An Algebraic Theory of Structured Objects

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Abstract

We present an algebraic theory of structured objects,[1] based on and generalizing Aczel's theory of form systems [2]. Notions of identity of structured objects and of transformations of systems of such objects are discussed. A generalization of Aczel's [2] representation theorem is proven.

1 Introduction

We develop an algebraic theory of structured objects, artifacts or otherwise, well-founded or not, based on and generalizing Aczel's theory of *form systems* [2]. Aczel's theory, further developed by Lunnon in her doctoral thesis [9], has been originally conceived as part of a long term project to provide a mathematical framework for *situation theory* and it has been in fact set to motion by an unpublished paper [5] of Jon Barwise, proposing a formal sketch of a model for situation theory.

Considerations from situation theory lead to a need to regard the objects of the theory as *structured objects*, objects within which other objects may occur as their *components*. The component-of relation, generalizing the membership relation, need not be well-founded, in fact applications of situation theory as in Barwise and Etchemendy [7] would require an anti-founded relation, allowing for objects that are components of themselves.

[1]A very first draft of this paper [8] was written some years ago, while I was still a graduate student. I have greatly benefited from discussions with Jon Barwise and have also had the good fortune to discuss things with Peter Aczel and Rachel Lunnon. It was not until I read [10] that my interest in the subject was revitilized, even though the direction taken in [10] is quite different from that taken here. This report is a completion of these old notes of mine, written in the hope that it will contribute to the discussion on formal theories of structured objects.

Aczel's theory of form systems formalizes the intuitive idea of a universe of structured objects, well founded (wf) or anti-founded (af), under an operation of *replacement* of components of an object by other objects.

We generalize Aczel's theory of form systems to the theory of what we have called in [8] *pre-form systems* and now call them simply *systems of objects*. What we have found missing in the original theory is (1) a discussion of appropriate concepts of *identity* of structured objects, (2) a study of adequate notions of *transformations* of systems of structured objects and (3) some more restraint view on *permissible* replacement maps. We take up these issues here, exploring natural alternatives to notions of identity of objects that arise by experimenting with objects with the means available, namely by replacement of components by other components. Transformations of systems of objects, we insist, should respect both replacement and components but they should also reflect identity of the abstract behavior of objects under replacement experiments. Discussing these issues leads us to a pleasant algebraic theory of systems of structured objects. We prove existence of free systems and derive from this a representation theorem that generalizes the representation theorem of [2].

As in [2] and unlike the direction taken in [10] we do not seek to provide a *theory*, first-order or otherwise, of universes of structured objects. Rather, we aim at modeling our pre-theoretic intuitions about structured objects, their components and change they may undergo due to replacement of components, by describing a formal model, some kind of *replacement algebra* that adequately, we believe, reflects our basic intuitions.

2 Systems of Structured Objects

To fix a context for discussion let us denote by V the class of all objects of our metatheory. This may include sets, atoms, structured physical objects and whatever the reader's ontological views allow for. For a structured object a, we denote by Ca the set of all objects that appear as components of a. However, in different contexts different components maps C may be considered. If σ is a map defined on the components of a we write $\sigma.a$ for the object obtained, intuitively speaking, by simultaneously replacing every component x of a by σx. In [2] a liberal view is taken, allowing for arbitrary replacements without imposing any constraint that the object σx replacing x as a component of a must be of the "same kind" as x. Thus, if a happens to be a physical object then $\sigma.a$ may or may not be physically realizable. Without assuming any preset notion of "sameness" we impose some restrictions on permissible replacements, thus diverging from and generalizing the approach of [2]. To make things more precise, let us suppose some given class X of *parts*, or *components*.

Definition 2.1 *A* system of objects *with parts from the class X (a system* over X*) is a structure $\mathcal{A} = (A, C_A, S_A, ._A)$ where (dropping the subscript A for simplicity)*

1. *A is a class of structured objects,*

2. *C is the components map and Ca is a subset of X for each object $a \in A$,*

3. *S is a collection of maps $\sigma : Ca \to X$, where $a \in A$, with a partial composition map (denoted by concatenation) and such that*

   3.1. for each $a \in A, \sigma \in S$, if $\sigma : Ca \to X$, then $\sigma.a$ is also in A

   3.2. for each $a \in A$, the identity map $id_{Ca} \in S$

   3.3. if $\sigma, \tau \in S$, $\tau : C(\sigma.a) \to X$, then $\tau\sigma \in S$

   and where

4. *the replacement operation $._A$ and the components map C satisfy the following axioms:*

   4.1. $C(\sigma.a) = \{\sigma x | x \in Ca\}$

   4.2. if $\sigma = id_{Ca}$, then $\sigma.a = a$

   4.3. $\tau.(\sigma.a) = \tau\sigma.a$, for all $\sigma, \tau \in S$, $\sigma : Ca \to X$, $\tau : C(\sigma.a) \to X$.

Membership of a replacement map σ in the set S is thus our notion of a *permissible* replacement. The *form systems* over some class X of [2] are exactly the systems of objects over X where S is the collection of all maps $\sigma : Ca \to X$, for $a \in A$. An *ontology* is defined in [2] as a form system over the class V of all objects. We recall also from [2] that an *elementary universe* is a form system $\mathcal{A} = (A, C, .)$ over the set A. Some simple examples will help fix the ideas.

Example 1 The *Instantiation Systems* of [12] are examples of systems of objects. Replacement maps are there called *instantiations*, objects are referred to as *terms* and their components are taken from a set Var of items called variables. Some finiteness conditions are imposed in [12] which makes instantiation systems a special case of systems of objects.

Example 2 Let Σ be a signature, that is to say a set of operation symbols with prescribed arities, and let T be the set of all closed Σ-terms. If $t = f(t_1, \ldots, t_n)$, for some n-ary $f \in \Sigma$, then let $Ct = \{t_1, \ldots, t_n\}$. If $\sigma : Ct \to T$ is a map, then let $\sigma.t = f(\sigma t_1, \ldots, \sigma t_n)$. We may let equality of terms be pure syntactic equality or else assume an equational theory Θ and declare $s = t$ just in case $\vdash_\Theta s = t$. Depending on what we are interested in, we may allow for all possible replacements or impose restrictions. For example, permissible replacement maps may be taken to be the maps σ such that for any $t \in dom(\sigma)$ we have $t = \sigma t$ (which is more interesting when we interpret equality as $\vdash_\Theta t = \sigma t$). In any case $\tau.(\sigma.t) = \tau\sigma.t$ and the rest of the axioms also trivially hold.[2]

Example 3 Let HF be the set of well-founded, hereditarily finite sets, that is sets s that are finite and such that every member of their transitive closure $Tc(s)$ is finite. In the cummulative hierarchy the well-founded hereditarily finite sets are exactly the sets of rank less than ω, so that $HF = R(\omega)$. Let $Cs = s$ and $X = HF$. If $\sigma : Cs \to X$, then $\sigma.s = \{\sigma x | x \in s\} \in HF$. This gives us an example of an elementary universe in the sense of [2].

Example 4 For a more mundaine example, let A be the set of all blocks in a Lego toy that can be possibly formed out of a given collection L of basic items of fixed shapes and colours. In speaking of possible blocks we do not mean to refer to object-types but rather to concrete particulars differentiated by the time-interval of their existence. For example suppose we form a block out of four pieces, then take it apart and form an identical block with the same pieces again. At the moment, we count these as two different objects. We will discuss the question of identity in a minute.

Now if u is such a block we let $Cu \subseteq L$ be the set of basic items used in the construction of the block. A replacement map is permissible only if it replaces a basic item by another basic item of the same shape but not necessarily of the same colour. If $\sigma : Cu \to L$, then $\sigma.u$ has the obvious meaning.

Example 5 To outline the boundaries of the theory we give a non-example. Let M be a proper class of atoms and V_M the class of all well-founded M-sets, namely sets with atoms from M possibly occurring in their build-up.

[2] It goes without saying that we assume that the theory Θ is well-behaved with respect to the signature. In other words a replacement theorem holds: if $\vdash_\Theta s_i = t_i$, then $\vdash_\Theta f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$.

For an M-set s let Cs be the set of atoms x such that $x \in s$ or there is some set $s' \in Tc(s)$ such that $x \in s'$. Now let $X = V_M \cup M$. Given a map $\sigma : Cs \to X$ define $\sigma.s$ by $\in$-induction:

$$\sigma.s = \{\sigma x | x \in s \cap M\} \cup \{\sigma.s' | s' \in s \cap V_M\}$$

This fails to be a form system or a system of objects in the sense of Definition 2.1 because the axiom $C(\sigma.s) = \{\sigma x | x \in Cs\}$ does not hold for any σ that assigns a pure set to atoms in Cs.

Example 4 is an example of a system of objects which is not a form system, as we have imposed restrictions on the permissible replacement maps. It justifies, we think, our generalizing the theory of form systems to that of systems of objects in the sense of Definition 2.1. Restrictions on replacement can be imposed by introducing an explicit typing of objects and their components, as well as of the replacement maps. An investigation along these lines has been carried out in [9]. The approach we take abstracts away from an awkward explicit typing but maintains the basic idea of not granting to all possible replacement maps the status of a permissible map.

2.1 Identity and Transformation

Suppose given a system $\mathcal{A}$. Objects in $\mathcal{A}$ change as a result of replacement actions. On the other hand, we should be able to think of the system $\mathcal{A}$ itself as being tranformed into some other system as a result of simultaneously transforming all objects in the system.

Example 6 Suppose our system consists of all cars of a certain make and model. Replacement of parts by parts of the same make results in another car of the same make and model. Suppose, however, all cars of that make and model turn out to be defective: their ignition system involves a serious risk of fire with potentially life-threatening consequences. A new part is manifactured and replacement of the old part with the new is offered free of charge. Our system of objects has thus been transformed.

To model our intuitions of structured objects we thus need to extend our treatment and provide for *transformations* of systems. There is a question, however, as to just what a legitimate transformation should be when the subject is approached in the abstract. This relates, we think, to an intricate question: that of the identity of objects through change due to replacement actions. Some systems of ojects have an intrinsic relation of identity. For example, in the system of hereditarily finite sets (Example 3) identity of objects is pure extensional identity of sets. When either intensional objects or physical objects such as artifacts constitute the universe of a system of objects identity is not a straightforward issue. One option is to postulate some relation R of identity on the system and then modify the presentation of a system so as to axiomatize the interaction of identity and replacement. Another option, which is the one we take here, is to classify various notions of identity that naturally arise in the system itself. What we are concerned with can, perhaps, better be described in a pragmatic and experimental language. Given a system of objects we can "experiment" with them with the means that we have available and that is to say by replacing components. What we would like to have is some notion of an abstract behavior of an object through this experimentation. Roughly then, we can construe two objects as being of the same type, identical, if they exhibit the same abstract behavior.

As it turns out there are different notions of identity we can formulate. We discuss two natural options below. To simplify the discussion we often avoid explicit mention of what the domain of a replacement map σ is when this can be unambiguously inferred from the context of the discussion. For example, in stating something like $\forall\sigma\exists\tau\ \sigma.a = \tau.b$ we really mean to say that for any permissible σ with domain the set Ca, there exists a permissible τ with domain Cb such that $\sigma.a = \tau.b$.

By abstact identity of objects in a system $\mathcal{A}$ we mean identity of the behaviors of the objects under the operation of replacement. A general notion of abstract identity may be taken to be a binary relation $\sim$ on A such that

$$a \sim b \text{ iff } (\forall\sigma\exists\tau\ \sigma.a \sim \tau.b \text{ and } \forall\tau\exists\sigma\ \sigma.a \sim \tau.b) \tag{1}$$

Call this double implication the condition (I). The intuition should be clear: Two objects a and b are to be deemed abstractly identical just in case every way to change one of them by a permissible replacement of components can be matched by a way to change the other, resulting again in abstractly identical objects. Note that identity of abstract behaviors is thus dependent on the collection S of available replacement experiments. This concept of identity is very broad and it covers a number of particular cases.

Example 7 In [12], two objects a, b are deemed of the same type, denoted by $a \simeq b$ just in case there exist instantiations σ and τ such that $a = \sigma.b$ and $b = \tau.a$. Identity is thus construed as the possibility for mutual reduction of each object to the other by replacement of components. Assuming we have a broad notion of identity $\sim$ satisfying condition (I), it is immediate that $a \simeq b$ implies $a \sim b$, for any objects a and b.

There is a notion of *bisimilarity* in the literature on process algebras that our concept of abstract identity is a generalization of. Unfortunately, as process languages cannot be described as systems of objects we cannot make the connection more clear.

Obviously, now, we cannot take (1) as a definition of $\sim$ because of the circularity involved. However, there is a standard way around this problem.

Definition 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be systems of objects (not necessarily over the same class of components). A binary relation $\mathcal{R}$ from A to B is a* pre-identity *iff for any $a \in A$ and $b \in B$, $a\mathcal{R}b$ implies*

- $\forall\sigma\exists\tau\ \sigma.a\mathcal{R}\tau.b$, *and*
- $\forall\tau\exists\sigma\ \sigma.a\mathcal{R}\tau.b$.

Let $\mathcal{F}$ be the operator on binary relations R from A to B defined by

$$\mathcal{F}(R) = \{(a,b)|\forall\sigma\exists\tau\ \sigma.aR\tau.b \text{ and } \forall\tau\exists\sigma\ \sigma.aR\tau.b\}$$

Then $\mathcal{F}$ is clearly monotone and a relation R is a pre-identity just in case $R \subseteq \mathcal{F}(R)$. Let $\sim$ be the largest fixed point of $\mathcal{F}$. Explicitly,

$$\sim = \bigcup\{R \subseteq A \times B|\ R \subseteq \mathcal{F}(R)\}$$

Lemma 2.3 *The relation $\sim$ is a (in fact, the largest) pre-identity and it satisfies condition (I).*

Proof: That $\sim$ is a pre-identity follows from the way we constructed this relation. For condition (I), the direction from left to right is straightforward. For the converse, let R be the binary relation defined by

$$aRb \text{ iff } \forall\sigma\exists\tau\ \sigma.a \sim \tau.b \text{ and } \forall\tau\exists\sigma\ \sigma.a \sim \tau.b$$

It is enough to verify that R is a pre-identity. So assume aRb holds. Given σ, let τ be such that $\sigma.a \sim \tau.b$. Then we have that for any σ' there is some τ' such that $\sigma'.(\sigma.a) \sim \tau'.(\tau.b)$. Conversely, for any τ' we can find σ' such that $\sigma'.(\sigma.a) \sim \tau'.(\tau.b)$. Thus $\sigma.aR\tau.b$ holds by definition of R. We may then conclude that R is a pre-identity. Hence $\sim$ satisfies condition (I). $\square$

Identity as $\sim$ is a very broad notion and it is probably best to think of it as reflecting structural similarity of objects. Objects that are identified by $\sim$ need not even have the same components and this is perhaps too liberal a notion of identity for many examples. Consider Example 3. It should be clear that the relation of equinumerosity of (hereditarily finite) sets satisfies condition (I). In a sense, then, it abstracts away too much structure.

We now define a more stringent notion of abstract identity, requiring that identity in the new sense implies that the two objects are built on the same set of components (but of course not at the same time, if temporal considerations are relevant to some particular case, like the Lego toy example). For our new notion of identity, denoted by $\equiv$, we would like to have

$$a \equiv b \text{ iff } Ca = Cb \text{ and } \forall \sigma\ \sigma.a \equiv \sigma.b \tag{2}$$

We call this condition (II). As this cannot be taken for a definition of $\equiv$, we proceed again as we did for the relation $\sim$.

Definition 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be systems of objects. A binary relation Ψ from A to B is a* congruence *iff $a\Psi b$ implies that $Ca = Cb$ and for all σ, $\sigma.a\Psi\sigma.b$.*

If $\mathcal{G}$ is the operator on binary relations from A to B such that

$$\mathcal{G}(R) = \{(a,b) |\ Ca = Cb \text{ and } \forall \sigma\ \sigma.aR\sigma.b\}$$

then clearly $\mathcal{G}$ is monotone and a relation Ψ is a congruence just in case $\Psi \subseteq \mathcal{G}(\Psi)$. We then let $\equiv$ be the largest fixpoint of $\mathcal{G}$. Explicitly,

$$\equiv = \bigcup \{\Psi \subseteq A \times B | \Psi \subseteq \mathcal{G}(\Psi)\}$$

As for the relation $\sim$ we can verify (by similar argument) the following lemma.

Lemma 2.5 *The relation $\equiv$ is a (in fact, the largest) congruence from A to B and a refinement of $\sim$. Furthermore, it satisfies condition (II).* $\square$

To get some intuition on what sense of identity is captured by $\sim$ and $\equiv$ we return to the Lego toy example (Example 4). There are two intuitive notions of abstract identity we can have. We may say that temporal instances of the "same" object are to be identified. We alluded to that when we first described the example. Given some object constructed from the basic items

x, y, z, decompose the object and then recompose "it" again at a different time, using exactly the same basic items x, y, z. Regarding objects strictly as particulars as we do, we are forced to see the two instances as distinct objects. This leaves us with the need for a notion of abstract identity that counts the two instances as the same object.

The second intutive notion of identity is that of two objects being copies of each other. We may say that the blocks a and b are copies of each other when they have exactly the same structure and they are composed by basic items that are copies of each other. Basic items are to be considered copies of each other if they have the same shape, but not necessarily the same colour (of course we can change the convention and require sameness of colour as well).

Given the constraints we have imposed on replacement maps it should be clear that the relation $\sim$ formalizes the second intuitive notion of abstract identity, which is probably better described as structural similarity. On the other hand, $\equiv$ captures the first, more stringent notion of identity.

Some Other Options: If the system $\mathcal{A} = (A, C, S, .)$ is a universe (a system of objects over A), then it is natural to consider a further notion of identity, in-between $\equiv$ and $\sim$.

Definition 2.6 *Define a relation $R \subseteq A \times A$ to be a* partial-identity *if aRb implies*

- $\forall a' \in Ca \exists b' \in Cb \; a'Rb'$
  $\forall b' \in Cb \exists a' \in Ca \; a'Rb'$
- $\forall \sigma \exists \tau \; \sigma.aR\tau.b$
  $\forall \tau \exists \sigma \; \sigma.aR\tau.b$

We may then let $\approx$ be the union of all partial identities and verify that $\approx$ itself is a partial identity. It is clear that $\equiv \subseteq \approx \subseteq \sim$.

We can also relativize the definitions of pre-identity and congruence for arbitrary systems to some given relation $r \subseteq X \times Y$ on components. Thus a pre-identity can be defined as a relation $R \subseteq (A \cup X) \times (B \cup Y)$ such that xRy iff xry and then by requiring that aRb satisfies conditions similar to these of Definition 2.6. We will not explore these notions further but perhaps they may be useful for potential applications.

2.2 Transformations of Systems of Objects

Our concept of a permissible transformation of a system $\mathcal{A}$ (over some class X) to a system $\mathcal{B}$ (over some class Y) must reflect what we perceive to be important in the structure of systems of objects. Hence transformations must be well-behaved with respect to components and replacement and they must reflect abstract identity of objects. We do not require that transformations should only be allowed for systems over the *same* class of components as this seems to be an undue restriction. This complicates the question of specifying what a transformation should be, since we have to provide both a map θ taking an object $a \in A$ to some object $\theta a \in B$ as well as a map that changes components from X to such from Y. There is the option of doing the latter globally, by assuming a map $i : X \to Y$, or locally by assuming a family of maps i_a, one for each object $a \in A$, such that $i_a : C_A a \to Y$. As the global option is a special case of the pointwise option (take the restriction $i_a = i|_{Ca}$) we prefer to first describe the general notion of a transformation.[3] The most significant difference of the two views is that in the global view i is taken to be a function from X to Y while in the local view it is a relation $i \subseteq X \times Y$ allowing for the same component to be changed in different ways depending on the object it is a component of.

Definition 2.7 *A* **transformation** *(homomorphism) of systems of objects* $\mathcal{A} = (A, C_A, S_A, \cdot_A) \to \mathcal{B} = (B, C_B, S_B, \cdot_B)$*, over classes* X *and* Y *respectively, is a pair* (θ, i) *such that* $\theta : A \to B$ *and* i *is a family of maps* $i = (i_a)_{a \in A}$*, where* $i_a : C_A a \to Y$*, and the following hold:*

- *preservation of components:* $C_B(\theta a) = i_a(C_A a) = \{i_a(x) | \; x \in C_A a\}$
- *for each* $a \in A$ *and map* $\sigma \in S_A$*,* $\sigma : C_A a \to X$*, there is a unique map* $\sigma^i \in S_B$*,* $\sigma^i : C_B(\theta a) \to Y$ *satisfying* $i_{\sigma.a} \circ \sigma = \sigma^i \circ i_a$*, and*
- *preservation of replacement:* $\sigma^i \cdot_B (\theta a) = \theta(\sigma \cdot_A a)$*.*

Composition $\mathcal{A} \xrightarrow{(\theta,i)} \mathcal{B} \xrightarrow{(\phi,j)} \mathcal{C}$ *is defined by* $(\phi, j) \circ (\theta, i) = (\phi\theta, ji)$*, where* $(ji)_a = j_{\theta a} \circ i_a$*.*

The conditions of preservation of components and of replacement should be intuitively clear. The second condition is a technical requirement as we need to make sure that components are changed in a coherent way.

[3] There are also some technical arguments for the local version, based on some simple category-theoretic considerations, detailed in [8].

Remark 2.8 Our definition of a legitimate transformation imposes strong restrictions. It is justified by our desire to investigate transformations that respect both components and replacement while also reflecting abstract identity of objects. There are of course contexts where the objectives may be different. For example, intuitively every replacement map σ may be thought of as inducing a transformation of a system of objects. Given a system of objects over X consider all maps π, where $dom(\pi)$ is a subset of X. Given an object a, let $\pi.a = \pi|_a.a$ where

$$\pi|_a x = \begin{cases} \pi x & \text{if } x \in dom(\pi) \cap Ca \\ x & \text{if } x \in Ca \setminus dom(\pi) \end{cases}$$

For the purpose of this remark we may drop the restriction to permissible maps. Then we can think of π as inducing a transformation of the system $\mathcal{A}$, $\pi : \mathcal{A} \to \mathcal{B}$, where $B = \{\pi.a | a \in A\}$ and the replacement operation and component maps in $\mathcal{B}$ are as in $\mathcal{A}$. The transformation is the pair (π, π), where π is once viewed as a map acting on objects and then also as a map changing components $i = \pi = (\pi|_a)_{a \in A}$. Preservation of components is no problem. But an arbitrary replacement map π will fail in general our definition of a legitimate transformation as we will not be able to find the unique map π^i required in the definition. We may relax requirements as follows. Given an arbitrary object a and a replacement map $\sigma : Ca \to X$, call π compatible with σ iff

$$\forall x, y \in Ca \; \pi x = \pi y \Longrightarrow \pi(\sigma x) = \pi(\sigma y)$$

Given a and σ the assumption of compatibility implies that there exists a map π^i as required in our definition. Simply let $\pi^i : C(\pi.a) \to X$ be the map $\pi^i(\pi x) = \pi(\sigma x)$.

It may be of interest to relax the definition of a legitimate transformation by making preservation of replacement depend on some compatibility condition. This approach is taken in [4]. We will maintain here the requirement for strict preservation of replacement for two reasons. First, replacement is the backbone of the structures we have called systems of objects. The components map is secondary and it arises only because we regard replacement maps concretely as functions. The abstract structure of a system of objects consists in some monoid-like set S of items we call replacement maps and an action of S to a set A of structured objects $S \times A \to A$. If anything is to be preserved then it seems that this should be the action of replacement.

On the other hand, there is no compelling reason why we should want to model our notion of transformation on the behavior of replacement maps when considered as transformations. The functional behavior of replacement maps in the way components are changed is not always desirable as our next example demonstrates.

Example 8 Consider a collection of human individuals, the citizens of an imaginary state, a fixed set of tasks to be accomplished and committees formed to undertake these tasks. The structured objects we consider are all the possible committees that can be formed for the given set of tasks.The components map delivers the set of individuals making up a committee. Membership of an individual to a committee changes over time for various reasons. We assume that every citizen is eligible for membership to any committee and thus all replacement maps are permissible. Suppose Charles Smith is in the committees for energy preservation and for the protection of the environment and that he wishes to resign from both. After replacing Mr Smith from these two committees we have a new system of committees. In the global view of transformations of systems of objects Charles Smith should be replaced by the *same* individual in both committees. This seems to be unduely restrictive, however, hence there is potential usefulness in considering the more general class of transformations we have described in Definition 2.7.

We have gone a good way towards satisfying our requirements as the maps we have described as legitimate transformations are well-behaved with respect to both components and replacement. But we would also like for a legitimate transformation θ to reflect identity of abstract behavior of objects, in the sense that for any two objects $a, a' \in A$, $\theta a = \theta a'$ only if a and a' are abstractly identical. This is a minimal criterion by which transformations respect identity of objects. The reason for the failure of reflecting identities is that the second condition in the definition is too weak. We strengthen it in the following:

Definition 2.9 $(\theta, i) : \mathcal{A} \to \mathcal{B}$ *is a* full transformation *if it satisfies the conditions of Definition 2.7 and, in addition, the following holds*

- *for all replacement maps* $\rho \in S_B$, *if* $\rho : C_B(\theta a) \to Y$ *for some* $a \in A$, *then there is some replacement map* $\sigma \in S_A$ *such that* $\sigma : C_A a \to X$ *and* $\rho ._B (\theta a) = \theta(\sigma ._A a)$.

Of course in the light of the other conditions on transformations this is equivalent to saying that every replacement map $\rho \in S_B$ defined on the components of an object of the form θa in B is of the form σ^i, for some $\sigma \in S_A$. In other words, in transforming a system of objects abstract identity is reflected provided that in the system $\{\theta a | a \in A\}$ no "new" experiments have been added. Every replacement experiment on θa is the reflection of some replacement experiment on a. Though strong, the requirement seems to be natural.

Lemma 2.10 *Full transformations respect abstract identity in the sense that $\theta a = \theta b$ only if $a \sim b$.*

Proof: Enought to show that the relation R defined by aRb iff $\theta a = \theta b$ is a pre-identity. Suppose aRb and let $\sigma : Ca \to X$. Then $\theta(\sigma.a) = \sigma^i.(\theta a) = \sigma^i.(\theta b)$. Let $\sigma^i = \rho$. Since θ is full, there is a replacement map $\tau : Cb \to X$ such that $\rho = \tau^i$. Thus, $\theta(\sigma.a) = \rho.(\theta b) = \theta(\tau.b)$, hence $\sigma.aR\tau.b$. $\square$

The refinement $\equiv$ of $\sim$, however, is not necessarily respected even by full transformations. Thus, if for certain applications $\equiv$ is our desired notion of identity of abstract behavior then further restrictions need to be imposed.

Definition 2.11 $(\theta, i) : \mathcal{A} \to \mathcal{B}$ *is a* normal transformation *if it satisfies the conditions of Definition 2.7 and, in addition, i is an injective function $i : X \hookrightarrow Y$.*

Lemma 2.12 *Normal transformations reflect $\equiv$-identity of objects.*

Proof: Enough to show that the relation R defined by aRb iff $\theta a = \theta b$ is a congruence. If $\theta a = \theta b$, then $C(\theta a) = \{ix | x \in Ca\} = \{iy | y \in Cb\} = C(\theta b)$. Since i is an injection $Ca = Cb$ follows. If $\sigma : Ca \to X$, let $\tau = \sigma$ and observe that $\theta(\sigma.a) = \theta(\sigma.b)$, hence we may conclude that $\sigma.aR\tau.b$ holds. Thus $a \equiv b$. $\square$

When restricting to systems over the same class X of components it is useful to consider a special class of transformations, defined below.

Definition 2.13 *A* standard *transformation $\mathcal{A} \to \mathcal{B}$ of systems over the same class X is a transformation (θ, i) where $i_a = id_{Ca}$, for each $a \in A$.* $\square$

When referring to standard transformations we will not make mention of the map i (since $i_a = id_{Ca}$) and regard it simply as a map $\theta : \mathcal{A} \to \mathcal{B}$.

Lemma 2.14 *Standard transformations reflect $\equiv$-identity of objects.* $\square$

3 Representation of Systems of Objects

We develop in this section the algebraic theory of systems of objects, concluded with the Representation Theorem (Theorem 3.12) for systems of objects over some fixed class X. We show that every system is isomorphic to a quotient of a restriction of a free ontology. We discuss first the operations of restriction and quotient. To make use of quotients we establish a Homomorphism Theorem (Theorem 3.4). We then turn to proving existence of free ontologies and form systems thus leading to our representation theorems.

Restriction: We will have use of two operations of restriction. The simplest one is to restrict to a class X of components, introduced in [2]. This operation will be very useful in the proof of the Representation Theorem for Form Systems 3.11. If $\mathcal{A}$ is a system over the class X of components then the restriction $\mathcal{A}|Y$ to a subclass $Y \subseteq X$ is the new system with universe of objects $A[Y] = \{a \in A | Ca \subseteq Y\}$ and permissible replacement maps $S[Y] = \{\sigma \in S | \sigma \subseteq Y \times Y\}$. The components map and the replacement operation are the obvious restrictions of the corresponding maps in $\mathcal{A}$. In particular, we will have use of the restriction of *ontologies*, that is form systems (all replacement maps are perimissible) over the class V of all objects.

For the second restriction operation define first a *partial monoid* of functions as a set of functions $\sigma : dom(\sigma) \rightarrow V$ such that if $\tau, \sigma \in S$ and $dom(\tau) = rng(\sigma)$, then $\tau\sigma \in S$. Furthermore, for each $\sigma \in S$, both the left and the right identities $id_{dom(\sigma)}, id_{rng(\sigma)}$ are in S.

Given a system $\mathcal{A} = (A, C, S_{_A}, .)$ and a partial monoid $S \subseteq S_{_A}$ let $\mathcal{A}|S$ be the system with

- universe of objects $\mathcal{A}[S] = \{a \in A | id_{Ca} \in S\}$
- components map $C_{_S}a = Ca$
- replacement $\sigma ._{_S} a = \sigma . a$

For representation purposes we will be only interested in restrictions $\mathcal{U}|S$ of ontologies to partial monoids. □

Quotient Systems: These are systems obtained by factoring out by congruences, Definition 2.4. Since systems may be large, that is to say their universes may be proper classes, we need to make sure that we have available some form of a quotient existence principle. Thus we assume global choice, which allows us to pick representatives from possibly proper classes of congruent objects. If $\mathcal{A}$ is a system of objects over some class X and Θ is a congruence on A we let $[a]_\Theta$ (or simply $[a]$ when no confusion is possible) be a representative of the congruence class of a. Since $a\Theta b$ implies $Ca = Cb$, we may let $C[a] = Ca$. Furthermore, since $a\Theta b$ implies that $\sigma.a\Theta\sigma.b$, for any permissible replacement map $\sigma : Ca \to X$, the replacement operation can be defined by $\sigma.[a] = [\sigma.a]$. Strictly speaking quotient systems are not unique. However, uniqueness up to isomorphism can be established.

Proposition 3.1 *Let $\mathcal{A}$ be a system and Θ a congruence on A. Let $[.]$ and $[.]'$ be choice functions selecting representatives of the congruence classes. Let $\mathcal{A}_\Theta$ and $(\mathcal{A}_\Theta)'$ be the two quotient systems obtained. Then the standard transformation $\phi : \mathcal{A}_\Theta \to (\mathcal{A}_\Theta)'$ is an isomorphism, where $\phi([a]) = [a]'$.*

Proof: The only interesting point is preservation of replacement. However, given $\sigma : C[a] \to X$, let $\sigma^i = \sigma$ and observe that

$$\phi(\sigma.[a]) = \phi([\sigma.a]) = [\sigma.a]' = \sigma.[a]' = \sigma.\phi([a])$$

The rest is immediate. □

In the sequel we will feel free to refer to *the* quotient system $\mathcal{A}_\Theta$ since any two such are isomorphic. □

Operations of product and disjoint sum can be defined in the natural way. In defining disjoint sum we have to take "copies" of the original systems to make sure that the operation of replacement in the new system is well defined. We point out the following:

Proposition 3.2 *If $j : X \simeq Y$ is a bijection and $\mathcal{U}$ is an ontology, then there is a normal isomorphism $\mathcal{U}|X \simeq \mathcal{U}|Y$.*

Proof: Let $(\theta, j) : \mathcal{U}|X \to \mathcal{U}|Y$ be defined by $j_a = j|_{Ca}$ and $\theta a = j_a.a$ for each $a \in U[X]$. Since j is injective, given $\sigma : Ca \to V$ we can define $\sigma^j = j_{\sigma.a} \circ \sigma \circ (j_a)^{-1}$. Thus (θ, j) is a legitimate normal transformation. But so is also the map $(\phi, i) : \mathcal{U}|Y \to \mathcal{U}|X$, where $i = j^{-1}$ and $\phi b = (j^{-1})_b.b$. Given $a \in U[X], b \in U[Y]$ we clearly have $\phi\theta a = a$ and $\theta\phi b = b$, hence $\mathcal{U}|X \simeq \mathcal{U}|Y$. □

Corollary 3.3 *Assume global choice. Then for every ontology $\mathcal{U}$ and cardinal κ the form system $\mathcal{U}|\kappa$ is the unique, up to normal isomorphism, X-form* *system with $|X| = \kappa$. In particular, if X is a proper class, then there is a normal isomorphism $\mathcal{U} \simeq \mathcal{U}|X$.* □

In view of the Representation Theorem, form systems over sets X, Y of the same cardinality are normally isomorphic.

We turn now to establishing a Homomorphism Theorem (Theorem 3.4). Next we prove existence of free systems (Theorem 3.6, 3.9) for an appropriate notion of freedom (Definition 3.5).

Theorem 3.4 (Homomorphism Theorem) *Let $\theta : \mathcal{A} \to \mathcal{B}$ be a standard transformation of systems over the class X of components and Θ a congruence on A such that $\Theta \subseteq ker(\theta)$, that is to say $a\Theta a'$ implies $\theta a = \theta a'$. Let $\mathcal{A}_\Theta$ be the quotient by Θ and $\pi : \mathcal{A} \to \mathcal{A}_\Theta$ the standard epimorphism $\pi a = [a]$. Then there exists a unique standard transformation $\hat{\theta} : \mathcal{A}_\Theta \to \mathcal{B}$ such that $\hat{\theta} \circ \pi = \theta$.*

Furthermore, $\hat{\theta}$ is an isomorphism iff θ is surjective and $\Theta = ker(\theta)$.

Proof: The transformation $\hat{\theta}$ is simply defined by $\hat{\theta}([a]) = \theta a$. By the assumption that $\Theta \subseteq ker(\theta)$, $\hat{\theta}$ is well-defined. Now suppose that $\hat{\theta} : \mathcal{A}_\Theta \cong \mathcal{B}$. Then clearly θ must be surjective. If $\theta a = \theta b$, then $\hat{\theta}([a]) = \hat{\theta}([b])$, hence $[a] = [b]$, that is to say $a\Theta b$ holds. The converse is immediate, too. □

By a *signature* we mean, as in [2], a pair (Ω, α) where Ω is a class and for each $\omega \in \Omega$, $\alpha\omega$ is a set. However, for technical reasons we need to also consider here transformations of signatures, which we define by analogy to transformations for systems of objects. Thus $(\theta, i) : (\Omega, \alpha) \to (\Omega', \alpha')$ is a morphism of signatures if $\theta : \Omega \to \Omega'$ and $i = (i_\omega)_{\omega\in\Omega}$ is a collection of maps $i_\omega : \alpha\omega \to V$ such that $\alpha'(\theta\omega) = \{i_\omega x | x \in \alpha\omega\}$. A *standard* morphism of signatures is a morphism (θ, i) where $i_\omega = id_{\alpha\omega}$. When (θ, i) is standard we simply refer to it as the morphism $\theta : (\Omega, \alpha) \to (\Omega', \alpha')$. Technically, we have two distinct categories of signatures, depending on what signature-maps we consider. We let Sgn be the category of signatures with standard signature morphisms and Sgn* the category of signatures with the more general notion of map described above. Similarly, we let On be the category of ontologies with standard ontology transformations and On* the category of ontologies with the more general notion of transformation.

3.1 Ontologies

For a given signature (Ω, α), the *signature ontology* $\mathcal{U}_\Omega$ is defined in [2] as the ontology $\mathcal{U}_\Omega = (\Omega[V], C, .)$, where

- $\Omega[V] = \{(\omega, f) | \omega \in \Omega \text{ and } f : \alpha\omega \to V\}$
- $C(\omega, f) = \{fx | x \in \alpha\omega\} = rng(f)$
- $\sigma.(\omega, f) = (\omega, \sigma f)$, if $\sigma : rng(f) \to V$.

Given any system $\mathcal{A} = (A, C, S, .)$ of objects we may take (A, C) as its underlying signature. We denote the map $\mathcal{A} \mapsto (A, C)$ by $|\mathcal{A}|$. Note that the map $|.|$ acts on transformations of systems, too, delivering transformations of signatures (by just forgetting properties about replacement). Specifying in our particular context the notion of free objects we have the following:

Definition 3.5 *An ontology $\mathcal{U}$ in* ON* *is* free over a signature (Ω, α) *in* SGN* *if there is a signature map $(\phi, i) : (\Omega, \alpha) \to |\mathcal{U}|$ such that for any ontology $\mathcal{U}'$ and signature map $(\theta, j) : (\Omega, \alpha) \to |\mathcal{U}'|$ there is a unique ontology transformation $(\hat{\theta}, \hat{j}) : \mathcal{U} \to \mathcal{U}'$ such that $(\hat{\theta}, \hat{j}) \circ (\phi, i) = (\theta, j)$. Similarly for $\mathcal{U}$ in* ON *and (Ω, α) in* SGN, *in which case we restrict to standard morphisms.*

Theorem 3.6 (Free Ontologies) *For every signature (Ω, α) there is an ontology $\mathcal{U}$ free over (Ω, α).*

Proof: We give the proof for the case where general ontology and signature morphisms are considered. The proof for the restriction to standard transfromations is similar and simpler.

Given (Ω, α), let $\mathcal{U}_\Omega$ be the signature ontology and let $(\phi, i) : (\Omega, \alpha) \to |\mathcal{U}_\Omega|$ be the map $j_\omega = id_{\alpha\omega}$ and $\phi\omega = (\omega, j_\omega)$.

Now let $\mathcal{U}$ be any ontology and $(\theta, j) : (\Omega, \alpha) \to |\mathcal{U}|$ a signature map. Define the ontology transformation $(\hat{\theta}, \hat{j}) : \mathcal{U}_\Omega \to \mathcal{U}$ by $\hat{j}_{(\omega,\sigma)} = j_\omega$, if $\sigma = id_{\alpha\omega}$ and otherwise let $\hat{j}_{(\omega,\sigma)} = id_{rng(\sigma)}$.

Define also $\sigma^{\hat{j}} : C(\phi\omega) \to V$ by $\sigma^{\hat{j}} \circ j_\omega = \sigma$, if $\sigma \neq id_{\alpha\omega}$ and otherwise let $\sigma^{\hat{j}} = id_{rng(\sigma)}$.

Finally, define $\hat{\theta}(\omega, \sigma) = \sigma^{\hat{j}}.\theta\omega$.

Verification that $(\hat{\theta}, \hat{j})$ is an ontology transformation is immediate and the equation $(\hat{\theta}\phi, \hat{j}i) = (\theta, j)$ is easily seen to hold. Uniqueness of the ontology transformation $(\hat{\theta}, \hat{j})$ with the prescribed property is also easy to see. $\square$

By uniqueness of free objects, up to isomorphism, when they exist we can conclude that

Corollary 3.7 *The free ontologies over a signature* (Ω, α) *are exactly the ontologies isomorphic to the signature ontology* $\mathcal{U}_\Omega$. $\square$

Theorem 3.8 (Ontology Representation) *For every ontology* $\mathcal{U}$ *there is a signature* (Ω, α) *and a congruence* Θ *on the signature ontology* $\mathcal{U}_\Omega$ *such that there is a standard isomorphism* $\mathcal{U} \cong \mathcal{U}_{\Omega,\Theta}$.

Proof: $\mathcal{U}_{\Omega,\Theta}$ is the quotient of the signature ontology $\mathcal{U}_\Omega$ when factored out by the congruence Θ. For the proof, given an ontology $\mathcal{U} = (U, C, .)$ let (Ω, α) be the signature $|\mathcal{U}| = (U, C)$ and $\mathcal{U}_\Omega$ the signature ontology. Since the identity is a morphism $(\Omega, \alpha) \to |\mathcal{U}|$ and $\mathcal{U}_\Omega$ is free over (Ω, α) there must be a (unique) morphism $(\hat{\theta}, \hat{j}) : \mathcal{U}_\Omega \to \mathcal{U}$. It is easy to see that this morphism is surjective. Let then $\Theta = ker(\hat{\theta})$. By the Homomorphism Theorem (Theorem 3.4) it follows that $\mathcal{U}_{\Omega,\Theta} \cong \mathcal{U}$. It is also clear that $(\hat{\theta}, \hat{j})$ is a standard morphism since both the identity $(\Omega, \alpha) \to |\mathcal{U}|$ and the morphism $(\phi, i) : (\Omega, \alpha) \to |\mathcal{U}_\Omega|$ are standard. $\square$

3.2 Form Systems

We dealt with ontologies first because this case is quite simple. In this section we turn to considering form systems over some fixed class X. Again, depending on what transformations we consider we distinguish between the categories X-Fs, with standard transformations, and X-Fs*, with the general notion of transformation. To prove existence of free form systems and representation we restrict the class of signatures to the X-*bounded* signatures, that is to say signatures (Ω, α) such that for each $\omega \in \Omega$, $\alpha\omega$ can be injected into X. If $\mathcal{A} = (A, C, .)$ is a form system over X, then its underlying signature $|\mathcal{A}| = (A, C)$ is obviously X-bounded. The definition of what it means for a form system (over X) to be free over an X-bounded signature (Ω, α) is completely analogous to Definition 3.5. Without further ado we state and prove:

Theorem 3.9 (Free Form Systems) *For every* X*-bounded signature* (Ω, α)*, there is an* X*-form system* $\mathcal{A}$ *free over* (Ω, α).

Proof: If (Ω, α) is X-bounded, we may in fact assume that $\alpha\omega \subseteq X$, for each $\omega \in \Omega$. For if not let $i_\omega : \alpha\omega \hookrightarrow X$ be the injections and consider the signature (Ω, α'), where $\alpha'\omega = \{\imath_\omega x | x \in \alpha\omega\}$. The two signatures are isomorphic and so we may as well assume at the outset that $\alpha\omega \subseteq X$.

Given the signature (Ω, α), let $\mathcal{U}_\Omega$ be the signature ontology free over (Ω, α) and consider the restriction $\mathcal{U}_\Omega|X$. By a completely analogous argument to that in the proof of Theorem 3.6 we can verify that $\mathcal{U}_\Omega|X$ is free over the X-bounded signature (Ω, α). □

Corollary 3.10 *The free X-form systems over the X-bounded signature (Ω, α) are exactly the systems isomorphic to the system $\mathcal{U}_\Omega|X$.* □

Theorem 3.11 (Representation of Form Systems) *For every X-form system $\mathcal{A}$, there is an X-bounded signature (Ω, α) and a congruence Θ on the system $\mathcal{U}_\Omega|X$ such that there is a standard isomorphism $\mathcal{A} \cong (\mathcal{U}_\Omega|X)_\Theta$.*
□

3.2.1 General Systems of Objects

We will prove here directly a representation theorem without detouring through a proof of existence of free systems. We can define a suitable notion of an S-bounded signature, for a partial monoid S, as a signature (Ω, α) such that for every $\omega \in \Omega$ the trivial replacement map $\sigma = id_{\alpha\omega} \in S$. We can then proceed, in principle at least, as we did for the case of ontologies and form systems and derive a result on free systems of objects over a given S-bounded signature. The interested reader might want to carry out the details. Here we constrain ourselves to the following:

Theorem 3.12 (Representation of Systems of Objects) *For every system $\mathcal{A} = (A, C, S, .)$, there is a (in fact, an S-bounded) signature (Ω, α) and a congruence Θ on the restriction $\mathcal{U}_\Omega|S$ such that there is a standard isomorphism $\mathcal{A} \cong (\mathcal{U}_\Omega|S)_\Theta$.*

Proof: Given $\mathcal{A}$, let $|\mathcal{A}| = (\Omega, \alpha)$ be its underlying signature (A, C) and consider the restriction $\mathcal{U}_\Omega|S$ of the signature ontology $\mathcal{U}_\Omega$. The universe of objects in $\mathcal{U}_\Omega|S$ consists of pairs (a, σ), $a \in A = \Omega$ and $dom(\sigma) = \alpha a = Ca$. Let $\pi : \mathcal{U}_\Omega|S \to \mathcal{A}$ be the map $\pi(a, \sigma) = \sigma.a$. Then π is a standard morphism. Satisfaction of the requirement for components of Definition 2.7 is obviously satisfied since

$$C'(a, \sigma) = rng(\sigma) = \{\sigma x | x \in \alpha a = Ca\}$$

Given $\tau \in S$ with $dom(\tau) = rng(\sigma)$, $\pi(\tau.(a, \sigma)) = \pi(a, \tau\sigma) = \tau.(\sigma.a)$. Hence a map $\tau^i = \tau$ exists such that the replacement requirement of Definition

2.7 is satisfied. In fact τ is the unique such map since if ρ were another one it should satisfy $i_{\tau.a} \circ \tau = \rho \circ i_a$. Given that π is a standard map, the components maps i_a are identities and thereby $\rho = \tau$.

Now clearly π is surjective, since for each $a \in A$ the pair (a, id_{Ca}) is in the universe $\mathcal{U}_\Omega[S]$ of the system $\mathcal{U}_\Omega|S$. Let then $\Theta = ker(\pi)$. By the Homomorphism Theorem 3.4 it follows that π is a standard isomorphism $\pi : \mathcal{A} \cong (\mathcal{U}_\Omega|S)_\Theta$. □

Summary

We have developed a model for our pre-theoretic intuitions of structured objects subject to change under permissible replacement of components. Our notion of a system of objects generalizes that of a form system presented in [2]. We approached the question of identity of objects through change describing the question in an experimental-like language. The general idea is that objects are to be classified as of the same type (abstractly identical) if they exhibit the same abstract behavior under replacement experiments. We distinguished some notions of identity, $\equiv, \approx$ and $\sim$, where $\equiv$ is a refinement of $\approx$ and $\approx$ a refinement of $\sim$. Systems of objects are, themselves, entities subject to change. We introduced a broad notion of permissible transformations that respect both components and replacement experiments. We also investigated further restrictions on transformations that will guarantee that abstract identity of objects is reflected. Systems of objects can be regarded as *replacement algebras*. It is then natural to raise some purely algebraic questions, such as the question of representation, also raised in [2]. For ontologies and form systems we obtained our representation results by essentially algebraic means, proving first existence of free systems and a homomorphism theorem. A representation theorem for form systems was first given in [2]. Our proof is different (and much shorter!). We also generalized the result here to a representation for arbitrary systems of objects. An essentially algebraic development for ontologies was also started in an Appendix in [2]. Ontologies are there regarded as some kind of many-sorted algebras. We have taken a much simpler approach here that, nevertheless, allows us to recapture and strengthen results of [2].

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