Refutation of Some Arguments Against my Disproof of Bell's Theorem

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In a couple of recent preprints Moldoveanu has suggested that there are errors in my disproof of Bell's theorem. Here I show that this claim is false. In particular, I show that my local-realistic framework is incorrectly and misleadingly presented in both of his preprints. In addition there are a number of serious mathematical and conceptual errors in his discussion of my framework. For example, contrary to his claim, my framework is manifestly non-contextual. In particular, quantum correlations are understood within it as purely topological effects, not contextual effects.

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I. INTRODUCTION

Bell's theorem [1] is based on an assumption that in any correlation experiment local measurement results can be described by functions of the form

$$\pm\, 1 \,=\, \mathscr{A}(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \mathcal{I} \subseteq \mathrm{I\!R}\,, \tag{1}$$

with $\mathrm{I\!R}^3 \ni \mathbf{n}$ representing a space of experimental contexts, $\lambda \in \Lambda$ representing a complete initial state of the system, and $\mathcal{I} \subseteq \mathrm{I\!R}$ representing the set of all possible measurement results in question [1]. Elsewhere I have shown that this assumption is false [2][3][4][5][6][7][8][9][10]. Elementary topological scrutiny reveals that no such local function—or its probabilistic counterpart $P(\mathscr{A}\,|\,\mathbf{n},\,\lambda)$—is capable of providing a complete account of every possible measurement result, even for the simplest of quantum systems. Unless enumerated by functions of the topologically correct form

$$\pm\, 1 \,=\, \mathscr{A}(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^3 \hookrightarrow \mathrm{I\!R}^4, \tag{2}$$

with their codomain S^3 being a parallelized 3-sphere, it is not possible to account for every possible measurement result for any two-level quantum system [5]. More specifically, unless the measurement results of Alice and Bob are represented by the equatorial points of a parallelized 3-sphere, the completeness criterion of EPR is not satisfied, and then there is no meaningful Bell's theorem to begin with [4][5][6][7][8][10]. In fact, naïvely replacing the *simply-connected* codomain S^3 in the above function by a *totally-disconnected* set $S^0 \equiv \{-1,\,+1\}$, as routinely done within all Bell type arguments, is a guaranteed way of introducing incompleteness in the accounting of measurement results from the start [5][6]. As I have argued elsewhere [2][3][4][5][6][7][8][9][10], the *only* unambiguously complete way of accounting for every possible measurement result locally is by means of standardized variables or bivectors of the form

$$\mathrm{I\!R}^4 \hookleftarrow S^3 \supset S^2 \ni \boldsymbol{\mu}\cdot\mathbf{n} \,\equiv\, \lambda\, I\cdot\mathbf{n} \,=\, \pm\, 1 \text{ about } \mathbf{n} \in \mathrm{I\!R}^3 \subset \mathrm{I\!R}^4, \tag{3}$$

for such bivectors intrinsically represent the equatorial points of a parallelized 3-sphere. Here $\boldsymbol{\mu} = \lambda\, I$ is a hidden variable or the complete state of the system, with $\lambda \equiv \pm\, 1$ being a fair coin and $I = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ being the fundamental volume form of the physical space. In statistical terms these bivectors then represent the standard scores, which are mathematical counterparts of the actually observed raw scores: $+1$ or -1 [2][4]. Moreover, once parallelized by a field of such bivectors (and their extensions to $\mathrm{I\!R}^4$), a 3-sphere remains as closed under multiplication of its points as the 0-sphere: $\{-1,\,+1\}$. As a result, setting the codomain of the standard scores to be the space of bivectors—which is isomorphic to an equatorial 2-sphere within a parallelized 3-sphere—guarantees that the locality or factorizability condition of Bell is automatically satisfied, for any number of measurement settings. The resulting model [2][4] of the EPR-Bohm correlations is therefore complete, local, and realistic, in the precise senses defined by EPR and Bell [6].

The informal picture that emerges from these formal arguments is as follows: EPR-Bohm correlations are telling us that we live in a parallelized 3-sphere, which differs from our usual conception of the physical space as $\mathrm{I\!R}^3$ only by a single point at infinity [4][5]. All measurement results, such as $\mathscr{A}(\mathbf{a},\,\lambda) = \pm\,1$, $\mathscr{B}(\mathbf{b},\,\lambda) = \pm\,1$, etc., are simply detections of one of the two possible orientations—or one of the two possible senses of local rotations—of this 3-sphere, predetermined by the initial state $\lambda = \pm\,1$. In other words, the hidden variable in this picture is the initial orientation of the physical space itself, which predetermines all possible outcomes at all possible measurement directions in the EPR-Bohm scenario. As a result, the measurement results are not contextual in any sense. For example, for a given

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orientation λ of the 3-sphere, the actual value $\mathscr{A}$ observed at a given direction $\mathbf{a}$, whether it is $+1$ or -1, does not change at all if, say, the measurement setting is changed from $\mathbf{a}$ to $\mathbf{a}'$. Why do the correlations between such purely random, non-contextual outcomes then turn out to be sinusoidal rather than linear? The reason has to do with the fact that there is a non-trivial twist in the Hopf fibration of the 3-sphere [4][10], and this twist is responsible for making the correlations more disciplined (or stronger) than linear. Consequently, in this picture the EPR-Bohm correlations are ontologically no different from those between the changing colors of Dr. Bertlmann's socks discussed by Bell [11].

As elegant, compelling, and conceptually appealing this picture is, in a pair of recent preprints Moldoveanu has questioned its validity by suggesting that there are errors in my disproof of Bell's theorem [12][13]. In what follows I show that this claim is false. The errors are in fact in Moldoveanu's understanding of my local-realistic framework. Contrary to what is claimed in his preprints, my work on the subject, [2][3][4][5][6][7][8][9][10], is perfectly cogent and error free. In what follows I bring out a number of mathematical and conceptual errors in his discussion of my work.

II. ORIENTATION OF THE 3-SPHERE IS NOT A CONVENTION BUT A HIDDEN VARIABLE

The first issue raised by Moldoveanu has to do with the hidden variable duality relation I have used in most of my papers as a convenient calculational tool:

$$\mathbf{a} \wedge \mathbf{b} = \boldsymbol{\mu} \cdot (\mathbf{a} \times \mathbf{b}) \equiv \lambda \, I \cdot (\mathbf{a} \times \mathbf{b}) \,. \tag{4}$$

Here $\boldsymbol{\mu} = \lambda I$ is a hidden variable of the model, with $\lambda \equiv \pm 1$ as a fair coin and $I = \mathbf{e}_x \mathbf{e}_y \mathbf{e}_z$ the standard volume form of the physical space. It represents the two alternately possible orientations of the 3-sphere, or equivalently the two alternately possible handedness of the bivector subalgebra representing the points of the 3-sphere. If we separate out the two independent instances of λ, then the combined duality relation (4) splits into two independent relations, with $\mathbf{a} \wedge \mathbf{b} = +I \cdot (\mathbf{a} \times \mathbf{b})$ representing the duality relation for the right-handed bivector basis and $\mathbf{a} \wedge \mathbf{b} = -I \cdot (\mathbf{a} \times \mathbf{b})$ representing the duality relation for the left-handed bivector basis. Although this differs from the usual practice in geometric algebra, the combined duality relation provides considerable computational ease in my model, provided one does not misunderstand its correct meaning. It leads to the following identity, which is a very convenient and powerful mathematical tool with rich physical and geometrical meaning, and plays a central role in most of my papers:

$$(\,\boldsymbol{\mu} \cdot \mathbf{a})(\,\boldsymbol{\mu} \cdot \mathbf{b}) \,=\, -\,\mathbf{a} \cdot \mathbf{b} \,-\, \boldsymbol{\mu} \cdot (\mathbf{a} \times \mathbf{b}), \tag{5}$$

Since the numbers $(\,\boldsymbol{\mu} \cdot \mathbf{a})$ and $(\,\boldsymbol{\mu} \cdot \mathbf{b})$ are identified in the model as the statistically pertinent standard scores (as opposed to actually observed raw scores, $+1$, -1, etc.), the EPR correlation follows at once form the above identity:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,=\, \lim_{n \,\gg\, 1} \left[\frac{1}{n} \sum_{i\,=\,1}^{n} \,(\,\boldsymbol{\mu}^i \cdot \mathbf{a})(\,\boldsymbol{\mu}^i \cdot \mathbf{b}) \right] \,=\, -\,\mathbf{a} \cdot \mathbf{b} \,-\, \lim_{n \,\gg\, 1} \left[\frac{1}{n} \sum_{i\,=\,1}^{n} \,\lambda^i \, I \cdot (\mathbf{a} \times \mathbf{b}) \right] \,=\, -\,\mathbf{a} \cdot \mathbf{b} \,+\, 0 \,. \tag{6}$$

Note also that since $(\mathbf{a} \times \mathbf{b})$ is an exclusive direction yielding a null result, the last summation is in fact unnecessary.

Rather surprisingly, this convenient mathematical tool has managed to confuse many people over the years for reasons that are only partly understandable. To be sure, in the standard practice the duality relation such as $\mathbf{a} \wedge \mathbf{b} = +I \cdot (\mathbf{a} \times \mathbf{b})$ is a fixed convention, and holds true whether one uses right-handed vector basis or left-handed. It simply tells us how the plane of $\mathbf{a} \wedge \mathbf{b}$ is related to its orthogonal vector $\mathbf{a} \times \mathbf{b}$. It should be remembered, however, that, although based on geometric algebra, mine is primarily a hidden variable model. Moreover, it should be remembered that the essence of my argument depends on the double cover property of the physical space [10], and therefore I am working primarily within the bivector subalgebra of the Clifford algebra $Cl_{3,0}$ [2][4]. To make this clear, let me derive the indefinite duality relation (4) once again from the first principles, and show how it fits into the identity (5).

Consider a right-handed frame of ordered basis bivectors, $\{\boldsymbol{\beta}_x,\, \boldsymbol{\beta}_y,\, \boldsymbol{\beta}_z\}$, and the corresponding bivector subalgebra

$$\boldsymbol{\beta}_j \, \boldsymbol{\beta}_k \,=\, -\,\delta_{jk} \,-\, \epsilon_{jkl} \, \boldsymbol{\beta}_l \tag{7}$$

of the Clifford algebra $Cl_{3,0}$. The latter is a linear vector space $\mathbb{R}^8$ spanned by the orthonormal basis

$$\{1,\;\; \mathbf{e}_x,\, \mathbf{e}_y,\, \mathbf{e}_z,\;\; \mathbf{e}_x \wedge \mathbf{e}_y,\, \mathbf{e}_y \wedge \mathbf{e}_z,\, \mathbf{e}_z \wedge \mathbf{e}_x,\;\; \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z\}, \tag{8}$$

where δ_{jk} is the Kronecker delta, ϵ_{jkl} is the Levi-Civita symbol, the indices $j,\, k,\, l = x,\, y,$ or z are cyclic indices, and

$$\boldsymbol{\beta}_j \,=\, \mathbf{e}_k \wedge \mathbf{e}_l \,=\, I \cdot \mathbf{e}_j \,. \tag{9}$$

Eq. (7) is a standard expression of bivector subalgebra, routinely used in geometric algebra to define the right-handed frame of basis bivectors [14]. From (7) it is easy to verify the familiar properties of the basis bivectors, such as

$$(\boldsymbol{\beta}_x)^2 \,=\, (\boldsymbol{\beta}_y)^2 \,=\, (\boldsymbol{\beta}_z)^2 \,=\, -\,1 \tag{10}$$

$$\text{and} \quad \boldsymbol{\beta}_x \, \boldsymbol{\beta}_y \,=\, -\,\boldsymbol{\beta}_y \, \boldsymbol{\beta}_x \quad \text{etc.} \tag{11}$$

Moreover, it is easy to verify that the bivectors satisfying the subalgebra (7) form a right-handed frame of basis bivectors. To this end, right-multiply both sides of Eq. (7) by $\boldsymbol{\beta}_l$, and then use the fact that $(\boldsymbol{\beta}_l)^2 = -1$ to arrive at

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\,\boldsymbol{\beta}_l \,=\, +1\,. \tag{12}$$

The fact that this ordered product yields a positive value confirms that $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$ indeed forms a right-handed frame of basis bivectors. This is a universally accepted convention, found in any textbook on geometric algebra [14].

Suppose now $\mathbf{a} = a_j\,\mathbf{e}_j$ and $\mathbf{b} = b_k\,\mathbf{e}_k$ are two unit vectors in $\mathrm{I\!R}^3$, where the repeated indices are summed over x, y, and z. Then the right-handed basis equation (7) leads to

$$\{\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\} \,=\, -\,a_j\,b_k\,\delta_{jk} \,-\, \epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l\,, \tag{13}$$

which, together with (9), is equivalent to

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, I\cdot(\mathbf{a}\times\mathbf{b}), \tag{14}$$

where $I = \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ is the standard trivector. Geometrically this identity describes all points of a parallelized 3-sphere.

Let us now consider a left-handed frame of ordered basis bivectors, which we also denote by $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$. It is important to recognize, however, that there is no *a priori* way of knowing that this new basis frame is in fact left-handed. To ensure that it is indeed left-handed we must first make sure that it is an ordered frame by requiring that its basis elements satisfy the bivector properties delineated in Eqs. (10) and (11). Next, to distinguish this frame from the right-handed frame defined by equation (12), we must require that its basis elements satisfy the property

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k\,\boldsymbol{\beta}_l \,=\, -\,1\,. \tag{15}$$

One way to ensure this is to multiply every non-scalar element in (8) by a minus sign. Then, instead of (9), we have

$$\boldsymbol{\beta}_j \,=\, -\,\mathbf{e}_k\wedge\mathbf{e}_l \,=\, (\,-\,I\,)\cdot(\,-\,\mathbf{e}_j\,) \,=\, I\cdot\mathbf{e}_j\,, \tag{16}$$

and the condition (15) is automatically satisfied. As is well known, this was the condition imposed by Hamilton on his unit quaternions, which we now know are nothing but a left-handed set of basis bivectors [14]. Indeed, it can be easily checked that the basis bivectors satisfying the properties (10), (11), (15), and (16) compose the subalgebra

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \,=\, -\,\delta_{jk} \,+\, \epsilon_{jkl}\,\boldsymbol{\beta}_l\,. \tag{17}$$

Conversely, it is easy to check that the basis bivectors defined by this subalgebra do indeed form a left-handed frame. To this end, right-multiply both sides of Eq. (17) by $\boldsymbol{\beta}_l$, and then use the property $(\boldsymbol{\beta}_l)^2 = -1$ to verify Eq. (15). As is well known, this subalgebra is generated by the unit quaternions originally proposed by Hamilton [14]. It is routinely used in the textbook treatments of angular momenta, but without mentioning the fact that it defines nothing but a left-handed set of basis bivectors. It may look more familiar if we temporarily change notation and rewrite Eq. (17) as

$$\mathbf{J}_j\,\mathbf{J}_k \,=\, -\,\delta_{jk} \,+\, \epsilon_{jkl}\,\mathbf{J}_l\,. \tag{18}$$

More importantly (and especially since Moldoveanu seems to have missed this point), I stress once again that there is no way to set apart the left-handed frame of basis bivectors from the right-handed frame without appealing to the intrinsically defined distinguishing conditions (12) and (15), or equivalently to the corresponding subalgebras (7) and (17). Note also that at no time within my framework the two subalgebras (7) and (17) are mixed in any way, either physically or mathematically. They merely play the role of two distinct and alternative hidden variable possibilities.

Suppose now $\mathbf{a} = a_j\,\mathbf{e}_j$ and $\mathbf{b} = b_k\,\mathbf{e}_k$ are two unit vectors in $\mathrm{I\!R}^3$, where the repeated indices are summed over x, y, and z. Then the left-handed basis equation (17) leads to

$$\{\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,b_k\,\boldsymbol{\beta}_k\,\} \,=\, -\,a_j\,b_k\,\delta_{jk} \,+\, \epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l \tag{19}$$

which, together with (16), is equivalent to

$$(\,I\cdot\mathbf{a})(\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,+\, I\cdot(\mathbf{a}\times\mathbf{b}), \tag{20}$$

where I is the standard trivector. Once again, geometrically this identity describes all points of a parallelized 3-sphere.

It is important to note, however, that there is a sign difference in the second term on the RHS of the identities (14) and (20). The algebraic meaning of this sign difference is of course clear from the above discussion, and it has been discussed extensively in most of my papers, with citations to prior literature [4][5][7]. But from the perspective of my model a more important question is: What does this sign difference mean *geometrically*? To bring out its geometric meaning, let us rewrite the identities (14) and (20) as

$$(+I\cdot\mathbf{a})(+I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (+I\,)\cdot(\mathbf{a}\times\mathbf{b}) \tag{21}$$

and

$$(-I\cdot\mathbf{a})(-I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (-I\,)\cdot(\mathbf{a}\times\mathbf{b}), \tag{22}$$

respectively. The geometrical meaning of the two identities is now transparent if we recall that the bivector $(+I\cdot\mathbf{a})$ represents a counterclockwise rotation about the **a**-axis, whereas the bivector $(-I\cdot\mathbf{a})$ represents a clockwise rotation about the **a**-axis. Accordingly, both identities interrelate the points of a unit parallelized 3-sphere, but the identity (21) interrelates points of a positively oriented 3-sphere whereas the identity (22) interrelates points of a negatively oriented 3-sphere. In other words, the 3-sphere represented by the identity (21) is oriented in the counterclockwise sense, whereas the 3-sphere represented by the identity (22) is oriented in the clockwise sense. These two alternative orientations of the 3-sphere is then the random hidden variable $\lambda = \pm 1$ (or the initial state $\lambda = \pm 1$) within my model.

Given this geometrical picture, it is now easy to appreciate that identity (21) corresponds to the physical space characterized by the trivector $+I$, whereas identity (22) corresponds to the physical space characterized by the trivector $-I$ [15]. This is further supported by the evident fact that, apart from the choice of a trivector, the identities (21) and (22) represent one and the same subalgebra. Moreover, there is clearly no *a priori* reason for Nature to choose $+I$ as a fundamental trivector over $-I$. Either choice provides a perfectly legitimate representation of the physical space, and neither is favored by Nature. Consequently, instead of characterizing the physical space by fixed basis (8), we can start out with two alternatively possible characterizations of the physical space by the *hidden* basis

$$\{1,\;\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z,\;\mathbf{e}_x\wedge\mathbf{e}_y,\,\mathbf{e}_y\wedge\mathbf{e}_z,\,\mathbf{e}_z\wedge\mathbf{e}_x,\;\lambda\,(\,\mathbf{e}_x\wedge\mathbf{e}_y\wedge\mathbf{e}_z\,)\}, \tag{23}$$

where $\lambda = \pm 1$. Although these considerations and the physical motivations behind them have been the starting point of my program (see, for example, discussions in Refs. [3], [7], and [10]), Moldoveanu has overlooked them completely. In fact, as we shall see in the next section, his arguments entirely depend on misidentifying the objective hidden variable λ with the subjective choice of reference handedness to be made by the experimenter for computational purposes.

Exploiting the natural freedom of choice in characterizing S^3 by either $+I$ or $-I$, we can now combine the identities (21) and (22) into a single hidden variable equation (at least for the computational purposes):

$$(\,\lambda I\cdot\mathbf{a})(\,\lambda I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (\,\lambda I\,)\cdot(\mathbf{a}\times\mathbf{b}), \tag{24}$$

where $\lambda = \pm 1$ now specifies the orientation of the 3-sphere. It is important to recognize that the difference between the trivectors $+I$ and $-I$ in this equation primarily reflects the difference in the handedness of the bivector basis $\{\boldsymbol{\beta}_x,\,\boldsymbol{\beta}_y,\,\boldsymbol{\beta}_z\}$, and not in the handedness of the vector basis $\{\,\mathbf{e}_x,\,\mathbf{e}_y,\,\mathbf{e}_z\}$. This should be evident from the foregone arguments, but let us bring this point home by considering the following change in the handedness of the vector basis:

$$+\,I \,=\, \mathbf{e}_x\mathbf{e}_y\mathbf{e}_z \;\;\longrightarrow\;\; (+\,\mathbf{e}_x)(-\,\mathbf{e}_y)(+\,\mathbf{e}_z) \,=\, -\,(\,\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\,) \,=\, -\,I\,. \tag{25}$$

Such a change does not induce a change in the handedness of the bivector basis, since it leaves the product $\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z$ unchanged. This can be easily verified by recalling that $\boldsymbol{\beta}_x \equiv I\cdot\mathbf{e}_x$, $\boldsymbol{\beta}_y \equiv I\cdot\mathbf{e}_y$, and $\boldsymbol{\beta}_z \equiv I\cdot\mathbf{e}_z$, and consequently

$$+\,1 \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \;\;\longrightarrow\;\; (-\,I\,)\cdot(+\,\mathbf{e}_x)\,(-\,I\,)\cdot(-\,\mathbf{e}_y)\,(-\,I\,)\cdot(+\,\mathbf{e}_z) \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \,=\, +1\,. \tag{26}$$

Conversely, a change in the handedness of bivector basis does not necessarily affect a change in the handedness of vector basis, but leads instead to

$$+\,1 \,=\, \boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z \;\;\longrightarrow\;\; (\,-\,\boldsymbol{\beta}_x)\,(\,-\,\boldsymbol{\beta}_y)\,(\,-\,\boldsymbol{\beta}_z) \,=\, -\,(\,\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z\,) \,=\, -1\,, \tag{27}$$

which in turn leads us back to equation (24) via equations (14) and (20). Thus the sign difference between the trivectors $+I$ and $-I$ captured in equation (24) arises from the sign difference in the product $\boldsymbol{\beta}_x\,\boldsymbol{\beta}_y\,\boldsymbol{\beta}_z$ and not from that in the product $\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$. It is therefore of very different geometrical significance [4]. It corresponds to the difference between two possible orientations of the 3-sphere mentioned above. It is also important to keep in mind that the combined equation (24) is simply a convenient shortcut for representing two completely independent initial states of the system, one corresponding to the counterclockwise orientation of the 3-sphere and the other corresponding to the clockwise orientation of the 3-sphere. Moreover, at no time these two alternative possibilities are mixed during the course of an experiment. They represent two independent physical scenarios, corresponding to two independent runs of the experiment. If we now use the notation $\boldsymbol{\mu} = \lambda\, I$, then the combined identity (24) takes the convenient form

$$(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}). \tag{28}$$

It should now be abundantly clear where the indefinite duality relation (4) originates from. It is simply a convenient shortcut describing the two alternate hidden variable possibilities encapsulated in this identity. Every mathematician knows that what Bourbaki calls "abuse of notation" can, when handled with care, greatly illuminate what would otherwise be a confusing situation. I do not claim that the indefinite duality relation (4) is illuminating the situation all that much, but it does facilitate considerable ease in intricate computations [5][6]. Thus Moldoveanu's alarm about this relation is much ado about nothing. In any case, if one remains uncomfortable about using the identity (28), then there is always the option of working directly with the bivector basis themselves, as I have done in my one-page paper [2]. Starting with equations (7) and (17) we first write the basic hidden variable equation of the model as

$$\boldsymbol{\beta}_j\,\boldsymbol{\beta}_k \;=\; -\,\delta_{jk}\;-\;\lambda\,\epsilon_{jkl}\,\boldsymbol{\beta}_l\,, \tag{29}$$

with $\lambda = \pm\,1$ as a fair coin representing the two possible orientations of the 3-sphere. To see the equivalence of this equation with the hidden variable identity (28), let $\mathbf{a} = a_j\,\mathbf{e}_j$ and $\mathbf{b} = b_k\,\mathbf{e}_k$ be two unit vectors in $\mathrm{I\!R}^3$. We then have

$$\{\,\lambda\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,\lambda\,b_k\,\boldsymbol{\beta}_k\,\} \;=\; -\,a_j\,b_k\,\delta_{jk}\;-\;\lambda\,\epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l\,, \tag{30}$$

which is equivalent to the identity (28) with $(\,\boldsymbol{\mu}\cdot\mathbf{a}) \equiv \{\,\lambda\,a_j\,\boldsymbol{\beta}_j\,\}$, $\;\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b}) \equiv \{\,\lambda\,\epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l\,\}$, etc. As a result, the standardized variables $(\,\boldsymbol{\mu}\cdot\mathbf{a}) \equiv \{\,\lambda\,a_j\,\boldsymbol{\beta}_j\,\}$ and $(\,\boldsymbol{\mu}\cdot\mathbf{b}) \equiv \{\,\lambda\,b_k\,\boldsymbol{\beta}_k\,\}$ immediately give rise to the EPR correlation:

$$\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\,a_j\,\boldsymbol{\beta}_j\,\}\,\{\,\lambda^i\,b_k\,\boldsymbol{\beta}_k\,\}\right] \;=\; -\,a_j\,b_j\;-\;\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\,\epsilon_{jkl}\,a_j\,b_k\,\boldsymbol{\beta}_l\,\}\right] \;=\; -\,a_j\,b_j\;+\;0\;=\;-\,\mathbf{a}\cdot\mathbf{b}\,, \tag{31}$$

$$\text{and}\qquad \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\,a_j\,\boldsymbol{\beta}_j\,\}\right] \;=\;0\;=\; \lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\{\,\lambda^i\,b_k\,\boldsymbol{\beta}_k\,\}\right]. \tag{32}$$

It is important to remember that what is being summed over here are points of a parallelized 3-sphere representing the outcomes of completely independent experimental runs in an EPR-Bohm experiment. In statistical terms what these results are then showing is that correlation between the raw numbers $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}) \,=\, (-\,I\cdot\mathbf{a}\,)\,(+\,\boldsymbol{\mu}\cdot\mathbf{a}\,) \,=\, \pm\,1 \in S^3$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}) \,=\, (+\,I\cdot\mathbf{b}\,)\,(+\,\boldsymbol{\mu}\cdot\mathbf{b}\,) \,=\, \pm\,1 \in S^3$ is $-\,\mathbf{a}\cdot\mathbf{b}$. According to Bell's theorem this is mathematically impossible. Further physical, mathematical, and statistical details of this "impossible" result can be found in Refs. [2] and [4].

III. HANDEDNESS OF THE BIVECTORS IS NOT A CONVENTION BUT AN INITIAL EPR STATE

In his preprint [13] Moldoveanu suggests that the above results are incorrect because I have used the duality relation (4) to derive them. That this claim is false should be clear from the above discussion, but let us try to get to the heart of his misconception. This is revealed form the following statement he makes, starting on the first page of [13]:

> Even without spelling in detail the error, it is easy to see that the exterior product term should not vanish on any handedness average because handedness is just a paper convention on how to consistently make computations. For example one can apply the same incorrect argument to complex numbers because there is the same freedom to choose the sign of $\sqrt{-1}$ based on the two dimensional coordinate handedness in this case. Then one can compute the average of let's say $z = 3 + 2i$ for a fair coin random distribution of handedness and arrive at the incorrect answer: $< z >= 3$ instead of $< z >= z$.

Now, to begin with, complex numbers do not have handedness. Thus his toy example badly misses the mark from the start. In fact the problems with it go far deeper, but let us play along. If one treats handedness as merely a "paper convention"—say a choice between $z = 3 + 2i$ and $z = 3 - 2i$ for performing computations—then of course one must maintain the same convention for all experimental runs to obtain the correct answer, and that answer can only be $< z >= z$. However, in my model handedness of the bivectors basis, or more precisely the orientation of the physical space S^3, is not a convention but a hidden variable λ. It is not a choice that is made by an experimenter but by Nature herself, as an initial or complete state of a given pair of particles. In other words, in my model the alternating handedness, "$z = 3 + 2i$" or "$z = 3 - 2i$", determine which two of the four event detectors are triggered for a given pair of particles. Consequently the correct answer for the average within my model cannot possibly be $< z >= z$ but $< z >= 3$. Thus, indeed, even without spelling out his error in detail we can see where Moldoveanu has gone wrong.

One might think that it is the unconventional nature of my framework that might have misled Moldoveanu into making such an elementary mistake. Reading his preprints suggests otherwise, however. For instance, his argument seems to overlook the fact that, since each specific instance of the hidden variable λ—i.e., each specific initial orientation of the 3-sphere—specifies an initial state of the EPR pair in my model, it corresponds to a physical scenario quite independent of the previous or subsequent initial state of the pair. Thus, as in Bell's own local model [16], Alice makes a series of measurements of *different* particles, not repeated measurements of the same particle. Each particle thus begins with a different initial state (or a different common cause) λ, which interacts with Alice's analyzer through the bivector $\{\, a_j\, \boldsymbol{\beta}_j(\lambda)\,\}$ to produce her measurement outcome $\mathscr{A}(\mathbf{a},\,\lambda)$. No mixing of algebra of any kind occurs during this process since what is averaged over are real numbers observed independently by Alice and Bob.

IV. FAILURE OF BELL'S THEOREM IMPLIES VINDICATION OF THE EPR ARGUMENT

One of several misreadings of my work by Moldoveanu is reflected in his claim that my counterexample to Bell's theorem is based on assumptions different from those of Bell. To see that this claim is false, let us recall the bare essentials of Bell's assumptions. These appear no more starkly than in Bell's own replies to his critics [16]. Bell insists that it is *not* possible to find local functions of the form

$$\mathscr{A}(\mathbf{a},\,\lambda)\,=\,+1\;\;\text{or}\;\;-1 \tag{33}$$

$$\text{and}\;\;\mathscr{B}(\mathbf{b},\,\lambda)\,=\,+1\;\;\text{or}\;\;-1 \tag{34}$$

which can give the correlation of the form

$$\langle\,\mathscr{A}\mathscr{B}\,\rangle\,=\,-\,\mathbf{a}\cdot\mathbf{b}\,, \tag{35}$$

where the measurement setting $\mathbf{b}$ of a particular polarizer has no effect on what happens, $\mathscr{A}$, in a remote region, and likewise that the measurement setting $\mathbf{a}$ has no effect on $\mathscr{B}$. "*This is the theorem*", he insists (my emphasis).

Now the first thing to recall here is that if this theorem is false, then there is nothing to impede the conclusion by EPR: *the description of physical reality provided by quantum mechanics is incomplete*. Moreover, demonstration of incompleteness of any theory for a single physical scenario is sufficient to demonstrate the incompleteness of that theory for *all* physical scenarios. Thus failure of Bell's theorem as stated above necessitates incompleteness of quantum mechanics as a whole [6]. It seems, however, that Moldoveanu has not appreciated the force of this logic. All other "impossibility proofs", no matter how elaborate, are powerless against the logic of the EPR argument. Consequently, irrespective of the arguments of the previous sections, my one-page paper [2] by itself is more than sufficient to vindicate EPR, repudiate Bell, and show that quantum mechanics is necessarily an incomplete theory of nature. This is because the variables $\mathscr{A}(\mathbf{a},\,\lambda)=\pm\,1$ and $\mathscr{B}(\mathbf{b},\,\lambda)=\pm\,1$ defined in equations (1) and (2) of that paper, together with the correlation between them obtained in equation (7), decisively and unambiguously contradict the assertion made by Bell in equations (33) to (35) above. What is more, they do so *irrespective of the interpretation attached to the mathematical terms involved in the calculation of the correlation in* [2]. It is therefore quite puzzling how anyone who has studied my paper and understood its logic (as well as that of EPR) can continue to believe in Bell's theorem.

V. THE LOCAL-REALISTIC FRAMEWORK IN QUESTION IS STRICTLY NON-CONTEXTUAL

Now the topological underpinning of my model is quite different from that of usually considered local hidden variable theories, which are implicitly expected to be contextual in general. Usually one expects the numbers $\mathscr{A}$ and $\mathscr{B}$ to change when the directions of measurements $\mathbf{a}$ and $\mathbf{b}$ are changed. This informal expectation, however, is profoundly misguided. No such local contextual change is ever observed in the actual experiments, or even predicted by quantum mechanics. Locally one only finds purely random measurement outcomes with no hidden order within them. The variations that are predicted and observed are only in the correlation between the numbers $\mathscr{A}$ and $\mathscr{B}$, and not in the randomness of the numbers $\mathscr{A}$ and $\mathscr{B}$ themselves [17]. And that is precisely what is predicted by my model.

Nevertheless, despite the fact that my framework is manifestly non-contextual, Moldoveanu has claimed that it is contextual. To appreciate the evident falsity of this claim, let us have a closer look at the relationship between the statistically pertinent standard scores $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$ and the actually observed raw scores $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})$:

$$S^3\ni\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})\,=\,(-\,I\cdot\mathbf{a}\,)\,(+\,\boldsymbol{\mu}\cdot\mathbf{a}\,)\,=\,\begin{cases}+\,1 & \text{if}\quad \boldsymbol{\mu}\,=\,+\,I\\ -\,1 & \text{if}\quad \boldsymbol{\mu}\,=\,-\,I\end{cases} \tag{36}$$

and

$$S^3\ni\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})\,=\,(+\,I\cdot\mathbf{b}\,)\,(+\,\boldsymbol{\mu}\cdot\mathbf{b}\,)\,=\,\begin{cases}-\,1 & \text{if}\quad \boldsymbol{\mu}\,=\,+\,I\\ +\,1 & \text{if}\quad \boldsymbol{\mu}\,=\,-\,I\,,\end{cases} \tag{37}$$

with equal probabilities for $\boldsymbol{\mu}$ being either $+\,I$ or $-\,I$. Note that $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})$, in addition to being manifestly realistic, are strictly *local* variables. Moreover, it is not difficult to see that they are manifestly non-contextual [18].

Alice's measurement result, although refers to her freely chosen context $\mathbf{a}$, depends only on the initial state $\boldsymbol{\mu}$; and likewise, Bob's measurement result, although refers to his freely chosen context $\mathbf{b}$, depends only on the initial state $\boldsymbol{\mu}$. In other words, all possible measurement results at all possible angles are determined entirely by the initial orientation of the 3-sphere specified by $\boldsymbol{\mu}$, and do not change when the local contexts are changed. This fact is so manifestly obvious from the above definitions of $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})$ that it makes one wonder why anyone would think my framework is contextual. Could it be because the standard scores $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$ somehow appear to be contextual? It is easy to check, however, that they are not:

$$\boldsymbol{\mu}\cdot\mathbf{n} = \begin{cases} +\,1 \text{ about } \mathbf{n} & \text{if } \boldsymbol{\mu} \,=\, +\,I, \\ -\,1 \text{ about } \mathbf{n} & \text{if } \boldsymbol{\mu} \,=\, -\,I. \end{cases} \tag{38}$$

Evidently, the values of the standard scores $\boldsymbol{\mu}\cdot\mathbf{n}$ also do not depend upon the experimental context $\mathbf{n}$. For instance, if the context is changed from $\mathbf{n}$ to $\mathbf{n}'$, we obtain

$$\boldsymbol{\mu}\cdot\mathbf{n}' = \begin{cases} +\,1 \text{ about } \mathbf{n}' & \text{if } \boldsymbol{\mu} \,=\, +\,I, \\ -\,1 \text{ about } \mathbf{n}' & \text{if } \boldsymbol{\mu} \,=\, -\,I, \end{cases} \tag{39}$$

and the corresponding raw scores remain exactly the same. This is not surprising, because, as stressed above, according to my model all measurement results are simply detecting the orientation of the 3-sphere specified by the initial state $\boldsymbol{\mu}$ (recall also that an answer to any quantum mechanical question can always be reduced to a set of Yes/No answers).

So, clearly, as far as the EPR-Bohm correlations are concerned, my model of the physical reality is not contextual. But what about more general quantum correlations? Could my local-realistic framework be contextual for general quantum correlations? Perhaps that is what Moldoveanu is hoping for [12]. It is not difficult to see however that my framework is manifestly non-contextual even for the most general case. This fact can be stated as a theorem:

Theorema Egregium:

> Every quantum mechanical correlation among a set of measurement results, such as $\mathscr{A}=\pm\,1$, $\mathscr{B}=\pm\,1$, $\mathscr{C}=\pm\,1$, etc., can be understood as a classical, local-realistic correlation among a set of points of a parallelized 7-sphere.

The proof of this theorem (or at least a sketch of it) can be found in section IVA of Ref. [5] and section VI of Ref. [6].

Now 7-sphere has a very rich topological structure. It happens to be homeomorphic to the space of unit octonions, which are well known to form the most general possible division algebra. In the language of fiber bundles one can view 7-sphere as a 4-sphere worth of 3-spheres. Each fiber of the 7-sphere is then a 3-sphere, and each one of these 3-spheres is a 2-sphere worth of circles. Thus the four parallelizable spheres—S^0, S^1, S^3, and S^7—can all be viewed as nested within a 7-sphere. The EPR-Bohm correlations can then be understood as correlations among the equatorial points of one of the fibers of this 7-sphere, as we saw in section II. Alternatively, the 7-sphere can be thought of as a 6-sphere worth of circles. Thus the above theorem can be framed entirely in terms of circles, each one of which described by a classical, octonionic spinor with a well-defined sense of rotation (i.e., whether it describes a clockwise rotation about a point within the 7-sphere or a counterclockwise rotation). This sense of rotation in turn defines a definite handedness (or orientation) about every point of the 7-sphere. If we designate this handedness by a random number λ, then local measurement results for any physical scenario can be represented by the raw scores of the form

$$S^7 \ni \mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}) \,=\, (-\,J\cdot\mathbf{N}(\mathbf{a})\,)\,(\,+\,\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{a})\,) \,=\, \begin{cases} +1 & \text{if} \quad \boldsymbol{\mu} \,=\, +\,J \\ -1 & \text{if} \quad \boldsymbol{\mu} \,=\, -\,J\,, \end{cases} \tag{40}$$

where $\mathbf{a}\in\mathrm{I\!R}^3$ and $\mathbf{N}(\mathbf{a})\in\mathrm{I\!R}^7$ are unit vectors and $\boldsymbol{\mu}=\lambda\,J$ is the hidden variable analogous to $\boldsymbol{\mu}=\lambda\,I$ with $I=\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z$ replaced by

$$J \,=\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_4 \,+\, \mathbf{e}_2\mathbf{e}_3\mathbf{e}_5 \,+\, \mathbf{e}_3\mathbf{e}_4\mathbf{e}_6 \,+\, \mathbf{e}_4\mathbf{e}_5\mathbf{e}_7 \,+\, \mathbf{e}_5\mathbf{e}_6\mathbf{e}_1 \,+\, \mathbf{e}_6\mathbf{e}_7\mathbf{e}_2 \,+\, \mathbf{e}_7\mathbf{e}_1\mathbf{e}_3\,. \tag{41}$$

The standard scores corresponding to the raw scores (40) are then given by $\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{a})$, which geometrically represent the equatorial points of a parallelized 7-sphere, just as $\boldsymbol{\mu}\cdot\mathbf{a}$ represented the equatorial points of a parallelized 3-sphere.

It is now clear that even in the most general case the variables representing measurement results in my framework are manifestly non-contextual. Just as before, Alice's measurement result, although refers to her freely chosen context $\mathbf{a}$, depends only on the initial state $\boldsymbol{\mu}=\lambda\,J$; and likewise, Bob's measurement result may refer to his freely chosen context $\mathbf{b}$, but would depend only on the initial state $\boldsymbol{\mu}=\lambda\,J$. In other words, all possible measurement results at all possible angles are determined entirely by the initial orientation of the 7-sphere specified by $\boldsymbol{\mu}=\lambda\,J$, and do not change when the local contexts are changed [1]. Moreover, the values of even the standard scores $\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{a})$ do not change when the local contexts are changed. To be sure, when the 3D context is changed, say from $\mathbf{a}$ to $\mathbf{a}'$, the corresponding 7D direction changes from $\mathbf{N}$ to $\mathbf{N}'$, but that does not at all affect the values of either the raw scores or the standard scores, because they are determined entirely by the initial orientation of the 7-sphere specified by $\boldsymbol{\mu}=\lambda\,J$.

VI. NON-COMMUTING STANDARD SCORES ARE MERELY CALCULATIONAL TOOLS

It is clear from the above discussion that my entire local-realistic framework is strictly non-contextual. Nevertheless, in his preprint [12] Moldoveanu has argued that it somehow "must be" contextual. His argument relies on some well known (but irrelevant) theorems (see below) and the non-commutativity of the bivectors, which, as we saw, plays the role of standard scores within my framework [4]. What is amiss in his argument, however, is the evident fact that non-commutativity enters in my framework only at the level of standard scores, not raw scores. In fact in either of his preprints there is no appreciation of the vital conceptual difference between the raw scores and standard scores, let alone the significance of this difference within my framework. Had he appreciated this conceptual difference (as explained, for example, in Ref. [4]), he would have recognized that non-commutativity of the standard scores within my framework—which he claims makes contextuality inevitable in general—is only an intermediate calculational tool. The actual eventualities, $\mathscr{A}$, $\mathscr{B}$, etc., (*i.e.*, the actual measurement results) always commute with each other,

$$[\,\mathscr{A},\,\mathscr{B}\,]\,=\,0\,,\quad\forall\;\mathbf{a},\;\mathbf{b},\;\text{and}\;\lambda\,,\tag{42}$$

because they are simply scalar numbers (*cf.* their definition (40) above). In statistical terms, these measurement results are raw scores, and the corresponding non-commuting variables—i.e., the bivectors $\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{a})$—are standard scores. The standard scores—or the standardized variables—indeed do not commute in general, but they are simply intermediate calculational tools, not something that is actually observed in the experiments. Therefore Moldoveanu is quite mistaken in building a case around the non-commutativity of such mathematical constructs. Moreover, the mystique of classical non-commutativity within my model completely evaporates when one notes that it can always be understood as a vector addition in a higher-dimensional space (see, for example, discussion below Eq. (38) in Ref. [6]). Thus non-commutativity within my model does not have the ontological significance Moldoveanu thinks it has.

Leaving aside this statistical misconception, let us see whether Moldoveanu's argument for the contextuality within my framework itself holds water. In fact his argument turns out to be both logically and conceptually incongruent even if we accept its premises (which are based on multiple misconceptions of my framework in any case). To be more specific, his argument relies on theorems against non-contextual hidden variable theories such as that by Kochen and Specker and some of its lesser known extensions, but without spelling out how exactly such theorems—based on discrete spaces of measurement results as they are—are applicable to my framework based on topology and continuity. In fact they are not at all applicable, because none of them even remotely address the topological concerns I have raised within the context of Bell's theorem and its variants [5][6]. Thus his attempt of trying to fit my topological framework within a preconceived conceptual box of contextuality is like trying to fit a square peg in a round hole.

VII. A SIMPLY-CONNECTED MODEL CANNOT BE SIMULATED BY ITS DESCRETIZED IMITATION

As we saw above, within my framework quantum correlations are explained as topological effects, not contextual effects, and that Moldoveanu has overlooked this obvious fact. This is reflected, for example, in his failed attempts to simulate my model on a computer by expecting some sort of contextual variation in the measurement results. The usual idea behind a computer simulation of EPR correlations is to program how a measurement function, say $\mathscr{A}(\mathbf{a},\,\lambda)$, changes when its context is changed, say from $\mathbf{a}$ to $\mathbf{a}'$, and likewise for the function $\mathscr{B}(\mathbf{b},\,\lambda)$. But in my model these functions do not change with their contexts at all. It is the topology of the physical space that brings about the sinusoidal correlation between $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$, not the contextual variations within $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ themselves. As counterintuitive as this may seem, that is what the mathematics of my model implies, and it matches exactly with the experimental evidence. Consequently, most unsophisticated simulation attempts are bound to fail.

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[1] Often in physics formal expositions end up obscuring the simplest of truths. Let me therefore try to redraw the above picture in homely terms of the science fiction novel Flatland [19]. Suppose we have proof that we are living in a parallelized 3-sphere rather than $\mathrm{I\!R}^3$. Then it would not surprise us that correlations between certain random binary events turn out to be sinusoidal rather than linear, because that is what the topology of the 3-sphere dictates [4][10]. More generally, if we had proof that we were living in a parallelized 7-sphere rather than $\mathrm{I\!R}^3$, then it would not surprise us that correlations between certain random binary events turn out to be stronger than linear, because that is what the topology of the 7-sphere dictates [5][6]. We would then not worry about contextuality or non-locality, but only about the topology of the 7-sphere. But that is precisely what my framework is suggesting. It is suggesting that quantum correlations are the evidence, not of non-locality or non-reality of any kind, but of the fact that we are living in a parallelized 7-sphere.

In fact, quite independently of Moldoveanu's failed attempts, in my view the whole fashion of simulating EPR correlations on a computer is completely wrong headed. It is based on serious misconceptions about the true physical and mathematical reasons for the existence of EPR correlations in Nature [5]. In all real-life demonstrations of the correlations, Alice and Bob are known to always observe truly random outcomes of their measurements: $\mathscr{A} = \pm 1$ and $\mathscr{B} = \pm 1$ [17]. Therefore, as correctly recognized by Bell, no local functions of the form $\mathscr{A}(\mathbf{a},\,\lambda)$ and $\mathscr{B}(\mathbf{b},\,\lambda)$ can reproduce the observed correlation, *unless the topological properties of the physical space itself are taken into account.* In the language of my model this means that one must first model the physical space, not as $\mathrm{I\!R}^3$, but as S^3, which differs from $\mathrm{I\!R}^3$ only by a single point at infinity [4]. By contrast, what is usually tried in attempts to simulate my model is a completely wrong headed approach, based on an implicit assumption that the numbers $\mathscr{A}$ and $\mathscr{B}$ are only apparently but not truly random, and if only one can somehow discover the correct functional dependence of these numbers on the disposition of the apparatus and hidden variables then the correct correlation between them would emerge. However, as Bell convincingly demonstrated long ago [1], one can never reproduce the sinusoidal correlation in this manner. For the EPR correlation are what they are because of the topological properties of the physical space itself [10], *not* because there exists some as-yet-uncovered hidden order in the randomness of $\mathscr{A}$ and $\mathscr{B}$. Moldoveanu would have saved himself a lot of time and effort had he appreciated this basic message of my framework.

In any case, a simply-connected model such as mine cannot possibly be either proved or disproved by its numerical simulation. A simulation of a model is an implementation of its analytical details, not an experiment that can either prove or disprove its validity. If reality can be so simply simulated then there would be no need for the staggeringly expensive actual experiments. Reality is mathematically far richer and profounder than what a computer can fathom.

VIII. COUNTING THE NUMBER OF TRIVECTORS HAS NO BEARING ON THE INTEGRATION

In his preprint [13] Moldoveanu makes a claim that "Isotropically weighted averages of non-scalar part of correlations and measurement outcomes cannot be both zero." To justify this claim he then enters into some strange counting of the $\boldsymbol{\mu}$'s in the integrations (18) and (19) of my primary paper [3]. He concludes from this counting that evaluation of both of these integrations cannot be right. However, while counting the $\boldsymbol{\mu}$'s he does not seem to realize that there is no operational difference between the non-scalar part of correlations and measurement outcomes. The non-scalar part of the correlation is just another possible measurement outcome, albeit along the exclusive direction $\mathbf{a}\times\mathbf{b}$. He also fails to take into account the obvious fact that the vector manifold in question over which the integrations are performed is related to the 3-sphere, which has a highly non-trivial topology reflected in the identity (17) of my paper. Moreover, he fails to recognize that equations (18) and (19) involve integrations over a random variable distributed over this non-trivial topology, and therefore a simple counting of the $\boldsymbol{\mu}$'s cannot possibly yield any insight into what the actual evaluation of a given integration would yield. In any case, for his conclusions he relies on one of his previous arguments which I have already refuted in section II above. Thus, in the light of the explicit and unambiguous results (31) and (32) obtained above, his intended argument here has neither any relevance nor any meaning for my model.

IX. TWIST IN THE CLIFFORD PARALLELS CANNOT BE REVEALED BY AN INCORRECT ROTOR

By now it is quite evident that much of Moldoveanu's discussion has nothing whatsoever to do with my model. A further illustration of this fact is his false claim that in Ref. [4] I have used an incorrect rotor to parallel transport a "bivector" (in fact I parallel transport a multivector, but never mind). What he fails to acknowledge, however, is that the parallel transport in question is part of a heuristic demonstration of an already proved result. By this stage in Ref. [4] I have rigorously derived the EPR correlation (as in equation (31) above), and already demonstrated how the correct combination of measurement outcomes, namely $++$, $--$, $+-$, and $-+$, arises due to a twist in the fibration of the 3-sphere, which is taken in my model as the physical space. The purpose of the heuristic demonstration is then to provide an additional intuitive understanding of how this twist brings about the strong quantum correlation.

The rotor I have used for this demonstration (with tildes on the vectors dropped to simplify notation) is of the form

$$\mathbf{a}\,\mathbf{b} \,=\, \mathcal{R}_{\mathbf{ab}} \,=\, \exp\left\{(\,I\cdot\mathbf{c}\,)\,\phi_{\mathbf{ab}}\right\}\,, \quad \text{with}\;\; \mathbf{c} := \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} \,\equiv\, \frac{\mathbf{a}\times\mathbf{a}'}{|\mathbf{a}\times\mathbf{a}'|} \,\equiv\, \frac{\mathbf{b}\times\mathbf{b}'}{|\mathbf{b}\times\mathbf{b}'|}\,, \tag{43}$$

where the respective angles $\psi_{\mathbf{aa}'}$ and $\psi_{\mathbf{bb}'}$ between $\mathbf{a}$ and $\mathbf{a}'$ and $\mathbf{b}$ and $\mathbf{b}'$ are assumed to be infinitesimally small. The reason for this particular choice of rotor has to do with the fact that it happens to be the correct choice to illustrate the twist in the Hopf fibration of the 3-sphere [20]. It is well known that this twist can be quantified by the relation

$$e^{i\psi_b} \,=\, e^{i\phi}\,e^{i\psi_a}\,, \tag{44}$$

where ψ_a and ψ_b are fiber coordinates above the two hemispheres of a 2-sphere (taken as the base manifold), ϕ is an angle parameterizing a thin strip around its equator, and $e^{i\phi}$ is the transition function gluing the two sections into a full 3-sphere [20]. It is easy to see from this equation that the fiber coordinates match perfectly at the angle $\phi = 0$ (modulo 2π), but differ from each other at all other intermediate angles. For instance, $e^{i\psi_a}$ and $e^{i\psi_b}$ differ by a minus sign at $\phi = \pi$. Now in the coordinate-free language of geometric algebra the above relation can be expressed as

$$\mathbf{b}\,\mathbf{b}' \,=\, \mathbf{a}\,\mathbf{b}\,\mathbf{a}\,\mathbf{a}' \tag{45}$$

for all $\mathbf{a}$'s and $\mathbf{b}$'s $\in \mathrm{I\!R}^3$, provided we identify the infinitesimal angles $\psi_{\mathbf{aa'}}$ and $\psi_{\mathbf{bb'}}$ between $\mathbf{a}$ and $\mathbf{a}'$ and $\mathbf{b}$ and $\mathbf{b}'$ with the fiber coordinates ψ_a and ψ_b, and the finite angle between $\mathbf{a}$ and $\mathbf{b}$ with the generator of the transition function $e^{i\phi}$. In the language of my model this generalized relation is then equivalent to equation (45) of Ref. [4]:

$$(+I\cdot\mathbf{b})\,(+\boldsymbol{\mu}\cdot\mathbf{b}')\,=\,\mathcal{R}_{\mathbf{ab}}\,\{(+I\cdot\mathbf{a})\,(+\boldsymbol{\mu}\cdot\mathbf{a}')\}\,. \tag{46}$$

It is now clear that the rotor I have used in Ref. [4] is *the* correct rotor for the purpose at hand. The rotor Moldoveanu has considered, on the other hand, has no relevance either for my model or for the demonstration of the twist under consideration. To justify his choice he goes on to produce arguments which further reveal his lack of appreciation, not only of the topology of the 3-sphere, but also of some of the most basic facts about geometric algebra. For instance he repeatedly identifies the quantity $[(+I\cdot\mathbf{a})\,(+\boldsymbol{\mu}\cdot\mathbf{a}')]$ as "bivector" when it is evidently a multivector (in fact a quaternion or a rotor). Consequently, his subsequent discussion is marred by nonsensical statements like "...the bivectors $[(+I\cdot\mathbf{a})\,(+\boldsymbol{\mu}\cdot\mathbf{a}')]$ and $[(+I\cdot\mathbf{b})\,(+\boldsymbol{\mu}\cdot\mathbf{b}')]$ are actually identical because they have the same orientation, magnitude, and sense of rotation." But they are not [4][5]. The two quaternions in question in fact represent two entirely different points of the 3-sphere [4]. These two points are anything but identical. What is more, because he thinks that $[(+I\cdot\mathbf{a})\,(+\boldsymbol{\mu}\cdot\mathbf{a}')]$ is a "bivector" he suggests a strange direction about which I should have parallel transported my "bivector." But his suggested direction has no relevance either for my model or for the 3-sphere.

X. NULL BIVECTOR IS A BIVECTOR THAT SUBTENDS NO MEANINGFUL AREA

As we just noted, in Ref. [4], in the course of the demonstration discussed above, I consider a bivector of the form

$$I\cdot\mathbf{c}\,,\qquad\text{with}\quad \mathbf{c} := \frac{\mathbf{a}\times\mathbf{a}'}{|\mathbf{a}\times\mathbf{a}'|}\,, \tag{47}$$

and state that in the limit $\mathbf{a}' \to \mathbf{a}$ the bivector $I\cdot\mathbf{c}$ reduces to a null bivector. Moldoveanu disputes this statement and produces a lengthy and convoluted argument to claim that the limit operation involved here is mathematically illegal. My statement, however, is trivially correct, and there is no illegal limiting operation of any kind involved in my reasoning. To see how trivial the issue is and how mistaken his argument is, let us define the following two vectors:

$$\widetilde{\mathbf{a}} \,=\, \frac{\mathbf{a}}{\sqrt{|\mathbf{a}\times\mathbf{a}'|}}\qquad\text{and}\qquad \widetilde{\mathbf{a}}' = \frac{\mathbf{a}'}{\sqrt{|\mathbf{a}\times\mathbf{a}'|}}\,. \tag{48}$$

We can now rewrite the bivector $I\cdot\mathbf{c}$ as

$$I\cdot\mathbf{c}\,=\,I\cdot(\widetilde{\mathbf{a}}\times\widetilde{\mathbf{a}}')\,=\,\widetilde{\mathbf{a}}\wedge\widetilde{\mathbf{a}}'. \tag{49}$$

Now it is true that in the limit $\mathbf{a}' \to \mathbf{a}$ not only the directions of the vectors $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$ tend to coincide but also their lengths tend to infinity. The question then is, whether or not the following statement is true:

$$\lim_{\mathbf{a}'\to\,\mathbf{a}}\;\widetilde{\mathbf{a}}\wedge\widetilde{\mathbf{a}}'\,=\,0\,. \tag{50}$$

The answer is that it is trivially true. Despite the fact that in the limit $\mathbf{a}' \to \mathbf{a}$ the lengths of the vectors $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$ tend to infinity, there can be no meaningful area spanned by the resulting vector $\widetilde{\mathbf{a}} = \widetilde{\mathbf{a}}'$, and hence what emerges in the limit is a null bivector. This is a standard understanding of null bivector found in any textbook [14]. Just as a null vector is a vector that has no meaningful length, a null bivector is a bivector that has no meaningful area.

But let us not rely on the authority of good books on geometric algebra. Let us instead see the absurdity of Moldoveanu's argument for ourselves. His claim—supported by lengthy and elaborate arguments—is that the bivector $\widetilde{\mathbf{a}}\wedge\widetilde{\mathbf{a}}'$ remains non-null even in the limit $\mathbf{a}' \to \mathbf{a}$. In other words, his argument is that the bivector $\widetilde{\mathbf{a}}\wedge\widetilde{\mathbf{a}}'$ remains non-null even when the vectors $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{a}}'$ are one and the same vector and the area spanned by it is zero. This is as absurd as claiming that a vector remains non-null even when the distance between its end points is zero.

In sum, because of various misconceptions about the basic concepts in geometric algebra, many of the categorical assertions made by Moldoveanu in his preprint [13] are simply wrong. In particular, there is nothing wrong with my statement in Ref. [4] that $I\cdot\mathbf{c}$ reduces to a null bivector in the limit $\mathbf{a}' \to \mathbf{a}$. Consequently, my argument starting from Eq. (42) and ending after Eq. (46) in that paper is an entirely cogent argument, illustrating how the illusion of quantum non-locality arises in the EPR case due to a twist in the Hopf fibration of 3-sphere. Could Moldoveanu's failure to appreciate this simple illustration be due to his failure to appreciate the topology of the 3-sphere itself?

XI. CORRELATION BETWEEN RAW SCORES IS GIVEN BY COVARIANCE BETWEEN STANDARD SCORES

At several junctures in the preprint [13] Moldoveanu reasons as follows: Since the outcomes of Alice and Bob are "always the same" for all directions according the definitions (36) and (37) above, it follows that the product of these outcomes, and hence the correlation between them, will always be equal to $+1$ regardless of the measurement directions chosen by Alice and Bob, contradicting the prediction $-\mathbf{a}\cdot\mathbf{b}$ of quantum mechanics. This conclusion, which is of course false on several counts, stems from a failure to appreciate the basic rules of statistical inference, not to mention the basic topology of the 3-sphere [4][5]. More specifically, what Moldoveanu has failed to recognize is that $\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu})$ and $\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu})$ are generated with *different* bivectorial scales of dispersion, and hence the correct correlation between them can be inferred only by calculating the covariation of the corresponding standard scores $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$:

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,=\,\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,\mathscr{A}(\mathbf{a},\,\boldsymbol{\mu}^i)\,\mathscr{B}(\mathbf{b},\,\boldsymbol{\mu}^i)\right]\,=\,\lim_{n\,\gg\,1}\left[\frac{1}{n}\sum_{i\,=\,1}^{n}\,(\,\boldsymbol{\mu}^i\cdot\mathbf{a})(\,\boldsymbol{\mu}^i\cdot\mathbf{b})\right]\,=\,-\,\mathbf{a}\cdot\mathbf{b}\,. \tag{51}$$

I have explained the relationship between raw scores and standard scores in great detail in Ref. [4], with explicit calculations for the optical EPR correlations observed in both Orsay and Innsbruck experiments [17]. Now we have already seen a different aspect of Moldoveanu's difficultly with these basic statistical concepts in section VI above. This is surprising, because the rules of correlation statistics were discovered by Galton and Pearson over a century ago [21], and today we learn about them in high-school. To be sure, I have used these rules within the setting of geometric algebra, but Moldoveanu appears to be familiar with the language of geometric algebra, so that does not quite explain his neglect of these rules. In any event, much of his discussion concerning the phenomenology of my model is marred by his evident failure to appreciate the distinction between raw scores and standard scores. This is compounded by his lack of appreciation that quantum correlations are understood within my framework as purely topological effects, not contextual effects. By contrast, a complete explanation of how these concepts work within my model and how they lead to predictions matching exactly those of quantum mechanics can be found in Ref. [4].

XII. WHAT IS ASSERTED IN MOLDOVEANU'S PREPRINTS IS NOT NECESSARILY WHAT IS TRUE

Moldoveanu's preprints contain numerous assertions about my framework that are simply not true. It may not be worthwhile to bring them all out, but let me highlight some examples here to show the extent of the problem:

1. It is asserted that I start my disproof with different assumptions from those of Bell. This is clearly false. It is clear from the first two equations of Ref. [2] that both Bell and I start with the variables $\mathscr{A}(\mathbf{a},\,\lambda)=\pm\,1$ and $\mathscr{B}(\mathbf{b},\,\lambda)=\pm\,1$. Thus there is no difference between the starting assumptions of my disproof and those of Bell.

2. It is asserted that my counterexample to Bell's theorem is not a strictly mathematical disproof. But of course it is. As we saw in section IV above, my counterexample [2] contradicts the mathematical claim made by Bell.

3. It is asserted that my interpretation of the EPR argument is "unusual." This is plainly false. I have faithfully followed the standard interpretation of EPR argument as offered, for example, by Clauser and Shimony [22] and by Greenberger *et al.* [23] (see especially footnote 10 of the latter). These are standard references in the subject.

4. It is asserted that I unjustifiably identify "completeness" with "parallelizability" within the context of EPR argument. This is an oversimplification of my rather subtle argument relating locality to division algebras [5].

5. It is asserted that my counterexample does not satisfy the conditions of remote context independence and remote outcome independence. But it certainly does, as can be readily seen from the very first two equations of Ref. [2].

6. It is asserted that my papers "suffer" from a convention ambiguity, and that I illegally mix conventions during computations. Nothing can be farther from the truth. There is no such ambiguity or mixing within my model.

7. It is asserted that associating a hidden variable to an abstract computation convention is unphysical. That is certainly true, but that is not what I am doing. As is clear from the discussions in sections II and III above, orientation of the physical space, which is the hidden variable λ in my model, has nothing to do with conventions.

8. It is asserted that I make an extraordinary claim without proof with regard to equation (149) of Ref. [6]. This is simply not true. I offer detailed proof of how my argument goes through, especially in the case of 7-sphere.

9. It is asserted that my model for the EPR correlation does not respect the detector swapping symmetry. That this is simply false can be seen at once from the definitions (36) and (37) used above for the local variables of Alice and Bob. Swapping the detectors $(-\,I\cdot\mathbf{a}\,)$ and $(+\,I\cdot\mathbf{b}\,)$ of Alice and Bob does not change either the measurement statistics or correlation. Even changing the sense of one of them only induces a statistically inconsequential sign change in the correlation. Further discussion on this issue can be found in Refs. [8] and [9].

10. It is asserted that my model predicts the same correlation regardless of the spin state of the particles in the EPR-Bohm experiment. That this is not true can be seen at once from the four examples discussed in Ref. [6]. In particular, the cosine correlation produced by the rotationally invariant singlet state corresponds to the correlation between two equatorial points of the 3-sphere in my model, whereas correlations produced by the rotationally non-invariant Hardy state correspond to correlations among a set of non-equatorial points of the 3-sphere. The GHZ correlations, on the other hand, correspond to the equatorial points of the 7-sphere instead of the 3-sphere. In general different quantum correlations correspond to different sets of points of the 7-sphere [6].

11. It is asserted that my model for the EPR-Bohm correlation does not satisfy the Malus's law for sequential spin measurements, and therefore it contradicts the experimental observations as well as the predictions of quantum mechanics for such measurements. This claim is demonstrably false. An explicit proof of the Malus's law within my model can be found in section V of Ref. [9], with further elaborations and detailed examples of specific spin cases in section III or Ref. [5]. The key ingredient in the proof is again the topology of the parallelized 3-sphere.

XIII. CONCLUSION

I have shown that much of the criticism of my work in Moldoveanu's preprints stems from incorrect understanding of my local-realistic framework. I have also shown that, contrary to the claims made in his preprints, my framework is manifestly non-contextual. In particular, quantum correlations are understood within it as purely topological effects, not contextual effects. Moreover, I have highlighted a number of conceptual and mathematical errors in Moldoveanu's discussion of my framework. Some of these errors are quite elementary. For example, in a couple of his arguments Moldoveanu identifies the quantity $[(+\,I\cdot\mathbf{a}\,)\,(+\,\boldsymbol{\mu}\cdot\mathbf{a}'\,)]$ as a bivector when it is evidently a multivector (in fact it is a quaternion or a rotor—cf. section IX above). This misleads him into developing an erroneous argument against my demonstration of the well known twist within a 3-sphere [4][20]. Moreover, much of Moldoveanu's analysis of my work appears to be an attempt to fit my framework into one preconceived conceptual box or another. The picture he thereby ends up creating has therefore little to do with my framework. Furthermore, some of the concerns raised by Moldoveanu are simply rewording of the concerns previously raised by others, which I have already addressed elsewhere [8]. Notwithstanding these difficulties, I have used this opportunity to further elucidate at least some aspects of my local-realistic framework. I hope this reassures the reader that my work is perfectly cogent and error free.

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