i} denote the set of all a_{-i}. Let $P(A_i)$ denote the set of all probability measures over A_i, and endow it with the topology of weak convergence. A mixed strategy for player i is a probability measure $\mu_i : A_i \rightarrow P(A_i)$.

Note that each pure strategy a_i corresponds to a degenerate mixed strategy. Finally, let $\mu = (\mu_1, \dots, \mu_n)$.

Assume that there is an interval $[m, M]$ such that all monetary payoffs, for any player, lie in this interval. Let $D[m, M]$ denote the set of probability distributions over monetary payments, and endow $D[m, M]$ with the topology of weak convergence. Let D^n denote the Cartesian product of n copies of $D[m, M]$, and endow D^n with the product topology. The *monetary payoff function* is a mapping $\beta : A \to D^n$. Let β_i denote the ith component of β. Then $\beta_i(a)$ is the probability distribution of the payoffs for player i when the players utilize the pure strategy combination a.

Having the payoff resulting from a pure strategy combination be a probability distribution is somewhat unusual. When players are expected utility maximizers and payoffs are given in utility values, payoffs are always deterministic. Here, though, payoffs are in terms of monetary amounts. These could be stochastic for several reasons. For example, the payoffs from the game could be inherently risky, such as when the payoffs are lottery tickets or assets with uncertain values. More realistically, at the time the players make their choices, there could be incomplete information, which makes the outcome of the game uncertain *ex ante*. When players make their bids in an auction, for example, they do not know the types of the other players, and the *ex ante* payoff from the game is a random variable, even when all the players use pure strategies and the value of the prize is deterministic.

We have yet to define monetary payoffs in the event that players use mixed strategies. Let $\pi : P(A) \to D$ be the probability distribution over payoffs induced when the players follow the mixed strategy combination μ and when the monetary payoffs from actions are given by the function β. Let δ_a denote the probability distribution over payoffs which arises when the pure strategy profile a is played with probability one. Notice4 that $\pi(\delta_a) = \beta(a)$. Whenever it can be done without confusion, we use the notation $\pi(a)$ to mean $\pi(\delta_a)$.

3. PREFERENCES AND BEST RESPONSES

To complete the description of the game, players' preferences over probability distributions must be specified. Each player has a preference function $V_i : D[m, M] \to \mathbb{R}$. A *strategic form game* can now be described as an $(n + 2)$-tuple $\Gamma = (A, \beta, V_i)_{i=1,\ldots,n}$. If V_i is linear, the player is an expected utility maximizer, while if V_i is nonlinear, the player is a non-expected utility maximizer. One particular form of nonlinearity is of interest here: the quasiconcavity or quasiconvexity of preferences.

The preference function V_i is (*strictly*) *quasiconcave* if $V_i(\lambda F + (1 - \lambda)F^*) \geqslant (>) \min\{V_i(F), V_i(F^*)\}$ for all $\lambda \in (0, 1)$, where $F, F^* \in D[m, M]$, $F \neq F^*$, and $\lambda F + (1 - \lambda_) F^*$ is a probability mixture of F and F^*. V_i is (*strictly*) *quasiconvex* if $V_i(\lambda F + (1 - \lambda)F^*) \leqslant (<) \max\{V_i(F), V_i(F^*)\}$ for all $\lambda \in (0, 1)$. If V_i is both quasiconcave and quasiconvex, it is said to satisfy *betweenness*.

In order to establish the existence of a pure-strategy equilibrium, it is first necessary to establish that players can play pure strategies as best responses to their opponents' pure strategies. This has not been an issue with expected utility players, because expected utility maximizers satisfy betweenness. When a player's preferences satisfy betweenness and that player is indifferent between two or more pure strategies, he is also indifferent over all possible mixtures of those pure strategies. Consequently, the set of best responses to a pure strategy combination must contain at least some pure strategies when preferences satisfy betweenness. In contrast, if a player's preference function is strictly quasiconcave, the player strictly prefers a mixture of indifferent pure strategies to the pure strategies themselves. In this case, then, the set of best responses to a pure strategy combination may not contain any pure strategies. This implies that to guarantee the existence of a pure- strategy equilibrium, preferences must be quasiconvex in the probabilities.

Formally, let

$$\psi_i(a_{-i}) = \{\mu_i \in P(A_i) | V_i(\pi_i(\mu_i, a_{-i})) \geqslant V_i(\pi_i(\mu_i', a_{-i})) \text{ for all } \mu_i' \in P(A_i)\}$$

be the best-response correspondence for player i. Note that this correspondence is only defined when the other players use pure strategies. While it is possible to write a best-response correspondence which applies when the other players use mixed strategies, it is not needed for the remainder of this paper.

LEMMA 1. *If V_i is continuous and quasiconvex, β_i is continuous, and A_i is compact, then $\psi_i(a_{-i})$ contains pure strategies for each $a_{-i} \in A_{-i}$.*

Proof. Suppose that there exists some profile a_{-i} such that $\psi_i(a_{-i})$ contains no pure strategies. By the continuity of V_i and β_i and the compactness of A_i (and hence of $P(A_i))$), $\psi_i(a_{-i})$ is nonempty. So, $\psi_i(a_{-i})$ must contain a mixed strategy $\mu_i(a_i)$. By quasiconvexity, there exists an α in the support of μ_i such that $V_i(\pi_i(\alpha, a_{-i})) \geqslant V_i(\pi_i(\mu_i, a_{-i}))$. Thus $\delta_\alpha \in \psi_i(a_{-i})$, contradicting the assumption of no pure strategies in the best-response set. □

The above discussion highlights how the betweenness assumption has affected the standard way of thinking about mixed strategies. When a player's preferences satisfy betweenness, the player only mixes if he is indifferent between the strategies he is mixing over. Put another way, there must be indifference among pure strategies before there can be a mixed strategy, and this fact has resulted in a common way to compute mixed strategies: a player chooses a mixed strategy to make his opponent indifferent between her pure strategies. Without betweenness this intuition disappears: the optimal mixture may entail placing positive probability on two pure strategies which are *not* indifferent if preferences are quasiconcave.

4. PURE STRATEGY EQUILIBRIA

We now turn to the issue of whether or not the strategic form game Γ possesses a pure-strategy equilibrium. In particular, we are interested in extending the standard results of Debreu, Glicksberg, and Fan to the case of non-expected utility maximizing players. Their result is that if all players are expected utility maximizers, if pure strategy sets are non-empty, convex and compact, and if each $V_i(\pi_i(a))$ is continuous in a and quasiconcave in a_i, then the game possesses a pure-strategy Nash equilibrium. Our main result is that the expected utility assumption of linearity in the probabilities can be replaced by quasiconvexity in the probabilities.

PROPOSITION 1. *Consider a strategic-form game* Γ. *If, for each* $i = 1, \ldots, n$,

(i) the pure strategy sets A_i *are nonempty, compact, and convex,*
(ii) the preference function V_i *is quasiconvex, and*
(iii) $V_i(\pi_i(a))$ *is continuous in* a *and quasiconcave in* a_i,

then there exists a pure-strategy Nash equilibrium.

Proof. Since $V_i(\pi_i(a_i, a_{-i}))$ is quasiconcave in a_i, ψ_i is convex-valued. Continuity of $V_i(\pi_i(a))$ coupled with compactness of A_i implies that ψ_i is nonempty, compact-valued and upper hemicontinuous.

Now let $R_i(a_{-i}) = \{a_i \in A_i | V_i(\pi_i(a_i, a_{-i})) \geqslant V_i(\pi_i(a_i', a_{-i}))$ for all $a_i' \in A_i\}$. That is, R_i is the best-response correspondence restricted to pure strategies. Because V_i is quasiconvex in the probabilities, R_i is nonempty by Lemma 1. Quasiconvexity of V_i also implies that if $\mu_i \in \psi_i(a_{-i})$ then $a_i \in \text{supp}(\mu_i)$ implies that $a_i \in R_i(a_{-i})$. Since ψ_i is compact-valued, convex-valued and upper hemicontinuous, so is R_i. Finally, the Cartesian product $R \equiv \times_{i=1,\ldots,n} R_i : A \to 2^A$ is therefore upper hemicontinuous, nonempty, compact, and convex. Since A is compact and convex, the Kakutani fixed point theorem can be applied, and there exists an $a^* \in A$ such that $a^* \in R(a^*)$. Then we have $a_i^* \in R_i(a_{-i}^*)$, which implies that $V_i(\pi_i(a_i^*, a_{-i}^*)) \geqslant V_i(\pi_i(a_i, a_{-i}^*))$ for all $a_i \in A_i$. Consequently, a^* is a pure-strategy Nash equilibrium. □

When players maximize rank-dependent expected utility, as in Chew, Karni, and Safra (1987) and Quiggin (1993), quasiconvexity of preferences is linked to risk aversion. In particular, the rank-dependent expected utility model states that there exist strictly increasing, continuous functions $u : \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \to [0, 1]$ with g onto such that

$$V(F) = \int u(x)\, \mathrm{d}(g \circ F)(x).$$

Chew, Karni, and Safra (1987) establish that the preference function V exhibits risk aversion if and only if both u and g are concave.

Chew (1985) shows that if g is concave then V must be quasiconvex over probability distributions (see also the discussion in Quiggin, 1993, pp. 120–121). Thus, when players maximize rank-dependent expected utility, condition (b) in Proposition 1 can be replaced by risk aversion:

COROLLARY 1. *Consider a strategic-form game* Γ *with rank-dependent expected utility-maximizing players. If, for each* $i = 1, \ldots, n$,

(i) the pure strategy sets A_i *are nonempty, compact and convex,*
(ii) the preference function V_i *is risk averse, and*
(iii) $V_i(\pi_i(a))$ *is continuous in* a *and quasiconcave in* a_i,

then there exists a pure-strategy Nash equilibrium.

5. DETERMINISTIC GAMES

In one special case it is possible to exploit the structure already built into the monetary game, namely the distinction between the payoff function β_i and the preference function V_i. The special case arises when the payoff function $\beta_i(a)$ yields a deterministic monetary value (i.e. a degenerate distribution over payoffs) for every pure strategy combination $a \in A$ and every $i = 1, \ldots, n$. To see why this makes a difference, consider what must happen for a player to respond to a pure strategy combination with a mixed strategy. Since the payoffs from pure strategies are deterministic money values, the payoff from the mixed strategy is a probability mixture of money values. For the player to strictly prefer the mixed strategy, so that no pure strategy equilibrium exists, it must be the case that the probability mixture is preferred to the monetary amounts over which the player mixes. This violates stochastic dominance preference, though. If the player exhibits first order stochastic dominance (FOSD) preference, the player prefers the highest payoff in the mixture to the mixture. Accordingly, Proposition 1 can be modified to replace the quasiconvexity of preferences with first order stochastic dominance preference.

Formally, the strategic-form game Γ is *deterministic* if $\beta(a)$ is a degenerate distribution in D^n for every $a \in A$. Examples of deterministic games include price- or quantity-setting games between oligopolists and many coordination games.

LEMMA 2. *If the strategic-form game* Γ *is deterministic,* V_i *is continuous and exhibits FOSD preference,* β_i *is continuous, and* A_i *is compact, then* $\psi_i(a_{-i})$ *contains pure strategies for each* $a_{-i} \in A_{-i}$.

Proof. Suppose that there exists some profile a_{-i} such that $\psi_i(a_{-i})$ contains no pure strategies. By the continuity of V_i and β_i and the compactness of A_i (and hence of $P(A_i)$)), $\psi_i(a_{-i})$ is nonempty. So $\psi_i(a_{-i})$ must contain a mixed strategy $\mu_i(a_i)$. Let α be the element in the support of μ_i which maximizes $\beta_i(\alpha, a_{-i})$. By FOSD preference, $V_i(\pi_i(\alpha, a_{-i})) \geqslant V_i(\pi_i(\mu_i, a_{-i}))$, so $\delta_\alpha \in \psi_i(a_{-i})$. Since there are no pure strategies in the best-response set, μ_i cannot be in the best-response set either, providing a contradiction. □

Lemma 2 establishes that in a deterministic game, FOSD preference guarantees that best-response correspondences contain pure strategies at every point. It is straightforward to adapt the proof of Proposition 1 to establish the following:[4]

PROPOSITION 2. *Consider a deterministic strategic-form game* Γ. *If, for each* $i = 1, \ldots, n$,

(i) the pure strategy sets A_i *are nonempty, compact, and convex,*
(ii) the preference function V_i *is continuous and exhibits FOSD preference, and*
(iii) $\beta_i(a)$ *is continuous in* a *and quasiconcave in* a_i,

then there exists a pure-strategy Nash equilibrium.

The conditions placed on the preference ordering in Proposition 4 are weaker than many of the conditions commonly assumed in the non-expected utility literature. For example, a common set of assumptions is that preferences are smooth, as in Machina (1982), and that all local utility functions are strictly increasing. Together these imply condition (ii) of Proposition 2. Because stronger restrictions on preferences are commonly assumed, the main restriction on the game is the restriction on the monetary payoff functions, that is, condition (iii) and the requirement that all payoffs from pure strategy combinations be nonstochastic.

6. CONCLUSION

This paper demonstrates that the expected utility requirement in standard pure-strategy equilibrium existence theorems can be replaced with quasiconvexity in the probabilities. We now turn to the issue of the validity of the quasiconvexity assumption. Experimental evidence on quasiconvexity vs. quasiconcavity is mixed, although Camerer and Ho (1994) find support for a complicated pattern with preferences tending toward quasiconvexity over gains and quasiconcavity over losses. The literature contains several theoretical papers pointing to the advantages of assuming one over the other. Crawford (1990) examines the existence of Nash equilibria in games with finite pure-strategy spaces. He finds that existence is guaranteed if preferences are quasiconcave in the probabilities, but that a different solution concept is needed if preferences are not quasiconcave. This result is, not surprisingly, exactly the opposite of ours. When the pure-strategy space is finite, players must be willing to mix to convexify the best-response correspondence. When a pure-strategy equilibrium is desired, though, one wants players to be either indifferent or averse to mixing.

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NOTES

1. Of course, previous results governing the existence of (possibly mixed-strategy) Nash equilibria in games with nonexpected utility players extend to our setting of convex action spaces.
2. The methods used in this paper can also be used to extend other pure-strategy equilibrium existence results, such as those of Dasgupta and Maskin (1986).
3. Crawford (1990) also analyzes monetary games, although he does not use this term.
4. Proposition 2 is a generalization of Lemma 1 in Ritzberger (1996).

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