On a Surprising Oversight by John S. Bell in the Proof of his Famous Theorem

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Bell inequalities are usually derived by assuming locality and realism, and therefore violations of the Bell-CHSH inequality are usually taken to imply violations of either locality or realism, or both. But, after reviewing an oversight by Bell, in the Corollary below we derive the Bell-CHSH inequality by assuming only that Bob can measure along vectors $\mathbf{b}$ and $\mathbf{b}'$ simultaneously while Alice measures along either $\mathbf{a}$ or $\mathbf{a}'$, and likewise Alice can measure along vectors $\mathbf{a}$ and $\mathbf{a}'$ simultaneously while Bob measures along either $\mathbf{b}$ or $\mathbf{b}'$, *without assuming locality.* The violations of the Bell-CHSH inequality therefore only mean impossibility of measuring along $\mathbf{b}$ and $\mathbf{b}'$ (or along $\mathbf{a}$ and $\mathbf{a}'$) simultaneously.

Consider – in slightly modern terms – the standard EPR-Bohm type experiment envisaged by John S. Bell to prove his famous theorem [1]. Alice is free to choose a detector direction $\mathbf{a}$ or $\mathbf{a}'$ and Bob is free to choose a detector direction $\mathbf{b}$ or $\mathbf{b}'$ to detect spins of the fermions they receive from a common source, at a space-like distance from each other. The objects of interest then are the bounds on the sum of possible averages put together in the manner of CHSH [2],

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\,, \tag{1}$$

with each average defined as

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,=\,\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^k)\;\mathscr{B}(\mathbf{b},\,\lambda^k)\right]\,\equiv\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,, \tag{2}$$

where $\mathscr{A}(\mathbf{a},\,\lambda^k)\equiv\mathscr{A}_k(\mathbf{a})=\pm1$ and $\mathscr{B}(\mathbf{b},\,\lambda^k)\equiv\mathscr{B}_k(\mathbf{b})=\pm1$ are the respective measurement results of Alice and Bob. Now, since $\mathscr{A}_k(\mathbf{a})=\pm1$ and $\mathscr{B}_k(\mathbf{b})=\pm1$, the average of their product is $-1\leqslant\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\leqslant+1$. As a result, we can immediately read off the upper and lower bounds on the string of the four averages considered above in (1):

$$-\,4\,\leqslant\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,+\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,+\,\Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,-\,\Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,\leqslant\,+4\,. \tag{3}$$

This should have been Bell's final conclusion. But he continued. And in doing so he overlooked something that is unjustifiable. He replaced the above sum of four separate averages of real numbers with the following single average:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})+\mathcal{E}(\mathbf{a},\,\mathbf{b}')+\mathcal{E}(\mathbf{a}',\,\mathbf{b})-\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\;\longrightarrow\;\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})+\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')+\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})-\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle. \tag{4}$$

As innocuous as this step may seem mathematically, it is in fact an illegitimate step physically, because what is being averaged on its RHS are *unobservable* and *unphysical* quantities. But it allows us to reduce the sum of four averages to

$$\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\big\{\,\mathscr{B}_k(\mathbf{b})+\mathscr{B}_k(\mathbf{b}')\,\big\}\,+\,\mathscr{A}_k(\mathbf{a}')\,\big\{\,\mathscr{B}_k(\mathbf{b})-\mathscr{B}_k(\mathbf{b}')\,\big\}\,\Big\rangle. \tag{5}$$

And since $\mathscr{B}_k(\mathbf{b})=\pm1$, if $|\mathscr{B}_k(\mathbf{b})+\mathscr{B}_k(\mathbf{b}')|=2$, then $|\mathscr{B}_k(\mathbf{b})-\mathscr{B}_k(\mathbf{b}')|=0$, and vice versa [3]. Consequently, using $\mathscr{A}_k(\mathbf{a})=\pm1$, it is easy to conclude that the absolute value of the above average cannot exceed 2, just as Bell concluded:

$$-\,2\,\leqslant\,\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,+\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,+\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,-\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,\leqslant\,+2\,. \tag{6}$$

Let us now try to understand why the replacement (4) above is illegal. To begin with, Einstein's (or even Bell's own) notion of local-realism does not demand this replacement. Since this notion is captured already in the definition of the measurement functions $\mathscr{A}(\mathbf{a},\,\lambda^k)$, the LHS of (4) satisfies the demand of local-realism perfectly well. To be sure, mathematically there is nothing wrong with a replacement of four separate averages with a single average. Indeed, every school child knows that the sum of averages is equal to the average of the sum. But this rule of thumb is not valid

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in the above case, because ($\mathbf{a}$, $\mathbf{b}$), ($\mathbf{a}$, $\mathbf{b}'$), ($\mathbf{a}'$, $\mathbf{b}$), and ($\mathbf{a}'$, $\mathbf{b}'$) are *mutually exclusive pairs of measurement directions*, corresponding to four *incompatible* experiments. Each pair can be used by Alice and Bob for a given experiment, for all runs 1 to n, but no two of the four pairs can be used by them simultaneously. This is because Alice and Bob do not have the ability to make measurements along counterfactually possible pairs of directions such as ($\mathbf{a}$, $\mathbf{b}$) and ($\mathbf{a}$, $\mathbf{b}'$) simultaneously. Alice, for example, can make measurements along $\mathbf{a}$ or $\mathbf{a}'$, but not along $\mathbf{a}$ *and* $\mathbf{a}'$ at the same time.

But this inconvenient fact is rather devastating for Bell's argument, because it means that his replacement (4) is illegitimate. Consider a specific run of the EPR-B experiment and the corresponding quantity being averaged in (4):

$$\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}) \,+\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}') \,+\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}) \,-\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,. \tag{7}$$

Here the index $k = 1$ now represents a specific run of the experiment. But since Alice and Bob have only two particles at their disposal for each run, only one of the four terms of the above sum is physically meaningful. In other words, the above quantity is physically meaningless, because Alice, for example, cannot align her detector along $\mathbf{a}$ and $\mathbf{a}'$ at the same time. And likewise, Bob cannot align his detector along $\mathbf{b}$ and $\mathbf{b}'$ at the same time. What is more, this will be true for all possible runs of the experiment, or equivalently for all possible pairs of particles. Which implies that all of the quantities listed below, as they appear in the average (5), are unobservable, and hence physically meaningless:

$$\begin{array}{l}
\mathscr{A}_1(\mathbf{a})\,\mathscr{B}_1(\mathbf{b}) \,+\, \mathscr{A}_1(\mathbf{a})\,\mathscr{B}_1(\mathbf{b}') \,+\, \mathscr{A}_1(\mathbf{a}')\,\mathscr{B}_1(\mathbf{b}) \,-\, \mathscr{A}_1(\mathbf{a}')\,\mathscr{B}_1(\mathbf{b}')\,, \\
\mathscr{A}_2(\mathbf{a})\,\mathscr{B}_2(\mathbf{b}) \,+\, \mathscr{A}_2(\mathbf{a})\,\mathscr{B}_2(\mathbf{b}') \,+\, \mathscr{A}_2(\mathbf{a}')\,\mathscr{B}_2(\mathbf{b}) \,-\, \mathscr{A}_2(\mathbf{a}')\,\mathscr{B}_2(\mathbf{b}')\,, \\
\mathscr{A}_3(\mathbf{a})\,\mathscr{B}_3(\mathbf{b}) \,+\, \mathscr{A}_3(\mathbf{a})\,\mathscr{B}_3(\mathbf{b}') \,+\, \mathscr{A}_3(\mathbf{a}')\,\mathscr{B}_3(\mathbf{b}) \,-\, \mathscr{A}_3(\mathbf{a}')\,\mathscr{B}_3(\mathbf{b}')\,, \\
\mathscr{A}_4(\mathbf{a})\,\mathscr{B}_4(\mathbf{b}) \,+\, \mathscr{A}_4(\mathbf{a})\,\mathscr{B}_4(\mathbf{b}') \,+\, \mathscr{A}_4(\mathbf{a}')\,\mathscr{B}_4(\mathbf{b}) \,-\, \mathscr{A}_4(\mathbf{a}')\,\mathscr{B}_4(\mathbf{b}')\,, \\
\quad\cdot \\
\quad\cdot \\
\quad\cdot \\
\mathscr{A}_n(\mathbf{a})\,\mathscr{B}_n(\mathbf{b}) \,+\, \mathscr{A}_n(\mathbf{a})\,\mathscr{B}_n(\mathbf{b}') \,+\, \mathscr{A}_n(\mathbf{a}')\,\mathscr{B}_n(\mathbf{b}) \,-\, \mathscr{A}_n(\mathbf{a}')\,\mathscr{B}_n(\mathbf{b}')\,.
\end{array}$$

But since each of the quantities above is physically meaningless, their average appearing on the RHS of (4), namely

$$\Big\langle\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}) \,+\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}') \,+\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}) \,-\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,, \tag{8}$$

is also physically meaningless. That is to say, no physical experiment can ever be performed – *even in principle* – that can meaningfully allow to measure or evaluate the above average, since none of the above list of quantities could have experimentally observable values [4]. Therefore the innocuous looking replacement (4) made by Bell is, in fact, illegal.

On the other hand, it is important to note that each of the four averages appearing on the LHS of replacement (4),

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^k)\,\,\mathscr{B}(\mathbf{b},\,\lambda^k)\right] \,\equiv\, \Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,, \tag{9}$$

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}') \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\lambda^k)\,\,\mathscr{B}(\mathbf{b}',\,\lambda^k)\right] \,\equiv\, \Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,, \tag{10}$$

$$\mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a}',\,\lambda^k)\,\,\mathscr{B}(\mathbf{b},\,\lambda^k)\right] \,\equiv\, \Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle\,, \tag{11}$$

$$\text{and}\quad \mathcal{E}(\mathbf{a}',\,\mathbf{b}') \,=\, \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{k\,=\,1}^{n}\,\mathscr{A}(\mathbf{a}',\,\lambda^k)\,\,\mathscr{B}(\mathbf{b}',\,\lambda^k)\right] \,\equiv\, \Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,, \tag{12}$$

is a perfectly well defined and observable physical quantity. Therefore the bounds (3) on their sum are quite harmless. These bounds of $\{-4,\,+4\}$, however, have never been violated in any experiment. Indeed, nothing can violate them.

In summary, Bell and his followers derive the upper bound of 2 on the CHSH string of averages by an illegal move. In the middle of their derivation they unjustifiably replace an observable, and hence physically meaningful quantity,

$$\Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle \,+\, \Big\langle\,\mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle \,+\, \Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b})\,\Big\rangle \,-\, \Big\langle\,\mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,, \tag{13}$$

with an experimentally unobservable, and hence physically entirely meaningless quantity (regardless of the method):

$$\Big\langle\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}) \,+\, \mathscr{A}_k(\mathbf{a})\,\mathscr{B}_k(\mathbf{b}') \,+\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}) \,-\, \mathscr{A}_k(\mathbf{a}')\,\mathscr{B}_k(\mathbf{b}')\,\Big\rangle\,. \tag{14}$$

If they do not make this illegitimate replacement, then the absolute upper bound on the CHSH string of averages is 4, not 2. And the absolute upper bound of 4 has never been exceeded – and can never exceed – in any experiment [5].

One may suspect that the above conclusion is perhaps an artifact of the discrete version, (2), of the expectation values $\mathcal{E}(\mathbf{a},\,\mathbf{b})$. Perhaps it can be ameliorated if we considered the CHSH sum (1) in the following continuous form:

$$\int_{\Lambda}\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,d\rho(\lambda)\,+\int_{\Lambda}\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\,d\rho(\lambda)\,+\int_{\Lambda}\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,d\rho(\lambda)\,-\int_{\Lambda}\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\,d\rho(\lambda)\,, \tag{15}$$

where Λ is the space of all hidden variables λ and $\rho(\lambda)$ is the probability distribution of λ [1]. Written in this form, it is now easy to see that the above CHSH sum of expectation values is both mathematically and physically identical to

$$\int_{\Lambda}\Big[\,\mathscr{A}(\mathbf{a},\,\lambda)\,\big\{\,\mathscr{B}(\mathbf{b},\,\lambda)\,+\,\mathscr{B}(\mathbf{b}',\,\lambda)\,\big\}\,+\,\mathscr{A}(\mathbf{a}',\,\lambda)\,\big\{\,\mathscr{B}(\mathbf{b},\,\lambda)\,-\,\mathscr{B}(\mathbf{b}',\,\lambda)\,\big\}\Big]\;d\rho(\lambda)\,. \tag{16}$$

But since the above two integral expressions are identical to each other, we can use the second expression without loss of generality to prove that the criterion of reality used by Bell is unreasonably restrictive compared to that of EPR.

To begin with, expression (16) involves an integration over fictitious quantities like $\mathscr{A}(\mathbf{a},\,\lambda)\,\{\mathscr{B}(\mathbf{b},\,\lambda)\pm\mathscr{B}(\mathbf{b}',\,\lambda)\}$. These quantities are not parts of the space of all possible measurement outcomes such as $\mathscr{A}(\mathbf{a},\,\lambda)$, $\mathscr{A}(\mathbf{a}',\,\lambda)$, $\mathscr{B}(\mathbf{b},\,\lambda)$, $\mathscr{B}(\mathbf{b}',\,\lambda)$, *etc.*; because that space — although evidently closed under multiplication — is *not* closed under addition. Since each function $\mathscr{B}(\mathbf{b},\,\lambda)$ is by definition either $+1$ or -1, their sum such as $\mathscr{B}(\mathbf{b},\,\lambda)+\mathscr{B}(\mathbf{b}',\,\lambda)$ can only take values from the set $\{-2,\,0,\,+2\}$, and therefore it is not a part of the unit 2-sphere representing the space of all possible measurement results. Consequently, the quantities $\mathscr{A}(\mathbf{a},\,\lambda)\,\{\mathscr{B}(\mathbf{b},\,\lambda)\pm\mathscr{B}(\mathbf{b}',\,\lambda)\}$ appearing in the integrand of (16) do not themselves exist, despite the fact that $\mathscr{A}(\mathbf{a},\,\lambda)$, $\mathscr{A}(\mathbf{a}',\,\lambda)$, $\mathscr{B}(\mathbf{b},\,\lambda)$ and $\mathscr{B}(\mathbf{b}',\,\lambda)$ exist, at least counterfactually, in accordance with the hypothesis of local-realism. This is analogous to the fact that the set $\mathcal{O}:=\{1,2,3,4,5,6\}$ of all possible outcomes of a die throw is not closed under addition. For example, the sum $3+6$ is not a part of the set $\mathcal{O}$.

But there is also a much more serious physical problem with Bell's version of reality. As noted above, the quantities $\mathscr{A}(\mathbf{a},\,\lambda)\,\{\mathscr{B}(\mathbf{b},\,\lambda)\pm\mathscr{B}(\mathbf{b}',\,\lambda)\}$ are not physically meaningful in *any* possible physical world, classical or quantum. That is because $\mathscr{B}(\mathbf{b},\,\lambda)$ and $\mathscr{B}(\mathbf{b}',\,\lambda)$ can coexist with $\mathscr{A}(\mathbf{a},\,\lambda)$ only counterfactually, since $\mathbf{b}$ and $\mathbf{b}'$ are mutually exclusive directions. If $\mathscr{B}(\mathbf{b},\,\lambda)$ coexists with $\mathscr{A}(\mathbf{a},\,\lambda)$, then $\mathscr{B}(\mathbf{b}',\,\lambda)$ cannot coexist with $\mathscr{A}(\mathbf{a},\,\lambda)$, and vice versa. But in the proof of his theorem Bell presumes both $\mathscr{B}(\mathbf{b},\,\lambda)$ and $\mathscr{B}(\mathbf{b}',\,\lambda)$ to coexist with $\mathscr{A}(\mathbf{a},\,\lambda)$ simultaneously. That is analogous to being in New York and Miami at exactly the same time. But no reasonable criterion of reality can justify such an unphysical demand. The EPR criterion of reality most certainly does not demand any such thing.

In conclusion, since the two integrands of (16) are physically meaningless, the bounds of -2 and $+2$ on (15) are also physically meaningless [6]. They are mathematical curiosities, without any relevance for the question of local realism.

Corollary: *It is not possible to be in two places at once.*

It is instructive to consider the converse of the above argument. Consider the following hypothesis: *It is possible — at least momentarily — to be in two places at once — for example, in New York and Miami — at exactly the same time.*

From this hypothesis it follows that in a world in which it is possible to be in two places at once, it would be possible for Bob to detect a component of spin along two mutually exclusive directions, say $\mathbf{b}$ *and* $\mathbf{b}'$, at exactly the same time as Alice detects a component of spin along the direction $\mathbf{a}$, or $\mathbf{a}'$. If we denote the measurement functions of Alice and Bob by $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$, respectively, then we can posit that in such a world it would be possible for the measurement event like $\mathscr{A}(\mathbf{a},\,\lambda)$ observed by Alice to coexist with both the measurement events $\mathscr{B}(\mathbf{b},\,\lambda)$ and $\mathscr{B}(\mathbf{b}',\,\lambda)$ that are otherwise only counterfactually observable by Bob, where λ is the initial state of the singlet system. Therefore, hypothetically, we can represent such a simultaneous event observed by Alice and Bob by a random variable

$$X(\mathbf{a},\,\mathbf{b},\,\mathbf{b}',\,\lambda)\,:=\,\mathscr{A}(\mathbf{a},\,\lambda)\,\{\mathscr{B}(\mathbf{b},\,\lambda)\,+\,\mathscr{B}(\mathbf{b}',\,\lambda)\}\,=\,+2,\ \text{or}\ 0,\ \text{or}\ -2\,, \tag{17}$$

notwithstanding the fact that there are in fact only two localized particles available to Alice and Bob for each run of their EPR-Bohm type experiment. It is also worth stressing here that in our familiar macroscopic world (after all $\mathbf{a}$ and $\mathbf{b}$ are macroscopic directions) such a bizarre spacetime event is never observed, because the measurement directions $\mathbf{a}$ and $\mathbf{b}$ freely chosen by Alice and Bob are mutually exclusive macroscopic measurement directions in physical space.

Likewise, nothing prevents Alice and Bob in such a bizarre world to simultaneously observe an event represented by

$$Y(\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)\,:=\,\mathscr{A}(\mathbf{a}',\,\lambda)\,\{\mathscr{B}(\mathbf{b},\,\lambda)\,-\,\mathscr{B}(\mathbf{b}',\,\lambda)\}\,=\,+2,\ \text{or}\ 0,\ \text{or}\ -2\,. \tag{18}$$

And of course nothing prevents Alice and Bob in such a bizarre world to simultaneously observe the sum of the above two events as a single event (*i.e.*, four simultaneous clicks of their four detectors), represented by the random variable

$$Z(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)\,:=\,X(\mathbf{a},\,\mathbf{b},\,\mathbf{b}',\,\lambda)\,+\,Y(\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)\,=\,+2\;\text{ or }\;-2\,. \tag{19}$$

Consider now a large number of such initial states λ and corresponding simultaneous events like $Z(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)$. We can then calculate the expected value of such an event occurring in this bizarre world, by means of the integral

$$\int_\Lambda Z(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)\;\,d\rho(\lambda)\,=\,\int_\Lambda\,\Big[\,\mathscr{A}(\mathbf{a},\,\lambda)\,\big\{\,\mathscr{B}(\mathbf{b},\,\lambda)\,+\,\mathscr{B}(\mathbf{b}',\,\lambda)\,\big\}\,+\,\mathscr{A}(\mathbf{a}',\,\lambda)\,\big\{\,\mathscr{B}(\mathbf{b},\,\lambda)\,-\,\mathscr{B}(\mathbf{b}',\,\lambda)\,\big\}\Big]\;d\rho(\lambda)\,, \tag{20}$$

where Λ is the space of all hidden variables λ and $\rho(\lambda)$ is the corresponding normalized probability measure of $\lambda \in \Lambda$.

Note that we are assuming nothing about the hidden variables λ. They can be as non-local as we do not like. They can be functions of $\mathscr{A}$ and $\mathscr{B}$, as well as of $\mathbf{a}$ and $\mathbf{b}$. In which case we would be dealing with a highly non-local model:

$$\Lambda \ni \lambda = f\,(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\mathscr{A},\,\mathscr{B})\,. \tag{21}$$

Next we ask: What are the upper and lower bounds on the expected value (20)? The answer is given by Eq. (19). Since $Z(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\lambda)$ can only take two values, -2 and $+2$, the bounds on its integration over $\rho(\lambda)$ are necessarily

$$-\,2\,\leqslant\int_\Lambda\,\Big[\,\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,+\,\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\,+\,\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,-\,\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\Big]\;d\rho(\lambda)\;\leqslant+2\,. \tag{22}$$

But using the addition property of anti-derivatives this expected value can be written as a sum of four expected values,

$$\int_\Lambda\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,d\rho(\lambda)\,+\int_\Lambda\mathscr{A}(\mathbf{a},\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\,d\rho(\lambda)\,+\int_\Lambda\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b},\,\lambda)\,d\rho(\lambda)\,-\int_\Lambda\mathscr{A}(\mathbf{a}',\,\lambda)\,\mathscr{B}(\mathbf{b}',\,\lambda)\,d\rho(\lambda)\,, \tag{23}$$

despite our allowing of $\lambda\,(\mathbf{a},\,\mathbf{a}',\,\mathbf{b},\,\mathbf{b}',\,\mathscr{A},\,\mathscr{B})$ *to be non-local.* As a result (22) can be written in a recognizable form as

$$-\,2\leqslant\,\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\leqslant+2\,. \tag{24}$$

Note that the *only* hypothesis used to derive these stringent bounds of $\pm\,2$ is the one stated above: *It is possible – at least momentarily – to be in two places at once.* Locality was never assumed; nor was the realism of EPR compromised.

Now we perform the experiments and find that our results exceed the bounds of $\pm\,2$ we found in (24) theoretically:

$$-\,2\sqrt{2}\,\leqslant\,\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\leqslant+2\sqrt{2}\,. \tag{25}$$

Consequently, we conclude that the hypothesis we started out with must be false: We do not actually live in a bizarre world in which it is possible – even momentarily – to be in New York and Miami at exactly the same time. This is what Bell proved. He proved that we do not live in such a bizarre world. But EPR never demanded, nor hoped that we do.

To summarize our Corollary, Bell inequalities are usually derived by assuming locality and realism, and therefore violations of the Bell-CHSH inequality are usually taken to imply violations of either locality or realism, or both. But we have derived the Bell-CHSH inequality above by assuming only that Bob can measure along the directions $\mathbf{b}$ and $\mathbf{b}'$ simultaneously while Alice measures along either $\mathbf{a}$ or $\mathbf{a}'$, and likewise Alice can measure along the directions $\mathbf{a}$ and $\mathbf{a}'$ simultaneously while Bob measures along either $\mathbf{b}$ or $\mathbf{b}'$, *without assuming locality.* The violations of the Bell-CHSH inequality therefore simply confirm the impossibility of measuring along $\mathbf{b}$ and $\mathbf{b}'$ (or along $\mathbf{a}$ and $\mathbf{a}'$) simultaneously.

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