

UCLA

UCLA Previously Published Works

Title

Theories without the tree property of the second kind

Permalink

<https://escholarship.org/uc/item/4h93b5rh>

Journal

Annals of Pure and Applied Logic, 165(2)

ISSN

0168-0072

Author

Chernikov, Artem

Publication Date

2014-02-01

DOI

10.1016/j.apal.2013.10.002

Peer reviewed

THEORIES WITHOUT THE TREE PROPERTY OF THE SECOND KIND

ARTEM CHERNIKOV

ABSTRACT. We initiate a systematic study of the class of theories without the tree property of the second kind — NTP_2 . Most importantly, we show: the burden is “sub-multiplicative” in arbitrary theories (in particular, if a theory has TP_2 then there is a formula with a single variable witnessing this); NTP_2 is equivalent to the generalized Kim’s lemma and to the boundedness of ist -weight; the dp -rank of a type in an arbitrary theory is witnessed by mutually indiscernible sequences of realizations of the type, after adding some parameters — so the dp -rank of a 1-type in any theory is always witnessed by sequences of singletons; in NTP_2 theories, simple types are co-simple, characterized by the co-independence theorem, and forking between the realizations of a simple type and arbitrary elements satisfies full symmetry; a Henselian valued field of characteristic $(0, 0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively), so in particular any ultraproduct of p -adics is NTP_2 ; adding a generic predicate to a geometric NTP_2 theory preserves NTP_2 .

INTRODUCTION

The aim of this paper is to initiate a systematic study of theories without the tree property of the second kind, or NTP_2 theories. This class was defined by Shelah implicitly in [She90] in terms of a certain cardinal invariant κ_{inp} (see Section 2) and explicitly in [She80], and it contains both simple and NIP theories. There was no active research on the subject until the recent interest in generalizing methods and results of stability theory to larger contexts, necessitated for example by the developments in the model theory of important algebraic examples such as algebraically closed valued fields [HHM08].

We give a short overview of related results in the literature. The invariant κ_{inp} , the upper bound for the number of independent partitions, was considered by Tsuboi in [Tsu85] for the case of stable theories. In [Adl08] Adler defines burden, by relativizing κ_{inp} to a fixed partial type, makes the connection to weight in simple theories and defines strong theories. Burden in the context of NIP theories, where it is called dp -rank, was already introduced by Shelah in [She05] and developed further in [OU11, KOU, KSed]. Results about forking and dividing in NTP_2 theories were established in [CK12]. In particular, it was proved that a formula forks over a model if and only if it divides over it (see Section 4). Some facts about ordered inp -minimal theories and

The author was supported by the Marie Curie Initial Training Network in Mathematical Logic - MALOA - From MAtheMatical LOgic to Applications, PITN-GA-2009-238381.

groups (that is with $\kappa_{\text{inp}}^1 = 1$) are proved in [Goo10, Sim11]. In [BY11, Theorem 4.13] Ben Yaacov shows that if a structure has IP, then its randomization (in the sense of continuous logic) has TP_2 . Malliaris [Mal12] considers TP_2 in relation to the saturation of ultra-powers and the Keisler order. In [Cha08] Chatzidakis observes that ω -free PAC fields have TP_2 .

A brief description of the results in this paper.

In Section 2 we introduce inp-patterns, burden, establish some of their basic properties and demonstrate that burden is sub-multiplicative: that is, if $\text{bdn}(a/C) < \kappa$ and $\text{bdn}(b/aC) < \lambda$, then $\text{bdn}(ab/C) < \kappa \times \lambda$. As an application we show that the value of the invariant of a theory $\kappa_{\text{inp}}(T)$ does not depend on the number of variables used in the computation. This answers a question of Shelah from [She90] and shows in particular that if T has TP_2 , then some formula $\phi(x, y)$ with x a singleton has TP_2 . It remains open whether burden in NTP_2 theories is actually sub-additive.

In Section 3 we describe the place of NTP_2 in the classification hierarchy of first-order theories and the relationship of burden to dp-rank in NIP theories and to weight in simple theories. We also recall some combinatorial “structure / non-structure” dichotomy due to Shelah, and discuss the behavior of the SOP_n hierarchy restricting to NTP_2 theories.

Section 4 is devoted to forking (and dividing) in NTP_2 theories. After discussing strictly invariant types, we give a characterization of NTP_2 in terms of the appropriate variants of Kim’s lemma, local character and bounded weight relatively to strict non-forking. As an application we consider theories with dependent dividing (i.e. whenever $p \in S(N)$ divides over $M \prec N$, there some $\phi(x, a) \in p$ dividing over M and such that $\phi(x, y)$ is NIP) and show that any theory with dependent dividing is NTP_2 . Finally we observe that the the analysis from [CK12] generalizes to a situation when one is working inside an NTP_2 type in an arbitrary theory.

A famous equation of Shelah “NIP = stability + dense linear order” turned out to be a powerful ideological principle, at least at the early stages of the development of NIP theories. In this paper the equation “ $\text{NTP}_2 = \text{simplicity} + \text{NIP}$ ” plays an important role. In particular, it seems very natural to consider two extremal kinds of types in NTP_2 theories (and in general) — simple types and NIP types. While it is perfectly possible for an NTP_2 theory to have neither, they form important special cases and are not entirely understood.

In section 5 we look at NIP types. In particular we show that the results of the previous section on forking localized to a type combined with honest definitions from [CS13] allow to omit the global NTP_2 assumption in the theorem of [KSed], thus proving that dp-rank of a type in arbitrary theory is always witnessed by mutually indiscernible sequences of its realizations, after adding some parameters (see Theorem 5.3). We also observe that in an NTP_2 theory, a type is NIP if and only if every extension of it has only boundedly many global non-forking extensions.

In Section 6 we consider simple types (defined as those types for which every completion satisfies the local character), first in arbitrary theories and then in NTP_2 . While it is more or less immediate that on the set of realizations of a simple type forking satisfies all the properties of forking in simple theories, the interaction between the realizations of a simple type and arbitrary tuples seems more intricate. We establish full symmetry between realizations of a simple type and arbitrary elements, answering a question of Casanovas in the case of NTP_2 theories (showing that simple types are co-simple, see Definition 6.7). Then we show that simple types are characterized as those satisfying the co-independence theorem and that co-simple stably embedded types are simple (so in particular a theory is simple if and only if it is NTP_2 and satisfies the independence theorem).

Section 7 is devoted to examples. We give an Ax-Kochen-Ershov type statement: a Henselian valued field of characteristic $(0,0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively). This is parallel to the result of Delon for NIP [Del81], and generalizes a result of Shelah for strong dependence [She05]. It follows that valued fields of Hahn series over pseudo-finite fields are NTP_2 . In particular, every theory of an ultra-product of p -adics is NTP_2 (and in fact of finite burden). We also show that expanding a geometric NTP_2 theory by a generic predicate (Chatzidakis-Pillay style [CP98]) preserves NTP_2 .

Acknowledgments. I am grateful to Itai Ben Yaacov, Itay Kaplan and Martin Hils for multiple discussions around the topics of the paper. I would also like to thank Hans Adler and Enrique Casanovas for their interest in this work and for suggesting nice questions. Finally, I thank Ehud Hrushovski and the referee for some corrections.

1. PRELIMINARIES

As usual, we will be working in a monster model \mathbb{M} of a complete first-order theory T . We will not be distinguishing between elements and tuples unless explicitly stated.

Definition 1.1. We will often be considering collections of sequences $(\bar{a}_\alpha)_{\alpha < \kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i < \lambda}$ (where each $a_{\alpha,i}$ is a tuple, maybe infinite). We say that they are *mutually indiscernible* over a set C if \bar{a}_α is indiscernible over $C\bar{a}_{\neq\alpha}$ for all $\alpha < \kappa$. We will say that they are *almost mutually indiscernible* over C if \bar{a}_α is indiscernible over $C\bar{a}_{<\alpha} (a_{\beta,0})_{\beta > \alpha}$. Sometimes we call $(a_{\alpha,i})_{\alpha < \kappa, i < \lambda}$ an *array*. We say that $(\bar{b}_\alpha)_{\alpha < \kappa'}$ is a *sub-array* of $(\bar{a}_\alpha)_{\alpha < \kappa}$ if for each $\alpha < \kappa'$ there is $\beta_\alpha < \kappa$ such that \bar{b}_α is a sub-sequence of \bar{a}_{β_α} . We say that an array is *mutually indiscernible* (almost mutually indiscernible) if rows are mutually indiscernible (resp. almost mutually indiscernible). Finally, an array is *strongly indiscernible* if it is mutually indiscernible and in addition the sequence of rows $(\bar{a}_\alpha)_{\alpha < \kappa}$ is an indiscernible sequence.

The following lemma will be constantly used for finding indiscernible arrays.

Lemma 1.2. (1) *For any small set C and cardinal κ there is λ such that:*

If $A = (a_{\alpha,i})_{\alpha < n, i < \lambda}$ is an array, $n < \omega$ and $|a_{\alpha,i}| \leq \kappa$, then there is an array $B = (b_{\alpha,i})_{\alpha < n, i < \omega}$ with rows mutually indiscernible over C and such that every finite sub-array of B has the same type over C as some sub-array of A .

(2) *Let C be small set and $A = (a_{\alpha,i})_{\alpha < n, i < \omega}$ be an array with $n < \omega$. Then for any finite $\Delta \in L(C)$ and $N < \omega$ we can find Δ -mutually indiscernible sequences $(a_{\alpha,i_{\alpha,0}}, \dots, a_{\alpha,i_{\alpha,N}}) \subset \bar{a}_\alpha$, $\alpha < n$.*

Proof. (1) Let $\lambda_0 = \kappa + |T| + |C|$, $\lambda_{n+1} = \beth_{(2^{\lambda_n})^+}$ and let $\lambda = \sum_{n < \omega} \lambda_n$. Now assume that we are given an array $A = (a_{\alpha,i})_{\alpha < n, i < \lambda}$, and let $\bar{a}_\alpha = (a_{\alpha,i})_{i < \lambda_\alpha}$. By the Erdős-Rado theorem (see e.g. [BY03, Lemma 1.2]) and the choice of λ_α 's we can find a sequence $\bar{a}'_{n-1} = (a'_{n-1,i})_{i < \omega}$ which is indiscernible over $\bar{a}_{< n-1}$ and such that every finite subsequence of \bar{a}'_{n-1} has the same type over $\bar{a}_{< n-1}$ as some finite subsequence of \bar{a}_{n-1} . Next, as $|\bar{a}_{< n-2} \cup \bar{a}'_{n-1}| \leq \lambda_{n-3}$ it follows by Erdős-Rado that we can find some sequence $\bar{a}'_{n-2} = (a'_{n-2,i})_{i < \omega}$ which is indiscernible over $\bar{a}_{< n-2} \bar{a}'_{n-1}$ and such that every finite subsequence of it has the same type over $\bar{a}_{< n-2} \bar{a}'_{n-1}$ as some subsequence of \bar{a}_{n-2} . Continuing in the same manner we get sequences $\bar{a}'_{n-1}, \bar{a}'_{n-2}, \dots, \bar{a}'_0$ and it is easy to check from the construction that they are mutually indiscernible and give rows of an array satisfying (1).

(2) By a repeated use of the finite Ramsey theorem, see [CH12, Lemma 3.5(3)] for details. \square

Lemma 1.3. *Let $(\bar{a}_\alpha)_{\alpha < \kappa}$ be almost mutually indiscernible over C . Then there are $(\bar{a}'_\alpha)_{\alpha < \kappa}$, mutually indiscernible over C and such that $\bar{a}'_\alpha \equiv_{C a_{\alpha,0}} \bar{a}_\alpha$ for all $\alpha < \kappa$.*

Proof. By Lemma 1.2, taking an automorphism, and compactness (see [CH12, Lemma 3.5(2)] for details). \square

Definition 1.4. Given a set of formulas Δ , let $R(\kappa, \Delta)$ be the minimal length of a sequence of singletons sufficient for the existence of a Δ -indiscernible sub-sequence of length κ . In particular, for finite Δ we have:

- (1) $R(\omega, \Delta) = \omega$ — by infinite Ramsey theorem,
- (2) $R(n, \Delta) < \omega$ for every $n < \omega$ — by finite Ramsey theorem,
- (3) $R(\kappa^+, \Delta) \leq \beth_\omega(\kappa)$ for any infinite κ — by Erdős-Rado theorem.

Remark 1.5. Let (\bar{a}_i) be a mutually indiscernible array over A . Then it is still mutually indiscernible over $\text{acl}(A)$.

Fact 1.6. (see e.g. [HP11]) *Let $p(x)$ be a global type invariant over a set C (that is $\phi(x, a) \in p$ if and only if $\phi(x, \sigma(a)) \in p$ for any $\sigma \in \text{Aut}(\mathbb{M}/C)$). For any set $D \supseteq C$, and an ordinal α , let the sequence $\bar{c} = \langle c_i \mid i < \alpha \rangle$ be such that $c_i \models p|_{D c_{< i}}$. Then \bar{c} is indiscernible over D and its type over*

D does not depend on the choice of \bar{c} . Call this type $p^{(\alpha)}|_D$, and let $p^{(\alpha)} = \bigcup_{D \supseteq C} p^{(\alpha)}|_D$. Then $p^{(\alpha)}$ also does not split over C .

Finally, we assume some acquaintance with the basics of simple (e.g. [Cas07]) and NIP (e.g. [Adl08]) theories.

2. BURDEN AND κ_{inp}

Let $p(x)$ be a (partial) type.

Definition 2.1. An *inp-pattern* in $p(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha < \kappa, i < \omega}$, $\phi_\alpha(x, y_\alpha)$ and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, a_{\alpha,i})\}_{i < \omega}$ is k_α -inconsistent, for each $\alpha < \kappa$
- $\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\}_{\alpha < \kappa} \cup p(x)$ is consistent, for any $f : \kappa \rightarrow \omega$.

The *burden* of $p(x)$, denoted $\text{bdn}(p)$, is the supremum of the depths of all inp-patterns in $p(x)$. By $\text{bdn}(a/C)$ we mean $\text{bdn}(tp(a/C))$.

Obviously, $p(x) \subseteq q(x)$ implies $\text{bdn}(p) \geq \text{bdn}(q)$ and $\text{bdn}(p) = 0$ if and only if p is algebraic. Also notice that $\text{bdn}(p) < \infty \Leftrightarrow \text{bdn}(p) < |T|^+$ by compactness.

First we observe that it is sufficient to look at mutually indiscernible inp-patterns.

Lemma 2.2. *For $p(x)$ a (partial) type over C , the following are equivalent:*

- (1) *There is an inp-pattern of depth κ in $p(x)$.*
- (2) *There is an array $(\bar{a}_\alpha)_{\alpha < \kappa}$ with rows mutually indiscernible over C and $\phi_\alpha(x, y_\alpha)$ for $\alpha < \kappa$ such that:*
 - $\{\phi_\alpha(x, a_{\alpha,i})\}_{i < \omega}$ is inconsistent for every $\alpha < \kappa$
 - $p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha < \kappa}$ is consistent.
- (3) *There is an array $(\bar{a}_\alpha)_{\alpha < \kappa}$ with rows almost mutually indiscernible over C with the same properties.*

Proof. (1) \Rightarrow (2) is a standard argument using Lemma 1.2 and compactness, (2) \Rightarrow (3) is clear and (3) \Rightarrow (1) is an easy reverse induction plus compactness. \square

We will need the following technical lemma.

Lemma 2.3. *Let $(\bar{a}_\alpha)_{\alpha < \kappa}$ be a mutually indiscernible array over C and b given. Let $p_\alpha(x, a_{\alpha,0}) = tp(b/a_{\alpha,0}C)$, and assume that $p^\infty(x) = \bigcup_{\alpha < \kappa, i < \omega} p_\alpha(x, a_{\alpha,i})$ is consistent. Then there are $(\bar{a}'_\alpha)_{\alpha < \kappa}$ such that:*

- (1) $\bar{a}'_\alpha \equiv_{a_{\alpha,0}C} \bar{a}_\alpha$ for all $\alpha < \kappa$
- (2) $(\bar{a}'_\alpha)_{\alpha < \kappa}$ is a mutually indiscernible array over Cb .

Proof. It is sufficient to find b' such that $b' \equiv_{a_{\alpha,0}C} b$ for all $\alpha < \kappa$ and $(\bar{a}_\alpha)_{\alpha < \kappa}$ is mutually indiscernible over $b'C$ (then applying an automorphism over C to conclude). Let $b^\infty \models p^\infty(x)$. By Lemma 1.2, for any finite $\Delta \in L(C)$, $S \subseteq \kappa$ and $n < \omega$, there is a $\Delta(b^\infty)$ -mutually indiscernible sub-array $(a'_{\alpha,i})_{\alpha \in S, i < n}$ of $(\bar{a}_\alpha)_{\alpha \in S}$. Let σ be an automorphism over C sending $(a'_{\alpha,i})_{\alpha \in S, i < n}$ to $(a_{\alpha,i})_{\alpha \in S, i < n}$ and $b' = \sigma(b^\infty)$. Then $(a_{\alpha,i})_{\alpha \in S, i < n}$ is $\Delta(b')$ -mutually indiscernible and $b' \models \bigcup_{\alpha \in S} p_\alpha(x, a_{\alpha,0})$, so $b' \equiv_{a_{\alpha,0}C} b$. Conclude by compactness. \square

Next lemma provides a useful equivalent way to compute the burden of a type.

Lemma 2.4. *The following are equivalent for a partial type $p(x)$ over C :*

- (1) *There is no inp-pattern of depth κ in p .*
- (2) *For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha < \kappa}$, an almost mutually indiscernible array over C , there is $\beta < \kappa$ and \bar{a}' indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.*
- (3) *For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha < \kappa}$, a mutually indiscernible array over C , there is $\beta < \kappa$ and \bar{a}' indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.*

Proof. (1) \Rightarrow (2): So let $(\bar{a}_\alpha)_{\alpha < \kappa}$ be almost mutually indiscernible over C and $b \models p(x)$ given. Let $p_\alpha(x, a_{\alpha,0}) = \text{tp}(b/a_{\alpha,0}C)$ and let $p_\alpha(x) = \bigcup_{i < \omega} p_\alpha(x, a_{\alpha,i})$.

Assume that p_α is inconsistent for each α , by compactness and indiscernibility of \bar{a}_α over C there is some $\phi_\alpha(x, a_{\alpha,0}c_\alpha) \in p_\alpha(x, a_{\alpha,0})$ with $c_\alpha \in C$ such that $\{\phi_\alpha(x, a_{\alpha,i}c_\alpha)\}_{i < \omega}$ is k_α -inconsistent. As $b \models \{\phi_\alpha(x, a_{\alpha,0}c_\alpha)\}_{\alpha < \kappa}$, by almost indiscernibility of $(\bar{a}_\alpha)_{\alpha < \kappa}$ over C and Lemma 2.2 we find an inp-pattern of depth κ in p – a contradiction.

Thus $p_\beta(x)$ is consistent for some $\beta < \kappa$. Then we can find \bar{a}' which is indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$ by Lemma 2.3.

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (1): Assume that there is an inp-pattern of depth κ in $p(x)$. By Lemma 2.2 there is an inp-pattern $(\bar{a}_\alpha, \phi_\alpha, k_\alpha)_{\alpha < \kappa}$ in $p(x)$ with $(\bar{a}_\alpha)_{\alpha < \kappa}$ a mutually indiscernible array over C . Let $b \models p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha < \kappa}$. On the one hand $b \models \phi_\alpha(b, a_{\alpha,0})$, while on the other $\{\phi_\alpha(x, a_{\alpha,i})\}_{i < \omega}$ is inconsistent, thus it is impossible to find an \bar{a}'_α as required for any $\alpha < \kappa$. \square

Theorem 2.5. *If there is an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\text{tp}(b_1b_2/C)$, then either there is an inp-pattern of depth κ_1 in $\text{tp}(b_1/C)$ or there is an inp-pattern of depth κ_2 in $\text{tp}(b_2/b_1C)$.*

Proof. Assume not. Without loss of generality $C = \emptyset$, and let $(\bar{a}_\alpha)_{\alpha \in \kappa_1 \times \kappa_2}$ be a mutually indiscernible array, where we consider the product $\kappa_1 \times \kappa_2$ lexicographically ordered. By induction on $\alpha < \kappa_1$ we choose \bar{a}'_α and $\beta_\alpha \in \kappa_2$ such that:

- (1) \bar{a}'_α is indiscernible over $b_2\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$.
- (2) $\text{tp}(\bar{a}'_\alpha/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}) = \text{tp}(\bar{a}_{(\alpha,\beta_\alpha)}/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)})$.

(3) $\bar{a}'_{\leq \alpha} \cup \bar{a}_{\geq (\alpha+1,0)}$ is a mutually indiscernible array.

Assume we have managed up to α , and we need to choose \bar{a}'_{α} and β_{α} . Let $D = \bar{a}'_{< \alpha} \bar{a}_{\geq (\alpha+1,0)}$. As $(\bar{a}_{(\alpha,\delta)})_{\delta \in \kappa_2}$ is a mutually indiscernible array over D (by assumption in the case $\alpha = 0$ and by (3) of the inductive hypothesis in the other cases) and there is no inp-pattern of depth κ_2 in $\text{tp}(b_2/D)$, by Lemma 2.4(3) there is some $\beta_{\alpha} < \kappa_2$ and \bar{a}'_{α} indiscernible over $b_2 D$ (which gives us (1)) such that $\text{tp}(\bar{a}'_{\alpha}/a_{(\alpha,\beta_{\alpha}),0}D) = \text{tp}(\bar{a}_{(\alpha,\beta_{\alpha})}/a_{(\alpha,\beta_{\alpha}),0}D)$ (which together with the inductive hypothesis gives us (2) and (3)).

So we have carried out the induction. Now it is easy to see by (1), noticing that the first elements of \bar{a}'_{α} and $\bar{a}_{(\alpha,\beta_{\alpha})}$ are the same by (2), that $(\bar{a}'_{\alpha})_{\alpha < \kappa_1}$ is an almost mutually indiscernible array over b_2 . By Lemma 1.3, we may assume that in fact $(\bar{a}'_{\alpha})_{\alpha < \kappa_1}$ is a mutually indiscernible array over b_2 .

As there is no inp-pattern of depth κ_1 in $\text{tp}(b_1/b_2)$, by Lemma 2.4 there is some $\gamma < \kappa_1$ and \bar{a} indiscernible over $b_1 b_2$ and such that $\bar{a} \equiv_{a'_{\gamma,0}} \bar{a}'_{\gamma} \equiv_{a_{(\gamma,\beta_{\gamma}),0}} \bar{a}_{(\gamma,\beta_{\gamma})}$. As $(\bar{a}_{\alpha})_{\alpha \in \kappa_1 \times \kappa_2}$ was arbitrary, by Lemma 2.4(3) this implies that there is no inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\text{tp}(b_1 b_2)$. \square

Corollary 2.6. “Sub-multiplicativity” of burden: If $\text{bdn}(a_i) < k_i$ for $i < n$ with $k_i \in \omega$, then $\text{bdn}(a_0 \dots a_{n-1}) < \prod_{i < n} k_i$.

In the case of NIP theories it is known that burden is not only sub-multiplicative, but actually sub-additive, i.e. $\text{bdn}(ab) \leq \text{bdn}(a) + \text{bdn}(b)$ (by [KOU] and Fact 3.8). Similarly, burden is sub-additive in simple theories because of the sub-additivity of weight and Fact 3.10. This motivates the following conjecture:

Conjecture 2.7. *Burden is sub-additive in NTP₂ theories.*

We also ask if burden is sub-additive in arbitrary theories.

Definition 2.8. For $n < \omega$, we let $\kappa_{\text{inp}(T)}^n$ be the first cardinal κ such that there is no inp-pattern $(\bar{a}_{\alpha}, \phi_{\alpha}(x, y_{\alpha}), k_{\alpha})$ of depth κ with $|x| \leq n$. And let $\kappa_{\text{inp}(T)} = \sup_{n < \omega} \kappa_{\text{inp}(T)}^n$. Notice that $\kappa_{\text{inp}(T)}^m \geq \kappa_{\text{inp}(T)}^n \geq n$ for all $n < m$, just because of having the equality in the language, and thus $\kappa_{\text{inp}(T)} \geq \aleph_0$.

We can use Theorem 2.5 to answer a question of Shelah [She90, Ch. III, Question 7.5].

Corollary 2.9. $\kappa_{\text{inp}(T)} = \kappa_{\text{inp}(T)}^n = \kappa_{\text{inp}(T)}^1$, as long as $\kappa_{\text{inp}(T)}^n$ is infinite for some $n < \omega$.

3. NTP₂ AND ITS PLACE IN THE CLASSIFICATION HIERARCHY

The aim of this section is to (finally) define NTP₂, describe its place in the classification hierarchy of first-order theories and what burden amounts to in the more familiar situations.

Definition 3.1. A formula $\phi(x, y)$ has TP_2 if there is an array $(a_{\alpha, i})_{\alpha, i < \omega}$ such that $\{\phi(x, a_{\alpha, i})\}_{i < \omega}$ is 2-inconsistent for every $\alpha < \omega$ and $\{\phi(x, a_{\alpha, f(\alpha)})\}_{\alpha < \omega}$ is consistent for any $f : \omega \rightarrow \omega$. Otherwise we say that $\phi(x, y)$ is NTP_2 , and T is NTP_2 if every formula is.

Lemma 3.2. *The following are equivalent for T :*

- (1) Every formula $\phi(x, y)$ with $|x| \leq n$ is NTP_2 .
- (2) $\kappa_{\text{inp}}^n(T) \leq |T|^+$.
- (3) $\kappa_{\text{inp}}^n(T) < \infty$.
- (4) $\text{bdn}(b/C) < |T|^+$ for all b and C , with $|b| = n$.

Proof. (1) \Rightarrow (2): Assume we have a mutually indiscernible inp-pattern $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha < |T|^+}$ of depth $|T|^+$. By pigeon-hole we may assume that $\phi_\alpha(x, y_\alpha) = \phi(x, y)$ and $k_\alpha = k$. Then by Ramsey and compactness we may assume in addition that (\bar{a}_α) is a strongly indiscernible array. If $\{\phi(x, a_{\alpha, 0}) \wedge \phi(x, a_{\alpha, 1})\}_{\alpha < n}$ is inconsistent for some $n < \omega$, then taking $b_{\alpha, i} = a_{n\alpha, i} a_{n\alpha+1, i} \dots a_{n\alpha+n-1, i}$, $(\bigwedge_{i < n} \phi(x, y_i), \bar{b}_\alpha, 2)_{\alpha < \omega}$ is an inp-pattern. Otherwise $\{\phi(x, a_{\alpha, 0}) \wedge \phi(x, a_{\alpha, 1})\}_{\alpha < \omega}$ is consistent, then taking $b_{\alpha, i} = a_{\alpha, 2i} a_{\alpha, 2i+1}$ we conclude that $(\phi(x, y_1) \wedge \phi(x, y_2), \bar{b}_\alpha, \lceil \frac{k}{2} \rceil)_{\alpha < \omega}$ is an inp-pattern. Repeat if necessary.

The other implications are clear by compactness. \square

Remark 3.3. (1) implies (2) is from [Adl07].

It follows from the lemma and Theorem 2.9 that if T has TP_2 , then some formula $\phi(x, y)$ with $|x| = 1$ has TP_2 . From Lemma 7.1 it follows that if $\phi_1(x, y_1)$ and $\phi_2(x, y_2)$ are NTP_2 , then $\phi_1(x, y_1) \vee \phi_2(x, y_2)$ is NTP_2 . This, however, is the only Boolean operation preserving NTP_2 (see Example 3.13).

Definition 3.4. [Adler] T is called *strong* if there is no inp-pattern of infinite depth in it. It is clearly a subclass of NTP_2 theories.

Proposition 3.5. *If $\phi(x, y)$ is NIP, then it is NTP_2 .*

Proof. Let $(a_{\alpha, j})_{\alpha, j < \omega}$ be an array witnessing that $\phi(x, y)$ has TP_2 . But then for any $s \subseteq \omega$, let $f(\alpha) = 0$ if $\alpha \in s$, and $f(\alpha) = 1$ otherwise. Let $d \models \{\phi(x, a_{\alpha, f(\alpha)})\}$. It follows that $\phi(d, a_{\alpha, 0}) \Leftrightarrow \alpha \in s$. \square

We recall the definition of dp-rank (e.g. [KOU]):

Definition 3.6. We let the dp-rank of p , denoted $\text{dprk}(p)$, be the supremum of κ for which there are $b \models p$ and mutually indiscernible over C (a set containing the domain of p) sequences $(\bar{a}_\alpha)_{\alpha < \kappa}$ such that none of them is indiscernible over bC .

Fact 3.7. *The following are equivalent for a partial type $p(x)$ (by Ramsey and compactness):*

- (1) $\text{dprk}(p) > \kappa$.
- (2) *There is an ict-pattern of depth κ in $p(x)$, that is $(\bar{a}_i, \varphi_i(x, y_i), k_i)_{i < \kappa}$ such that $p(x) \cup \{\varphi_i(x, a_{i,s(i)})\}_{i < \kappa} \cup \{\varphi_i(x, a_{i,j})\}_{s(i) \neq j < \kappa}$ is consistent for every $s : \kappa \rightarrow \omega$.*

It is easy to see that every inp-pattern with mutually indiscernible rows gives an ict-pattern of the same depth. On the other hand, if T is NIP then every ict-pattern gives an inp-pattern of the same depth (see [Adl07, Section 3]). Thus we have:

- Fact 3.8.** (1) *For a partial type $p(x)$, $\text{bdn}(p) \leq \text{dprk}(p)$. And if $p(x)$ is an NIP type, then $\text{bdn}(p) = \text{dprk}(p)$*
- (2) *T is strongly dependent $\Leftrightarrow T$ is NIP and strong.*

Proposition 3.9. *If T is simple, then it is NTP_2 .*

Proof. Of course, inp-pattern of the form $(\bar{a}_\alpha, \phi(x, y), k)_{\alpha < \omega}$ witnesses the tree property. \square

Moreover,

Fact 3.10. [Adl07, Proposition 8] *Let T be simple. Then the burden of a partial type is the supremum of the weights of its complete extensions. And T is strong if and only if every type has finite burden.*

Definition 3.11. [Shelah] $\phi(x, y)$ is said to have TP_1 if there are $(a_\eta)_{\eta \in \omega < \omega}$ and $k \in \omega$ such that:

- $\{\phi(x, a_{\eta|i})\}_{i \in \omega}$ is consistent for any $\eta \in \omega^\omega$
- $\{\phi(x, a_{\eta_i})\}_{i < k}$ is inconsistent for any mutually incomparable $\eta_0, \dots, \eta_{k-1} \in \omega^{<\omega}$.

Fact 3.12. [She90, III.7.7, III.7.11] *Let T be NTP_2 , $q(y)$ a partial type and $\phi(x, y)$ has TP witnessed by $(a_\eta)_{\eta \in \omega < \omega}$ with $a_\eta \models q$, and such that in addition $\{\phi(x, a_{\eta|i})\}_{i \in \omega} \cup p(x)$ is consistent for any $\eta \in \omega^\omega$. Then some formula $\psi(x, \bar{y}) = \bigwedge_{i < k} \phi(x, y_i) \wedge \chi(x)$ (where $\chi(x) \in p(x)$) has TP_1 , witnessed by (b_η) with $b_\eta \subseteq q(\mathbb{M})$ and such that $\{\phi(x, b_{\eta|i})\}_{i \in \omega} \cup p(x)$ is consistent.*

It is not stated in exactly the same form there, but immediately follows from the proof. See [Adl07, Section 4] and [KKS12, Theorem 6.6] for a more detailed account of the argument. See [KK11] for more details on NTP_1 .

Example 3.13. Triangle-free random graph (i.e. the model companion of the theory of graphs without triangles) has TP_2 .

Proof. We can find $(a_{ij}b_{ij})_{i,j < \omega}$ such that $R(a_{ij}, b_{ik})$ for every i and $j \neq k$, and this are the only edges around. But then $\{xRa_{ij} \wedge xRb_{ij}\}_{j < \omega}$ is 2-inconsistent for every i as otherwise it would have created a triangle, while $\{xRa_{if(i)} \wedge xRb_{if(i)}\}_{i < \omega}$ is consistent for any $f : \omega \rightarrow \omega$. Note that

the formula xRy is NTP_2 , thus demonstrating that a conjunction of two NTP_2 formulas need not be NTP_2 . \square

A similar argument shows that the theory of a K_n -free random graph has TP_2 for all $n \geq 3$. In fact it is known that the triangle-free random graph is rosy and 2-dependent (in the sense of [She07]), thus there is no implication between rosiness and NTP_2 , and between k -dependence and NTP_2 for $k > 1$.

3.1. On the SOP_n hierarchy restricted to NTP_2 theories. We recall the definition of SOP_n for $n \geq 2$ from [She96, Definition 2.5]:

Definition 3.14. (1) Let $n \geq 3$. We say that a formula $\phi(x, y)$ has SOP_n if there are $(a_i)_{i \in \omega}$ such that:

- (a) There is an infinite chain: $\models \phi(a_i, a_j)$ for all $i < j < \omega$,
 - (b) There are no cycles of length n : $\models \neg \exists x_0 \dots x_{n-1} \bigwedge_{j=i+1(\text{ mod } n)} \phi(x_i, x_j)$.
- (2) $\phi(x, y)$ has SOP_2 if and only if it has TP_1 .
- (3) For a theory T , $\text{SOP} \Rightarrow \dots \Rightarrow \text{SOP}_{n+1} \Rightarrow \text{SOP}_n \Rightarrow \dots \Rightarrow \text{SOP}_3 \Rightarrow \text{SOP}_2 \Rightarrow \text{TP}$.
- (4) By Fact 3.12 we see that restricting to NTP_2 theories, the last 2 items coincide.

The following are the standard examples showing that the SOP_n hierarchy is strict for $n \geq 3$:

Example 3.15. [She96, Claim 2.8]

- (1) For $n \geq 3$, let T_n be the model completion of the theory of directed graphs (no self-loops or multiple edges) with no directed cycles of length $\leq n$. Then it has SOP_n but not SOP_{n+1} .
- (2) For odd $n \geq 3$, the model completion of the theory of graphs with no odd cycles of length $\leq n$, has SOP_n but not SOP_{n+1} .
- (3) Consider the model companion of a theory in the language $(\langle n, l \rangle)_{l \leq n}$ saying:
 - (a) $x \langle n, m-1 \rangle y \rightarrow x \langle n, m \rangle y$,
 - (b) $x \langle n, n \rangle y$,
 - (c) $\neg(x \langle n, n-1 \rangle x)$,
 - (d) if $l + k + 1 = m \leq n$ then $x \langle n, l \rangle y \wedge y \langle n, k \rangle z \rightarrow x \langle n, m \rangle z$.

It eliminates quantifiers.

However, all these examples have TP_2 .

Proof. (1) Let $\phi(x, y_1 y_2) = xRy_1 \wedge y_2Rx$. For $i \in \omega$ we choose sequence $(a_{i,j} b_{i,j})_{j \in \omega}$ such that $\models R(a_{i,j}, b_{i,k})$ and $R(b_{i,j}, a_{i,k})$ for all $j < k \in \omega$, and these are the only edges around — it is possible as no directed cycles are created. Now for any i , if there is $c \models \phi(x, a_{i,0} b_{i,0}) \wedge \phi(x, a_{i,1} b_{i,1})$, then we would have a directed cycle $c, b_{i,0}, a_{i,1}$ of length 3 — a contradiction. On the other hand,

given and $i_0 < \dots < i_n$ and j_0, \dots, j_n there has to be an element $a \models \bigwedge_{\alpha \leq n} \phi(x, a_{i_\alpha, j_\alpha} b_{i_\alpha, j_\alpha})$ as there are no directed cycles created. Thus $\phi(x, y_1 y_2)$ has TP_2 .

(2) and (3) Similar. \square

This naturally leads to the following question:

Problem 3.16. Is the SOP_n hierarchy strict restricting to NTP_2 theories?

In [She90, Exercise III.7.12] Shelah suggests an example of a theory satisfying $\text{NTP}_2 + \text{NSOP}$ which is not simple. However, his example doesn't seem to work.

4. FORKING IN NTP_2

In [Kim01, Theorem 2.4] Kim gives several equivalents to the simplicity of a theory in terms of the behavior of forking and dividing.

Fact 4.1. *The following are equivalent:*

- (1) T is simple.
- (2) $\phi(x, a)$ divides over A if and only if $\{\phi(x, a_i)\}_{i < \omega}$ is inconsistent for every Morley sequence $(a_i)_{i < \omega}$ over A .
- (3) Dividing in T satisfies local character.

In this section we show an analogous characterization of NTP_2 . But first we recall some facts about forking and dividing in NTP_2 theories and introduce some terminology.

- Definition 4.2.**
- (1) A type $p(x) \in S(C)$ is *strictly invariant* over A if it is Lascar invariant over A and for any small $B \subseteq C$ and $a \models p|_B$, we have that $\text{tp}(B/aA)$ does not divide over A (we can replace “does not divide” by “does not fork” $C = \mathbb{M}$). For example, a definable type or a global type which is both an heir and a coheir over M , are strictly invariant over M .
 - (2) We will write $a \downarrow_c^{\text{ist}} b$ when $\text{tp}(a/bc)$ can be extended to a global type $p(x)$ strictly invariant over A .
 - (3) We say that $(a_i)_{i < \omega}$ is a strict Morley sequence over A if it is indiscernible over A and $a_i \downarrow_A^{\text{ist}} a_{< i}$ for all $i < \omega$.
 - (4) As usual, we will write $a \downarrow_c^u b$ if $\text{tp}(a/bc)$ is finitely satisfiable in c , $a \downarrow_c^d b$ ($a \downarrow_c^f b$) if $\text{tp}(a/bc)$ does not divide (resp. does not fork) over c .
 - (5) We write $a \downarrow_c^i b$ if $\text{tp}(a/bc)$ can be extended to a global type $p(x)$ Lascar invariant over c . We point out that if $a \downarrow_c^i b$ and $(b_i)_{i < \omega}$ is a c -indiscernible sequence with $b_0 = b$, then it is actually indiscernible over a .
 - (6) If T is simple, then $\downarrow^i = \downarrow^{\text{ist}}$. And if T is NIP, then $\downarrow^i = \downarrow^f$.

- (7) We say that a set A is an *extension base* if every type over A has a global non-forking extension. Every model is an extension base (because every type has a global coheir). A theory in which every set is an extension base is called *extensible*.

Strictly invariant types exist in any theory (but it is not true that every type over a model has a global extension which is strictly invariant over the same model). In fact, there are theories in which over any set there is some type without a global strictly invariant extension (see [CKS12]).

Lemma 4.3. *Let $p(x)$ be a global type invariant over A , and let $M \supset A$ be $|A|^+$ -saturated. Then p is strictly invariant over M .*

Proof. It is enough to show that p is an heir over M . Let $\phi(x, c) \in p$. By saturation of M , $\text{tp}(c/A)$ is realized by some $c' \in M$. But as p is invariant over A , $\phi(x, c') \in p$ as wanted. \square

One of the main uses of strict invariance is the following criterion for making indiscernible sequences mutually indiscernible without changing their type over the first elements.

Lemma 4.4. *Let $(\bar{a}_i)_{i < \kappa}$ and C be given, with \bar{a}_i indiscernible over C and starting with a_i . If $a_i \downarrow_C^{\text{ist}} a_{<i}$, then there are mutually C -indiscernible $(\bar{b}_i)_{i < \kappa}$ such that $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.*

Proof. Enough to show for finite κ by compactness. So assume we have chosen $\bar{a}'_0, \dots, \bar{a}'_{n-1}$, and lets choose \bar{a}'_n . As $a_n \downarrow_C^{\text{ist}} a_{<n}$, there are $\bar{a}''_0 \dots \bar{a}''_{n-1} \equiv_{C a_0 \dots a_{n-1}} \bar{a}'_0 \dots \bar{a}'_{n-1}$ and such that $a_n \downarrow_C^{\text{ist}} \bar{a}''_{<n}$. As $a_n \downarrow_C^i \bar{a}''_{<n, \neq j}$ for $j < n$, it follows by the inductive assumption and Definition 4.2(5) that \bar{a}''_j is indiscernible over $a_n \bar{a}''_{\neq j}$. On the other hand $\bar{a}''_0 \dots \bar{a}''_{n-1} \downarrow_C^f a_n$, and so by basic properties of forking there is some $\bar{a}'_n \equiv_{C a_n} \bar{a}_n$ indiscernible over $\bar{a}''_0, \dots, \bar{a}''_{n-1}$. Conclude by Lemma 1.3. \square

Remark 4.5. This argument is essentially from [She09, Section 5].

We recall a result about forking and dividing in NTP_2 theories from [CK12].

Fact 4.6. [CK12] *Let T be NTP_2 and $M \models T$.*

- (1) *Every $p \in S(M)$ has a global strictly invariant extension.*
- (2) *For any a , $\phi(x, a)$ divides over M if and only if $\phi(x, a)$ forks over M , if and only if for every $(a_i)_{i < \omega}$, a strict Morley sequence in $\text{tp}(a/M)$, $\{\phi(x, a_i)\}_{i < \omega}$ is inconsistent.*
- (3) *In fact, just assuming that A is an extension base, we still have that $\phi(x, a)$ does not divide over A if and only if $\phi(x, a)$ does not fork over A .*

4.1. Characterization of NTP_2 . Now we can give a method for computing the burden of a type in terms of dividing with each member of an \downarrow^{ist} -independent sequence.

Lemma 4.7. *Let $p(x)$ be a partial type over C . The following are equivalent:*

- (1) *There is an inp-pattern of depth κ in $p(x)$.*

(2) There is $d \models p(x)$, $D \supseteq C$ and $(a_\alpha)_{\alpha < \kappa}$ such that $a_\alpha \downarrow_D^{\text{ist}} a_{<\alpha}$ and $d \not\downarrow_D^d a_\alpha$ for all $\alpha < \kappa$.

Proof. (1) \Rightarrow (2): Let $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha < \kappa}$ be an inp-pattern in $p(x)$ with (\bar{a}_α) mutually indiscernible over C . Let $q_\alpha(\bar{y}_\alpha)$ be a non-algebraic type finitely satisfiable in \bar{a}_α and extending $\text{tp}(a_{\alpha 0}/C)$. Let $M \supseteq C(\bar{a}_\alpha)_{\alpha < \kappa}$ be $(|C| + \kappa)^+$ -saturated. Then q_α is strictly invariant over M by Lemma 4.3. For $\alpha, i < \kappa$ let $b_{\alpha, i} \models q_\alpha \upharpoonright_{M(b_{\alpha, j})_{\alpha < \kappa, j < i}(b_{\beta, i})_{\beta < \alpha}}$. Let $e_\alpha = b_{\alpha, \alpha}$. Now we have:

- $e_\alpha \downarrow_M^{\text{ist}} e_{<\alpha}$: as $e_\alpha \models q_\alpha \upharpoonright_{e_{<\alpha} M}$.
- there is $d \models p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha < \kappa}$: it is easy to see by construction that for any $\Delta \in L(C)$ and $\alpha_0 < \dots < \alpha_{n-1} < \kappa$, if $\models \Delta(e_{\alpha_0}, \dots, e_{\alpha_{n-1}})$, then $\models \Delta(a_{\alpha_0, i_0}, \dots, a_{\alpha_{n-1}, i_{n-1}})$ for some $i_0, \dots, i_{n-1} < \omega$. By assumption on $(\bar{a}_\alpha)_{\alpha < \kappa}$ and compactness it follows that $p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha < \kappa}$ is consistent.
- $\phi_\alpha(x, e_\alpha)$ divides over M : notice that $(b_{\alpha, \alpha+i})_{i < \omega}$ is an M -indiscernible sequence starting with e_α , as $b_{\alpha, \alpha+i} \models q_\alpha \upharpoonright_{M(b_{\alpha, \alpha+j})_{j < i}}$ and q_α is finitely satisfiable in M . As $\text{tp}(\bar{b}_\alpha)$ is finitely satisfiable in \bar{a}_α , we conclude that $\{\phi_\alpha(x, b_{\alpha, \alpha+i})\}_{i < \omega}$ is k_α -inconsistent.

(2) \Rightarrow (1): Let $d \models p(x)$, $D \supseteq C$ and $(a_\alpha)_{\alpha < \kappa}$ such that $a_\alpha \downarrow_D^{\text{ist}} a_{<\alpha}$ and $d \not\downarrow_D^f a_\alpha$ for all $\alpha < \kappa$ be given. Let $\phi_\alpha(x, a_\alpha) \in \text{tp}(d/a_\alpha D)$ be a formula dividing over D , and let \bar{a}_α indiscernible over D and starting with a_α witness it. By Lemma 2.2 we can find a $(\bar{a}'_\alpha)_{\alpha < \kappa}$, mutually indiscernible over D and such that $\bar{a}'_\alpha \equiv_{a_\alpha D} \bar{a}_\alpha$. It follows that $\{\phi_\alpha(x, y_\alpha), \bar{a}'_\alpha\}_{\alpha < \kappa}$ is an inp-pattern of depth κ in $p(x)$. \square

Definition 4.8. We say that dividing satisfies *generic local character* if for every $A \subseteq B$ and $p(x) \in S(B)$ there is some $A' \subseteq B$ with $|A'| \leq |T|^+$ and such that: for any $\phi(x, b) \in p$, if $b \downarrow_A^{\text{ist}} A'$, then $\phi(x, b)$ does not divide over AA' .

Of course, the local character of dividing implies the generic local character. We are ready to prove the main theorem of this section.

Theorem 4.9. *The following are equivalent:*

- (1) T is NTP₂.
- (2) T has absolutely bounded \downarrow^{ist} -weight: for every M , b and $(a_i)_{i < |T|^+}$ with $a_i \downarrow_M^{\text{ist}} a_{<i}$, $b \downarrow_M^d a_i$ for some $i < |T|^+$.
- (3) T has bounded \downarrow^{ist} -weight: for every M there is some κ_M such that given b and $(a_i)_{i < \kappa_M}$ with $a_i \downarrow_M^{\text{ist}} a_{<i}$, $b \downarrow_M^d a_i$ for some $i < \kappa_M$.
- (4) T satisfies “Kim’s lemma”: for any $M \models T$, $\phi(x, a)$ divides over M if and only if $\{\phi(x, a_i)\}_{i < \omega}$ is inconsistent for every strict Morley sequence over M .
- (5) Dividing in T satisfies generic local character.

Proof. (1) implies (2): Assume that there are M , b and $(a_i)_{i < |T|^+}$ with $a_i \downarrow_M^{\text{ist}} a_{<i}$ and $b \not\downarrow_M^d a_i$ for all i . But then by Lemma 4.7 $\text{bdn}(b/M) \geq |T|^+$, thus T has TP₂ by Lemma 3.2.

(2) implies (3) is clear.

(1) implies (4): by Fact 4.6(1)+(2).

(4) implies (3): assume that we have M , b and $(a_i)_{i < \kappa}$ such that, letting $\kappa = \beth_{(2|M|)^+}$, $a_i \downarrow_M^{\text{ist}} a_{<i}$ and $b \not\downarrow_M^d a_i$ for all $i < \kappa$. We may assume that dividing is always witnessed by the same formula $\phi(x, y)$. Extracting an M -indiscernible sequence $(a'_i)_{i < \omega}$ from $(a_i)_{i < \kappa}$ by Erdős-Rado, we get a contradiction to (4) as $\{\phi(x, a'_i)\}_{i < \omega}$ is still consistent, (a'_i) is a strict Morley sequence over M and $\phi(x, a'_0)$ divides over M .

(3) implies (1): Assume that $\varphi(x, y)$ has TP_2 , let $A = (\bar{a}_\alpha)_{\alpha < \omega}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i < \omega}$ be a strongly indiscernible array witnessing it (so rows are mutually indiscernible and the sequence of rows is indiscernible). Let $M \supset A$ be some $|A|^+$ -saturated model, and assume that κ_M is as required by (3). Let $\lambda = \beth_{(2|M|)^+}$ and $\mu = (2^{2^\lambda})^+$. Adding new elements and rows by compactness, extend our strongly indiscernible array to one of the form $(\bar{a}_\alpha)_{\alpha \in \omega + \mu^*}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i \in \lambda}$. By all the indiscernibility around it follows that $\bar{a}_\alpha \downarrow_A^u \bar{a}_{<\alpha}$ for all $\alpha < \mu$. As there can be at most 2^{2^λ} global types from $S_\lambda(\mathbb{M})$ that are finitely satisfiable in A , without loss of generality there is some $p(\bar{x}) \in S_\lambda(\mathbb{M})$ finitely satisfiable in A and such that $\bar{a}_\alpha \models p(\bar{x})|_{A\bar{a}_{<\alpha}}$.

By Lemma 4.3, $p(\bar{x})$ is strictly invariant over M . We choose $(\bar{b}_\alpha)_{\alpha < \kappa_M}$ such that $\bar{b}_\alpha \models p|_{M\bar{b}_{<\alpha}}$.

By the choice of λ and Erdős-Rado, for each $\alpha < \kappa_M$ there is $i_\alpha < \lambda$ and \bar{d}_α such that \bar{d}_α is an M -indiscernible sequence starting with $b_{\alpha i_\alpha}$ and such that type of every finite subsequence of it is realized by some subsequence of \bar{b}_α . Now we have:

- $d_{\alpha 0} \downarrow_M^{\text{ist}} d_{<\alpha 0}$ (as $d_{\alpha 0} = b_{\alpha i_\alpha}$ and $\bar{b}_\alpha \downarrow_M^{\text{ist}} \bar{b}_{<\alpha}$),
- $\varphi(x, d_{\alpha 0})$ divides over M (as \bar{d}_α is M -indiscernible and $\{\varphi(x, d_{\alpha i})\}_{i \in \omega}$ is inconsistent by construction),
- $\{\varphi(x, d_{\alpha 0})\}_{\alpha < \kappa_M}$ is consistent (follows by construction).

Taking some $c \models \{\varphi(x, d_{\alpha 0})\}_{\alpha < \kappa_M}$ we get a contradiction to (3).

(5) implies (2): Let $p(x) = \text{tp}(b/B)$ with $B = M \cup \bigcup_{i < |T|^+} a_i$. Letting $A = M$, it follows by generic local character that there is some $A' \subseteq B$ with $|A'| \leq |T|$, such that $b \downarrow_{MA'}^d a$ for any $a \in B$ with $a \downarrow_M A'$. Let $i \in |T|$ be such that $i > \{j : a_j \in A'\}$. Then $a_i \downarrow_M^{\text{ist}} A$, but also $b \not\downarrow_{MA'}^d a_i$ (by left transitivity as $A' \downarrow_M^d a_i$ and $b \not\downarrow_M^d a_i$) — a contradiction.

(1) implies (5): Let $p(x) \in S(B)$ and $A \subseteq B$ be given. By induction on $i < |T|^+$ we try to choose $a_i \in B$ and $\varphi_i(x, a_i) \in p$ such that $a_i \downarrow_A^{\text{ist}} a_{<i}$ and $\varphi_i(x, a_i)$ divides over $a_{<i}A$. But then by Lemma 4.7 $\text{bdn}(b/A) \geq |T|^+$, thus T has TP_2 by Lemma 3.2. So we had to get stuck, and letting $A' = \bigcup a_i$ witnesses the generic local character. \square

Remark 4.10. (1) The proof of the equivalences shows that in (2) and (3) we may replace $a \downarrow_C^{\text{ist}} b$ by “ $\text{tp}(a/bC)$ extends to a global type which is both an heir and a coheir over C ”.

- (2) From the proof one immediately gets a similar characterization of strongness. Namely, the following are equivalent:
- (a) T is strong.
 - (b) For every M , finite (or even singleton) b and $(a_i)_{i < \omega}$ with $a_i \downarrow_M^{\text{ist}} a_{<i}$, $b \downarrow_M^d a_i$ for some $i < \omega$.
 - (c) For every $A \subseteq B$ and $p(x) \in S(B)$ there is some *finite* $A' \subseteq B$ such that: for any $\phi(x, b) \in p$, if $b \downarrow_A^{\text{ist}} A'$, then $\phi(x, b)$ does not divide over AA' .

If we are working over a somewhat saturated model and consider only small sets, then we actually have the generic local character with respect to \downarrow^u in the place of \downarrow^{ist} .

Lemma 4.11. *Let $(\bar{a}_i)_{i < \kappa}$ and C be given, \bar{a}_i starting with a_i . If \bar{a}_i is indiscernible over $\bar{a}_{<i}C$ and $a_i \downarrow_C^i a_{<i}$, then $(\bar{a}_i)_{i < \kappa}$ is almost mutually indiscernible over C .*

Proposition 4.12. *Let T be NTP_2 . Let M be κ -saturated, $p(x) \in S(M)$ and $A \subset M$ of size $< \kappa$. Then there is $A \subseteq A' \subset M$ of size $< \kappa$ such that for any $\phi(x, a) \in p$, if $a \downarrow_A^i A'$ then $\phi(x, a)$ does not fork over A' .*

Proof. Assume not, then we can choose inductively on $\alpha < |T|^+$:

- (1) $\bar{a}_\alpha \subseteq M$ such that $a_{\alpha,0} \downarrow_A^i A_\alpha$ and \bar{a}_α is A_α -indiscernible, $A_\alpha = A \cup \bigcup_{\beta < \alpha} \bar{a}_\beta$.
- (2) $\phi_\alpha(x, y_\alpha)$ such that $\phi_\alpha(x, a_{\alpha,0}) \in p$ and $\{\phi_\alpha(x, a_{\alpha,i})\}_{i < \omega}$ is inconsistent.

(1) is possible by saturation of M . But then by Lemma 4.11, $(\bar{a}_\alpha)_{\alpha < |T|^+}$ are almost mutually indiscernible. \square

4.2. Dependent dividing.

Definition 4.13. We say that T has *dependent dividing* if given $M \preceq N$ and $p(x) \in S(N)$ dividing over M , then there is a dependent formula $\phi(x, y)$ and $c \in N$ such that $\phi(x, c) \in p$ and $\phi(x, c)$ divides over M .

Proposition 4.14. (1) *If T has dependent dividing, then it is NTP_2 .*

(2) *If T has simple dividing, then it is simple.*

Proof. (1) In fact we will only use that dividing is always witnessed by an instance of an NTP_2 formula. Assume that T has TP_2 and let $\phi(x, y)$ witness this. Let T_{Sk} be a Skolemization of T , $\phi(x, y)$ still has TP_2 in T_{Sk} . Then as in the proof of Theorem 4.9, for any κ we can find $(b_i)_{i < \kappa}$, a and M such that $a \models \{\phi(x, b_i)\}_{i < \kappa}$, $\phi(x, b_i)$ divides over M and $\text{tp}(b_i/b_{<i}M)$ has a global heir-coheir over M , all in the sense of T_{Sk} . Taking $M_i = \text{Sk}(Mb_i) \models T$, and now working in T , we still have that $a \not\downarrow_M^d M_i$ and $M_i \downarrow_M^{\text{ist}} M_{<i}$ (as $\text{tp}(M_i/M_{<i}M)$ still has a global heir-coheir over M). But then for each i we find some $d_i \in M_i$ and NTP_2

formulas $\phi_i(x, y_i) \in L$ such that $a \models \{\phi_i(x, d_i)\}$ and $\phi_i(x, d_i)$ divides over M , witnessed by \bar{d}_i starting with d_i . We may assume that $\phi_i = \phi'$, and this contradicts ϕ' being NTP_2 .

(2) Similar argument shows that if T has simple dividing, then it is simple. \square

Of course, if T is NIP, then it has dependent dividing, and for simple theories it is equivalent to the stable forking conjecture. It is natural to ask if every NTP_2 theory T has dependent dividing.

4.3. Forking and dividing inside an NTP_2 type.

Definition 4.15. A partial type $p(x)$ over C is said to be NTP_2 if the following does not exist: $(\bar{a}_\alpha)_{\alpha < \omega}$, $\phi(x, y)$ and $k < \omega$ such that $\{\phi(x, a_{\alpha i})\}_{i < \omega}$ is k -inconsistent for every $\alpha < \omega$ and $\{\phi(x, a_{\alpha f(\alpha)})\}_{\alpha < \omega} \cup p(x)$ is consistent for every $f : \omega \rightarrow \omega$. Of course, T is NTP_2 if and only if every partial type is NTP_2 . Also notice that if $p(x)$ is NTP_2 , then every extension of it is NTP_2 and that $q((x_i)_{i < \kappa}) = \bigcup_{i < \kappa} p(x_i)$ is NTP_2 (follows from Theorem 2.5).

For the later use we will need a generalization of the results from [CK12] working inside a partial NTP_2 type, and with no assumption on the theory.

Lemma 4.16. *Let $p(x)$ be an NTP_2 type over M . Assume that $p(x) \cup \{\phi(x, a)\}$ divides over M , then there is a global coheir $q(x)$ extending $\text{tp}(a/M)$ such that $p(x) \cup \{\phi(x, a_i)\}_{i < \omega}$ is inconsistent for any sequence $(a_i)_{i < \omega}$ with $a_i \models q|_{a_{< i}M}$.*

Proof. The proof of [CK12, Lemma 3.12] goes through. \square

Lemma 4.17. *Assume that $\text{tp}(a_i/C) = p(x)$ for all i and that $\text{tp}(a_i/a_{< i}C)$ has a strictly invariant extension to $p(\mathbb{M}) \cup C$. Then there are mutually C -indiscernible $(\bar{b}_i)_{i < \kappa}$ such that $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.*

Proof. The assumption is sufficient for the proof of Lemma 4.4 to work. \square

Lemma 4.18. *Let $p(x)$ over M be NTP_2 , $a \in p(\mathbb{M})$, $c \in M$ and assume that $p(x) \cup \{\phi(x, ac)\}$ divides over M . Assume that $\text{tp}(a/M)$ has a strictly invariant extension $p'(y) \in S(p(\mathbb{M}))$. Then for any $(a_i)_{i < \omega}$ such that $a_i \models p'|_{a_{< i}M}$, $p(x) \cup \{\phi(x, a_i c)\}_{i < \omega}$ is inconsistent.*

Proof. Let $(\bar{a}_0 c)$ with $a_{0,0} = a_0$ be an M -indiscernible sequence witnessing that $p(x) \cup \{\phi(x, a_0 c)\}$ divides over M . Let \bar{a}_i be its image under an M -automorphism sending a_0 to a_i . By Lemma 4.4(2) we can find $(\bar{b}_i)_{i < \omega}$ mutually indiscernible over M and with $\bar{b}_i \equiv_{a_i M} \bar{a}_i$. By the choice of \bar{b}_i 's and compactness, there is some $\psi(x) \in p(x)$ such that $\{\psi(x) \wedge \phi(x, b_{i,j} c)\}_{j < \omega}$ is k -inconsistent for all $i < \omega$. It follows that $p(x) \cup \{\phi(x, a_i c)\}_{i < \omega}$ is inconsistent as p is NTP_2 . \square

We need a version of the Broom lemma localized to an NTP_2 type.

Lemma 4.19. *Let $p(x)$ be an NTP_2 type over M and $p'(x)$ be a partial global type invariant over M . Suppose that $p(x) \cup p'(x) \vdash \bigvee_{i < n} \phi_i(x, c)$ and each $\phi_i(x, c)$ divides over M . Then $p(x) \cup p'(x)$ is inconsistent.*

Proof. Follows from the proof of [CK12, Lemma 3.1]. \square

Corollary 4.20. *Let $p(x)$ be an NTP_2 type over M and $a \in p(\mathbb{M})$. Then $\text{tp}(a/M)$ has a strictly invariant extension $p'(x) \in S(p(\mathbb{M}) \cup M)$.*

Proof. Following the proof of [CK12, Proposition 3.7] but using Lemma 4.19 in place of the Broom lemma. \square

And finally,

Proposition 4.21. *Let $p(x)$ be an NTP_2 type over M , $a \in p(\mathbb{M}) \cup M$ and assume that $\{\phi(x, a)\} \cup p(x)$ does not divide over M . Then there is $p'(x) \in S(p(\mathbb{M}) \cup M)$ which does not divide over M and $\{\phi(x, a)\} \cup p(x) \subset p'(x)$.*

Proof. By compactness, it is enough to show that if $p(x) \cup \{\phi(x, ac)\} \vdash \bigvee_{i < n} \phi_i(x, a_i c_i)$ with $a, a_i \in p(\mathbb{M})$ and $c, c_i \in M$, then $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide over M for some $i < n$. As in the proof of [CK12, Corollary 3.16], let $(a^j a_0^j \dots a_{n-1}^j)_{j < \omega}$ be a strict Morley sequence in $\text{tp}(aa_0 \dots a_{n-1}/M)$, which exists by Lemma 4.20. Notice that $(a^j c a_0^j c_0 \dots a_{n-1}^j c_{n-1})_{j < \omega}$ is still indiscernible over M . Then $p(x) \cup \{\phi(x, a^j c)\}_{j < \omega}$ is consistent, which implies that $p(x) \cup \{\phi_i(x, a_i^j c_i)\}_{j < \omega}$ is consistent for some $i < n$. But then by Lemma 4.18, $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide over M — as wanted. \square

5. NIP TYPES

Let T be an arbitrary theory.

- Definition 5.1.**
- (1) A partial type $p(x)$ over C is called NIP if there is no $\phi(x, y) \in L$, $(a_i)_{i \in \omega}$ with $a_i \models p(x)$ and $(b_s)_{s \subseteq \omega}$ such that $\models \phi(a_i, b_s) \Leftrightarrow i \in s$.
 - (2) The roles of a 's and b 's in the definition are interchangeable. It is easy to see that any extension of an NIP type is again NIP, and that the type of several realizations of an NIP type is again NIP.
 - (3) $p(x)$ is NIP $\Leftrightarrow \text{dprk}(p) < |T|^+ \Leftrightarrow \text{dprk}(p) < \infty$ (see Definition 3.6).

Lemma 5.2. *Let $p(x)$ be an NIP type.*

- (1) *Let $\bar{a} = (a_\alpha)_{\alpha < \kappa}$ be an indiscernible sequence over A with a_α from $p(\mathbb{M})$, and c be arbitrary. If $\kappa = (|a_\alpha| + |c|)^+$, then some non-empty end segment of \bar{a} is indiscernible over Ac .*

- (2) Let $(\bar{a}_\alpha)_{\alpha < \kappa}$ be mutually indiscernible (over \emptyset), with $\bar{a}_\alpha = (a_{\alpha i})_{i < \lambda}$ from $p(\mathbb{M})$. Assume that $\bar{a} = (a_{0i} a_{1i} \dots)_{i < \lambda}$ is indiscernible over A . Then $(\bar{a}_\alpha)_{\alpha < \kappa}$ is mutually indiscernible over A .

Standard proofs of the corresponding results for NIP theories go through, see e.g. [Adl08].

5.1. Dp-rank of a type is always witnessed by an array of its realizations. In [KSed] Kaplan and Simon demonstrate that inside an NTP_2 theory, dp-rank of a type can always be witnessed by mutually indiscernible sequences of realizations of the type. In this section we show that the assumption that the theory is NTP_2 can be omitted, thus proving the following general theorem with no assumption on the theory.

Theorem 5.3. *Let $p(x)$ be an NIP partial type over C , and assume that $\text{dprk}(p) \geq \kappa$. Then there is $C' \supseteq C$, $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha < \kappa}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i < \omega}$ such that:*

- $a_{\alpha i} \models p(x)$ for all α, i
- $(\bar{a}_\alpha)_{\alpha < \kappa}$ are mutually indiscernible over C'
- None of \bar{a}_α is indiscernible over bC' .
- $|C'| \leq |C| + \kappa$.

Corollary 5.4. *It follows that dp-rank of a 1-type is always witnessed by mutually indiscernible sequences of singletons.*

We will use the following result from [CS13, Proposition 1.1]:

Fact 5.5. *Let $p(x)$ be a (partial) NIP type, $A \subseteq p(\mathbb{M})$ and $\phi(x, c)$ given. Then there is $\theta(x, d)$ with $d \in p(\mathbb{M})$ such that:*

- (1) $\theta(A, d) = \phi(A, c)$,
- (2) $\theta(x, d) \cup p(x) \rightarrow \phi(x, c)$.

We begin by showing that the burden of a dependent type can always be witnessed by mutually indiscernible sequences from the set of its realizations.

Lemma 5.6. *Let $p(x)$ be a dependent partial type over C of burden $\geq \kappa$. Then we can find $(\bar{d}_\alpha)_{\alpha < \kappa}$ witnessing it, mutually indiscernible over C and with $\bar{d}_i \subseteq p(\mathbb{M}) \cup C$.*

Proof. Let λ be large enough compared to $|C|$. Assume that $\text{bdn}(p) \geq \kappa$, then by compactness we can find $(\bar{b}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{i < \kappa}$ such that $\bar{b}_\alpha = (b_{\alpha i})_{i < \lambda}$, $\{\phi_\alpha(x, b_{\alpha i})\}_{\alpha < \kappa}$ is k_α -inconsistent and $p(x) \cup \{\phi_\alpha(x, b_{\alpha f(\alpha)})\}_{i < \kappa}$ is consistent for every $f : \kappa \rightarrow \lambda$, let a_f realize it. Set $A = \{a_f\}_{f \in \lambda^\kappa} \subseteq p(\mathbb{M})$.

By Fact 5.5, let $\theta_{\alpha i}(x, d_{\alpha i})$ be an honest definition of $\phi_\alpha(x, b_{\alpha i})$ over A (with respect to $p(x)$), with $d_{\alpha i} \in p(\mathbb{M})$. As λ is very large, we may assume that $\theta_{\alpha i} = \theta_\alpha$.

Now, as $\theta_\alpha(x, d_{\alpha i}) \cup p(x) \rightarrow \phi_\alpha(x, b_{\alpha i})$, it follows that there is some $\psi_\alpha(x, c) \in p$ such that letting $\chi_\alpha(x, y_1 y_2) = \theta_\alpha(x, y_1) \wedge \psi_\alpha(x, y_2)$, $\{\chi_\alpha(x, d_{\alpha i} c_\alpha)\}_{i < \omega}$ is k_α -inconsistent.

On the other hand, $\{\chi_\alpha(x, d_{\alpha f(\alpha)} c_\alpha)\}_{\alpha < \kappa} \cup p(x)$ is consistent, as the corresponding a_f realizes it. Thus this array still witnesses that burden of p is at least κ . \square

We will also need the following lemma.

Lemma 5.7. *Let $p(x)$ be an NIP type over $M \models T$*

- (1) *Assume that $a \in p(\mathbb{M}) \cup M$ and $\phi(x, a)$ does not divide over M , then there is a type $q(x) \in S(p(\mathbb{M}) \cup M)$ invariant under M -automorphisms and with $\phi(x, a) \in q$.*
- (2) *Let $p'(x) \supset p(x)$ be an M invariant type such that $p^{(\omega)}$ is an heir-coheir over M . If $(a_i)_{i < \omega}$ is a Morley sequence in p' and indiscernible over bM with $b \in p(\mathbb{M})$, then $\text{tp}(b/MI)$ has an M -invariant extension in $S(p(\mathbb{M}) \cup M)$.*

Proof. (1) As NIP type is in particular an NTP_2 type, by Lemma 4.21 we find a type $q(x) \in S(p(\mathbb{M}))$ which doesn't divide over M and such that $\phi(x, a) \in q$. It is enough to show that $q(x)$ is Lascar-invariant over M . Assume that we have an M -indiscernible sequence $(a_i)_{i < \omega}$ in $p(\mathbb{M})$ such that $\phi(x, a_0) \wedge \neg\phi(x, a_1) \in q$. But then $\{\phi(x, a_{2i}) \wedge \phi(x, a_{2i+1})\}_{i < \omega}$ is inconsistent, so q divides over M — a contradiction. Easy induction shows the same for a_0 and a_1 at Lascar distance n .

(2) By Lemma 4.18 and (1). \square

Now for the *proof of Theorem 5.3*. The point is that first the array witnessing dp-rank of our type $p(x)$ can be dragged inside the set of realizations of p by Lemma 5.6. Then, combined with the use of Proposition 5.7 instead of the unrelativized version, the proof of Kaplan and Simon [KSed, Section 3.2] goes through working inside $p(\mathbb{M})$.

Problem 5.8. Is the analogue of Lemma 5.6 true for the burden of an arbitrary type in an NTP_2 theory?

We include some partial observations to justify it.

Proposition 5.9. *The answer to the Problem 5.8 is positive in the following cases:*

- (1) *T satisfies dependent forking (so in particular if T is NIP).*
- (2) *T is simple.*

Proof. (1): Recall that if $\text{bdn}(p) \geq \kappa$, then we can find $(b_i)_{i < \kappa}$, $a \models p$ and $M \supseteq C$ such that $a \not\downarrow_M^d b_i$ and $b_i \downarrow_M^{\text{ist}} b_{< i}$. Notice that $p(x)$ still has the same burden in the sense of a Skolemization T^{Sk} . Choose inductively $M_i \supseteq M \cup b_i$ such that $M_i \downarrow_M^{\text{ist}} b_{< i}$, let $M_i = \text{Sk}(M \cup b_i)$. Let $\phi(x, b_i)$

witness this dividing with $\phi(x, y)$ an NIP formula, we can make \bar{b}_i mutually indiscernible. Now the proof of Lemma 5.6 goes through.

(2): Let $p(x) \in S(A)$, $a \models p(x)$ and let $(b_i)_{i < \kappa}$ independent over A , with $a \not\downarrow_A b_i$. Without loss of generality $A = \emptyset$. Consider $\text{tp}(a/b_0)$ and take $I = (a_i)_{i < |T|^+}$ such that $a \frown I$ is a Morley sequence in it. By extension and automorphism we may assume $b_{>0} \downarrow_{ab_0} I$, together with $a \downarrow_{b_0} I$ implies $b_{>0} \downarrow_{b_0} I$, thus $b_{>0} \downarrow I$ (as $b_{>0} \downarrow b_0$).

Assume that I is a Morley sequence over \emptyset , then by simplicity $a_i \downarrow b_0$ for some i , contradicting $a_i \equiv_{b_0} a$ and $a \not\downarrow b_0$. Thus by indiscernibility $a \not\downarrow a_{<n}$ for some n , while $\{a_{<n}\} \cup b_{>0}$ is an independent set.

Repeating this argument inductively and using the fact that the burden of a type in a simple theory is the supremum of the weights of its completions (Fact 3.10) allows to conclude. \square

5.2. NIP types inside an NTP_2 theory. We give a characterization of NIP types in NTP_2 theories in terms of the number of non-forking extensions of its completions.

Theorem 5.10. *Let T be NTP_2 , and let $p(x)$ be a partial type over C . The following are equivalent:*

- (1) p is NIP.
- (2) Every $p' \supseteq p$ has boundedly many global non-forking extensions.

Proof. (1) \Rightarrow (2): A usual argument shows that a non-forking extension of an NIP type is in fact Lascar-invariant (see Lemma 5.7), thus there are only boundedly many such.

(2) \Rightarrow (1): Assume that $p(x)$ is not NIP, that is there are $I = (b_i)_{i \in \omega}$ such that such that for any $s \subseteq \omega$, $p_s(x) = p(x) \cup \{\phi(x, b_i)\}_{i \in s} \cup \{\neg\phi(x, b_i)\}_{i \notin s}$ is consistent. Let $q(y)$ be a global non-algebraic type finitely satisfiable in I . Let $M \supseteq IC$ be some $|IC|^+$ -saturated model. It follows that $q^{(\omega)}$ is a global heir-coheir over M by Lemma 4.3. Take an arbitrary cardinal κ , and let $J = (c_i)_{i \in \kappa}$ be a Morley sequence in q over M . We claim that for any $s \subseteq \kappa$, $p_s(x)$ does not divide over M . First notice that $p_s(x)$ is consistent for any s , as $\text{tp}(J/M)$ is finitely satisfiable in I . But as for any $k < \omega$, $(c_{ki}c_{k(i+1)} \dots c_{k(i+1)-1})_{i < \omega}$ is a Morley sequence in $q^{(k)}$, together with Fact 4.6 this implies that $p_s(x)|_{c_0 \dots c_{k-1}}$ does not divide over M for any $k < \omega$, thus by indiscernibility of J , $p_s(x)$ does not divide over M , thus has a global non-forking extension by Fact 4.6.

As there are only boundedly many types over M , there is some $p' \in S(M)$ extending p , with unboundedly many global non-forking extensions. \square

Remark 5.11. (2) \Rightarrow (1) is just a localized variant of an argument from [CKS12].

6. SIMPLE TYPES

6.1. Simple and co-simple types. Simple types, to the best of our knowledge, were first defined in [HKP00, §4] in the form of (2).

Definition 6.1. We say that a partial type $p(x) \in S(A)$ is *simple* if it satisfies any of the following equivalent conditions:

- (1) There is no $\phi(x, y)$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k < \omega$ such that: $\{\phi(x, a_{\eta i})\}_{i < \omega}$ is k -inconsistent for every $\eta \in \omega^{<\omega}$ and $\{\phi(x, a_{\eta i})\}_{i < \omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$.
- (2) Local character: If $B \supseteq A$ and $p(x) \subseteq q(x) \in S(B)$, then $q(x)$ does not divide over AB' for some $B' \subseteq B$, $|B'| \leq |T|$.
- (3) Kim's lemma: If $\{\phi(x, b)\} \cup p(x)$ divides over $B \supseteq A$ and $(b_i)_{i < \omega}$ is a Morley sequence in $\text{tp}(b/B)$, then $p(x) \cup \{\phi(x, b_i)\}_{i < \omega}$ is inconsistent.
- (4) Bounded weight: Let $B \supseteq A$ and $\kappa \geq \beth_{(2|B|)^+}$. If $a \models p(x)$ and $(b_i)_{i < \kappa}$ is such that $b_i \downarrow_B^f b_{<i}$, then $a \downarrow_B^d b_i$ for some $i < \kappa$.
- (5) For any $B \supseteq A$, if $b \downarrow_B^f a$ and $a \models p(x)$, then $a \downarrow_B^d b$.

Proof.

- (1) \Rightarrow (2): Assume (2) fails, then we choose $\phi_\alpha(x, b_\alpha) \in q(x)$ k_α -dividing over $A \cup B_\alpha$, with $B_\alpha = \{b_\beta\}_{\beta < \alpha} \subseteq B$, $|B_\alpha| \leq |\alpha|$ by induction on $\alpha < |T|^+$. Then w.l.o.g. $\phi_\alpha = \phi$ and $k_\alpha = k$. Now construct a tree in the usual manner, such that $\{\phi(x, a_{\eta i})\}_{i < \omega}$ is inconsistent for any $\eta \in \omega^{<\omega}$ and $\{\phi(x, a_{\eta i})\}_{i < \omega} \cup p(x)$ is consistent for any $\eta \in \omega^\omega$.
- (2) \Rightarrow (3): Let $I = (|T|^+)^*$, and $(b_i)_{i \in I}$ be Morley over B in $\text{tp}(b/B)$. Assume that $a \models p(x) \cup \{\phi(x, b_i)\}_{i \in I}$. By (2), $\text{tp}(a/(b_i)_{i \in I} B)$ does not divide over $B(b_i)_{i \in I_0}$ for some $I_0 \subseteq I$, $|I_0| \leq |T|$. Let $i_0 \in I$, $i_0 < I_0$. Then $(b_i)_{i \in I_0} \downarrow_B^f b_{i_0}$, and thus $\phi(x, b_{i_0})$ divides over BI_0 - a contradiction.
- (3) \Rightarrow (4): Assume not, then by Erdős-Rado and finite character find a Morley sequence over B and a formula $\phi(x, y)$ such that $\models \phi(a, b_i)$ and $\phi(x, b_i)$ divides over B , contradiction to (3).
- (4) \Rightarrow (5): For κ as in (4), let $I = (b_i)_{i < \kappa}$ be a Morley sequence over B , indiscernible over Ba and with $b_0 = b$. By (4), $a \downarrow_B^d b_i$ for some $i < \kappa$, and so $a \downarrow_B^d b$ by indiscernibility.
- (5) \Rightarrow (1): Let $(b_\eta)_{\eta \in \omega^{<\omega}}$ witness the tree property of $\phi(x, y)$, such that $\{\phi(x, b_{\eta i})\}_{i < \omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$. Then by Ramsey and compactness we can find $(b_i)_{i \leq \omega}$ indiscernible over a , $\models \phi(a, b_i)$ and $\phi(x, b_i)$ divides over $b_{<i}A$. Taking $B = A \cup \{b_i\}_{i < \omega}$ we see that $a \not\downarrow_B^d b_\omega$, while $b_\omega \downarrow_B^f a$ (as it is finitely satisfiable in B by indiscernibility) - a contradiction to (5).

□

Remark 6.2. Let $p(x) \in S(A)$ be simple.

- (1) Any $q(x) \supseteq p(x)$ is simple.
- (2) Let $p(x) \in S(A)$ be simple and $C \subseteq p(\mathbb{M})$. Then $\text{tp}(C/A)$ is simple.

Proof. (1): Clear, for example by (1) from the definition.

(2): Let $C = (c_i)_{i \leq n}$, and we show that for any $B \supseteq A$, if $b \downarrow_B^f C$, then $C \downarrow_B^d b$ by induction on the size of C . Notice that $b \downarrow_{Bc_{<n}}^f c_n$ and $c_n \models p$, thus $c_n \downarrow_{Bc_{<n}}^d b$. By the inductive assumption $c_{<n} \downarrow_B^d b$, thus $c_{\leq n} \downarrow_B^d b$. \square

We give a characterization in terms of local ranks.

Proposition 6.3. *The following are equivalent:*

- (1) $p(x)$ is simple in the sense of Definition 6.1.
- (2) $D(p, \Delta, k) < \omega$ for any finite Δ and $k < \omega$.

Proof. Standard proof goes through. \square

Lemma 6.4. *Let $p(x) \in S(A)$ be simple, $a \models p(x)$ and $B \supseteq A$ arbitrary. Then $a \downarrow_{B_0}^f B$ for some $|B_0| \leq |T|^+$.*

Proof. Standard proof using ranks goes through. \square

It follows that in the Definition 6.1 we can replace everywhere “dividing” by “forking”.

Lemma 6.5. *Let $p(x) \in S(A)$ be simple. If A is an extension base, then $\{\phi(x, c)\} \cup p(x)$ forks over A if and only if it divides over A .*

Proof. Assume that $\{\phi(x, c)\} \cup p(x)$ does not divide over A , but $\{\phi(x, c)\} \cup p(x) \vdash \bigvee_{i < n} \phi_i(x, c_i)$ and each of $\phi_i(x, c_i)$ divides over A . As A is an extension base, let $(c_i c_{0,i} \dots c_{n-1,i})$ be a Morley sequence in $\text{tp}(cc_0 \dots c_{n-1}/A)$. As $p(x) \cup \{\phi(x, c)\}$ does not divide over A , let $a \models p(x) \cup \{\phi(x, c_i)\}$, but then $p(x) \cup \{\phi_i(x, c_{i,j})\}_{j < \omega}$ is consistent for some $i < n$, contradicting Kim’s lemma. \square

Problem 6.6. Let $q(x)$ be a non-forking extension of a complete type $p(x)$, and assume that $q(x)$ is simple. Does it imply that $p(x)$ is simple?

Unlike stability or NIP, it is possible that $\phi(x, y)$ does not have the tree property, while $\phi^*(x', y') = \phi(y', x')$ does. This forces us to define a dual concept.

Definition 6.7. A partial type $p(x)$ over A is *co-simple* if it satisfies any of the following equivalent properties:

- (1) No formula $\phi(x, y) \in L(A)$ has the tree property witnessed by some $(a_\eta)_{\eta \in \omega < \omega}$ with $a_\eta \subseteq p(\mathbb{M})$.
- (2) Every type $q(x) \in S(BA)$ with $B \subseteq p(\mathbb{M})$ does not divide over AB' for some $B' \subseteq B$, $|B'| \leq (|A| + |T|)^+$.

- (3) Let $(a_i)_{i < \omega} \subseteq p(\mathbb{M})$ be a Morley sequence over BA , $B \subseteq p(\mathbb{M})$ and $\phi(x, y) \in L(A)$. If $\phi(x, a_0)$ divides over BA then $\{\phi(x, a_i)\}_{i < \omega}$ is inconsistent.
- (4) Let $B \subseteq p(\mathbb{M})$ and $\kappa \geq \beth_{(2^{|B|+|A|})^+}$. If $(b_i)_{i < \kappa} \subseteq p(\mathbb{M})$ is such that $b_i \downarrow_{AB}^f b_{<i}$ and a arbitrary, then $a \downarrow_{AB}^d b_i$ for some $i < \kappa$.
- (5) For $B \subseteq p(\mathbb{M})$, if $a \models p$ and $a \downarrow_{AB}^f b$, then $b \downarrow_{AB}^d a$.

Proof. Similar to the proof in Definition 6.1. \square

Remark 6.8. It follows that if $p(x)$ is a co-simple type over A and $B \subseteq p(\mathbb{M})$, then any $q(x) \in S(AB)$ extending p is co-simple (while adding the parameters from outside of the set of solutions of p may ruin co-simplicity).

It is easy to see that T is simple \Leftrightarrow every type is simple \Leftrightarrow every type is co-simple. What is the relation between simple and co-simple in general?

Example 6.9. There is a co-simple type over a model which is not simple.

Proof. Let T be the theory of an infinite triangle-free random graph, this theory eliminates quantifiers. Let $M \models T$, $m \in M$ and consider $p(x) = \{xRm\} \cup \{\neg xRa\}_{a \in M \setminus \{m\}}$ - a non-algebraic type over M . As there can be no triangles, if $a, b \models p(x)$ then $\neg aRb$. It follows that for any $A \subseteq p(\mathbb{M})$ and any B , $B \not\downarrow_M^d A \Leftrightarrow B \cap A \neq \emptyset$. So $p(x)$ is co-simple, for example by checking the bounded weight (Definition 6.7(4)).

For each $\alpha < \omega$, take $(b'_{\alpha,i}, b''_{\alpha,i})_{i < \omega}$ such that $b'_{\alpha,i} R b''_{\alpha,j}$ for all $i \neq j$, and no other edges between them or to elements of M . Then $\{xRb'_{\alpha,i} \wedge xRb''_{\alpha,i}\}_{i < \omega}$ is 2-inconsistent for every α , while $p(x) \cup \left\{ xRb'_{\alpha,\eta(\alpha)} \wedge xRb''_{\alpha,\eta(\alpha)} \right\}_{\alpha < \omega}$ is consistent for every $\eta : \omega \rightarrow \omega$. Thus $p(x)$ is not simple by Definition 6.1(1). \square

However, this T has TP_2 by Example 3.13.

Problem 6.10. Is there a simple, non co-simple type in an arbitrary theory?

6.2. Simple types are co-simple in NTP_2 theories. In this section we assume that T is NTP_2 (although some lemmas remain true without this restriction). In particular, we will write \downarrow to denote non-forking/non-dividing when working over an extension base as they are the same by Fact 4.6(3).

Lemma 6.11. *Weak chain condition:* Let A be an extension base, $p(x) \in S(A)$ simple. Assume that $a \models p(x)$, $I = (b_i)_{i < \omega}$ is a Morley sequence over A and $a \downarrow_A b_0$. Then there is an aA -indiscernible $J \equiv_{Ab_0} I$ satisfying $a \downarrow_A J$.

Proof. Let $a \models \phi(x, b_0)$, then $\{\phi(x, b_0)\} \cup p(x)$ does not divide over A .

Claim. $\{\phi(x, b_0) \wedge \phi(x, b_1)\} \cup p(x)$ does not divide over A .

Proof. As $p(x)$ satisfies Definition 6.1(3), $(b_{2i}b_{2i+1})_{i < \omega}$ is a Morley sequence over A and $\{\phi(x, b_i)\}_{i < \omega} \cup p(x)$ is consistent. \square

By iterating the claim and compactness, we conclude that $\bigcup_{i < \omega} p(x, b_i)$ does not divide over A , where $p(x, b_0) = \text{tp}(a/b_0)$. As A is an extension base and forking equals dividing, there is $a' \models \bigcup_{i < \omega} p(x, b_i)$ satisfying $a' \perp_A I$. By Ramsey, compactness and the fact that $a'b_i \equiv_A ab_0$ we find a sequence as wanted. \square

Remark 6.12. In fact, in [BC] we demonstrate that in an NTP_2 theory this lemma holds over extension bases with I just an indiscernible sequence, not necessarily Morley.

Lemma 6.13. *Let A be an extension base, $p \in S(A)$ simple. For $i < \omega$, Let \bar{a}_i be a Morley sequence in $p(x)$ over A starting with a_i , and assume that $(a_i)_{i < \omega}$ is a Morley sequence in $p(x)$. Then we can find $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ such that $(\bar{b}_i)_{i < \omega}$ are mutually indiscernible over A .*

Proof. W.l.o.g. $A = \emptyset$.

First observe that by simplicity of p , $\{a_i\}_{i < \omega}$ is an independent set.

For $i < \omega$, we choose inductively \bar{b}_i such that:

- (1) $\bar{b}_i \equiv_{a_i} \bar{a}_i$
- (2) \bar{b}_i is indiscernible over $a_{>i}\bar{b}_{<i}$
- (3) $a_{>i+1}\bar{b}_{\leq i} \perp a_{i+1}$
- (4) $a_{\geq i+1} \perp \bar{b}_{\leq i}$

Base step: As $a_{>0} \perp a_0$ and $\text{tp}(a_{>0})$ is simple by Remark 6.2 and Lemma 6.11, we find an $a_{>0}$ -indiscernible $\bar{b}_0 \equiv_{a_0} \bar{a}_0$ with $a_{>0} \perp \bar{b}_0$.

Induction step: Assume that we have constructed $\bar{b}_0, \dots, \bar{b}_{i-1}$. By (3) for $i-1$ it follows that $a_{>i}\bar{b}_{<i} \perp a_i$. Again by Remark 6.2 and Lemma 6.11 we find an $a_{>i}\bar{b}_{<i}$ -indiscernible sequence $\bar{b}_i \equiv_{a_i} \bar{a}_i$ such that $a_{>i}\bar{b}_{<i} \perp \bar{b}_i$.

We check that it satisfies (3): As all tuples are inside $p(\mathbb{M})$, we can use symmetry, transitivity and $\perp^d = \perp^f$ freely. And so, $a_{>i+1}a_{i+1}\bar{b}_{<i} \perp \bar{b}_i \Rightarrow a_{>i+1}\bar{b}_{<i} \perp_{a_{i+1}} \bar{b}_i + a_{>i+1}\bar{b}_{<i} \perp a_{i+1}$ (as $a_{>i+1} \perp a_{i+1}$ and $\bar{b}_{<i} \perp a_{\geq i+1}$ by (4) for $i-1$) $\Rightarrow a_{>i+1}\bar{b}_{<i} \perp \bar{b}_i a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{<i} \perp_{\bar{b}_i} a_{i+1} + \bar{b}_i \perp a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{\leq i} \perp a_{i+1}$.

We check that it satisfies (4): As $a_{>i}\bar{b}_{<i} \perp \bar{b}_i \Rightarrow a_{>i} \perp_{\bar{b}_{<i}} \bar{b}_i + a_{>i} \perp \bar{b}_{<i}$ by (4) for $i-1 \Rightarrow a_{>i} \perp \bar{b}_{\leq i}$.

Having chosen $(\bar{b}_i)_{i < \omega}$ we see that they are almost mutually indiscernible by (1) and (2). Conclude by Lemma 1.3. \square

Lemma 6.14. *Let T be NTP_2 , A an extension base and $p(x) \in S(A)$ simple. Assume that $\phi(x, a)$ divides over A , with $a \models p(x)$. Then there is a Morley sequence over A witnessing it.*

Proof. As A is an extension base, let $M \supseteq A$ be such that $M \downarrow_A^f a$. Then $\phi(x, a)$ divides over M . By Fact 4.6(1), there is a Morley sequence $(a_i)_{i < \omega}$ over M witnessing it (in particular $(a_i)_{i < \omega} \subseteq p(\mathbb{M})$). We show that it is actually a Morley sequence over A . Indiscernibility is clear, and we check that $a_i \downarrow_A a_{<i}$ by induction. As $a_i \downarrow_M a_{<i}$, $a_{<i} \downarrow_M a_i$ by simplicity of $\text{tp}(a_{<i}/M)$. Noticing that $M \downarrow_A^f a_i$, we conclude $a_{<i} \downarrow_A a_i$, so again by simplicity $a_i \downarrow_A a_{<i}$. \square

Proposition 6.15. *Let T be NTP_2 , A an extension base and $p(x) \in S(A)$ simple. Assume that $a \models p$ and $a \downarrow_A^f b$. Then $b \downarrow_A^d a$.*

Proof. Assume that there is $\phi(x, a) \in L(Aa)$ such that $\models \phi(b, a)$ and $\phi(x, a)$ divides over A . Let $(a_i)_{i < \omega}$ be a Morley sequence over A starting with a . Assume that $\{\phi(x, a_i)\}_{i < \omega}$ is consistent. Let \bar{a}_0 be a Morley sequence witnessing that $\phi(x, a_0)$ k -divides over A (exists by Lemma 6.14), and let \bar{a}_i be its image under an A -automorphism sending a_0 to a_i . By Lemma 6.13, we find $\bar{a}'_i \equiv_{a_i A} \bar{a}_i$, such that $(\bar{a}'_i)_{i < \omega}$ are mutually indiscernible. But then we have that $\{\phi(x, a_{i, \eta(i)})\}_{i < \omega}$ is consistent for any $\eta \in \omega^\omega$, while $\{\phi(x, a_{i, j})\}_{j < \omega}$ is k -inconsistent for any $i < \omega$ — contradiction to NTP_2 .

Now let $(a_i)_{i < \omega}$ be a Morley sequence over A starting with a and indiscernible over Ab . Then clearly $b \models \{\phi(x, a_i)\}_{i < \omega}$ for any $\phi(x, a) \in \text{tp}(b/aA)$, so by the previous paragraph $b \downarrow_A^d a$. \square

Lemma 6.16. *Let $p(x)$ be a partial type over A . Assume that $p(x)$ is not co-simple over A . Then there is some $M \supseteq A$, $a \models p(x)$ and b such that $a \downarrow_M^u b$ but $b \not\downarrow_M^d a$.*

Proof. So assume that $p(x)$ is not co-simple over A , then there is an $L(A)$ -formula $\phi(x, y)$ and $(a_\eta)_{\eta \in \omega < \omega} \subseteq p(\mathbb{M})$ witnessing the tree property. Let T^{Sk} be a Skolemization of T , then of course $\phi(x, y)$ and a_η still witness the tree property. As in the proof of (5) \Rightarrow (1) in Definition 6.7, working in the sense of T^{Sk} , we can find an Ab -indiscernible sequence $(a_i)_{i < \omega+1}$ in $p(x)$ such that $\phi(x, a_i)$ divides over $Aa_{<i}$ and $b \models \{\phi(x, a_i)\}_{i < \omega+1}$. Let $I = (a_i)_{i < \omega}$ and $\text{Sk}(AI) = M \models T$. It follows that $a_\omega \downarrow_M^u b$ (by indiscernibility) and that $b \not\downarrow_M^d a_\omega$ (as $M \subseteq \text{dcl}(Aa_{<\omega})$) — also in the sense of T , as dividing is witnessed by an L -formula $\phi(x, y)$. \square

Theorem 6.17. *Let T be NTP_2 , A an arbitrary set and assume that $p(x)$ over A is simple. Then $p(x)$ is co-simple over A .*

Proof. If $p(x)$ over A is not co-simple over A , then by Lemma 6.16 we find some $M \supseteq A$, $a \models p$ and b such that $a \downarrow_M^u b$, but $b \not\downarrow_M^d a$. As M is an extension base, it follows by Proposition 6.15 that $\text{tp}(a/M)$ is not simple, thus $p(x)$ is not simple by Remark 6.2(1) — a contradiction. \square

Corollary 6.18. *Let T be NTP_2 and $p(x) \in S(A)$ simple.*

- (1) If $a \models p(x)$ then $a \downarrow_A b \Leftrightarrow b \downarrow_A a$
- (2) Right transitivity: If $a \models p(x)$, $B \supseteq A$, $a \downarrow_A B$ and $a \downarrow_B C$ then $a \downarrow_A C$.

6.3. Independence and co-independence theorems.

In [Kim01] Kim demonstrates that if T has TP_1 , then the independence theorem fails for types over models, assuming the existence of a large cardinal. We give a proof of a localized and a dual versions, showing in particular that the large cardinal assumption is not needed.

Definition 6.19. Let $p(x)$ be (partial) type over A .

- (1) We say that $p(x)$ satisfies the independence theorem if for any $b_1 \downarrow_A^f b_2$ and $c_1 \equiv_A^{\text{Lstp}} c_2 \subseteq p(\mathbb{M})$ such that $c_1 \downarrow_A^f b_1$ and $c_2 \downarrow_A^f b_2$, there is some $c \downarrow_A^f b_1 b_2$ such that $c \equiv_{b_1 A} c_1$ and $c \equiv_{b_2 A} c_2$.
- (2) We say that $p(x)$ satisfies the co-independence theorem if for any $b_1 \downarrow_A^f b_2$ and $c_1 \equiv_A^{\text{Lstp}} c_2 \models p$ such that $b_1 \downarrow_A^f c_1$ and $b_2 \downarrow_A^f c_2$, there is some $c \models p$ such that $b_1 b_2 \downarrow_A^f c$ and $c \equiv_{A b_1} c_1$, $c \equiv_{A b_2} c_2$.

Of course, both the independence and the co-independence theorems hold in simple theories, but none of them characterizes simplicity.

Proposition 6.20. Let T be NTP_2 and $p(x)$ is a partial type over A .

- (1) If every $p'(x) \supseteq p$ with $p'(x) \in S(M)$, $M \supseteq A$ satisfies the co-independence theorem, then it is simple.
- (2) If $p(x)$ satisfies the independence theorem, then it is co-simple.

Proof. (1) Our argument is based on the proof of [Kim01, Proposition 2.5]. Without loss of generality $A = \emptyset$. Assume that p is not simple, then by Fact 3.12 there are some formula $\phi(x, y)$, $(a_\eta)_{\eta \in \omega < \omega}$ such that:

- $\{\phi(x, a_{\eta i})\}_{i \in \omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$.
- $\phi(x, a_\eta) \wedge \phi(x, a_{\eta'})$ is inconsistent for any incomparable $\eta, \eta' \in \omega < \omega$.

By compactness we can find a tree with the same properties indexed by $\kappa^{<\kappa}$, for a cardinal κ large enough. Let T^{Sk} be some Skolemization of T , and we work in the sense of T^{Sk} .

Claim. There is a sequence $(c_i d_i)_{i \in \omega}$ satisfying:

- (1) $\{\phi(x, c_i)\}_{i \in \omega} \cup p(x)$ is consistent.
- (2) c_i, d_i start an infinite sequence indiscernible over $c_{<i} d_{<i}$.
- (3) $\phi(x, d_i) \wedge \phi(x, d_j)$ is inconsistent for any $i \neq j \in \omega$.

Proof. By induction we choose $s_i \neq t_i \in \kappa$, $c_i = a_{s_1 \dots s_{i-1} s_i}$ and $d_i = a_{s_1 \dots s_{i-1} t_i}$ for some $s_i \neq t_i \in \kappa$ such that there is a $c_{<i} d_{<i}$ -indiscernible sequence starting with $a_{s_1 \dots s_{i-1} s_i}, a_{s_1 \dots s_{i-1} t_i}$ (exists by

Erdős-Rado as κ is large enough), so we get (2). From the assumption on $(a_\eta)_{\eta \in \kappa < \kappa}$ we get (1) as $s_0 \triangleleft s_0 s_1 \triangleleft s_0 s_1 s_2 \triangleleft \dots$ lie on the same branch in the tree order and (3) as $s_0 \dots s_{i-1} t_i$ and $s_0 \dots s_{i-1} s_i$ are incomparable in the tree order. \square

By compactness and Ramsey we can find a and $(c_i d_i)_{i \leq \omega+1}$ indiscernible over a , satisfying (1)–(3) and such that $a \models p(x) \cup \{\phi(x, c_i)\}$.

Let $M = \text{Sk}(c_i d_i)_{i < \omega}$, a model of T^{Sk} . Then we have $c_{\omega+1} \downarrow_M^u a$ and $d_\omega \downarrow_M^u c_{\omega+1}$ by indiscernibility. As $c_\omega d_\omega$ start an M -indiscernible sequence, there is $\sigma \in \text{Aut}(\mathbb{M}/M)$ sending c_ω to d_ω . Let $a' = \sigma(a)$, then $a' \equiv_M^{\text{Lstp}} a$, $d_\omega \downarrow_M^u a'$ (as $c_\omega \downarrow_M^u a$ by indiscernibility) and $\phi(a', d_\omega)$. But $\phi(x, c_{\omega+1}) \wedge \phi(x, d_\omega)$ is inconsistent by (3)+(2). As ϕ is an L -formula, M is in particular an L -model and \downarrow^u in the sense of T^{Sk} implies \downarrow^u in the sense of T , we get that the co-independence theorem fails for $p' = \text{tp}_L(a/M)$ in T .

(2) Similar. \square

Now we will show that in NTP_2 theories simple types satisfy the independence theorem over extension bases. We will need the following fact from [BC].

Fact 6.21. *Let T be NTP_2 and $M \models T$. Assume that $c \downarrow_M ab$, $b \downarrow_M a$, $b' \downarrow_M a$, $b \equiv_M b'$. Then there exists $c' \downarrow_M ab'$ and $c'b' \equiv_M cb$, $c'a \equiv_M ca$.*

Proposition 6.22. *Let T be NTP_2 and $p(x)$ a simple type over $M \models T$. Then it satisfies the independence theorem: assume that $e_1 \downarrow_M e_2$, $d_i \downarrow_M e_i$, $d_1 \equiv_M d_2 \models p(x)$. Then there is $d \downarrow_M e_1 e_2$ with $d \equiv_{e_i M} d_i$.*

Proof. First we find some $e'_1 \downarrow_M d_2 e_2$ and such that $e'_1 d_2 \equiv_M e_1 d_1$ (Let $\sigma \in \text{Aut}(\mathbb{M}/M)$ be such that $\sigma(d_1) = d_2$, then $\sigma(e_1) d_2 \equiv_M e_1 d_1$. By simplicity of $\text{tp}(d_1/M)$ and the assumption we get $e_1 \downarrow_M d_1$, which implies that $\sigma(e_1) \downarrow_M d_2$. Let e'_1 realize a non-forking extension to $d_2 e_2$). Then we also have $d_2 \downarrow_M e'_1 e_2$ (by transitivity and symmetry using simplicity of $\text{tp}(d_2/M)$).

Applying Fact 6.21 with $a = e_2, b = e'_1, b' = e_1, c = d_2$ we find some $d \downarrow_M e_1 e_2$, $d e_1 \equiv_M d_2 e'_1 \equiv_M d_1 e_1$ and $d e_2 \equiv_M d_2 e_2$ — as wanted. \square

We conclude with the main theorem of the section.

Theorem 6.23. *Let T be NTP_2 and $p(x)$ a partial type over A . Then the following are equivalent:*

- (1) $p(x)$ is simple (in the sense of Definition 6.1).
- (2) For any $B \supseteq A$, $a \models p$ and b , $a \downarrow_A^f b$ if and only if $b \downarrow_A^f a$.
- (3) Every extension $p'(x) \supseteq p(x)$ to a model $M \supseteq A$ satisfies the co-independence theorem.

Proof. (1) is equivalent to (2) is by Definitions 6.1 and Corollary 6.18.

(1) implies (3): By Proposition 6.22 and Corollary 6.18.

(3) implies (1) is by Proposition 6.20. \square

Problem 6.24. Is every co-simple type simple in an NTP_2 theory?

We point out that at least every co-simple *stably embedded* type (defined over a small set) is simple. Recall that a partial type $p(x)$ defined over A is called stably embedded if for any $\phi(\bar{x}, c)$ there is some $\psi(\bar{x}, y) \in L(A)$ and $d \in p(\mathbb{M})$ such that $p(\mathbb{M})^n \cap \phi(\bar{x}, c) = p(\mathbb{M})^n \cap \psi(\bar{x}, d)$. If $p(x)$ happens to be defined by finitely many formulas, it is easy to see by compactness that $\psi(\bar{x}, y)$ can be chosen to depend just on $\phi(\bar{x}, y)$, and not on c . But for an arbitrary type this is not true.

Proposition 6.25. *Let T be NTP_2 . Let $p(x)$ be a co-simple type over A and assume that p is stably embedded. Then $p(x)$ is simple.*

Proof. Assume $p(x)$ is not simple, and let $(a_\eta)_{\eta \in \omega^{<\omega}}$, k and $\phi(x, y)$ witness this. We may assume in addition that (a_η) is an indiscernible tree over A (that is, ss-indiscernible in the terminology of [KKS12], see Definition 3.7 and the proof of Theorem 6.6 there).

By the stable embeddedness assumption, there is some $\psi(x, z) \in L(A)$ and $b \subseteq p(\mathbb{M})$ such that $\psi(x, b) \cap p(\mathbb{M}) = \phi(x, a_\emptyset) \cap p(\mathbb{M})$. It follows by the indiscernibility over A that for every $\eta \in \omega^{<\omega}$ there is $b_\eta \subseteq p(\mathbb{M})$ satisfying $\psi(x, b_\eta) \cap p(\mathbb{M}) = \phi(x, a_\eta) \cap p(\mathbb{M})$.

As $\{\phi(x, a_{\emptyset i})\}_{i < \omega}$ is k -inconsistent, it follows that $\{\psi(x, b_{\emptyset i})\}_{i < \omega} \cup p(x)$ is k -inconsistent, thus $\{\psi(x, b_{\emptyset i})\}_{i < \omega} \cup \{\chi(x)\}$ is k -inconsistent for some $\chi(x) \in p$ by compactness and indiscernibility. Again by the indiscernibility over A we have that $\{\psi(x, b_{\eta i})\}_{i < \omega} \cup \{\chi(x)\}$ is k -inconsistent for every $\eta \in \omega^{<\omega}$. It is now easy to see that $\psi'(x, z) = \psi(x, z) \wedge \chi(x)$ and $(b_\eta)_{\eta \in \omega^{<\omega}}$ witness that $p(x)$ is not co-simple over A . \square

Remark 6.26. If $p(x)$ is actually a definable set, the argument works in an arbitrary theory since instead of extracting a sufficiently indiscernible tree (which seems to require NTP_2), we just use the uniformity of stable embeddedness given by compactness.

7. EXAMPLES

In this section we present some examples of NTP_2 theories. But first we state a general lemma which may sometimes simplify checking NTP_2 in particular examples.

Lemma 7.1.

- (1) *If $(\bar{a}_\alpha, \phi_{\alpha,0}(x, y_{\alpha,0}) \vee \phi_{\alpha,1}(x, y_{\alpha,1}), k_\alpha)_{\alpha < \kappa}$ is an inp-pattern, then $(\bar{a}_\alpha, \phi_{\alpha,f(\alpha)}(x, y_{\alpha,f(\alpha)}), k_\alpha)_{\alpha < \kappa}$ is an inp-pattern for some $f : \kappa \rightarrow \{0, 1\}$.*
- (2) *Let $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha < \kappa}$ be an inp-pattern and assume that $\phi_\alpha(x, a_{\alpha 0}) \leftrightarrow \psi_\alpha(x, b_\alpha)$ for $\alpha < \kappa$. Then there is an inp-pattern of the form $(\bar{b}_\alpha, \psi_\alpha(x, z_\alpha), k_\alpha)_{\alpha < \kappa}$.*

7.1. Adding a generic predicate. Let T be a first-order theory in the language L . For $S(x) \in L$ we let $L_P = L \cup \{P(x)\}$ and $T_{P,S}^0 = T \cup \{\forall x (P(x) \rightarrow S(x))\}$.

Fact 7.2. [CP98] *Let T be a theory eliminating quantifiers and \exists^∞ . Then:*

(1) $T_{P,S}^0$ has a model companion $T_{P,S}$, which is axiomatized by T together with

$$\forall \bar{z} \left[\exists \bar{x} \phi(\bar{x}, \bar{z}) \wedge (\bar{x} \cap \text{acl}_L(\bar{z}) = \emptyset) \wedge \bigwedge_{i < n} S(x_i) \wedge \bigwedge_{i \neq j < n} x_i \neq x_j \right] \rightarrow \\ \left[\exists \bar{x} \phi(\bar{x}, \bar{z}) \wedge \bigwedge_{i \in I} P(x_i) \wedge \bigwedge_{i \notin I} \neg P(x_i) \right]$$

for every formula $\phi(\bar{x}, \bar{z}) \in L$, $\bar{x} = x_0 \dots x_{n-1}$ and every $I \subseteq n$. It is possible to write it in first-order due to the elimination of \exists^∞ .

(2) $\text{acl}_L(a) = \text{acl}_{L_P}(a)$

(3) $a \equiv^{L_P} b \Leftrightarrow$ there is an isomorphism between L_P structures $f : \text{acl}(a) \rightarrow \text{acl}(b)$ such that $f(a) = b$.

(4) Modulo $T_{P,S}$, every formula $\psi(\bar{x})$ is equivalent to a disjunction of formulas of the form $\exists \bar{z} \phi(\bar{x}, \bar{z})$ where $\phi(\bar{x}, \bar{z})$ is a quantifier-free L_P formula and for any \bar{a}, \bar{b} , if $\models \phi(\bar{a}, \bar{b})$, then $\bar{b} \in \text{acl}(\bar{a})$.

Theorem 7.3. *Let T be geometric (that is, the algebraic closure satisfies the exchange property, and T eliminates \exists^∞) and NTP_2 . Then T_P is NTP_2 .*

Proof. Denote $a \downarrow_c^a b \Leftrightarrow a \notin \text{acl}(bc) \setminus \text{acl}(c)$. As T is geometric, \downarrow^a is a symmetric notion of independence, which we will be using freely from now on.

Let $(\bar{a}_i, \phi(x, y), k)_{i < \omega}$ be an inp-pattern, such that $(\bar{a}_i)_{i < \omega}$ is an indiscernible sequence and \bar{a}_i 's are mutually indiscernible in the sense of L_P , and ϕ an L_P -formula.

Claim. For any i , $\{a_{ij}\}_{j < \omega}$ is an \downarrow^a -independent set (over \emptyset) and $a_{ij} \notin \text{acl}(\emptyset)$.

Proof. By indiscernibility and compactness. □

Let $A = \bigcup_{i < \omega} \bar{a}_i$.

Claim. There is an infinite A -indiscernible sequence $(b_t)_{t < \omega}$ such that $b_t \models \{\phi(x, a_{i0})\}_{i < \omega}$ for all $t < \omega$.

Proof. First, there are infinitely many different b_t 's realizing $\{\phi(x, a_{i0})\}_{i < \omega}$, as $\{\phi(x, a_{i0})\}_{0 < i < \omega} \cup \{\phi(x, a_{0j})\}$ is consistent for any $j < \omega$ and $\{\phi(x, a_{0j})\}_{j < \omega}$ is k -inconsistent. Extract an A -indiscernible sequence from it. □

Let $p_i(x, a_{i0}) = \text{tp}_L(b_0/a_{i0})$.

Claim. For some/every $i < \omega$, there is $b \models \bigcup_{j < \omega} p_i(x, a_{ij})$ such that in addition $b \notin \text{acl}(A)$.

Proof. For any $N < \omega$, let

$$q_i^N(x_0 \dots x_{N-1}, a_{i0}) = \bigcup_{n < N} p_i(x_n, a_{i0}) \cup \{x_{n_1} \neq x_{n_2}\}_{n_1 \neq n_2 < N}$$

As $b_0 \dots b_{N-1} \models \bigcup_{i < \omega} q_i^N(x_0 \dots x_{N-1}, a_{i0})$ and T is NTP_2 , there must be some $i < \omega$ such that $\bigcup_{j < \omega} q_i^N(x_0 \dots x_{N-1}, a_{ij})$ is consistent for arbitrary large N (and by indiscernibility this holds for every i). Then by compactness we can find $b \models \bigcup_{j < \omega} p_i(x, a_{ij})$ such that in addition $b \notin \text{acl}(A)$. \square

Work with this fixed i . Notice that $b_0 a_{i0} \equiv^L b a_{ij}$ for all $j \in \omega$.

Claim. The following is easy to check using that \perp^a satisfies exchange.

- (1) $\text{acl}(A) \cap \text{acl}(a_{ij}b) = \text{acl}(a_{ij})$.
- (2) $\text{acl}(a_{ij}b) \cap \text{acl}(a_{ik}b) = \text{acl}(b)$ for $j \neq k$.

Now we conclude as in the proof of [CP98, Theorem 2.7]. That is, we are given a coloring P on \bar{a}_i . Extend it to a P_i -coloring on $\text{acl}(a_{ij}b)$ such that $a_{ij}b$ realizes $\text{tp}_{L_P}(a_{i0}b_0)$, and by the claim all P_i 's are consistent. Thus there is some b' such that $b_0 a_{i0} \equiv^{L_P} b' a_{ij}$ for all $j \in \omega$, in particular $b' \models \{\phi_i(x, a_{ij})\}$ — a contradiction. \square

Example 7.4. Adding a (directed) random graph to an o -minimal theory is NTP_2 .

Problem 7.5. Is it true without assuming exchange for the algebraic closure? Is κ_{inp} preserved? So in particular, is strongness preserved?

7.2. Valued fields. In this section we are going to prove the following theorem:

Theorem 7.6. *Let $\bar{K} = (K, \Gamma, k, v : K \rightarrow \Gamma, ac : K \rightarrow k)$ be a Henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language. Let $\kappa = \kappa_{\text{inp}}^1(k) \times \kappa_{\text{inp}}^1(\Gamma)$. Then $\kappa_{\text{inp}}^1(K) < R(\kappa + 2, \Delta)$ for some finite set of formulas Δ (see Definition 1.4). In particular:*

- (1) *If k is NTP_2 , then \bar{K} is NTP_2 (If K was TP_2 , then by Lemma 3.2 we would have $\kappa_{\text{inp}}^1(K) = \infty > \beth_\omega(|T|^+) > R(|T|^+ + 2, \Delta)$). Every ordered abelian group is NIP by [GS84], thus $\kappa_{\text{inp}}(\Gamma) \leq |T|$. But then the theorem implies $\kappa_{\text{inp}}^1(k) > |T|^+$, so k has TP_2).*
- (2) *If k and Γ are strong (of finite burden), then \bar{K} is strong (resp. of finite burden). The argument is the same as for (1) using Definition 1.4(1),(2).*

Example 7.7. (1) Hahn series over pseudo-finite fields are NTP_2 .

- (2) In particular, let $K = \prod_p \text{prime } \mathbb{Q}_p / \mathfrak{U}$ with \mathfrak{U} a non-principal ultra-filter. Then k is pseudo-finite, so has IP by [Dur80]. And Γ has SOP of course. It is known that the valuation rings of \mathbb{Q}_p are definable in the pure field language uniformly in p (see e.g. [Ax65]), thus the valuation ring is definable in K in the pure field language, so K has both IP and SOP in the pure field language. By Theorem 7.6 it is strong of finite burden, even in the larger Denef-Pas language. Notice, however, that the burden of K is at least 2 (witnessed by the formulas “ $ac(x) = y$ ”, “ $v(x) = y$ ” and infinite sequences of different elements in k and Γ).

Corollary 7.8. [She05] *If k and Γ are strongly dependent, then K is strongly dependent.*

Proof. By Delon's theorem [Del81], if k is NIP, then K is NIP. Conclude by Theorem 7.6 and Fact 3.8. \square

We start the proof with a couple of lemmas about the behavior of $v(x)$ and $ac(x)$ on indiscernible sequences which are easy to check.

Lemma 7.9. *Let $(c_i)_{i \in I}$ be indiscernible. Consider function $(i, j) \mapsto v(c_j - c_i)$ with $i < j$. It satisfies one of the following:*

- (1) *It is strictly increasing depending only on i (so the sequence is pseudo-convergent).*
- (2) *It is strictly decreasing depending only on j (so the sequence taken in the reverse direction is pseudo-convergent).*
- (3) *It is constant (we'll call such a sequence "constant").*

Contrary to the usual terminology we do not exclude index sets with a maximal element.

Lemma 7.10. *Let $(c_i)_{i \in I}$ be an indiscernible pseudo-convergent sequence. Then for any a there is some $h \in \bar{I} \cup \{+\infty, -\infty\}$ (where \bar{I} is the Dedekind closure of I) such that (taking c_∞ such that $I \cap c_\infty$ is indiscernible):*

For $i < h$: $v(c_\infty - c_i) < v(a - c_\infty)$, $v(a - c_i) = v(c_\infty - c_i)$ and $ac(a - c_i) = ac(c_\infty - c_i)$.

For $i > h$: $v(c_\infty - c_i) > v(a - c_\infty)$, $v(a - c_i) = v(a - c_\infty)$ and $ac(a - c_i) = ac(a - c_\infty)$.

Notice that in fact there is a finite set of formulas Δ such that these lemmas are true for Δ -indiscernible sequences. Fix it from now on, and let $\delta = R(\kappa + 2, \Delta)$ for $\kappa = \kappa_k \times \kappa_\Gamma$ with $\kappa_k = \kappa_{\text{inp}}^1(k)$ and $\kappa_\Gamma = \kappa_{\text{inp}}^1(\Gamma)$.

Lemma 7.11. *In K , there is no inp-pattern $(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha)_{\alpha < \delta}$ with mutually indiscernible rows such that x is a singleton and $\phi_\alpha(x, y_\alpha) = \chi_\alpha(v(x - y), y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x - y), y_\alpha^k)$, where $\chi_\alpha \in L_\Gamma$ and $\rho_\alpha \in L_k$.*

Proof. Assume otherwise, and let $d_{\alpha i} = c_{\alpha i} d_{\alpha i}^\Gamma d_{\alpha i}^k$ where $c_{\alpha i} \in K$ corresponds to y , $d_{\alpha i}^\Gamma \in \Gamma$ corresponds to y_α^Γ and $d_{\alpha i}^k \in k$ corresponds to y_α^k . By the choice of δ , there is a Δ -indiscernible sub-sequence of $(c_{\alpha 0})_{\alpha < \delta}$ of length $\kappa + 2$. Take a sub-array consisting of rows starting with these elements – it is still an inp-pattern of depth $\kappa + 2$ – and replace our original array with it. Let $c_{-\infty}$ and c_∞ be such that $c_{-\infty} \frown (c_{\alpha 0})_{\alpha < \kappa} \frown c_\infty$ is Δ -indiscernible and $(\bar{d}_\alpha)_{\alpha < \kappa}$ is a mutually indiscernible array over $c_{-\infty} c_\infty$ (so either find c_∞ by compactness if κ is infinite, or just let it be $c_{\kappa-1,0}$ and replace our array by $(\bar{d}_\alpha)_{\alpha < \kappa-1}$). Let $a \models \{\phi_\alpha(x, d_{\alpha 0})\}_{\alpha < \kappa+1}$.

Case 1. $(c_{\alpha 0})$ is pseudo-convergent. Let $h \in \{-\infty\} \cup \kappa + 1 \cup \{\infty\}$ be as given by Lemma 7.10.

Case 1.1. Assume $0 < h$. Then $v(a - c_{00}) = v(c_\infty - c_{00})$, $ac(a - c_{00}) = ac(c_\infty - c_{00})$. But then actually $c_\infty \models \phi(x, d_{00})$, and by indiscernibility of the array over c_∞ , $c_\infty \models \{\phi(x, d_{0i})\}_{i < \omega}$ — a contradiction.

Case 1.2: Thus $v(a - c_{\alpha 0}) = v(a - c_\infty)$, $ac(a - c_{\alpha 0}) = ac(a - c_\infty)$ and $v(a - c_\infty) < v(c_\infty - c_{\alpha 0})$ for all $0 < \alpha < \kappa + 1$.

Let $\chi'_\alpha(x', e_{\alpha i}^\Gamma) := \chi_\alpha(x', d_{\alpha i}^\Gamma) \wedge x' < v(c_\infty - c_{\alpha i})$ with $e_{\alpha i}^\Gamma = d_{\alpha i}^\Gamma \cup v(c_\infty - c_{\alpha i})$. Finally, for $\alpha < \kappa_\Gamma$ let $f_{\alpha i}^\Gamma = \bigcup_{\beta < \kappa_k} e_{\kappa_k \times \alpha + \beta, i}$ and $p_\alpha(x', f_{\alpha i}^\Gamma) = \left\{ \chi'_\beta(x', e_{\kappa_k \times \alpha + \beta, i}^\Gamma) \right\}_{\beta < \kappa_k}$. As $(f_{\alpha i}^\Gamma)$ is a mutually indiscernible array in Γ , $\{p_\alpha(x', f_{\alpha i}^\Gamma)\}_{\alpha < \kappa_\Gamma}$ is realized by $v(a - c_\infty)$ and $\kappa_{\text{inp}}^1(\Gamma) = \kappa_\Gamma$, there must be some $\alpha < \kappa_\Gamma$ and $a_\Gamma \in \Gamma$ such that (unwinding) $a_\Gamma \models \left\{ \chi'_\beta(x', e_{\kappa_k \times \alpha + \beta, i}^\Gamma) \right\}_{\beta < \kappa_k, i < \omega}$.

Analogously letting $\chi'_\beta(x', e_{\beta i}^k) := \rho_{\kappa_k \times \alpha + \beta}(x', d_{\kappa_k \times \alpha + \beta, i}^k)$, noticing that $(e_{\beta i}^k)_{\beta < \kappa_k, i < \omega}$ is an indiscernible array in k and $\kappa_k = \kappa_{\text{inp}}(k)$, there must be some $a_\rho \in k$ and $\beta < \kappa_k$ such that $a_\rho \models \{\chi'_\beta(x', e_{\beta i}^k)\}_{i < \omega}$.

Finally, take $a' \in K$ with $v(a' - c_\infty) = a_\Gamma \wedge ac(a' - c_\infty) = a_\rho$ and let $\gamma = \kappa_k \times \alpha + \beta$. As $a_\Gamma < v(c_\infty - c_{\gamma i})$ it follows that $v(a' - c_{\gamma i}) = v(a' - c_\infty)$ and $ac(a' - c_{\gamma i}) = ac(a' - c_\infty)$. But then $a' \models \{\phi_\gamma(x, d_{\gamma i})\}_{i < \omega}$ — a contradiction.

Case 2: (c_0^α) is decreasing — reduces to the first case by reversing the order of rows.

Case 3: (c_0^α) is constant.

If $v(a - c_{\alpha 0}) < v(c_\infty - c_{\alpha 0})$ ($= v(c_{\beta 0} - c_{\alpha 0})$ for $\beta \neq \alpha$) for some α , then $v(a - c_{\alpha 0}) = v(a - c_{\beta 0}) = v(a - c_\infty)$ for any β , and $ac(a - c_{\alpha 0}) = ac(a - c_\infty)$ for all α 's and it falls under case 1.2.

Next, there can be at most one α with $v(a - c_{\alpha 0}) > v(c_\infty - c_{\alpha 0})$ (if also $v(a - c_{\beta 0}) > v(c_\infty - c_{\beta 0})$ for some $\beta > \alpha$ then $v(c_\infty - c_{\beta 0}) = v(c_{\beta 0} - c_{\alpha 0}) = v(a - c_{\beta 0}) > v(c_\infty - c_{\beta 0})$, a contradiction). Throw the corresponding row away and we are left with the case $v(a - c_{\alpha 0}) = v(c_\infty - c_{\alpha 0}) = v(a - c_\infty)$ for all $\alpha < \kappa$. It follows by indiscernibility that $v(a - c_\infty) = v(c_\infty - c_{\alpha i})$ for all α, i . Notice that it follows that $ac(a - c_{\alpha 0}) \neq ac(c_\infty - c_{\alpha 0})$ and $ac(a - c_{\alpha 0}) = ac(a - c_\infty) + ac(c_\infty - c_{\alpha 0})$.

Let $\rho'_\alpha(x', e_{\alpha i}^k) := \rho_\alpha(x' - ac(c_\infty - c_{\alpha i}), d_{\alpha i}^k) \wedge x' \neq ac(c_\infty - c_{\alpha i})$ with $e_{\alpha i}^k = d_{\alpha i}^k \cup ac(c_\infty - c_{\alpha i})$. Notice that $ac(a - c_\infty) \models \{\rho'_\alpha(x', e_{\alpha 0}^k)\}$ and that $(e_{\alpha i}^k)$ is a mutually indiscernible array in k . Thus there is some $\alpha < \kappa$ and $a_k \models \{\rho'_\alpha(x', e_{\alpha i}^k)\}_{i < \omega}$.

Take $a' \in K$ such that $v(a' - c_\infty) = v(a - c_\infty) \wedge ac(a' - c_\infty) = a_k$. By the choice of a_k we have that $v(a' - c_\infty) = v(a - c_\infty) = v(c_\infty - c_{\alpha i})$ and that $ac(a' - c_\infty) \neq ac(c_\infty - c_{\alpha i})$, thus $v(a' - c_{\alpha i}) = v(a - c_\infty)$ and $ac(a' - c_{\alpha i}) = a_k + ac(c_\infty - c_{\alpha i})$. It follows that $a' \models \{\phi_\alpha(x, d_{\alpha i})\}_{i < \omega}$ — a contradiction. \square

Lemma 7.12. *In K , there is no inp-pattern $(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha)_{\alpha < \delta}$ such that x is a singleton and $\phi_\alpha(x, y_\alpha) = \chi_\alpha(v(x - y_1), \dots, v(x - y_n), y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x - y_1), \dots, ac(x - y_n), y_\alpha^k)$, where $\chi_\alpha \in L_\Gamma$ and $\rho_\alpha \in L_k$.*

Proof. We prove it by induction on n . The base case is given by Lemma 7.11. So assume that we have proved it for $n - 1$, and let $(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha)_{\alpha < \delta}$ be an inp-pattern with $\phi_\alpha(x, y_\alpha) = \chi_\alpha(v(x - y_1), \dots, v(x - y_n), y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x - y_1), \dots, ac(x - y_n), y_\alpha^k)$ and $d_{\alpha i} = c_{\alpha i}^1 \dots c_{\alpha i}^n d_{\alpha i}^\Gamma d_{\alpha i}^k$.

So let $a \models \{\phi_\alpha(x, d_{\alpha 0})\}_{\alpha < \delta}$. Fix some $\alpha < \delta$.

Case 1: $v(a - c_{\alpha 0}^1) < v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $v(a - c_{\alpha 0}^1) = v(a - c_{\alpha 0}^n)$ and $ac(a - c_{\alpha 0}^1) = ac(a - c_{\alpha 0}^n)$. We take

$$\begin{aligned} \phi'_\alpha(x, d'_{\alpha i}) &= (\chi_\alpha(v(x - c_{\alpha i}^1), \dots, v(x - c_{\alpha i}^1), d_{\alpha i}^\Gamma) \wedge v(x - c_{\alpha 0}^1) < v(c_{\alpha i}^n - c_{\alpha i}^1)) \\ &\quad \wedge \rho_\alpha(ac(x - c_{\alpha i}^1), \dots, ac(x - c_{\alpha i}^1), d_{\alpha i}^\rho) \end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1)$.

Case 2: $v(a - c_{\alpha 0}^1) > v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $v(a - c_{\alpha 0}^n) = v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and $ac(a - c_{\alpha 0}^n) = ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. Take

$$\begin{aligned} \phi'_\alpha(x, d'_{\alpha i}) &= (\chi_\alpha(v(x - c_{\alpha i}^1), \dots, v(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\Gamma) \wedge v(x - c_{\alpha 0}^n) > v(c_{\alpha i}^n - c_{\alpha i}^1)) \\ &\quad \wedge \rho_\alpha(ac(x - c_{\alpha i}^1), \dots, ac(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\rho) \end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Case 3: $v(a - c_{\alpha 0}^n) < v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and **Case 4:** $v(a - c_{\alpha 0}^n) > v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ are symmetric to the cases 1 and 2, respectively.

Case 5: $v(a - c_{\alpha 0}^1) = v(a - c_{\alpha 0}^n) = v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $ac(a - c_{\alpha 0}^n) = ac(a - c_{\alpha 0}^1) - ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. We take

$$\begin{aligned} \phi'_\alpha(x, d'_{\alpha i}) &= (\chi_\alpha(v(x - c_{\alpha i}^1), \dots, v(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\Gamma) \wedge v(x - c_{\alpha 0}^1) = v(c_{\alpha i}^n - c_{\alpha i}^1)) \\ &\quad \wedge (\rho_\alpha(ac(x - c_{\alpha i}^1), \dots, ac(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\rho) \wedge ac(x - c_{\alpha 0}^1) \neq ac(c_{\alpha i}^n - c_{\alpha i}^1)) \end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

In any case, we have that $\{\phi'_\alpha(x, d'_{\alpha i})\}_{i < \omega}$ is inconsistent, $\{\phi_\beta(x, d_{\beta, 0})\}_{\beta < \alpha} \cup \{\phi'_\alpha(x, d'_{\alpha 0})\} \cup \{\phi_\beta(x, d_{\beta 0})\}_{\alpha < \beta < \delta}$ is consistent, and $(\bar{d}_\beta)_{\beta < \alpha} \cup \{\bar{d}'_\alpha\} \cup (\bar{d}_\beta)_{\alpha < \beta < \delta}$ is a mutually indiscernible array. Doing this for all α by induction we get an inp-pattern of the same depth involving strictly less different $v(x - y_i)$'s — contradicting the inductive hypothesis. \square

Finally, we are ready to prove Theorem 7.6.

Proof. By the cell decomposition of Pas [Pas89], every formula $\phi(x, \bar{c})$ is equivalent to one of the form $\bigvee_{i < n} (\chi_i(x) \wedge \rho_i(x))$ where $\chi_i = \bigwedge \chi_j^i(v(x - c_j^i), \bar{d}_j^i)$ with $\chi_j^i(x, \bar{d}_j^i) \in L(\Gamma)$ and $\rho_i = \bigwedge \rho_j^i(ac(x - c_j^i), \bar{e}_j^i)$ with $\rho_j^i(x, \bar{e}_j^i) \in L(k)$. By Lemma 7.1, if there is an inp-pattern of depth κ with x ranging over K , then there has to be an inp-pattern of depth κ and of the form as in

Lemma 7.12, which is impossible. It is sufficient, as Γ and k are stably embedded with no new induced structure and are fully orthogonal. \square

Problem 7.13.

- (1) Can the bound on $\kappa_{\text{inp}}^1(K)$ given in Theorem 7.6 be improved? Specifically, is it true that $\kappa_{\text{inp}}^1(K) \leq \kappa_{\text{inp}}^1(k) \times \kappa_{\text{inp}}^1(\Gamma)$ in the ring language?
- (2) Determine the burden of $K = \prod_p \text{prime } \mathbb{Q}_p/\mathfrak{U}$ in the pure field language. In [DGL11] it is shown that each of \mathbb{Q}_p is dp-minimal, so combined with Fact 3.8 it has burden 1. Note that K is not inp-minimal in the Denef-Pas language, as the residue field is infinite, so $\{v(x) = v_i\}$, $\{ac(x) = a_i\}$ shows that the burden is at least 2. However, Hrushovski pointed out to me that the angular component is not definable in the pure ring language, thus the conjecture is that every ultraproduct of p -adics is of burden 1 in the pure ring (or RV) language.

REFERENCES

- [Adl07] Hans Adler. Strong theories, burden, and weight. *Preprint*, 2007.
- [Adl08] Hans Adler. An introduction to theories without the independence property. *Archive Math. Logic*, to appear, 2008.
- [Ax65] James Ax. On the undecidability of power series fields. *Proc. Amer. Math. Soc.*, 16:846, 1965.
- [BC] Ben Yaacov, Itai and Artem Chernikov. An independence theorem for NTP_2 theories. *Preprint*, *arXiv:1207.0289*.
- [BY03] Itay Ben-Yaacov. Simplicity in compact abstract theories. *J. Math. Log.*, 3(2):163–191, 2003.
- [BY11] Itai Ben Yaacov. On theories of random variables. *Israel Journal of Mathematics*, to appear, 2011.
- [Cas07] Enrique Casanovas. Simplicity simplified. *Rev. Colombiana Mat.*, 41(Número Especial):263–277, 2007.
- [CH12] Artem Chernikov and Martin Hils. Valued difference fields and NTP_2 . *Preprint*, *arXiv:1208.1341v2*, August 2012.
- [Cha08] Zoe Chatzidakis. Independence in (unbounded) PAC fields, and imaginaries. *Unpublished*, 2008.
- [CK12] Artem Chernikov and Itay Kaplan. Forking and dividing in NTP_2 theories. *J. Symbolic Logic*, 77(1):1–20, 2012.
- [CKS12] Artem Chernikov, Itay Kaplan, and Saharon Shelah. On non-forking spectra. *Preprint*, *arXiv:1205.3101*, 2012.
- [CP98] Z. Chatzidakis and A. Pillay. Generic structures and simple theories. *Annals of Pure and Applied Logic*, 95(1-3):71 – 92, 1998.
- [CS13] Artem Chernikov and Pierre Simon. Externally definable sets and dependent pairs. *Israel Journal of Mathematics*, 194(1):409–425, 2013.
- [Del81] Françoise Delon. Types sur $\mathbf{C}((X))$. In *Study Group on Stable Theories (Bruno Poizat), Second year: 1978/79 (French)*, pages Exp. No. 5, 29. Secrétariat Math., Paris, 1981.
- [DGL11] Alfred Dolich, John Goodrick, and David Lippel. Dp-minimality: basic facts and examples. *Notre Dame J. Form. Log.*, 52(3):267–288, 2011.
- [Dur80] Jean-Louis Duret. Les corps faiblement algebriquement clos non separablement clos ont la propriete d'indépendance. In Leszek Pacholski, Jędrzej Wierzejewski, and Alec Wilkie, editors, *Model Theory of*

- Algebra and Arithmetic*, volume 834 of *Lecture Notes in Mathematics*, pages 136–162. Springer Berlin / Heidelberg, 1980. 10.1007/BFb0090164.
- [Goo10] John Goodrick. A monotonicity theorem for DP-minimal densely ordered groups. *J. Symbolic Logic*, 75(1):221–238, 2010.
- [GS84] Y. Gurevich and P. H. Schmitt. The theory of ordered abelian groups does not have the independence property. *Transactions of the American Mathematical Society*, 284(1):pp. 171–182, 1984.
- [HHM08] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. *Stable domination and independence in algebraically closed valued fields*, volume 30 of *Lecture Notes in Logic*. Association for Symbolic Logic, Chicago, IL, 2008.
- [HKP00] Bradd Hart, Byunghan Kim, and Anand Pillay. Coordinatisation and canonical bases in simple theories. *The Journal of Symbolic Logic*, 65(1):pp. 293–309, 2000.
- [HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. *J. Eur. Math. Soc. (JEMS)*, 13(4):1005–1061, 2011.
- [Kim01] Byunghan Kim. Simplicity, and stability in there. *J. Symbolic Logic*, 66(2):822–836, 2001.
- [KK11] Byunghan Kim and Hyeung-Joon Kim. Notions around tree property 1. *Annals of Pure and Applied Logic*, 162(9):698 – 709, 2011.
- [KKS12] Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow. Tree indiscernibilities, revisited. *Preprint*, 2012.
- [KOU] Itay Kaplan, Alf Onshuus, and Alex Usvyatsov. Additivity of the dp-rank. *Transactions of the American Mathematical Society*, *accepted*.
- [KSed] Itay Kaplan and Pierre Simon. Witnessing dp-rank. *Notre Dame J. of Formal logic*, *accepted*.
- [Mal12] Maryanthe Malliaris. Hypergraph sequences as a tool for saturation of ultrapowers. *Journal of Symbolic Logic*, 77(1):195–223, 2012.
- [OU11] Alf Onshuus and Alexander Usvyatsov. On DP-minimality, strong dependence and weight. *J. Symbolic Logic*, 76(3):737–758, 2011.
- [Pas89] Johan Pas. Uniform p -adic cell decomposition and local zeta functions. *J. Reine Angew. Math.*, 399:137–172, 1989.
- [She80] Saharon Shelah. Simple unstable theories. *Ann. Math. Logic*, 19(3):177–203, 1980.
- [She90] S. Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, second edition, 1990.
- [She96] Saharon Shelah. Toward classifying unstable theories. *Ann. Pure Appl. Logic*, 80(3):229–255, 1996.
- [She05] Saharon Shelah. Strongly dependent theories. *Preprint*, *arXiv:math.LO/0504197*, 2005.
- [She07] Saharon Shelah. Definable groups for dependent and 2-dependent theories. *Preprint*, *arXiv:math/0703045*, 2007.
- [She09] Saharon Shelah. Dependent first order theories, continued. *Israel J. Math.*, 173:1–60, 2009.
- [Sim11] Pierre Simon. On dp-minimal ordered structures. *J. Symbolic Logic*, 76(2):448–460, 2011.
- [Tsu85] Akito Tsuboi. On the number of independent partitions. *J. Symbolic Logic*, 50(3):809–814, 1985.