What Really Sets the Upper Bound on Quantum Correlations?

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The discipline of parallelization in the manifold of all possible measurement results is shown to be responsible for the existence of all quantum correlations, with the upper bound of $2\sqrt{2}$ on their strength stemming from the maximum of possible torsion within all norm-composing parallelizable manifolds. A profound interplay is thus uncovered between the existence and strength of quantum correlations and the parallelizability of the spheres S^0, S^1, S^3, and S^7 necessitated by the four real division algebras. In particular, parallelization within a unit 3-sphere is shown to be responsible for the existence of EPR and Hardy type correlations, whereas that within a unit 7-sphere is shown to be responsible for the existence of all GHZ type correlations. Moreover, parallelizability in general is shown to be equivalent to the completeness criterion of EPR, in addition to necessitating the locality condition of Bell. It is therefore shown to predetermine both the local outcomes as well as the quantum correlations among the remote outcomes, dictated by the infinite factorizability of points within the spheres S^3 and S^7. The twin illusions of quantum entanglement and non-locality are thus shown to stem from the topologically incomplete accountings of the measurement results.

I. INTRODUCTION

Despite their ostensible cogency, all Bell type arguments are fundamentally flawed from their very inception [1][2][3]. They are based on circular reasoning, stemming from the topologically naïve assumption that functions of the form

$$A(\mathbf{n},\,\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow\mathcal{I}\subseteq\mathrm{I\!R} \tag{1}$$

can provide complete determination of every possible measurement result concerning a given physical system, with $\mathbf{n}\in\mathrm{I\!R}^3$ representing a direction of measurement, $\lambda\in\Lambda$ representing a complete initial state of the system, and $\mathcal{I}\subseteq\mathrm{I\!R}$ representing the set of all possible measurement results in question. This assumption, however, is demonstrably false. Elementary topological scrutiny reveals that no such function—or its probabilistic counterpart $P(A\,|\,\mathbf{n},\,\lambda)$—is capable of providing a complete account of every possible measurement result, even for the simplest of the quantum systems. As we have shown elsewhere [4][5][6][7][8][9], unless enumerated by local functions of the topologically correct form

$$A(\mathbf{n},\,\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^2\subset S^3\hookrightarrow\mathrm{I\!R}^4, \tag{2}$$

with their codomain S^2 being a *simply-connected* equatorial 2-sphere within a parallelized 3-sphere (composed of numbers $+1$ or -1), it is not possible to account for every possible measurement result for any two-level quantum system. More precisely, unless the measurement results of Alice and Bob are represented by the equatorial points of two parallelized 3-spheres, the completeness criterion of EPR is not satisfied, and then there is no meaningful Bell's theorem to begin with [4][5][8]. In fact, naïvely replacing the *simply-connected* codomain $S^2\subset S^3$ in the above function by a *totally-disconnected* set $S^0\equiv\{-1,\,+1\}$, as routinely done within all Bell type arguments, is a guaranteed way of introducing incompleteness in the accounting of measurement results from the very start [5][8]. Moreover, any probabilistic reformulation of prescription (1)—as popularized by Wigner [10] and Bell [11]—cannot respect the completeness criterion of EPR, for the probabilistic rules of inference are inherently incapable of guaranteeing a complete specification of every *individual* physical system. Worse still, all such probabilistic counterparts $P(A\,|\,\mathbf{n},\,\lambda)$ of $A(\mathbf{n},\,\lambda)$ surreptitiously presuppose vector-algebraic models of the Euclidean space, which we have shown to be both physically and topologically incomplete [5][8]. In fact—as we shall soon show—the *only* unambiguously complete way of local-realistically accounting for every possible measurement result is by means of unit bivectors of the form

$$\mathrm{I\!R}^4\hookleftarrow S^3\supset S^2\ni\boldsymbol{\mu}\cdot\mathbf{n}\,=\,\pm\,1\;\text{ about }\;\mathbf{n}\in\mathrm{I\!R}^3\subset\mathrm{I\!R}^4 \tag{3}$$

as we have argued [6][5][7][8], for such bivectors intrinsically represent the equatorial points of a parallelized 3-sphere. Moreover, once parallelized by a field of such bivectors (and their extensions to $\mathrm{I\!R}^4$), a 3-sphere remains as closed

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under multiplication of its points as the 0-sphere: $\{-1, +1\}$. As a result, setting the codomain of the function $A(\mathbf{n}, \lambda)$ to be the space of bivectors—which is isomorphic to an equatorial 2-sphere within a parallelized 3-sphere—guarantees that the locality or factorizability condition of Bell is automatically satisfied, for any number of measurement settings:

$$(A_{\mathbf{a}}\, B_{\mathbf{b}}\, C_{\mathbf{c}}\, D_{\mathbf{d}}\, \dots)(\boldsymbol{\mu}) : S^2 \times S^2 \times S^2 \times S^2 \dots \longrightarrow S^3 \;\text{ implies}$$
$$S^3 \ni (A_{\mathbf{a}}\, B_{\mathbf{b}}\, C_{\mathbf{c}}\, D_{\mathbf{d}}\, \dots)(\boldsymbol{\mu}) \,=\, A_{\mathbf{a}}(\boldsymbol{\mu})\, B_{\mathbf{b}}(\boldsymbol{\mu})\, C_{\mathbf{c}}(\boldsymbol{\mu})\, D_{\mathbf{d}}(\boldsymbol{\mu}) \,\dots \text{ for all } A_{\mathbf{a}}(\boldsymbol{\mu}),\, B_{\mathbf{b}}(\boldsymbol{\mu}),\, C_{\mathbf{c}}(\boldsymbol{\mu}),\, D_{\mathbf{d}}(\boldsymbol{\mu}) \dots \in S^2. \quad (4)$$

It is then easy to show that [4], although the incomplete local functions (1) can only give rise to linear correlations,

$$\mathcal{E}(\mathbf{a}, \mathbf{b}) \,=\, \int_{\Lambda} A(\mathbf{a}, \lambda)\, B(\mathbf{b}, \lambda)\; d\rho(\lambda) \,=\, -1 + \frac{2}{\pi}\, \cos^{-1}(\mathbf{a}\cdot\mathbf{b})\,, \quad (5)$$

the topologically *complete* local functions (2) *can* and *must* give rise to the super-linear EPR-Bohm correlations [5],

$$\mathcal{E}(\mathbf{a}, \mathbf{b}) \,=\, \int_{\Lambda} A(\mathbf{a}, \boldsymbol{\mu})\, B(\mathbf{b}, \boldsymbol{\mu})\; d\rho(\boldsymbol{\mu}) \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \quad (6)$$

contrary to the prevalent belief that no local-realistic theory can reproduce quantum mechanical predictions. In fact, as we have shown elsewhere [4], the local functions (2) lead to violations of the CHSH inequalities [12] of the form

$$|\mathcal{E}(\mathbf{a}, \mathbf{b}) + \mathcal{E}(\mathbf{a}, \mathbf{b}') + \mathcal{E}(\mathbf{a}', \mathbf{b}) - \mathcal{E}(\mathbf{a}', \mathbf{b}')| \,\leq\, 2\sqrt{1 - (\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})} \,\leq\, 2\sqrt{2}\,, \quad (7)$$

in quantitatively precise agreement with the predictions of quantum mechanics—angle by angle, direction by direction. What is more, the quantum mechanical predictions of even rotationally non-invariant states such as the GHZ states [2] and Hardy state [3]—and indeed the quantum mechanical predictions of any arbitrary entangle state—can be reproduced *exactly* within such a local-realistic framework, as we have demonstrated elsewhere [4][5][6][7][8]. All that is required for this purpose is to replace each incomplete map (1) with a topologically complete map of the form

$$A(\mathbf{n}, \lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \Sigma, \quad (8)$$

where Σ is the *closed* topological space of all possible measurement results for a given physical system. In fact, as we shall see, it is both physically and mathematically incorrect to take the codomain of the function $A(\mathbf{n}, \lambda)$ anything other than the space of all possible measurement results (*i.e.*, both actual as well as counterfactual results) [4][5].

In this paper we wish to go a step further and identify the true local-realistic reason behind the existence and strength of *all* quantum correlations. This will in turn lead us to identify the true reason behind the existence of the upper bound on the strength of all quantum correlations. To this end, we shall first identify the parallelizability of the 3-sphere—or equivalently the triviality of its tangent bundle—as the *raison d'être* for the EPR-Bohm correlations. In particular, we shall show that the deviation in strength of the correlations from linear, (5), to super-linear, (6), is nothing but a measure of Cartan torsion within the parallelized 3-sphere. That is to say, while the linear correlations reflect the vanishing torsion of the trivially parallelized flat Euclidean space, the super-linear correlations reflect the maximum strength of the non-vanishing torsion within a non-trivially parallelized 3-sphere. More generally, we shall show that the upper bound on the strength of quantum correlations is set by the maximum of possible torsions in all possible norm-composing parallelizable manifolds, considered as possible spaces of all possible measurement results for any quantum system—*i.e.*, considered as the codomains of the Bell type functions $A(\mathbf{n}, \lambda) : \mathrm{I\!R}^3 \times \Lambda \rightarrow \Sigma$. This reveals a profound interplay between the existence and strength of quantum correlations and the parallelizability of the spheres S^0, S^1, S^3, and S^7, which are the only possible norm-composing parallelizable manifolds permitted by the existence of the four real division algebras: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The latter fact stems from some powerful and well known mathematical theorems, with far-reaching consequences for the entire edifice of mathematics and physics [13]. On the basis of these theorems, we shall prove that the upper bound on the strength of quantum correlations exists because of topological reasons, *regardless of quantum mechanics*, and that *local* causality—dictated by the discipline of parallelization within the manifold of all possible measurement results—is all that is necessary to understand it.

Conversely, our analysis will make it plain that it is topologically impossible for any Bell type map (1) to constitute a manifold of all possible measurement results, even for the simplest of the quantum systems. Such a naïve map would therefore necessarily fail to satisfy the completeness criterion of EPR, giving rise to the illusion of non-locality. The essential mathematical reason for this is the fact that parallelizability is a deeply topological concept, best understood in the language of fiber bundles [14][15][16]. It disciplines not only the local points (*i.e.*, actual measurement results) within the set of all possible measurement results, but also their neighborhood relations with other local points, whether realized actually or counterfactually. For this reason the prevalent belief in "quantum non-locality"—with its topologically unscrupulous treatment of the set of all possible measurement results—is necessarily false. It stems from circular reasoning, arising from the intrinsic incompleteness of all Bell type maps $A(\mathbf{n}, \lambda) : \mathrm{I\!R}^3 \times \Lambda \rightarrow \mathcal{I} \subseteq \mathrm{I\!R}$.

(At this stage the reader may wish to skip to the concluding section to find a summary of our main results.)

II. COMPLETING THE INCOMPLETE ACCOUNTING BY BELL

To appreciate these facts, let us look at them a little more closely. Let T_pS^3 denote the tangent space to a 3-sphere at a point p. Then the tangent bundle of S^3 can be expressed as

$$\mathrm{T}S^3 = \bigcup_{p\,\in\, S^3} \{p\} \times T_pS^3. \tag{9}$$

Now this tangent bundle happens to be *trivial*:

$$\mathrm{T}S^3 \equiv S^3 \times \mathrm{I\!R}^3. \tag{10}$$

And as we shall soon appreciate, it is this elementary topological fact, and *not* quantum entanglement, that is truly responsible for both the existence and strength of the EPR-Bohm type correlations.

The triviality of the tangent bundle $\mathrm{T}S^3$ means that the 3-sphere is *parallelizable.* A k-dimensional manifold is said to be parallelizable if it admits k vector fields that are linearly-independent everywhere. Thus on a 3-sphere we can always find three linearly-independent vector fields that are nowhere vanishing [15]. These can then be used to define a basis of a tangent space at each of its points. As a result, a single coordinate chart can be defined on a 3-sphere that fixes each of its points uniquely. Informally, a manifold is said to be parallelizable if it is possible to set all of its points in a smooth flowing motion at the same time, in *any* direction. Rather astoundingly, this turns out to be possible only for the 0-, 1-, 3-, and 7-spheres [13][17]. Thus parallelizability of these spheres happens to be an exceptionally special topological property. One way to appreciate it is by considering a manifold that is not parallelizable. For example, it is not possible to set every point of a 2-sphere in a smooth flowing motion, even in *one* direction. However you may try, there will always remain at least one fixed point—a pole—that will refuse to move. This makes it impossible, for example, to cover the Earth with a single coordinate chart. For similar reasons, parallelizability of the 3-sphere, or equivalently the triviality of its tangent bundle, turns out to be indispensable for respecting the completeness criterion of EPR. And since this criterion is *the* starting point of Bell's theorem, understanding the parallelizability of 3-sphere turns out to be indispensable for understanding the topological error involved in all Bell type arguments.

To appreciate this in full detail, recall that according to the completeness criterion of EPR

every element of the physical reality must
have a counterpart in the physical theory.

Motivated by the EPR argument [18], Bell naïvely thought that one could provide a complete specification of the values of all possible elements of reality (*i.e.*, of all possible measurement results) by means of functions of the form

$$A(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \mathcal{I} \subseteq \mathrm{I\!R}. \tag{11}$$

But, as we have discussed elsewhere [4][5][8], it is not possible to provide a complete account of all possible measurement results by means of such a function, unless its codomain is homeomorphic to $S^2 \subset S^3$. For suppose we unpack it as

$$A(\mathbf{n},\,\lambda) : \begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \cdot \\ \mathbf{n}_j \\ \cdot \\ \cdot \end{pmatrix} \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_k \\ \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} A(\mathbf{n}_1,\,\lambda_2) = +1 \\ A(\mathbf{n}_2,\,\lambda_1) = -1 \\ \cdot \\ A(\mathbf{n}_j,\,\lambda_k) = +1 \\ \cdot \\ \cdot \end{pmatrix}. \tag{12}$$

The question then is: What should be the codomain on the RHS of this expression? The answer is not too difficult to discern, but it requires us to recall both the logic of the EPR argument (as done in Ref. [4]) and what is meant by a *function* in mathematics [19]. In particular, it requires us to recall that a function is not defined in mathematics until its codomain is precisely specified [19]. Now in the standard EPR-Bell case there are infinitely many possible spin components that could be measured by either Alice or Bob—one corresponding to each direction $\mathbf{n} \in \mathrm{I\!R}^3$. Thus there is a one-to-one correspondence between the set of all obtainable results and the points of a unit 2-sphere defined by $||\,\mathbf{n}\,|| = 1$. That is to say, *the set of all possible measurement results*—both actual and counterfactual—is homeomorphic to a unit 2-sphere. And according to the reality criterion of EPR there would then exist an element of reality corresponding to each of these results. Moreover, if a function such as $A(\mathbf{n},\,\lambda)$ is to provide a *complete* accounting of values for all of these elements of reality (as presumed by Bell), then it ought to be valid for all possible measurement results, not just a handful of them. But it is evident from the above equation that no function of the form $A(\mathbf{n},\,\lambda)$ can specify all possible measurement results, unless it is a bijective function with 2-sphere as its codomain. In other words, no function of the form $A(\mathbf{n},\,\lambda)$ can satisfy the completeness criterion of EPR, unless its codomain is homeomorphic to a 2-sphere. For suppose that—following Bell—we assume the codomain of $A(\mathbf{n},\,\lambda)$ to be a totally disconnected set $\{-1,\,+1\}$. To begin with, this choice would reduce the function to many-to-one; but never mind. What is worse is that the set of all possible measurement results will then be a disconnected set (in the topological sense [20]). This is fine for any finite number of measurement results, but not for *all* possible measurement results, because a disconnected set of numbers cannot be rendered homeomorphic to a simply-connected set (such as S^2). As we just saw, the set of all possible measurement results is a 2-sphere, which is a compact, simply-connected set, and it is well known that such a set cannot be put in one-to-one correspondence with even $\mathbb{R}^2$ or $\mathbb{R}$, let alone a disconnected set of numbers made out of ± 1 (cf. Refs. [4] and [5]). For the same reason, with any $\mathcal{I} \subseteq \mathbb{R}$ as its codomain, the function $A(\mathbf{n},\,\lambda)$ will always leave at least one measurement result unaccounted for [4][5]. That is to say, it would be topologically impossible to account for every possible measurement result by means of a function like $A(\mathbf{n},\,\lambda)$, unless its codomain is homeomorphic to a 2-sphere. In sum, the correct functional form of $A(\mathbf{n},\,\lambda)$ cannot be inferred by considering only a finite number of measurement results if the criterion of completeness is to be respected. A careful topological consideration of the physical scenario involving all possible measurement results is inevitable. And that clearly dictates that the codomain of $A(\mathbf{n},\,\lambda)$ must at least be homeomorphic to a 2-sphere [4][5].

Actually, even setting the codomain of $A(\mathbf{n},\,\lambda)$ to a 2-sphere is not enough to guarantee completeness, because, as we noted earlier, 2-sphere is not a parallelizable sphere. It is not possible to set every single point of a 2-sphere in smooth flowing motion simultaneously, and as a result it is not possible to cover the 2-sphere with a single coordinate chart. As is well known to every cartographer, such a chart would become indeterminate at least at one point (what is the longitude at Earth's North Pole, for example?). By contrast, 1-sphere does not lead to such a problem. We can always find a single coordinate chart that fixes each point of S^1 uniquely. The problem in the case of 2-sphere is a manifestation of the well known "hairy ball theorem" in algebraic topology, which states that no non-vanishing continuous tangent vector field (a Killing field) can exist on any even-dimensional sphere. For our concerns, however, there is a simple way out of this problem. What we must ensure is that the codomain of the Bell type function $A(\mathbf{n},\,\lambda)$ for the EPR-Bohm case is an equatorial 2-sphere within a unit 3-sphere. Then, since 3-sphere is an odd-dimensional sphere, the hairy ball theorem would not prevent us from finding a coordinate chart that specifies each of its points uniquely—and consequently each of the points of its equatorial 2-sphere uniquely—allowing us to satisfy the completeness criterion of EPR unambiguously.[1] What would permit this of course is the fact that 3-sphere is parallelizable [21].

The parallelizability of the 3-sphere is not guaranteed for all of its representations, however, since there are more than one ways to embed one space into another. For example, the standard Cantor set and Antoine's necklace are two homeomorphic subsets of $\mathbb{R}^3$, but their configurations are topologically quite distinct from one another, because of the manner in which they are situated within $\mathbb{R}^3$ [22]. Similarly, one may consider embedding S^3 into $\mathbb{R}^4$ as

$$X_0^2 \,+\, X_1^2 \,+\, X_2^2 \,+\, X_3^2 \,=\, 1, \tag{13}$$

but this will not do if ordinary vector basis in $\mathbb{R}^4$ are used for this purpose. The resulting representation of S^3 will not necessarily be parallelized. That is, given three linearly-independent vector fields forming a basis of the tangent space at one point of S^3, it will not always be possible to find three linearly-independent vector fields forming a basis of the tangent space at every other point of S^3. Therefore, in order to find a representation of S^3 that renders it parallelizable, we shall have to spell out the precise mathematical definition of parallelizability in a greater detail.

To this end, let $V_p \in T_pM$ and $V_q \in T_qM$ be two tangent vectors defined, respectively, at two arbitrary points p and q of a manifold M. In analogy with the flat Euclidean case, these vectors are said to be parallel to each other if the components of V_p in the basis of T_pM are equal to the components of V_q in the basis of T_qM [15][23]. Then, for a general manifold, the possibility of continuously transporting a basis of T_pM to those of T_qM allows us to introduce a notion of parallelity for such vectors that is *absolute* in the sense that it is not dependent on the path connecting p and q. More precisely, a Riemannian manifold M is said to admit *absolute parallelism* if it is possible to define parallelity of two directions at two different points independently of the coordinates chosen, so that (1) every geodesic of M is parallel to itself at all of its points, and (2) the angle between a pair of tangent directions at one point on M is equal to the angle between the pair of parallelly transported tangent directions at any other point of M [24]. Thus the parallelism so defined is *conformal*, or angle preserving[2], as a result of being absolute. Moreover, it gives

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[1] Similar considerations show that for the three- and four-particle GHZ states EPR-completeness cannot be satisfied unless the codomain of the corresponding function $A(\mathbf{n},\,\lambda)$ is taken to be the equatorial 6-sphere contained within a unit 7-sphere, since 7-sphere is also a parallelizable sphere, and therefore can be coordinated just as unproblematically as the 3-sphere discussed here (cf. Refs. [4] and [21]).

[2] It is worth noting here that this conformality of absolute parallelism is what is responsible for the rotational invariance of the singlet state—i.e., the fact that EPR correlations depend *only* on the angle between the directions chosen by Alice and Bob and nothing else.

rise to a new connection on M that (1) leaves the metric tensor invariant, (2) has the same geodesics as the original connection, and (3) preserves the vanishing of the Riemann curvature tensor. In fact, if M is simply-connected, then the vanishing of the curvature tensor is both necessary and sufficient for the absolute parallelism defined above [25]:

$$R^{\alpha}{}_{\beta\,\gamma\,\delta} \,=\, \partial_\gamma\,\Omega^{\alpha}_{\beta\,\delta} \,-\, \partial_\delta\,\Omega^{\alpha}_{\beta\,\gamma} \,+\, \Omega^{\alpha}_{\sigma\,\gamma}\,\Omega^{\sigma}_{\beta\,\delta} \,-\, \Omega^{\alpha}_{\sigma\,\delta}\,\Omega^{\sigma}_{\beta\,\gamma} \,=\, 0 \tag{14}$$

with respect to the asymmetric connection

$$\Omega^{\gamma}_{\alpha\,\beta} \,=\, \Gamma^{\,\gamma}_{\alpha\,\beta} \,+\, \mathcal{T}^{\,\gamma}_{\alpha\,\beta}\,, \tag{15}$$

where $\Gamma^{\,\gamma}_{\alpha\,\beta}$ is the symmetric Levi-Civita connection and $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ is the totally antisymmetric torsion tensor. Thus, for simply-connected manifolds flatness is equivalent to parallelizability. The vanishing of the curvature tensor guarantees the path-independence of the parallel transport, which in turn guarantees the existence of a set of linearly-independent tangent vectors at every point of the manifold [15][23]. The parallel transport of any arbitrary vector defined on M can then be viewed simply as its translation on M (either left or right). For the special case of spaces with vanishing torsion, $\mathcal{T}^{\,\gamma}_{\alpha\,\beta} \equiv 0$, one sets $\Omega^{\gamma}_{\alpha\,\beta} = \Gamma^{\,\gamma}_{\alpha\,\beta}$, and the vanishing of the curvature tensor then leads to the flat Euclidean spaces. On the other hand, as Einstein noted in the context of his unified field theory, *there exist continua admitting absolute parallelism that are nevertheless not Euclidean.* For such continua the torsion tensor does not vanish. The 3-sphere was one of the first examples of such an absolutely parallelizable space with non-vanishing torsion, discovered by Clifford [23]. The auto-parallel geodesics in this case are non-co-spherical great circles, called Clifford parallels, with remarkable topological properties [26]. In fact, parallelized 3-sphere is entirely made up of such "skewed" great circles. Because of the non-vanishing torsion, these circles twist around each other, and yet remain parallel to each other all along, with each circle threading through every other in a highly intricate fashion [27]. Intuitively, then, absolutely parallelizable spaces closely resemble the familiar Euclidean space—in the sense that their curvature tensors vanish identically, and yet in many respects they are profoundly different spaces from the flat Euclidean space.

In the light of these extraordinary features of S^3, the reader ought to be struck by the naïvety of Bell's choice of a local prescription. Clearly, no simpleminded function like (1) with a totally disconnected codomain S^0 can provide a complete account of all possible measurement results constituting S^3. Neither can any probabilistic reformulation of equation (1) do justice to the topological subtleties inherent in the parallelizability of S^3. Only by explicitly finding a representation of the 3-sphere that satisfies all of the conditions of absolute parallelity specified above—i.e., explicitly finding a field of absolutely parallel tangent vectors well defined at every point of the 3-sphere—can the EPR criterion of completeness be respected. For only then can every point of the 3-sphere be unambiguously coordinated, and only then can the value of every possible element of reality be uniquely predicted, by means of the prescription

$$A(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^2 \subset S^3 \hookrightarrow \mathrm{I\!R}^4. \tag{16}$$

Fortunately, it turns out to be possible to find such an unambiguous representation of the 3-sphere by taking the basis of the vector $\mathbf{X}$ defined in (13) to satisfy the quaternionic [3] (or Clifford-algebraic) product rules (*cf.* pp 220 of Ref. [15])—*i.e.*, by rendering $\mathbf{X}$ to be a spinorial vector field

$$\mathbf{X} \,=\, X_0 \,+\, X_1\,(\mathbf{e}_2 \wedge \mathbf{e}_3) \,+\, X_2\,(\mathbf{e}_3 \wedge \mathbf{e}_1) \,+\, X_3\,(\mathbf{e}_1 \wedge \mathbf{e}_2), \tag{17}$$

with the bivector (or spinor) basis in $\mathrm{I\!R}^4$ [5][6][7][8]:

$$\{\,1,\;\; \mathbf{e}_2 \wedge \mathbf{e}_3\,,\;\; \mathbf{e}_3 \wedge \mathbf{e}_1\,,\;\; \mathbf{e}_1 \wedge \mathbf{e}_2\,\} \,\equiv\, \{\,1,\;\; I\cdot\mathbf{e}_1\,,\;\; I\cdot\mathbf{e}_2\,,\;\; I\cdot\mathbf{e}_3\,\}, \tag{18}$$

where $I := \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the fundamental trivector of the geometric algebra. The properties of this representation can be easily checked as follows. Suppose we are given a tangent space at the tip of a vector $\mathbf{X}_0 = (X_0,\,0,\,0,\,0) \in \mathrm{I\!R}^4$ spanned by these basis so that any arbitrary tangent bivector at the tip of $\mathbf{X}_0$ can be expressed as

$$I\cdot\mathbf{n} \,=\, n_1\,\mathbf{e}_2 \wedge \mathbf{e}_3 \,+\, n_2\,\mathbf{e}_3 \wedge \mathbf{e}_1 \,+\, n_3\,\mathbf{e}_1 \wedge \mathbf{e}_2\,. \tag{19}$$

Then the tangent bases $(\beta_1(\mathbf{X}),\;\beta_2(\mathbf{X}),\;\beta_3(\mathbf{X}))$ at any other $\mathbf{X} \in \mathrm{I\!R}^4$ can be found by taking a geometric product of the above basis with $\mathbf{X}$ using the bivector subalgebra

$$(I\cdot\mathbf{e}_j)\,(I\cdot\mathbf{e}_k) \,=\, -\,\delta_{jk} \,-\sum_{l=1}^{3}\epsilon_{jkl}\;(I\cdot\mathbf{e}_l)\,, \tag{20}$$

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[3] It is worth stressing here that within the geometric framework used in Refs. [4] to [8] as well as in the present work, there is nothing complex, imaginary, or "non-real" about quaternions and octonions. They are *real* geometric quantities, on par with real numbers [28].

which gives

$$\begin{aligned}
\beta_1(\mathbf{X}) &= (\mathbf{e}_2 \wedge \mathbf{e}_3)\,\mathbf{X} \\
&= -X_1 + X_0\,(\mathbf{e}_2 \wedge \mathbf{e}_3) + X_3\,(\mathbf{e}_3 \wedge \mathbf{e}_1) - X_2\,(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&= (-X_1,\ X_0,\ X_3,\ -X_2), \\
\beta_2(\mathbf{X}) &= (\mathbf{e}_3 \wedge \mathbf{e}_1)\,\mathbf{X} \\
&= -X_2 - X_3\,(\mathbf{e}_2 \wedge \mathbf{e}_3) + X_0\,(\mathbf{e}_3 \wedge \mathbf{e}_1) + X_1\,(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&= (-X_2,\ -X_3,\ X_0,\ X_1), \\
\beta_3(\mathbf{X}) &= (\mathbf{e}_1 \wedge \mathbf{e}_2)\,\mathbf{X} \\
&= -X_3 + X_2\,(\mathbf{e}_2 \wedge \mathbf{e}_3) - X_1\,(\mathbf{e}_3 \wedge \mathbf{e}_1) + X_0\,(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&= (-X_3,\ X_2,\ -X_1,\ X_0).
\end{aligned} \tag{21}$$

It is easy to check that the bases $(\beta_1(\mathbf{X}),\ \beta_2(\mathbf{X}),\ \beta_3(\mathbf{X}))$ are indeed orthonormal for all $\mathbf{X}$ with respect to the usual inner product in $\mathrm{I\!R}^4$, with each of the three $\beta_i(\mathbf{X})$ also being orthogonal to $\mathbf{X} = (X_0,\ X_1,\ X_2,\ X_3)$, and thus define a tangent space at the tip of that $\mathbf{X}$. Moreover, by explicitly calculating connection coefficients it can be checked that the Riemann curvature tensor does indeed vanish for these bases (*cf.* pp 220 of Ref. [15]), rendering the resulting parallelism of S^3 absolute. This is of course not surprising, since what is effected by the geometric products here is a left-translation of the basis at $\mathbf{X_0}$ to basis at $\mathbf{X}$ by means of parallel transport that is manifestly path-independent, rendering the 3-sphere flat: $R^{\alpha}{}_{\beta\,\gamma\,\delta} = 0$. What is more, this procedure of finding orthonormal tangent bases at different points of S^3 can be repeated *ad infinitum*, providing a continuous field of absolutely parallel spinorial tangent vectors at every point of S^3. That is, given the bases $(\beta_1(\mathbf{X}),\ \beta_2(\mathbf{X}),\ \beta_3(\mathbf{X}))$ at the tip of some vector $\mathbf{X} \in \mathrm{I\!R}^4$, the bases at the tip of any other vector $\mathbf{Y} \in \mathrm{I\!R}^4$ can be obtained by computing

$$(\beta_1(\mathbf{Y}),\ \beta_2(\mathbf{Y}),\ \beta_3(\mathbf{Y})) \,=\, (\beta_1(\mathbf{X})\,\mathbf{Y},\ \beta_2(\mathbf{X})\,\mathbf{Y},\ \beta_3(\mathbf{X})\,\mathbf{Y}), \tag{22}$$

and so on for *all* points of S^3. This amounts to generating a continuous, orthonormality preserving, left-translation of the basis at $\mathbf{X}$ to basis at $\mathbf{Y}$, for *all* pairs of vectors $\mathbf{X}$ and $\mathbf{Y}$. Consequently, each point of S^3 is now characterized by a spinorial vector of the form (17), representing the smooth flowing motion of that point, without any singularities, discontinuities, or fixed points hindering its coordinatization.[4] And as we discussed above, such a singularity-free coordinatization of S^3 is an indispensable prerequisite for the fulfilment of the completeness criterion of EPR.

Actually, a parallelized 3-sphere has much more to offer than simply providing a prerequisite for the completeness criterion. Parallelization also renders the 3-sphere *closed* under multiplication of its points, as already seen in the above derivation. As we have discussed elsewhere [4][5], closed-ness under multiplication is a very powerful property of the parallelized spheres, permitting fulfillment of the factorizability condition of Bell. Using the subalgebra (20) it is easy to show that if $\mathbf{X}$ and $\mathbf{Y}$ are two absolutely parallel spinorial unit vectors on S^3 of the form (17), then so is their geometric product $\mathbf{Z} = \mathbf{X}\mathbf{Y}$, for *all* $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. In other words, any $\mathbf{Z}$ can be factorized into a product of $\mathbf{X}$ and $\mathbf{Y}$ (in fact into a product of any number of unit vectors, including infinitely many of them). Moreover, since any spinorial unit vector in $\mathrm{I\!R}^4$ satisfies the normalization condition $||\mathbf{X}|| = 1$, the space of all such vectors $\mathbf{X}$ is homeomorphic to a unit 3-sphere, with each pair of vectors satisfying the property $||\mathbf{X}\,\mathbf{Y}|| = ||\mathbf{X}||\;||\mathbf{Y}||$. The latter property confirms that this 3-sphere not only remains closed under multiplication, but also possesses a multiplicative inverse for each of its points, rendering it equivalent to a normed division algebra. And this normed division algebra is nothing but the quaternionic algebra (20), or the bivector subalgebra we have used in the derivations of (21) and (22). Thus parallelization of a 3-sphere not only consolidates completeness, but also necessitates local causality:

$$\begin{aligned}
\textbf{completeness} \iff \textbf{parallelization} &\implies \textbf{factorizability} \\
&\implies \textbf{local causality}
\end{aligned}$$

Algebraically what brings about these remarkable implications is the bivector subalgebra (20), which we have used for parallelizing the spinorial vectors (17) everywhere on S^3. Geometrically, on the other hand, it is the flatness of

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[4] Think of a Chinese army marching in unison, not on a plane, and not vertically, but horizontally, on the surface of a three-dimensional sphere embedded in a four-dimensional cube. In his book on quantum theory [29] Peres alludes to the fact that quantum correlations are more disciplined than their classical counterparts. While we do not agree with the quantum/classical distinction here, we agree with his assertion. The discipline he alludes to is precisely the discipline of absolutely parallel spinor fields on S^3. It is this discipline that is ultimately responsible for the quantum correlations (although Peres presumably had the discipline of quantum entanglement in mind).

the parallelized 3-sphere, $R^{\alpha}{}_{\beta\,\gamma\,\delta} = 0$, that is responsible for these implications. On the equator of this parallelized 3-sphere, which is of course a 2-sphere, the spinorial vectors reduce to pure bivectors, as can be easily checked. Given two such bivectors representing two points of the equatorial 2-sphere, say $+\,I\cdot\mathbf{a}$ and $+\,I\cdot\mathbf{b}$, the bivector subalgebra (20) leads to the crucial identity:

$$(+\,I\cdot\mathbf{a})(+\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (+\,I)\cdot(\mathbf{a}\times\mathbf{b}), \tag{23}$$

provided we use the duality relation $\mathbf{a}\wedge\mathbf{b} \,=\, +\,I\cdot(\mathbf{a}\times\mathbf{b})$. The RHS of this identity is simply a different expression of the full spinorial vector (17), and represents a non-equatorial point of the 3-sphere. In other words, it represents an absolutely parallel spinorial vector characterizing a generic (i.e., in general non-equatorial) point of the 3-sphere. Analogously, for the left-handed subalgebra represented by $-\,I$ we have the left-handed identity [5]

$$(-\,I\cdot\mathbf{a})(-\,I\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, (-\,I)\cdot(\mathbf{a}\times\mathbf{b}), \tag{24}$$

along with the left-handed duality relation $\mathbf{a}\wedge\mathbf{b} \,:=\, -\,I\cdot(\mathbf{a}\times\mathbf{b})$. These two identities can now be combined into a single hidden variable equation relating the points of S^3,

$$(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\,, \tag{25}$$

along with the combined duality relation $\mathbf{a}\wedge\mathbf{b} \,:=\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})$. Then the complete state of the EPR-Bohm system can be taken to be $\boldsymbol{\mu} = \pm\,I$, specifying the right-handed (+) or left-handed (−) orthonormal frame $\{\,\mathbf{e}_1,\,\mathbf{e}_2,\,\mathbf{e}_3\}$ in $\mathrm{I\!R}^3$. The identity (25) thus provides an unambiguous characterization of every single point of the 3-sphere, devoid of singularities, discontinuities, or fixed points, with each point represented by an absolutely parallel spinorial vector of "uncontrollable" sense (clockwise or counterclockwise). This can be verified by noting that the space of all bivectors $\boldsymbol{\mu}\cdot\mathbf{n}$ is isomorphic to a unit 2-sphere defined by $||\mathbf{n}||^2 = 1$, since

$$||\boldsymbol{\mu}\cdot\mathbf{n}||^2 = (-\,\boldsymbol{\mu}\cdot\mathbf{n})(+\,\boldsymbol{\mu}\cdot\mathbf{n}) = -\,\boldsymbol{\mu}^2\,\mathbf{n}\,\mathbf{n} = \mathbf{n}\,\mathbf{n} = \mathbf{n}\cdot\mathbf{n} = ||\mathbf{n}||^2 = 1 \tag{26}$$

for any unit vector $\mathbf{n}\in\mathrm{I\!R}^3$. Thus every bivector $\boldsymbol{\mu}\cdot\mathbf{n}$ represents an intrinsic point of a unit 2-sphere, regardless of whether $\boldsymbol{\mu} = +\,I$ or $\boldsymbol{\mu} = -\,I$. The left hand side of the identity (25) is thus a product of two points of this 2-sphere. The right hand side, on the other hand, represents a point, not of a 2-sphere, but 3-sphere. This can be recognized by noting that $||\,-\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})\,||^2 = \mathbf{p}\cdot\mathbf{p} = 1$ for a unit vector $\mathbf{p}\in\mathrm{I\!R}^4$, and so the space of all multivectors $-\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})$ is indeed isomorphic to a unit 3-sphere. The two sides of the identity (25) thus relate two equatorial points of the 3-sphere to a non-equatorial point of the 3-sphere, and play the central role in the local-realistic model of Refs. [4] to [9]. In particular, Eq. (19) of Ref. [6] (or equivalently Eq. (23) of Ref. [5]), namely

$$\mathcal{E}(\mathbf{a},\,\mathbf{b}) = \int_{\Lambda}(\,\boldsymbol{\mu}\cdot\mathbf{a}\,)(\,\boldsymbol{\mu}\cdot\mathbf{b}\,)\;d\boldsymbol{\rho}(\boldsymbol{\mu}) \,=\, -\,\mathbf{a}\cdot\mathbf{b}\,, \tag{27}$$

follows at once from the identity (25), providing the correct local-realistic correlations between the points $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$ of the equatorial 2-sphere. More generally, all sixteen predictions of the rotationally non-invariant Hardy state can also be shown to follow from this identity, as correlations among the *non*-equatorial points of the 3-sphere [4]. And it is crucial to remember that all of these correlations stem from the discipline of parallelization in the 3-sphere.

III. WHAT WOULD ALICE OBSERVE AT HER DETECTOR?

The main message of the previous section is that the *only* way to satisfy the completeness criterion of EPR within Bell's local-realistic framework is by representing the measurement results as intrinsic points of an equatorial 2-sphere within a parallelized 3-sphere—i.e., by setting

$$A(\mathbf{n},\,\lambda) \,=\, \boldsymbol{\mu}\cdot\mathbf{n}\,, \tag{28}$$

which is a *definite* and *real* geometric quantity [7][28] such that

$$\mathrm{I\!R}^4 \hookleftarrow S^3 \supset S^2 \ni \boldsymbol{\mu}\cdot\mathbf{n} \,=\, \pm\,1 \;\text{ about }\; \mathbf{n}\in\mathrm{I\!R}^3\subset\mathrm{I\!R}^4, \tag{29}$$

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[5] See Ref. [5] for a more complete discussion of these and other related features of the model of Ref. [6].

where $\boldsymbol{\mu} = \pm\, I = \pm\, (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$ is the complete state of the EPR system. In other words, the only complete way to represent the measurement results is by local variables of the form

$$A(\mathbf{n},\,\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^2 \subset S^3 \hookrightarrow \mathrm{I\!R}^4. \tag{30}$$

Moreover, it is very important to appreciate that—despite appearances—the 3-vector $\mathbf{n} \in \mathrm{I\!R}^3$ is not an intrinsic part of the bivector $\boldsymbol{\mu} \cdot \mathbf{n}$, but belongs to the space $\mathrm{I\!R}^3$ "dual" to the space S^2 of bivectors [7][9]. Consequently, all Alice would see at her detector is one of the two possible *definite* outcomes, $+1$ or -1, which would have been predetermined, depending on whether the composite system started out in the complete state $\boldsymbol{\mu} = +\, I$ or $\boldsymbol{\mu} = -\, I$:

$$\boldsymbol{\mu} \cdot \mathbf{n} = \begin{cases} +\,1 \text{ about } \mathbf{n} & \text{if } \boldsymbol{\mu} \,=\, +\, I, \\ -\,1 \text{ about } \mathbf{n} & \text{if } \boldsymbol{\mu} \,=\, -\, I. \end{cases} \tag{31}$$

To be sure, what is actually observed by Alice is simply a "click" of a detector about some direction $\mathbf{n}$, and this "click" is then recorded as either $+1$ or -1 in her notebook, *together with the direction* $\mathbf{n}$ *about which it occurred*. And this is precisely what is encoded by the unit bivector $\boldsymbol{\mu} \cdot \mathbf{n}$, for that is the sum total of all the attributes any bivector of the form $\boldsymbol{\mu} \cdot \mathbf{n}$ can possess [7]. In other words, operationally the information content in the bivector $\boldsymbol{\mu} \cdot \mathbf{n}$ is identical to what is recorded by Alice in her notebook [4]. What is more, as we saw in the previous section, mathematically the *only* correct way of representing this information is by means of a unit bivector of the form $\boldsymbol{\mu} \cdot \mathbf{n}$. Physically, on the other hand, the observed "click" is best understood as a detection of the sense (clockwise or counter-clockwise) of a pure binary rotation (in fact unit bivectors in Clifford algebra simply represent pure binary rotations in the physical 3-space [28]). Thus the local-realistic variables specified in equation (28) are operationally no different from the standard variables assumed by Bell [1], apart from being *complete*. In other words, what differs between our variables and those postulated by Bell is their topologies—$S^2 \subset S^3$ versus $\mathcal{I} \subseteq \mathrm{I\!R}$, complete versus incomplete—not what is actually being detected or recorded by Alice. The correlations between two such variables, $\boldsymbol{\mu} \cdot \mathbf{a}$ and $\boldsymbol{\mu} \cdot \mathbf{b}$, will then be necessarily super-linear, because of the remarkable topological properties of the parallelized 3-sphere.

To understand this, let us consider a bivector $\boldsymbol{\mu} \cdot \mathbf{n}$ in an otherwise empty universe, with a given definite state $\boldsymbol{\mu} = \pm\, I$. It is then clear from the properties of such a bivectors [28] that—whatever the choice of $\mathbf{n}$—the rotational sense of $(+\, I \cdot \mathbf{n})$ will always be counterclockwise about $\mathbf{n}$ and clockwise about its negative, whereas that of $(-\, I \cdot \mathbf{n})$ will always be clockwise about $\mathbf{n}$ and counterclockwise about its negative. Given these inevitabilities, how can one ever see anything other than linear correlations predicted by Bell? The answer lies in the fact that bivectors are not isolated objects, but represent *relative* rotations within the geometrical constraints of our physical space. In particular, the bivectors Alice could observe are meaningful only as solutions of the parallelizing identity

$$(\,\boldsymbol{\mu} \cdot \mathbf{a})(\,\boldsymbol{\mu} \cdot \mathbf{a}') \,=\, -\,\mathbf{a} \cdot \mathbf{a}' \,-\, \boldsymbol{\mu} \cdot (\mathbf{a} \times \mathbf{a}'). \tag{32}$$

This of course is simply a local-realistic analogue of the familiar identity from quantum mechanics,

$$(i\boldsymbol{\sigma} \cdot \mathbf{a})(i\boldsymbol{\sigma} \cdot \mathbf{a}') \,=\, -\,\mathbf{a} \cdot \mathbf{a}'\, \mathbb{1} \,-\, i\,\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{a}'), \tag{33}$$

but with major ontological differences [6]. To be sure, the two identities—(32) and (33)—are simply two different representations of one and the same algebra, namely the quaternionic subalgebra of the Clifford algebra $Cl_{3,0}$, but the identity (33) is a complex-valued matrix representation of this subalgebra, whereas the identity (32) is its *real*-valued multivector representation [5][6][8]. The latter thus describes the strictly local-realistic structure of binary rotations in physical space, discovered by Rodrigues and Hamilton. In particular, while the identity (33) is an operator relation, meaningful only within the context of a Hilbert space and the rest of the formalism of quantum mechanics, the identity (32) is a purely geometric relation among directed numbers of *definite* values, as made explicit in the equations (29) and (31) above. More importantly, both of these identities simply constrain how rotations within our physical space are allowed to compose, *relative to one another*. This can be understood most transparently by rewriting equation (32) as

$$\exp\left\{(\,\boldsymbol{\mu} \cdot \mathbf{a})\frac{\pi}{2}\right\}\, \exp\left\{(\,\boldsymbol{\mu} \cdot \mathbf{a}'\,)\frac{\pi}{2}\right\} \,=\, -\, \exp\left\{(\,\boldsymbol{\mu} \cdot \mathbf{a}'')\, \theta_{\mathbf{a}\,\mathbf{a}'}\right\}, \tag{34}$$

where $\mathbf{a}'' := \mathbf{a} \times \mathbf{a}' / |\mathbf{a} \times \mathbf{a}'|$ and $\theta_{\mathbf{a}\,\mathbf{a}'}$ is the angle between $\mathbf{a}$ and $\mathbf{a}'$. Now the first thing this equation brings out is that bivectors $\boldsymbol{\mu} \cdot \mathbf{a}$ and $\boldsymbol{\mu} \cdot \mathbf{a}'$ are nothing but binary rotations (i.e., rotations by angle π) about the axes $\mathbf{a}$ and $\mathbf{a}'$ respectively, whereas their composition is a non-binary rotation by angle $2\,\theta_{\mathbf{a}\,\mathbf{a}'}$ about the orthogonal axis $\mathbf{c}$. But what is more important to note is that the rotations on the LHS of the above equation are *counterclockwise* rotations about $\mathbf{a}$ and $\mathbf{a}'$, whereas their composition on the RHS is a *clockwise* rotation about $\mathbf{c}$. Thus rotations of the same sense do not necessarily compose a net rotation of that same sense. In general rotations in Clifford algebra are represented by rotors $R = \exp\{(\boldsymbol{\mu} \cdot \mathbf{a})\, \Phi\}$, with bivectors acting as unit imaginaries (i.e., as i of the complex numbers). The sense of each rotor can then be inferred from its own sign. In other words, R and $-R$ describe the same rotation—i.e., the same starting and end points of the movement—but with opposite senses. In our case, since the axis $\mathbf{a}$ chosen by Alice is in no way privileged, and the bivector $\boldsymbol{\mu}\cdot\mathbf{a}$ she observes is necessarily a solution of the above equation, we immediately see that there are at least four alternatives possible for the senses of various bivectors she could observe:

$$\left[-\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a})\frac{\pi}{2}\right\}\right]\left[-\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a}')\frac{\pi}{2}\right\}\right] = -\exp\{(\boldsymbol{\mu}\cdot\mathbf{a}'')\,\theta_{\mathbf{a}\,\mathbf{a}'}\}\,, \tag{35}$$

$$\left[+\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a})\frac{\pi}{2}\right\}\right]\left[-\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a}')\frac{\pi}{2}\right\}\right] = +\exp\{(\boldsymbol{\mu}\cdot\mathbf{a}'')\,\theta_{\mathbf{a}\,\mathbf{a}'}\}\,, \tag{36}$$

$$\left[-\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a})\frac{\pi}{2}\right\}\right]\left[+\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a}')\frac{\pi}{2}\right\}\right] = +\exp\{(\boldsymbol{\mu}\cdot\mathbf{a}'')\,\theta_{\mathbf{a}\,\mathbf{a}'}\}\,, \tag{37}$$

$$\left[+\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a})\frac{\pi}{2}\right\}\right]\left[+\exp\left\{(\boldsymbol{\mu}\cdot\mathbf{a}')\frac{\pi}{2}\right\}\right] = -\exp\{(\boldsymbol{\mu}\cdot\mathbf{a}'')\,\theta_{\mathbf{a}\,\mathbf{a}'}\}\,. \tag{38}$$

What these alternatives show is that the sense of the bivector $\boldsymbol{\mu}\cdot\mathbf{a}$—although necessarily either definitely positive or definitely negative—is not fixed by fixing only the sense of $\boldsymbol{\mu}$, along with a direction $\mathbf{a}$. It depends on the senses of at least two other bivectors, namely $\boldsymbol{\mu}\cdot\mathbf{a}'$ and $\boldsymbol{\mu}\cdot\mathbf{a}''$, about two other directions, namely $\mathbf{a}'$ and $\mathbf{a}''$. This is easier to appreciate for orthogonal directions. Suppose we set $\mathbf{a}=\mathbf{e}_x$, $\mathbf{a}'=\mathbf{e}_y$, and $\mathbf{a}''=\mathbf{e}_z$. Then, for the fixed initial state $\boldsymbol{\mu}=+I$, the above set of alternative possibilities for Alice take a simpler form:

$$[(-I)\cdot(+\mathbf{e}_x)]\,[(-I)\cdot(+\mathbf{e}_y)] = (-I)\cdot(+\mathbf{e}_z) \tag{39}$$

$$[(+I)\cdot(+\mathbf{e}_x)]\,[(-I)\cdot(+\mathbf{e}_y)] = (+I)\cdot(+\mathbf{e}_z) \tag{40}$$

$$[(-I)\cdot(+\mathbf{e}_x)]\,[(+I)\cdot(+\mathbf{e}_y)] = (+I)\cdot(+\mathbf{e}_z) \tag{41}$$

$$[(+I)\cdot(+\mathbf{e}_x)]\,[(+I)\cdot(+\mathbf{e}_y)] = (-I)\cdot(+\mathbf{e}_z). \tag{42}$$

Thus what is observed by Alice along the direction $\mathbf{e}_x$ very much depends on what she could have observed along the directions $\mathbf{e}_y$ and $\mathbf{e}_z$, had she chosen those directions instead. It is also important to note that all of these are *local, counterfactual* directions. The remote directions $\mathbf{b}$ chosen by Bob have no baring on what Alice observes along her own local directions. Thus what is illustrated here is the geometry of Alice's own 2-sphere of possible outcomes.

Let us now see how things work in practice for both Alice and Bob. Suppose now we align the detectors for Alice and Bob to be in two mutually orthogonal directions, say $\mathbf{a}=\mathbf{e}_x$ for Alice and $\mathbf{b}=\mathbf{e}_y$ for Bob. And suppose the particles are prepared in the complete state $\boldsymbol{\mu}$ with 50/50 chance of each pair being in either the state $\boldsymbol{\mu}=+I$ or the state $\boldsymbol{\mu}=-I$. If the first pair of particles are in the state $\boldsymbol{\mu}=+I$, and if Alice observes the spin to be "up" along the direction $\mathbf{e}_x$, then what will Bob observe along the direction $\mathbf{e}_y$? Since the answer must come from the constraint (32) imposed by the parallelization of S^3, we will have two alternatives possible in this situation:

$$[(+I)\cdot(+\mathbf{e}_x)]\,[(+I)\cdot(+\mathbf{e}_y)] = (-I)\cdot(+\mathbf{e}_z)\,, \tag{43}$$

$$\text{or}\quad [(+I)\cdot(+\mathbf{e}_x)]\,[(-I)\cdot(+\mathbf{e}_y)] = (+I)\cdot(+\mathbf{e}_z)\,. \tag{44}$$

It is easy to verify that both of these possibilities are permitted by the parallelizing topological constraint (32). Thus, we will either have the outcomes (up, up) or the outcomes (up, down) if the initial state is $\boldsymbol{\mu}=+I$ and the detectors are fixed along the directions $\mathbf{a}=\mathbf{e}_x$ and $\mathbf{b}=\mathbf{e}_y$. Note that nothing would change even if Bob takes a third detector and places it along the direction $\mathbf{e}_z$, because we only have two particles to be detected, not three. The third possibility along $\mathbf{e}_z$ merely provides a counterfactual possibility: If Bob chooses to measure the spin of the second particle along the direction $\mathbf{e}_z$ instead of $\mathbf{e}_y$, then he would obtain spin "down" instead of "up" in the first case, and "up" instead of "down" in the second case.

Suppose now we send a second pair of particles towards the detectors, and suppose this second pair is in the state $\boldsymbol{\mu}=-I$. Then the analogs of the above two equations are:

$$[(-I)\cdot(+\mathbf{e}_x)]\,[(-I)\cdot(+\mathbf{e}_y)] = (+I)\cdot(+\mathbf{e}_z)\,, \tag{45}$$

$$\text{or}\quad [(-I)\cdot(+\mathbf{e}_x)]\,[(+I)\cdot(+\mathbf{e}_y)] = (-I)\cdot(+\mathbf{e}_z)\,, \tag{46}$$

and we will obtain the remaining two outcome pairs, (down, down) and (down, up). As a result, all four of the possibilities

$$\text{(up, up), (up, down), (down, down), and (down, up)} \tag{47}$$

will be observed by Alice and Bob if their detectors are fixed, respectively, in the directions $\mathbf{a}=\mathbf{e}_x$ and $\mathbf{b}=\mathbf{e}_y$, with the counterfactually possible direction being $\mathbf{e}_z$. Moreover, despite the tossup among the four possible pairs, the outcomes are clearly deterministic, because the net beable $A_{\mathbf{e}_x}B_{\mathbf{e}_y}C_{\mathbf{e}_z}$ (with $A_{\mathbf{e}_x}\equiv\boldsymbol{\mu}\cdot\mathbf{e}_x$, $B_{\mathbf{e}_y}\equiv\boldsymbol{\mu}\cdot\mathbf{e}_y$, and $C_{\mathbf{e}_z}\equiv\boldsymbol{\mu}\cdot\mathbf{e}_z$) has a definite value, $A_{\mathbf{e}_x}B_{\mathbf{e}_y}C_{\mathbf{e}_z}(\boldsymbol{\mu})$, for each specific state $\boldsymbol{\mu}$ (the value $+1$ for $\boldsymbol{\mu}=+I$, and -1 for $\boldsymbol{\mu}=-I$).

IV. UPPER BOUND IS SET BY THE MAXIMUM POSSIBLE TORSION IN THE SET OF ALL POSSIBLE OUTCOMES

In Section II we saw that the discipline of absolute parallelization is essential for the existence and strength of the EPR-Bohm correlations. Moreover, in Ref. [4] we have explicitly shown how the correlations exhibited by even the rotationally non-invariant states—such as the Hardy and GHZ states—stem from absolute parallelizations within the 3- and 7-spheres. Drawing from the topological lessons learned from these and other examples from our previous work [5][6][7][8], in this section we shall show that absolute parallelization is responsible also for the existence and strength of *all* quantum correlations, and moreover it imposes a strict upper bound of $2\sqrt{2}$ on their strength. Intuitively this is now easy to understand. As we saw in Section II, the completeness criterion of EPR is equivalent to the parallelization in the space of all possible measurement results, and the latter condition is equivalent to the vanishing of the Riemann curvature for this space [25]. Now it is clearly not possible to flatten any manifold more than what is dictated by the vanishing of its curvature tensor, and hence the condition $R^{\alpha}{}_{\beta\,\gamma\,\delta} = 0$ naturally imposes a constraint on the strength of possible correlations among its points. The corresponding parallelizing torsion $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ then naturally provides a measure of this strength, and the maximum of all possible parallelizing torsions within all possible parallelizable manifolds imposes an absolute upper bound—*i.e.*, the Tsirel'son bound—on the strength of all causally possible correlations.

A. When the Codomain Σ is an Arbitrary Manifold:

In order to see this in full generality, consider an arbitrary quantum state $|\Psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of arbitrary dimensions, which may or may not be finite. We impose no restrictions on either $|\Psi\rangle$ or $\mathcal{H}$, apart from their usual quantum mechanical meanings. In particular, the state $|\Psi\rangle$ can be as entangled as one may like, and the space $\mathcal{H}$ can be as large or small as one may like. Next consider a self-adjoint operator $\widehat{\mathcal{O}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)$ on this Hilbert space, parameterized by a number of local parameters, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, etc., with their usual contextual meaning [30] in any Bell type setup [12]. The quantum mechanical expectation value of this observable in the state $|\Psi\rangle$ is then given by:

$$\mathcal{E}_{_{Q.M.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) \,=\, \langle\Psi|\;\widehat{\mathcal{O}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots)\,|\Psi\rangle\,. \tag{48}$$

In Section VI of Ref. [4] we have shown how this expectation value can always be reproduced within our local-realistic framework [5][6][7]. Here is how the procedure works: One begins with a set of Bell type local functions of the form

$$A_{\mathbf{n}}(\lambda) : \begin{pmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \cdot \\ \mathbf{n}_j \\ \cdot \\ \cdot \end{pmatrix} \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_k \\ \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} A_{\mathbf{n}_1}(\lambda_2) \\ A_{\mathbf{n}_2}(\lambda_1) \\ \cdot \\ A_{\mathbf{n}_j}(\lambda_k) \\ \cdot \\ \cdot \end{pmatrix} \equiv \Sigma. \tag{49}$$

Now, as we discussed at length in Section II, the completeness criterion of EPR is equivalent to the parallelization within the codomain Σ of these functions. We therefore demand Σ to be a simply-connected, parallelizable manifold, representing the set of all possible measurement results at each local end of the experimental setup. The parallelizing torsion $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ would then be a measure of how much this codomain deviates from the flat Euclidean space $\mathrm{I\!R}^n$ (for which, of course, the value of $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ is identically zero). Consequently, the realistic correlations among the functions

$$A_{\mathbf{a}}(\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \Sigma, \quad B_{\mathbf{b}}(\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \Sigma, \quad C_{\mathbf{c}}(\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \Sigma, \quad D_{\mathbf{d}}(\lambda) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow \Sigma, \;\dots\,, \tag{50}$$

namely

$$\mathcal{E}_{_{L.R.}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\,\dots) \,=\, \int_{\Lambda} A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\,C_{\mathbf{c}}(\lambda)\,D_{\mathbf{d}}(\lambda)\,\dots\;\,d\rho(\lambda)\,, \tag{51}$$

would be a measure of this torsion. In particular, if the torsion is nonzero, then the correlations would be super-linear:

$$\text{Parallelizing Torsion }\;\mathcal{T}^{\,\gamma}_{\alpha\,\beta} \neq 0 \quad\Longleftrightarrow\quad \text{Quantum Correlations.}$$

So far this procedure is quite general [4]. Nothing prevents it from being valid for any arbitrary state $|\Psi\rangle$, and we shall soon see how it works in practice through examples. What is nontrivial, however, is to show that the correlations

(51) thus produced would also be locally causal. In other words, what we must show is that the joint beables such as $(A_{\mathbf{a}}\,B_{\mathbf{b}}\,C_{\mathbf{c}}\,D_{\mathbf{d}}\dots)(\lambda)$ corresponding to the operators $\widehat{\mathcal{O}}(\mathbf{a},\,\mathbf{b},\,\mathbf{c},\,\mathbf{d},\dots)$ can always be factorized into local parts as

$$\Sigma\,\ni\,(A_{\mathbf{a}}\,B_{\mathbf{b}}\,C_{\mathbf{c}}\,D_{\mathbf{d}}\,\dots)(\lambda)\,=\,A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\,C_{\mathbf{c}}(\lambda)\,D_{\mathbf{d}}(\lambda)\,\dots\,; \tag{52}$$

and conversely the product of the local beables must satisfy the map

$$[\,A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\,C_{\mathbf{c}}(\lambda)\,D_{\mathbf{d}}(\lambda)\,\dots]\,:\,\Sigma\times\Sigma\times\Sigma\times\Sigma\,\dots\,\longrightarrow\,\Sigma\,\ni\,(A_{\mathbf{a}}\,B_{\mathbf{b}}\,C_{\mathbf{c}}\,D_{\mathbf{d}}\,\dots)(\lambda)\,. \tag{53}$$

Then the locality or factorizability condition of Bell would be automatically satisfied, as we have shown in Ref. [4].

It turns out, however, that for a generic Σ parallelization is not sufficient to guarantee factorizability. To be sure, parallelization will give rise to super-linear correlations (provided the torsion is nonzero), but these correlations may or may not respect local causality unless we assume that Σ is a norm-composing manifold—*i.e.*, unless we assume that the norms of its points $A_{\mathbf{a}}(\lambda)$, $A_{\mathbf{a'}}(\lambda)$ *etc.* satisfy the following law of composition under multiplication [31]:

$$||\,A_{\mathbf{a}}(\lambda)\,A_{\mathbf{a'}}(\lambda)\,||\,=\,||\,A_{\mathbf{a}}(\lambda)\,||\;||\,A_{\mathbf{a'}}(\lambda)\,||. \tag{54}$$

For example, it can be easily checked that the norms of the points $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{a'}$ belonging to $S^2\subset S^3$ discussed in the previous section are composed under multiplication in this manner, since $(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{a'})\,=\,-\,\mathbf{a}\cdot\mathbf{a'}\,-\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a'})$ remains within S^3 (note that we have made no assumption about the dimensionality of Σ). Now it is well known that in the 1920's Cartan and Schouten [32] established the classification of all parallelizable Riemannian manifolds by generalizing the parallelism on the 3-sphere Clifford had discovered earlier, and later Wolf [25][33] extended their results to the pseudo-Riemannian manifolds. It follows from these results that a simply-connected irreducible Riemannian manifold admitting absolute parallelism is isometric to one of the following: the real line, a simple Lie group, or the 7-sphere. For our purposes, without loss of generality, if we now admit only those Σ's whose points are of unit norm,

$$||\,A_{\mathbf{n}}(\lambda)\,||\,=\,1, \tag{55}$$

with the condition (54) remained satisfied, then the above set of all possible simply-connected parallelizable manifolds reduces to the set of just three spheres: S^1, S^3, or S^7. Thus the strengths of all quantum correlations are constrained by the magnitudes of possible torsions within just these three parallelizable spheres. In fact, quite independently of the Cartan and Schouten classification, it can be shown that S^0, S^1, S^3, and S^7 are the only four spheres (out of infinitely many possible) that can be parallelized, given the condition (54). This was proved in 1958 by Kervaire [34], and later independently by Bott and Milnor [16]. This is a profound result, with far-reaching consequences for the entire edifice of mathematics and physics. For example, long before Cartan and Schouten it was proved by Hurwitz in 1898 [35] that any division algebra over the field of real numbers that possesses a norm satisfying the condition (54) must be the real, complex, quaternionic, or octonionic algebra of dimensions 1, 2, 4, and 8, respectively [13][17]. Subsequently Adams [36] proved that a smooth fibration of the sphere S^{2n-1} by $(n-1)$-sphere can occur only when $n=1,\ 2,\ 4$, or 8. This in turn implies that

$$S^{k-1}\hookrightarrow \mathrm{I\!R}^k \text{ is parallelizable iff } \mathbb{R}^k \text{ is a real division algebra.}$$

This can be understood as follows. If x and y are any two unit elements of one of the division algebras with norms satisfying the condition (54) and x is not an identity, then $xy\neq y$. That is, multiplication by x on the corresponding sphere moves every point of the sphere, and does so smoothly—i.e., without leaving any fixed points, singularities, or discontinuities. Hence the very properties defining the real division algebra imply that the corresponding unit sphere is parallelizable. Thus the theorems by Hurwitz, Adams, and others bring out a profound connection between the existence of the only possible real division algebras—namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$—and the parallelizability of the unit spheres S^0, S^1, S^3, and S^7. Moreover, once parallelized, S^0, S^1, S^3, and S^7 are the only four spheres that remain closed under multiplication of their points, and consequently setting any one of them as a possible codomain of the function $A(\mathbf{n},\,\lambda)$ would automatically satisfy the locality or factorizability condition of Bell:

$$\begin{gathered}(A_{\mathbf{a}}\,B_{\mathbf{b}})(\lambda):\,S^k\,\times\,S^k\,\longrightarrow\,S^k\;\;\text{implying}\\ S^k\ni(A_{\mathbf{a}}\,B_{\mathbf{b}})(\lambda)\,=A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\;\text{ for all }\;A_{\mathbf{a}}(\lambda),\,B_{\mathbf{b}}(\lambda)\,\in\,S^k\;\text{ and }\;k=0,\,1,\,3,\text{ or }7.\end{gathered} \tag{56}$$

In a series of explicit examples [4][5][6][7][8] we have already shown how this locality condition works in practice. In particular, in Ref. [4] we have shown how local functions of the form

$$A(\mathbf{n},\,\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6\subset S^7\hookrightarrow\mathrm{I\!R}^8, \tag{57}$$

can reproduce *exactly* the quantum mechanical correlations predicted by both the 3-particle and 4-particle GHZ states [2]. We have also shown how every quantum mechanical prediction of even the highly asymmetric Hardy state [3] can be reproduced *exactly*, by using a similar pair of local functions with 3-sphere as their codomain [4]. And of course we have shown the same for the standard rotationally invariant EPR states for both spin-1/2 and photon systems [5][6].

B. When the Codomain Σ is a Parallelized 3-Sphere:

Equipped with the spheres S^0, S^1, S^3, and S^7 as the only viable candidates for the space of all possible measurement results for any quantum mechanical system [6], we next proceed to demonstrate how the upper bound on all possible quantum correlations is set by the maximum of possible torsions within these spheres:

$$\text{Maximum of Torsion } \mathcal{T}^{\gamma}_{\alpha\,\beta} \neq 0 \quad \Longrightarrow \quad \text{The Upper Bound } 2\sqrt{2}.$$

To this end, let us consider the familiar string of expectation functionals studied by CHSH [12]; namely

$$\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\,. \tag{58}$$

As is well known, assuming that the distribution $\rho(\lambda)$ remains the same for all four of the functionals this string can be rewritten in terms of the products of the local functions as

$$\int_{\Lambda}\big\{\,A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda)\,+\,A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}'}(\lambda)\,+\,A_{\mathbf{a}'}(\lambda)\,B_{\mathbf{b}}(\lambda)\,-\,A_{\mathbf{a}'}(\lambda)\,B_{\mathbf{b}'}(\lambda)\,\big\}\;\,d\rho(\lambda)\,. \tag{59}$$

And since $A_{\mathbf{n}}(\lambda)$ and $B_{\mathbf{n}'}(\lambda)$ are two independent points belonging to two independent copies of Σ, they satisfy

$$[\,A_{\mathbf{n}}(\lambda),\,B_{\mathbf{n}'}(\lambda)\,]\,=\,0\quad\forall\;\mathbf{n}\;\text{and}\;\mathbf{n}'\,\in\,\mathrm{I\!R}^3 \tag{60}$$

(which is equivalent to assuming a null result—$C_{\mathbf{n}\times\mathbf{n}'}(\lambda)=0$—along the third exclusive direction $\mathbf{n}\times\mathbf{n}'$).

If we now square the integrand of Eq. (59), use the above commutation relations, and use the fact that, by definition, all local functions square to unity (the algebra goes through even when the squares of the local functions are allowed to be -1), then the absolute value of the CHSH string leads to the following form of variance inequality [7]:

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')|\,\leq\,\sqrt{\int_{\Lambda}\big\{\,4\,+\,[\,A_{\mathbf{a}}(\lambda),\,A_{\mathbf{a}'}(\lambda)\,]\,[\,B_{\mathbf{b}'}(\lambda),\,B_{\mathbf{b}}(\lambda)\,]\,\big\}\;d\rho(\lambda)\;}\,. \tag{61}$$

provided we assume that both associators

$$[\![\,A_{\mathbf{a}}(\lambda),\;A_{\mathbf{a}'}(\lambda),\;A_{\mathbf{a}''}(\lambda)]\!]\;:=\;\Big(A_{\mathbf{a}}(\lambda)\,A_{\mathbf{a}'}(\lambda)\Big)\,A_{\mathbf{a}''}(\lambda)\,-\,A_{\mathbf{a}}(\lambda)\Big(A_{\mathbf{a}'}(\lambda)\,A_{\mathbf{a}''}(\lambda)\Big) \tag{62}$$

$$\text{and}\quad[\![\,B_{\mathbf{b}}(\lambda),\;B_{\mathbf{b}'}(\lambda),\;B_{\mathbf{b}''}(\lambda)]\!]\;:=\;\Big(B_{\mathbf{b}}(\lambda)\,B_{\mathbf{b}'}(\lambda)\Big)\,B_{\mathbf{b}''}(\lambda)\,-\,B_{\mathbf{b}}(\lambda)\Big(B_{\mathbf{b}'}(\lambda)\,B_{\mathbf{b}''}(\lambda)\Big) \tag{63}$$

vanish identically. This can be easily checked for the case studied in the previous section; namely, for the choices $A_{\mathbf{a}}(\lambda)=\boldsymbol{\mu}\cdot\mathbf{a}$ and $B_{\mathbf{b}}(\lambda)=\boldsymbol{\mu}\cdot\mathbf{b}$, which are points of an equatorial 2-sphere within the parallelized 3-sphere:

$$[\![\,\boldsymbol{\mu}\cdot\mathbf{a},\;\boldsymbol{\mu}\cdot\mathbf{a}',\;\boldsymbol{\mu}\cdot\mathbf{a}''\,]\!]\,=\,0 \tag{64}$$

$$\text{and}\quad[\![\,\boldsymbol{\mu}\cdot\mathbf{b},\;\boldsymbol{\mu}\cdot\mathbf{b}',\;\boldsymbol{\mu}\cdot\mathbf{b}''\,]\!]\,=\,0\,, \tag{65}$$

where the products among the bivectors $\boldsymbol{\mu}\cdot\mathbf{a}$, $\boldsymbol{\mu}\cdot\mathbf{a}'$, $\boldsymbol{\mu}\cdot\mathbf{a}''$, *etc.* are the "de-factorizing" geometric products, such as

$$(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{a}')\,=\,-\,\mathbf{a}\cdot\mathbf{a}'\,-\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}'). \tag{66}$$

It is very important to appreciate here that neither the associators nor the commutators in the above equations have anything to do with quantum mechanics. They simply encode certain aspects of the geometry and topology of the parallelized 3-sphere. The commutators in equation (61), for instance, simply encode ordinary vector additions

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[6] One is of course free to leave the codomain Σ completely arbitrary, but the resulting correlations will then be weaker than quantum correlations, for without the discipline of parallelization within Σ there would be nothing to strengthen the correlations beyond the linear case [4]. Besides, as we saw in Sec. II, without parallelization both completeness and locality are compromised, especially since the latter is entailed by the factorizability within Σ. Thus S^0, S^1, S^3, and S^7 are the only viable options for producing strong correlations.

in the embedding space $\mathrm{I\!R}^4$ (for a complete discussion on this point, see Eqs. (36) to (40) of Ref. [4]). More pertinent to our concerns here, the commutators provide a geometric measure of the torsions within the two copies of $\Sigma = S^3$:

$$\mathcal{T}_{\mathbf{a}\,\mathbf{a}'} := \frac{1}{2}\left[\,A_{\mathbf{a}}(\lambda),\; A_{\mathbf{a}'}(\lambda)\right] \,=\, -\,A_{\mathbf{a}\times\mathbf{a}'}(\lambda) \tag{67}$$

$$\text{and}\quad \mathcal{T}_{\mathbf{b}\,\mathbf{b}'} := \frac{1}{2}\left[\,B_{\mathbf{b}}(\lambda),\; B_{\mathbf{b}'}(\lambda)\right] \,=\, -\,B_{\mathbf{b}\times\mathbf{b}'}(\lambda)\,. \tag{68}$$

This can be understood as follows. As we discussed in Section II, the set of all spinorial vectors $-\,\mathbf{a}\cdot\mathbf{a}' \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}')$ (or *real* quaternions) is isomorphic to a unit 3-sphere, and this 3-sphere is parallelized by these multivectors. Moreover, from equation (66) we see that the left multiplication of the bivector $\boldsymbol{\mu}\cdot\mathbf{a}'$ by the bivector $\boldsymbol{\mu}\cdot\mathbf{a}$ parallel transports $\boldsymbol{\mu}\cdot\mathbf{a}'$ to the multivector $-\,\mathbf{a}\cdot\mathbf{a}' \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}')$ on the 3-sphere, but the right multiplication of the bivector $\boldsymbol{\mu}\cdot\mathbf{a}'$ by the bivector $\boldsymbol{\mu}\cdot\mathbf{a}$ parallel transports $\boldsymbol{\mu}\cdot\mathbf{a}'$ to the multivector $-\,\mathbf{a}'\cdot\mathbf{a} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}'\times\mathbf{a})$ on the 3-sphere:

$$(\,\boldsymbol{\mu}\cdot\mathbf{a}')(\,\boldsymbol{\mu}\cdot\mathbf{a}) \,=\, -\,\mathbf{a}\cdot\mathbf{a}' \,+\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}'). \tag{69}$$

Now the 3-sphere is parallelized by these multivectors, so its Riemann curvature tensor is identically zero: $R^{\alpha}{}_{\beta\,\gamma\,\delta} = 0$. Therefore the difference between the RHS of Eq. (66) and the RHS of Eq. (69) has to be due to a non-vanishing torsion in the manifold (clearly, in a manifold with vanishing curvature, if the torsion is also vanishing then there is no reason for the left multiplication and right multiplication to give different results for the parallel transport). This bivectorial difference thus gives a measure of the parallelizing torsion in the 3-sphere:

$$\mathcal{T}_{\mathbf{a}\,\mathbf{a}'} := \frac{1}{2}\left[\,\boldsymbol{\mu}\cdot\mathbf{a},\, \boldsymbol{\mu}\cdot\mathbf{a}'\right] \,=\, -\,\boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{a}'). \tag{70}$$

Substituting for this torsion from equations (67) and (68) into inequality (61) then reduces the inequality to

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}')| \,\leq\, \sqrt{\int_{\Lambda}\left\{\,4 \,+\, \left[\,-\,2\,A_{\mathbf{a}\times\mathbf{a}'}(\lambda)\right]\left[\,-\,2\,B_{\mathbf{b}'\times\mathbf{b}}(\lambda)\right]\,\right\}\; d\rho(\lambda)}\,. \tag{71}$$

Next, using the identity $(\,\boldsymbol{\mu}\cdot\mathbf{a})(\,\boldsymbol{\mu}\cdot\mathbf{b}) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, \boldsymbol{\mu}\cdot(\mathbf{a}\times\mathbf{b})$, which in our generic notation takes the form

$$A_{\mathbf{a}}(\lambda)\,B_{\mathbf{b}}(\lambda) \,=\, -\,\mathbf{a}\cdot\mathbf{b} \,-\, C_{\mathbf{a}\times\mathbf{b}}(\lambda)\,, \tag{72}$$

the above inequality can be further simplified to

$$\begin{aligned}
|\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}')| &\leq \sqrt{\int_{\Lambda}\Big\{4 + 4\left[\,-\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b}) \,-\, C_{(\mathbf{a}\times\mathbf{a}')\times(\mathbf{b}'\times\mathbf{b})}(\lambda)\,\right]\Big\}\, d\rho(\lambda)} \\
&\leq \sqrt{\{4 - 4\,(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})\}\int_{\Lambda} d\rho(\lambda) \,-\, 4\int_{\Lambda} C_{(\mathbf{a}\times\mathbf{a}')\times(\mathbf{b}'\times\mathbf{b})}(\lambda)d\rho(\lambda)}
\end{aligned} \tag{73}$$

(this is a purely mathematically step, for now we are at the stage of comparing the observations of Alice and Bob). Now the last integral under the radical is proportional to the integral

$$\int_{\Lambda} C_{\mathbf{z}}(\lambda)\;\, d\rho(\lambda)\,, \quad \text{where }\; \mathbf{z} := \frac{(\mathbf{a}\times\mathbf{a}')\times(\mathbf{b}'\times\mathbf{b})}{||(\mathbf{a}\times\mathbf{a}')\times(\mathbf{b}'\times\mathbf{b})||}\,, \tag{74}$$

which vanishes identically for more than one reason. To begin with, it involves an average of the functions $C_{\mathbf{z}}(\lambda) = \pm\,1$ about $\mathbf{z}$, and hence is necessarily zero if the distribution $\rho(\lambda)$ remains uniform over Λ. Moreover, operationally the functions $C_{\mathbf{z}}(\lambda)$ themselves are necessarily zero, because they represent measurement results along the direction that is exclusive to the directions $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$. That is to say, any detector along the direction $\mathbf{z}$ would necessarily yield a null result, provided the detectors along the directions $\mathbf{a}$ or $\mathbf{a}'$ and $\mathbf{b}$ or $\mathbf{b}'$ have yielded non-null results. If, moreover, we assume that the distribution $\rho(\lambda)$ remains normalized on Λ, then the above inequality reduces to

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}')| \,\leq\, 2\,\sqrt{1 - (\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})}\,. \tag{75}$$

Finally, by noticing that

$$-1 \leq\; (\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b}) \;\leq +1\,, \tag{76}$$

we arrive at the inequalities

$$-\,2\sqrt{2}\;\leq\;\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\;\leq\;+\,2\sqrt{2}\,, \tag{77}$$

which are *exactly* the inequalities predicted by quantum mechanics, with the correct upper bounds at both ends. We have derived these inequalities entirely local-realistically however, by considering only the parallelizability of the space of all possible measurement results, which in this case we took to be a unit 3-sphere. Moreover, we have derived the inequalities without necessitating any averaging procedure involving the results in the third direction $C_{\mathbf{z}}(\lambda)$, and without needing to assume that the distribution of states $\rho(\lambda)$ remains uniform over Λ throughout the experiment.

Quantum mechanically, the case considered above is that of a rotationally invariant state, namely the EPR-Bohm state [6]. In general, however, a given two-particle state may not be rotationally invariant, and in that case all possible measurement results will not remain confined to the equatorial 2-sphere. A good example in which the measurement results are non-equatorial points of a unit 3-sphere is the Hardy state, which we have studied in detail elsewhere [4]:

$$|\Psi_{\mathbf{z}}\rangle\;=\;\frac{1}{\sqrt{1+\cos^2\theta}}\;\Big\{\cos\theta\Big(|\mathbf{z},\,+\rangle_1\otimes|\mathbf{z},\,-\rangle_2\,+\,|\mathbf{z},\,-\rangle_1\otimes|\mathbf{z},\,+\rangle_2\Big)\,-\,\sin\theta\Big(|\mathbf{z},\,+\rangle_1\otimes|\mathbf{z},\,+\rangle_2\Big)\Big\}\,. \tag{78}$$

If spin components of the particles are measured along the directions $\mathbf{a}$ and $\mathbf{a}'$ at one end of the observation station and along the directions $\mathbf{b}$ and $\mathbf{b}'$ at the other end, then this state leads to the following "asymmetrical" predictions:

$$\begin{aligned}
&\langle\Psi_{\mathbf{z}}\,|\,\mathbf{a}',\,+\rangle_1\,\otimes\,|\mathbf{b}\,,\,+\rangle_2\;=\;0\,,\\
&\langle\Psi_{\mathbf{z}}\,|\,\mathbf{a}\,,\,+\rangle_1\,\otimes\,|\mathbf{b}',\,+\rangle_2\;=\;0\,,\\
&\langle\Psi_{\mathbf{z}}\,|\,\mathbf{a}\,,\,-\rangle_1\,\otimes\,|\mathbf{b}\,,\,-\rangle_2\;=\;0\,,\\
\text{but}\;\;&\langle\Psi_{\mathbf{z}}\,|\,\mathbf{a}',\,+\rangle_1\,\otimes\,|\mathbf{b}',\,+\rangle_2\;=\;\frac{\sin\theta\,\cos^2\theta}{\sqrt{1\,+\,\cos^2\theta}}\;\neq\;0\,,
\end{aligned} \tag{79}$$

where θ is an arbitrary but known parameter (*i.e.*, a known common cause). The asymmetry of these predictions, stemming from the rotational non-invariance of the underlying quantum state, naturally leads one to believe that no local-realistic theory can reproduce them exactly. There is, however, nothing mysterious about these predictions. We have been able to reproduce, not only the above four predictions, but all sixteen predictions of the Hardy state, in our purely local-realistic framework [4]. They emerge simply as classical correlations among various *non*-equatorial points of a 3-sphere. More specifically, they can be reproduced *exactly* by using a set of complete local functions of the form

$$S^3\ni A_{\mathbf{a}}(\lambda)\;=\;\cos\alpha_{\mathbf{a}}\,+\,(\boldsymbol{\mu}\cdot\mathbf{a})\,\sin\alpha_{\mathbf{a}}\;=\;\pm1\;\;\text{about}\;\;\widetilde{\mathbf{a}}\in\mathrm{I\!R}^4 \tag{80}$$

$$\text{and}\quad S^3\ni B_{\mathbf{b}}(\lambda)\;=\;\cos\beta_{\mathbf{b}}\,+\,(\boldsymbol{\mu}\cdot\mathbf{b})\,\sin\beta_{\mathbf{b}}\;=\;\pm1\;\;\text{about}\;\;\widetilde{\mathbf{b}}\in\mathrm{I\!R}^4, \tag{81}$$

which in general represent non-equatorial points of a unit 3-sphere, reducing to the equatorial points $\boldsymbol{\mu}\cdot\mathbf{a}$ and $\boldsymbol{\mu}\cdot\mathbf{b}$ for right angles (some intuition for the geometry and topology of the 3-sphere would be helpful here, as described, for example, in Ref. [4]). Note that $A_{\mathbf{a}}(\lambda)B_{\mathbf{b}}(\lambda)$ is again a non-equatorial point of the 3-sphere, exhibiting its closed-ness under multiplication. Conversely, any given point of the 3-sphere can always be factorized into any number of such non-equatorial points of the 3-sphere (see Eqs. (53) and (54) of Ref. [4] for an explicit demonstration).

Returning to our main concerns, for such non-equatorial points the expressions (67) and (68) for the parallelizing torsion generalize to

$$\mathcal{T}_{\mathbf{a}\,\mathbf{a}'}:=\frac{1}{2}\,[\,A_{\mathbf{a}}(\lambda),\,A_{\mathbf{a}'}(\lambda)]\;=\;-\,\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a}'}\,A_{\mathbf{a}\times\mathbf{a}'}(\lambda) \tag{82}$$

$$\text{and}\quad\mathcal{T}_{\mathbf{b}\,\mathbf{b}'}:=\frac{1}{2}\,[\,B_{\mathbf{b}}(\lambda),\,B_{\mathbf{b}'}(\lambda)]\;=\;-\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b}'}\,B_{\mathbf{b}\times\mathbf{b}'}(\lambda)\,, \tag{83}$$

as can be readily checked. It is then straightforward to repeat the calculations from equation (71) onwards to arrive at the inequality

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')|\;\leq\;2\,\sqrt{1-(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})\,\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a}'}\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b}'}}\,. \tag{84}$$

Then, by noticing that

$$-1\leq\;(\mathbf{a}\times\mathbf{a}')\cdot(\mathbf{b}'\times\mathbf{b})\,\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a}'}\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b}'}\;\leq+1\,, \tag{85}$$

we once again arrive at the inequalities

$$-\,2\sqrt{2}\;\leq\;\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b}')\,+\,\mathcal{E}(\mathbf{a}',\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a}',\,\mathbf{b}')\;\leq\;+\,2\sqrt{2}\,. \tag{86}$$

Thus, it is not possible to exceed the upper bound on correlations set by quantum mechanics even when arbitrary, non-equatorial points of the 3-sphere are considered, as, for example, in reproducing the predictions of Hardy state.

C. When the Codomain Σ is a Parallelized 1-Sphere:

Instead of 3-sphere, if we now take 1-sphere to be the space Σ of all possible measurement results, then the upper bound on the CHSH inequalities cannot be exceeded beyond $|2|$. This is because, apart from being unphysical as a codomain, 1-sphere is a trivial one-dimensional manifold, with both its curvature and torsion vanishing. This can be readily seen by parameterizing it with rotors of the form $\exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\phi_a\} \equiv \cos\phi_a + (\boldsymbol{\mu}\cdot\mathbf{x})\sin\phi_a$, and noticing that

$$\exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\,\phi_a\}\exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\,\phi_{a'}\} \,=\, \exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\,(\phi_a+\,\phi_{a'})\} \,=\, \exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\,\phi_{a'}\}\exp\{(\boldsymbol{\mu}\cdot\mathbf{x})\,\phi_a\}. \tag{87}$$

In other words, unlike in the case of 3-sphere, in this case parallel transport by either left multiplication or right multiplication brings us to the same resulting point of the 1-sphere, because of the absence of both curvature and torsion in the manifold. Consequently (since $\mathcal{T}_{a\,a'}=0$), the generic inequality (61) in this case simply reduces to

$$-\,2 \;\leq\; \mathcal{E}(\mathbf{a},\,\mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\,\mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\,\mathbf{b}') \;\leq\; +\,2\,. \tag{88}$$

D. When the Codomain Σ is a Parallelized 0-Sphere:

Not surprisingly, the situation in the case of 0-sphere is even more fictitious. In this case the manifold of all possible measurement result is a totally-disconnected set of just two points, $S^0 \equiv \{-1,\,+1\}$, with no meaningful notion of curvature. The vanishing of the "torsion" is evident from the equality of the left and right multiplications of its points, $(+1)(-1) = (-1)(+1)$, giving $\mathcal{T}_{a\,a'}=0$. The choice of 0-sphere as Σ is thus even more naïve than the previous one, for in this case the manifold is not even simply-connected [4]. Although Bell made this choice in the very first equation of his paper [1], one only needs to recall some elementary concepts in topology to recognize that this is an *ad hoc* and unphysical choice that cannot satisfy the criterion of completeness set out by EPR [4][5][8]. In any case, contrary to Bell's assumptions, it should be amply clear from our discussion so far that such a choice—or even its more general envelope $\mathrm{I\!R}$—has nothing whatsoever to do with reality, local or otherwise [5]. It is therefore not surprising that in this case, since $\mathcal{T}_{a\,a'}=0$ as noted, the generic inequality (61) once again reduces to the trivial inequality (88).

E. When the Codomain Σ is a Parallelized 7-Sphere:

So far we have considered the parallelized spheres S^0, S^1, and S^3, and found that when any one of them happens to be the set of all possible measurement results, it is not possible to exceed the upper bound on correlations predicted by quantum mechanics [6]. The remaining parallelizable sphere however—the 7-sphere—happens to have the maximally nontrivial topological structure of all parallelizable spheres [21][4]. It is therefore particularly important to investigate whether the quantum upper bound can be exceeded when S^7 happens to be the codomain of the function $A(\mathbf{n},\,\lambda)$.

To this end, recall that, just as a parallelized 3-sphere is an S^2 worth of 1-spheres but with a twist in the manifold S^3 ($\neq S^2\times S^1$), a parallelized 7-sphere is an S^4 worth of 3-spheres but with a twist in the manifold S^7 ($\neq S^4\times S^3$). More precisely, just as S^3 is a nontrivial fiber bundle over S^2 with Clifford parallels S^1 as its linked fibers [27], S^7 is also a nontrivial fiber bundle, but over S^4, and with entire 3-dimensional spheres S^3 as its linked fibers [17]. Now it is the twist in the bundle S^3 that forces one to forgo the commutativity of the complex numbers (corresponding to the circle S^1) in favor of the non-commutativity of the quaternionic numbers [7] [27]. In other words, as we saw in Section II, 3-sphere cannot be parallelized by commuting complex numbers but only by non-commuting quaternionic numbers [15]. Analogously, the twist in the bundle S^7 forces one to forgo the associativity of the quaternionic numbers (corresponding to S^3) in favor of the non-associativity of the octonionic numbers. In other words, 7-sphere cannot be parallelized by the associative quaternionic numbers but only by the non-associative octonionic numbers [21]. And, of course, the reason why it can be parallelized at all is because its tangent bundle happens to be trivial (*cf.* Eq. (10)):

$$\mathrm{T}S^7 \,=\, \bigcup_{p\,\in\,S^7}\{p\}\times T_pS^7 \;\equiv\; S^7\times\mathrm{I\!R}^7. \tag{89}$$

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[7] Once again we emphasize that there is nothing "imaginary" or "non-real" about the quaternionic and octonionic numbers within the geometric framework used in the present work and in Refs. [4] to [8]. They are *real* geometric quantities, on par with the real numbers [28].

This lack of associativity means that, unlike the 3-sphere (which is homeomorphic to the group SU(2)), 7-sphere is not a group manifold, but forms only a quasi-group [31][37][4]. Now, at the algebraic level, there are two equivalent ways of dealing with the non-associativity of S^7. One way is to generalize the Lie algebra concept by abandoning the Jacobi identity in favor of a weaker structure, which leads to a non-associative algebra known as the Mal'tsev algebra [38][39]. This is not a convenient route for us, because it requires keeping track of non-vanishing associators in the calculations. There is however a more elegant way of dealing with the non-associativity of S^7, found in the literature on supergravity [40][41]. Instead of abandoning the Jacobi identity one maintains it rigorously, but at the price of relinquishing the invariance of the structure constants. The resulting algebra is associative, but of course it is still not a Lie algebra, because the structure constants now depend on the points of S^7. Here is how this works in practice:

Although the algebra of 7-sphere is non-associative whereas the entire edifice of Clifford-algebra is by definition associative, the 7-sphere can be represented by the algebra $Cl_{7,\,0}$ in almost the same way as the 3-sphere can be represented by the algebra $Cl_{3,\,0}$ [42][4]. One begins with the generalization of the basis (18) in $\mathrm{I\!R}^4$ to the basis

$$\{1,\, J\cdot\mathbf{e}_1,\, J\cdot\mathbf{e}_2,\, J\cdot\mathbf{e}_3,\, J\cdot\mathbf{e}_4,\, J\cdot\mathbf{e}_5,\, J\cdot\mathbf{e}_6,\, J\cdot\mathbf{e}_7\} \tag{90}$$

in $\mathrm{I\!R}^8$, where—instead of $I=\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3\equiv\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ as the fundamental trivector—we now take

$$J\,=\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_4\,+\,\mathbf{e}_2\mathbf{e}_3\mathbf{e}_5\,+\,\mathbf{e}_3\mathbf{e}_4\mathbf{e}_6\,+\,\mathbf{e}_4\mathbf{e}_5\mathbf{e}_7\,+\,\mathbf{e}_5\mathbf{e}_6\mathbf{e}_1\,+\,\mathbf{e}_6\mathbf{e}_7\mathbf{e}_2\,+\,\mathbf{e}_7\mathbf{e}_1\mathbf{e}_3 \tag{91}$$

as the fundamental trivector of our space. Note, however, that the choice of this trivector is by no means unique. Unlike the case in three dimensions where SO(3)—the rotation group of $\mathrm{I\!R}^3$—is the automorphism group of quaternions, in seven dimensions the group SO(7)—the rotation group of the subspace $\mathrm{I\!R}^7$ orthogonal to the identity in $\mathrm{I\!R}^8$—is *not* the automorphism group of octonions [42][17]. As is well known [31], the rotation group of octonions is actually a subgroup of SO(7), the smallest of the exceptional Lie groups G_2. This subgroup fixes a trivector J out of many possible—as in our representation above—whose choice then determines the product rule

$$(J\cdot\mathbf{e}_j)\,(J\cdot\mathbf{e}_k)\,=\,-\,\delta_{jk}\,-\sum_{l=1}^{7}f_{jkl}\,(J\cdot\mathbf{e}_l). \tag{92}$$

This rule is analogous to the one given in equation (20), but, instead of being components of an SO(3)-invariant tensor, the structure constants f_{jkl} are now components of a totally antisymmetric G_2-invariant tensor. Consequently, the basis bivectors now satisfy the following octonionic product rule:

$$(J\cdot\mathbf{e}_j)\,(J\cdot\mathbf{e}_{j+1})\,=\,(J\cdot\mathbf{e}_{j+3})\quad\text{with}\quad(J\cdot\mathbf{e}_{j+7})\,=\,(J\cdot\mathbf{e}_j), \tag{93}$$

which can be easily checked as such by substituting for the trivector from equation (91). Rather beautifully, each of the basic triples satisfying this rule generates a quaternionic subalgebra representing a 3-sphere, just as one would expect from the (local) fiber bundle decomposition of the 7-sphere into $S^4\times S^3$ [42]. Globally, however, $S^7\neq S^4\times S^3$, and consequently the algebra dictated by this rule is not associative (as can be checked easily). On the other hand, precisely because of this non-associativity the 7-sphere can be parallelized using the basis (90), analogously to how we parallelized the 3-sphere in Section II (*cf.* Eq. (21)). Since at every point $\boldsymbol{\xi}$ of S^7 the seven multivectors $(J\cdot\mathbf{e}_j)\,\boldsymbol{\xi}$ are mutually orthogonal and tangent to the sphere, they constitute nowhere vanishing orthonormal frame parallelizing the sphere [21]. Moreover, by explicitly calculating connection coefficients it can be checked that the Riemann curvature tensor does indeed vanish for the new bases, rendering the resulting parallelism of S^7 absolute [21].

Given a vector $\mathbf{N}\in\mathrm{I\!R}^7$ and the bivector basis (90), the generic bivector $J\cdot\mathbf{N}$ can be expanded in this basis as

$$J\cdot\mathbf{N}\,=\,N_1\,J\cdot\mathbf{e}_1\,+\,N_2\,J\cdot\mathbf{e}_2\,+\,N_3\,J\cdot\mathbf{e}_3\,+\,N_4\,J\cdot\mathbf{e}_4\,+\,N_5\,J\cdot\mathbf{e}_5\,+\,N_6\,J\cdot\mathbf{e}_6\,+\,N_7\,J\cdot\mathbf{e}_7\,. \tag{94}$$

It is worth recalling here that, although there is clearly isomorphism between the Euclidean vector space and the bivector space, a bivector is an *abstract entity of its own*, with properties quite distinct from those of a vector [28]. Given two such unit bivectors, say $J\cdot\mathbf{N}$ and $J\cdot\mathbf{N}'$, the bivector subalgebra (92) leads to the identity

$$(J\cdot\mathbf{N})(J\cdot\mathbf{N}')\,=\,-\,\mathbf{N}\cdot\mathbf{N}'\,-\,J\cdot(\mathbf{N}\times\mathbf{N}'), \tag{95}$$

provided we use the duality relation $\mathbf{N}\wedge\mathbf{N}'\,=\,J\cdot(\mathbf{N}\times\mathbf{N}')$. Crucially, the definition of the cross product here,

$$\mathbf{e}_j\times\mathbf{e}_k\,:=\,\sum_{l=1}^{7}f_{jkl}\,\mathbf{e}_l\,, \tag{96}$$

depends on the choice of J, and consequently the direction of the vector $\mathbf{N}\times\mathbf{N}'$ also depends on the choice of J. Unlike the case in 3-dimensions, in 7-dimensions there are many planes other than the linear span of $\mathbf{N}$ and $\mathbf{N}'$ giving the same direction as $\mathbf{N}\times\mathbf{N}'$ (i.e., there are more than one planes orthogonal to the direction $\mathbf{N}\times\mathbf{N}'$) [42]. If we let $\mathbf{N}$ and $\mathbf{N}'$ run through all of $\mathrm{I\!R}^7$, the image set of the simple bivectors $\mathbf{N}\wedge\mathbf{N}'$ is a manifold of dimension $2\times 7-3=11$, whereas the image set of $\mathbf{N}\times\mathbf{N}'$ is just $\mathrm{I\!R}^7$. Thus there is a great deal of freedom available to the planes $\mathbf{N}\wedge\mathbf{N}'$ to be distinct from one other and yet be orthogonal to $\mathbf{N}\times\mathbf{N}'$. Consequently, the duality relation $\mathbf{N}\wedge\mathbf{N}' = J\cdot(\mathbf{N}\times\mathbf{N}')$ is not a one-to-one correspondence, but only a method of associating a vector to a bivector. In terms of symmetry groups, this implies that, unlike the 3-dimensional cross product, which is invariant under all rotations of SO(3), the 7-dimensional cross product is not invariant under all of SO(7), but only under its subgroup G_2 that fixes the trivector.

As noted above [40], an elegant way of handling the non-uniqueness of the duality relation $\mathbf{N}\wedge\mathbf{N}' = J\cdot(\mathbf{N}\times\mathbf{N}')$ as well as the non-associativity of the algebra (93) is to let the structure constants depend on the points of S^7,

$$f_{jkl}(\boldsymbol{\xi}) \,=\, f_{jkl}(\boldsymbol{\xi}_o) \,-\, [\![\,(J\cdot\mathbf{e}_j),\; (J\cdot\mathbf{e}_k),\; \boldsymbol{\xi}\,]\!]\;\left(\boldsymbol{\xi}^\dagger\,(J\cdot\mathbf{e}_l)^\dagger\right) \tag{97}$$

(*cf.* Eq. (3.3) of Ref. [21]), so that the product rule (92) is generalized to

$$(J\cdot\mathbf{e}_j)\,(J\cdot\mathbf{e}_k) \,=\, -\,\delta_{jk} \,-\sum_{l=1}^{7} f_{jkl}(\boldsymbol{\xi})\;(J\cdot\mathbf{e}_l)\,. \tag{98}$$

Here the symbol $\dagger$ stands for the "reverse" operation of geometric algebra defined by $(\boldsymbol{\xi}_1^\dagger)^\dagger=\boldsymbol{\xi}_1$ and $(\boldsymbol{\xi}_1\boldsymbol{\xi}_2)^\dagger=\boldsymbol{\xi}_2^\dagger\boldsymbol{\xi}_1^\dagger$ (similar to the octonionic conjugation operation), and $\boldsymbol{\xi}_o$ is some fixed point on S^7, say the north or the south pole: $\boldsymbol{\xi}_o=\pm 1$. Evidently, at the fixed antipodal points $\boldsymbol{\xi}_o=\pm 1$ the associator in equation (97) would vanish, and the algebra (98) would reduce to the non-associative algebra of equations (91) to (93). At a general point $\boldsymbol{\xi}$ of S^7, on the other hand, the non-associativity is absorbed into the structure functions $f_{jkl}(\boldsymbol{\xi})$, and the algebra (98) is rendered associative. Unlike the structure constants f_{jkl} the structure functions $f_{jkl}(\boldsymbol{\xi})$ are not G_2-invariant, but extend to all of SO(7). In the language of group theory the choice between the structure constants f_{jkl} and the structure functions $f_{jkl}(\boldsymbol{\xi})$ can be understood as a choice between the two alternative coset decompositions of the 7-sphere [40]:

$$S^7 \,\cong\, \frac{\mathrm{Spin}(7)}{\mathrm{G}_2} \quad\text{ or }\quad S^7 \,\cong\, \frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}\,. \tag{99}$$

Here Spin(7) is the double cover of SO(7) (analogous to how $S^3\cong\mathrm{SU}(2)$ is the double cover of SO(3)). It is important to note also that the choice between these two coset decompositions of S^7—*i.e.*, between f_{jkl} and $f_{jkl}(\boldsymbol{\xi})$—is purely a matter of convenience. Nothing fundamental is either gained or lost in choosing one over the other [40][41].

Now, for a given pair of unit vectors $\mathbf{N},\,\mathbf{N}'\in\mathrm{I\!R}^7$ and the right-handed trivector $+\,J$, the algebra (98) at once leads to an identity analogous to (23),

$$(+\,J\cdot\mathbf{N})(+\,J\cdot\mathbf{N}') \,=\, -\,\mathbf{N}\cdot\mathbf{N}' \,-\, (+\,J)\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}'), \tag{100}$$

but with the definition of cross product depending on the points $\boldsymbol{\xi}$ of S^7,

$$\mathbf{e}_j\times_{\boldsymbol{\xi}}\mathbf{e}_k \,:=\, \sum_{l=1}^{7} f_{jkl}(\boldsymbol{\xi})\;\mathbf{e}_l\,. \tag{101}$$

Thus we now have a $\boldsymbol{\xi}$-dependent duality relation, $\mathbf{N}\wedge\mathbf{N}' = +\,J\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')$, exhibiting a $\boldsymbol{\xi}$-dependent association of the vectors $\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}'$ to the bivectors $\mathbf{N}\wedge\mathbf{N}'$ over the entire sphere. Analogously, for the left-handed subalgebra represented by $-\,J$, we have the left-handed identity

$$(-\,J\cdot\mathbf{N})(-\,J\cdot\mathbf{N}') \,=\, -\,\mathbf{N}\cdot\mathbf{N}' \,-\, (-\,J)\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}'), \tag{102}$$

along with the left-handed duality relation $\mathbf{N}\wedge\mathbf{N}' := -\,J\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')$. The two identities (100) and (102) can now be combined into a single hidden variable equation relating the points of S^7,

$$(\,\boldsymbol{\mu}\cdot\mathbf{N})(\,\boldsymbol{\mu}\cdot\mathbf{N}') \,=\, -\,\mathbf{N}\cdot\mathbf{N}' \,-\, \boldsymbol{\mu}\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')\,, \tag{103}$$

along with the combined duality relation $\mathbf{N}\wedge\mathbf{N}' := \boldsymbol{\mu}\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')$, with the complete initial state given by $\boldsymbol{\mu}=\pm\,J$ specifying the right-handed (+) or left-handed (−) orthonormal frame $\{\,\mathbf{e}_1,\,\mathbf{e}_2,\,\mathbf{e}_3,\,\mathbf{e}_4,\,\mathbf{e}_5,\,\mathbf{e}_6,\,\mathbf{e}_7\}$ in $\mathrm{I\!R}^7$. The identity (103) provides an unambiguous characterization of every single point of the 7-sphere, with each point represented by an absolutely parallel octonionic spinor of "uncontrollable" sense (clockwise or counterclockwise). This can be verified by noting that the space of all bivectors $\boldsymbol{\mu}\cdot\mathbf{N}$ is isomorphic to a unit 6-sphere defined by $||\mathbf{N}||^2 = 1$, since

$$||\boldsymbol{\mu}\cdot\mathbf{N}||^2 = (-\,\boldsymbol{\mu}\cdot\mathbf{N})(+\,\boldsymbol{\mu}\cdot\mathbf{N}) = -\,\boldsymbol{\mu}^2\,\mathbf{N}\,\mathbf{N} = \mathbf{N}\,\mathbf{N} = \mathbf{N}\cdot\mathbf{N} = ||\mathbf{N}||^2 = 1 \tag{104}$$

for any unit vector $\mathbf{N}\in\mathrm{I\!R}^7$. Thus every bivector $\boldsymbol{\mu}\cdot\mathbf{N}$ represents an intrinsic point of a unit 6-sphere, regardless of whether $\boldsymbol{\mu} = +\,J$ or $\boldsymbol{\mu} = -\,J$. The left hand side of the identity (103) is thus a product of two points of this 6-sphere. The right hand side, on the other hand, represents a point, not of a 6-sphere, but 7-sphere. This can be recognized by noting that $||\,-\,\mathbf{N}\cdot\mathbf{N}'\,-\,\boldsymbol{\mu}\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')\,||^2 = \mathbf{P}\cdot\mathbf{P} = 1$ for some unit vector $\mathbf{P}\in\mathrm{I\!R}^8$, and so the space of all multivectors $-\,\mathbf{N}\cdot\mathbf{N}'\,-\,\boldsymbol{\mu}\cdot(\mathbf{N}\times_{\boldsymbol{\xi}}\mathbf{N}')$ is indeed isomorphic to a unit 7-sphere. The two sides of the identity (103) thus relate two distinct points of an equatorial 6-sphere to a unique non-equatorial point of the 7-sphere.

Analogous to the 3-sphere case (*cf.* footnote 1 and Eq. (28)), we now represent all possible measurement results as intrinsic points of an equatorial 6-sphere within this parallelized 7-sphere, by setting

$$A(\mathbf{n},\,\lambda)\;=\;\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{n})\,, \tag{105}$$

which is a *definite* and *real* geometric quantity [7][28], with the complete state $\boldsymbol{\mu} = \pm\,J$ and $\mathbf{n}\in\mathrm{I\!R}^3$, such that

$$\mathrm{I\!R}^8 \hookleftarrow S^7 \supset S^6 \ni \boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{n})\;=\;\pm\,1\;\text{ about }\mathbf{n}\in\mathrm{I\!R}^3\subset\mathrm{I\!R}^7\subset\mathrm{I\!R}^8. \tag{106}$$

In other words, we represent the measurement results in this case by the complete local variables of the form

$$A(\mathbf{n},\,\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6\subset S^7\hookrightarrow\mathrm{I\!R}^8 \tag{107}$$

in such a manner that all Alice would see at her 3-dimensional detector is one of the two possible *definite* outcomes, $+1$ or -1. These outcomes would have been predetermined, depending on whether the composite system started out in the complete state $\boldsymbol{\mu} = +\,J$ or $\boldsymbol{\mu} = -\,J$:

$$\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{n})\;=\begin{cases}+\,1\;\text{ about }\mathbf{n}\in\mathrm{I\!R}^3\;\equiv\;+\,1\;\text{ about }\mathbf{N}(\mathbf{n})\in\mathrm{I\!R}^7 & \text{if }\boldsymbol{\mu}\;=\;+\,J,\\ -\,1\;\text{ about }\mathbf{n}\in\mathrm{I\!R}^3\;\equiv\;-\,1\;\text{ about }\mathbf{N}(\mathbf{n})\in\mathrm{I\!R}^7 & \text{if }\boldsymbol{\mu}\;=\;-\,J.\end{cases} \tag{108}$$

It is important to note here that—despite appearances—the 7-vector $\mathbf{N}(\mathbf{n})\in\mathrm{I\!R}^7$ is not an intrinsic part of the bivector $\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{n})$, but belongs to the space $\mathrm{I\!R}^7$ "dual" to the space S^6 of bivectors. In other words, the sum total of information contained in the bivector $\boldsymbol{\mu}\cdot\mathbf{N}(\mathbf{n})$ is operationally identical to what is actually recorded in the laboratory.

An explicit example may help here to understand this construction. In Ref. [4] we have shown how the quantum mechanical predictions of three- and four-particle GHZ states can be reproduced *exactly*, if a parallelized 7-sphere is taken to be the codomain of the function $A(\mathbf{n},\,\lambda)$. Here is how this works: The four-particle GHZ state is given by

$$|\Psi_{\mathbf{z}}\rangle\;=\;\frac{1}{\sqrt{2}}\Big\{|\mathbf{z},\,+\rangle_1\otimes|\mathbf{z},\,+\rangle_2\otimes|\mathbf{z},\,-\rangle_3\otimes|\mathbf{z},\,-\rangle_4\,-\,|\mathbf{z},\,-\rangle_1\otimes|\mathbf{z},\,-\rangle_2\otimes|\mathbf{z},\,+\rangle_3\otimes|\mathbf{z},\,+\rangle_4\Big\}, \tag{109}$$

which is a rotationally non-invariant state, with $\mathbf{z}$ as a privileged direction [2]. The quantum mechanical expectation value in this state—of finding the spin of particle 1 along $\mathbf{n}_1$, the spin of particle 2 along $\mathbf{n}_2$, etc.—is given by

$$\mathcal{E}^{\Psi_{\mathbf{z}}}_{Q.M.}(\mathbf{n}_1,\,\mathbf{n}_2,\,\mathbf{n}_3,\,\mathbf{n}_4)\;:=\;\langle\Psi_{\mathbf{z}}|\,\boldsymbol{\sigma}\cdot\mathbf{n}_1\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{n}_2\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{n}_3\,\otimes\,\boldsymbol{\sigma}\cdot\mathbf{n}_4\,|\Psi_{\mathbf{z}}\rangle. \tag{110}$$

Now it can be shown that there is a one-to-one correspondence between the EPR elements of reality corresponding to this state and the points of a unit 7-sphere [4]. In other words, the topological space of all possible measurement results for this state is a unit 7-sphere. Therefore we seek to construct four local maps of the form

$$A_{\mathbf{n}_1}(\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6,\quad B_{\mathbf{n}_2}(\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6,\quad C_{\mathbf{n}_3}(\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6,\quad\text{and}\quad D_{\mathbf{n}_4}(\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow S^6\,. \tag{111}$$

Moreover, just as the 0-, 1-, and 3-spheres discussed earlier, a parallelized 7-sphere remains closed under multiplication of its points, and hence the above maps will automatically preserve the locality condition of Bell:

$$\begin{gathered}(A_{\mathbf{n}_1}\,B_{\mathbf{n}_2}\,C_{\mathbf{n}_3}\,D_{\mathbf{n}_4}\,)(\lambda):\;S^6\,\times\,S^6\,\times\,S^6\,\times\,S^6\,\longrightarrow\,S^7\;\;\text{implying}\\ S^7\ni(A_{\mathbf{n}_1}\,B_{\mathbf{n}_2}\,C_{\mathbf{n}_3}\,D_{\mathbf{n}_4}\,)(\lambda)\;=\;A_{\mathbf{n}_1}(\lambda)\,B_{\mathbf{n}_2}(\lambda)\,C_{\mathbf{n}_3}(\lambda)\,D_{\mathbf{n}_4}(\lambda)\;\text{ for all }\;A_{\mathbf{n}_1}(\lambda),\,B_{\mathbf{n}_2}(\lambda),\,C_{\mathbf{n}_3}(\lambda),\,D_{\mathbf{n}_4}(\lambda)\;\in\;S^6.\end{gathered} \tag{112}$$

In fact, the product of any number of points of an equatorial 6-sphere will be a point of the 7-sphere, and, conversely, any point of a 7-sphere can always be factorized into any number of such points of the equatorial 6-sphere. Equipped with this powerful mathematical property, we take our local maps $A_{\mathbf{n}_1}(\lambda)$, $B_{\mathbf{n}_2}(\lambda)$, $C_{\mathbf{n}_3}(\lambda)$, and $D_{\mathbf{n}_4}(\lambda)$ to be the following four points of an equator of the parallelized 7-sphere:

$$S^7 \supset S^6 \ni A_{\mathbf{n}_1}(\lambda) \,=\, \pm 1 \;\text{ about the direction }\; \mathbf{N}(\mathbf{n}_1) \,:=\, (\,-n_{1x},\, +n_{1y},\, -n_{1z},\, 0,\, 0,\, 0,\, 0\,) \in \mathrm{I\!R}^7 \subset \mathrm{I\!R}^8, \tag{113}$$

$$S^7 \supset S^6 \ni B_{\mathbf{n}_2}(\lambda) \,=\, \pm 1 \;\text{ about the direction }\; \mathbf{N}(\mathbf{n}_2) \,:=\, (\,+n_{2x},\, +n_{2y},\, 0,\, +n_{2z},\, 0,\, 0,\, 0\,) \in \mathrm{I\!R}^7 \subset \mathrm{I\!R}^8, \tag{114}$$

$$S^7 \supset S^6 \ni C_{\mathbf{n}_3}(\lambda) \,=\, \pm 1 \;\text{ about the direction }\; \mathbf{N}(\mathbf{n}_3) \,:=\, (\,+n_{3x},\, +n_{3y},\, 0,\, 0,\, +n_{3z},\, 0,\, 0\,) \in \mathrm{I\!R}^7 \subset \mathrm{I\!R}^8, \tag{115}$$

$$S^7 \supset S^6 \ni D_{\mathbf{n}_4}(\lambda) \,=\, \pm 1 \;\text{ about the direction }\; \mathbf{N}(\mathbf{n}_4) \,:=\, (\,+n_{4x},\, -n_{4y},\, 0,\, 0,\, 0,\, -n_{4z},\, 0\,) \in \mathrm{I\!R}^7 \subset \mathrm{I\!R}^8, \tag{116}$$

with n_{1x}, n_{1y}, and n_{1z} being the components of $\mathbf{n}_1 \in \mathrm{I\!R}^3$; n_{2x}, n_{2y}, and n_{2z} being the components of $\mathbf{n}_2 \in \mathrm{I\!R}^3$; etc. Thus, with these identifications between the points of the equators S^2 of S^3 and S^6 of S^7, a specification of the experimental directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, and $\mathbf{n}_4$ in $\mathrm{I\!R}^3$ is equivalent to a specification of the directions $\mathbf{N}(\mathbf{n}_1)$, $\mathbf{N}(\mathbf{n}_2)$, $\mathbf{N}(\mathbf{n}_3)$, and $\mathbf{N}(\mathbf{n}_4)$ in $\mathrm{I\!R}^7$. Using such identifications, we can therefore rewrite the maps (111) as

$$A_{\mathbf{N}(\mathbf{n}_1)}(\lambda) : \mathrm{I\!R}^7\!\times\Lambda \longrightarrow S^7, \quad B_{\mathbf{N}(\mathbf{n}_2)}(\lambda) : \mathrm{I\!R}^7\!\times\Lambda \longrightarrow S^7, \quad C_{\mathbf{N}(\mathbf{n}_3)}(\lambda) : \mathrm{I\!R}^7\!\times\Lambda \longrightarrow S^7, \quad \text{and} \quad D_{\mathbf{N}(\mathbf{n}_4)}(\lambda) : \mathrm{I\!R}^7\!\times\Lambda \longrightarrow S^7, \tag{117}$$

thus completing our construction. Explicit calculations then show that (the reader is strongly urged to go through the calculations in Ref. [4]), in the spherical coordinates—with angles θ_1 and ϕ_2 representing the polar and azimuthal angles of the direction $\mathbf{n}_1$, etc.—the local-realistic expectation value for the four-particle GHZ system,

$$\mathcal{E}_{{}_{L.R.}}(\mathbf{n}_1,\, \mathbf{n}_2,\, \mathbf{n}_3,\, \mathbf{n}_4) \,=\, \int_{\Lambda} A_{\mathbf{N}(\mathbf{n}_1)}(\lambda)\, B_{\mathbf{N}(\mathbf{n}_2)}(\lambda)\, C_{\mathbf{N}(\mathbf{n}_3)}(\lambda)\, D_{\mathbf{N}(\mathbf{n}_4)}(\lambda)\; d\rho(\lambda)\,, \tag{118}$$

works out to be [4]

$$\mathcal{E}_{{}_{L.R.}}(\mathbf{n}_1,\, \mathbf{n}_2,\, \mathbf{n}_3,\, \mathbf{n}_4) \,=\, \cos\theta_1\, \cos\theta_2\, \cos\theta_3\, \cos\theta_4 \,-\, \sin\theta_1\, \sin\theta_2\, \sin\theta_3\, \sin\theta_4\, \cos\,(\,\phi_1 \,+\, \phi_2 \,-\, \phi_3 \,-\, \phi_4\,)\,. \tag{119}$$

This *exactly* matches the corresponding quantum mechanical prediction spelt out in Appendix F of Ref. [2].

Returning to our main concern of upper bound, we once again start with the CHSH type integral of local functions,

$$\int_{\Lambda} \big\{\, A_{\mathbf{N}(\mathbf{a})}(\lambda)\, B_{\mathbf{N}(\mathbf{b})}(\lambda) \,+\, A_{\mathbf{N}(\mathbf{a})}(\lambda)\, B_{\mathbf{N}(\mathbf{b}')}(\lambda) \,+\, A_{\mathbf{N}(\mathbf{a}')}(\lambda)\, B_{\mathbf{N}(\mathbf{b})}(\lambda) \,-\, A_{\mathbf{N}(\mathbf{a}')}(\lambda)\, B_{\mathbf{N}(\mathbf{b}')}(\lambda)\,\big\}\;\, d\rho(\lambda)\,, \tag{120}$$

and take $A_{\mathbf{N}(\mathbf{n})}(\lambda)$ and $B_{\mathbf{N}(\mathbf{n}')}(\lambda)$ to be two points belonging to two independent copies of the 7-sphere, so that

$$\big[\, A_{\mathbf{N}(\mathbf{n})}(\lambda),\, B_{\mathbf{N}(\mathbf{n}')}(\lambda)\,\big] \,=\, 0 \quad \forall\; \mathbf{n} \text{ and } \mathbf{n}' \,\in\, \mathrm{I\!R}^3, \tag{121}$$

which is equivalent to assuming a null result, $C_{\mathbf{N}(\mathbf{n})\times_{\boldsymbol{\xi}_3}\mathbf{N}(\mathbf{n}')}(\lambda) = 0$, along the third exclusive direction $\mathbf{N}(\mathbf{n}) \times_{\boldsymbol{\xi}_3} \mathbf{N}(\mathbf{n}')$. If we now square the integrand of Eq. (120), use the above commutation relations, and use the fact that, by definition, all local functions square to unity (the algebra goes through even when the squares are allowed to be -1), then the absolute value of the CHSH string of expectation values leads to the following form of variance inequality [7],

$$|\mathcal{E}(\mathbf{a},\, \mathbf{b}) \,+\, \mathcal{E}(\mathbf{a},\, \mathbf{b}') \,+\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}) \,-\, \mathcal{E}(\mathbf{a}',\, \mathbf{b}')| \,\leq\, \sqrt{\int_{\Lambda} \big\{\, 4 \,+\, \big[\, A_{\mathbf{N}(\mathbf{a})}(\lambda),\; A_{\mathbf{N}(\mathbf{a}')}(\lambda)\big]\, \big[\, B_{\mathbf{N}(\mathbf{b}')}(\lambda),\; B_{\mathbf{N}(\mathbf{b})}(\lambda)\big] \,\big\}\; d\rho(\lambda)}\,, \tag{122}$$

provided we assume the products of the local functions to be associative. The Mal'tsev algebra of 7-sphere is however non-associative, so this last assumption amounts to choosing the $\boldsymbol{\xi}$-dependent identity (103), and the corresponding $\boldsymbol{\xi}$-dependent cross product $\mathbf{N}(\mathbf{a}) \times_{\boldsymbol{\xi}_1} \mathbf{N}(\mathbf{a}')$, as we discussed earlier. With this practical choice of the coset structure, the remaining calculations proceed just as they did in the case of the 3-sphere. Thus, by rewrite the identity (103) as

$$A_{\mathbf{N}(\mathbf{a})}(\lambda)\, A_{\mathbf{N}(\mathbf{a}')}(\lambda) \,=\, -\,\mathbf{N}(\mathbf{a}) \cdot \mathbf{N}(\mathbf{a}') \,-\, A_{\mathbf{N}(\mathbf{a})\times_{\boldsymbol{\xi}_1}\mathbf{N}(\mathbf{a}')}(\lambda)\,, \tag{123}$$

$$\text{and} \quad B_{\mathbf{N}(\mathbf{b})}(\lambda)\, B_{\mathbf{N}(\mathbf{b}')}(\lambda) \,=\, -\,\mathbf{N}(\mathbf{b}) \cdot \mathbf{N}(\mathbf{b}') \,-\, B_{\mathbf{N}(\mathbf{b})\times_{\boldsymbol{\xi}_2}\mathbf{N}(\mathbf{b}')}(\lambda)\,, \tag{124}$$

we arrive at the following $\boldsymbol{\xi}$-dependent parallelizing torsions within the two copies of the 7-sphere:

$$\mathcal{T}_{\mathbf{N}(\mathbf{a})\mathbf{N}(\mathbf{a}')} := \frac{1}{2}\,\big[\, A_{\mathbf{N}(\mathbf{a})}(\lambda),\, A_{\mathbf{N}(\mathbf{a}')}(\lambda)\big] \,=\, -\, A_{\mathbf{N}(\mathbf{a})\times_{\boldsymbol{\xi}_1}\mathbf{N}(\mathbf{a}')}(\lambda) \tag{125}$$

$$\text{and} \quad \mathcal{T}_{\mathbf{N}(\mathbf{b})\mathbf{N}(\mathbf{b}')} := \frac{1}{2}\,\big[\, B_{\mathbf{N}(\mathbf{b})}(\lambda),\, B_{\mathbf{N}(\mathbf{b}')}(\lambda)\big] \,=\, -\, B_{\mathbf{N}(\mathbf{b})\times_{\boldsymbol{\xi}_2}\mathbf{N}(\mathbf{b}')}(\lambda)\,. \tag{126}$$

It is worth stressing here that this $\boldsymbol{\xi}$-dependence of the parallelizing torsion is *the* characteristic trait of the 7-sphere [41]. Unlike in the 3-sphere, the torsion in the 7-sphere does not remain constant over the whole of the manifold S^7 [21]. However, although the $\boldsymbol{\xi}$-dependence of the cross product appearing in the above expressions is not necessarily the same for Alice and Bob, the Pythagorean rule $||\,\mathbf{N(a)}\times_{\boldsymbol{\xi}}\mathbf{N(a')}\,|| \,=\, ||\,\mathbf{N(a)}\,||\,||\,\mathbf{N(a')}\,||\,\sin\{\mathbf{N(a)},\,\mathbf{N(a')}\}$ remains the same for both of them [42]. Substituting the above pair of torsions into the inequality (122) then simplifies it to

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})|\,\leq\,\sqrt{\int_{\Lambda}\Big\{\,4\,+\,\Big[-2\,A_{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}}(\lambda)\Big]\,\Big[-2\,B_{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}}(\lambda)\Big]\,\Big\}\;d\rho(\lambda)}\,. \tag{127}$$

And using the identity (103) once again in the form

$$A_{\mathbf{N(a)}}(\lambda)\,B_{\mathbf{N(b)}}(\lambda)\,=\,-\,\mathbf{N(a)}\cdot\mathbf{N(b)}\,-\,C_{\mathbf{N(a)}\times_{\boldsymbol{\xi}_3}\mathbf{N(b)}}(\lambda)\,, \tag{128}$$

the above inequality further simplifies to

$$\begin{aligned}
&|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})|\\
&\quad\leq\sqrt{\int_{\Lambda}4+4\Big[-\,\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\cdot\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}\,-\,C_{\big\{[\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}]\times_{\boldsymbol{\xi}_3}[\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}]\big\}}(\lambda)\,\Big]\,d\rho(\lambda)}\\
&\quad\leq\sqrt{\Big[4-4\,\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\cdot\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}\Big]\int_{\Lambda}d\rho(\lambda)\,-\,4\int_{\Lambda}C_{\big\{[\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}]\times_{\boldsymbol{\xi}_3}[\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}]\big\}}(\lambda)\;d\rho(\lambda)}
\end{aligned} \tag{129}$$

(this is a purely mathematically step, for now we are at the stage of comparing the observations of Alice and Bob). Now the last integral under the radical is proportional to the integral

$$\int_{\Lambda}C_{\mathbf{N(z)}}(\lambda)\;d\rho(\lambda)\,,\quad\text{where }\;\mathbf{N(z)}:=\frac{\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\times_{\boldsymbol{\xi}_3}\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}}{\big|\big|\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\times_{\boldsymbol{\xi}_3}\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}\big|\big|}\,, \tag{130}$$

which vanishes identically for more than one reason. To begin with, it involves an average of the binary functions $C_{\mathbf{N(z)}}(\lambda)=\pm\,1$ about $\mathbf{N(z)}$, and hence is necessarily zero for uniform distributions, given the equivalence of the $\mathrm{I\!R}^3$ and $\mathrm{I\!R}^7$ directions in our construction. Moreover, operationally the functions $C_{\mathbf{N(z)}}(\lambda)$ themselves are necessarily zero, because they represent measurement results along the direction that is exclusive to the directions $\mathbf{N(a)}$, $\mathbf{N(a')}$, $\mathbf{N(b)}$, and $\mathbf{N(b')}$. That is to say, any detector along the direction $\mathbf{N(z)}$ would necessarily yield a null result, provided the detectors along the directions $\mathbf{N(a)}$ or $\mathbf{N(a')}$ and $\mathbf{N(b)}$ or $\mathbf{N(b')}$ have yielded non-null results. If, moreover, we assume that the distribution $\rho(\lambda)$ remains normalized on the space Λ, then the above inequality reduces to

$$|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})|\,\leq\,2\sqrt{1-\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\cdot\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}}\,. \tag{131}$$

Finally, by noticing that $-1\leq\big\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\big\}\cdot\big\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\big\}\leq+1$, we arrive at the inequalities

$$-\,2\sqrt{2}\;\leq\;\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})\;\leq\;+\,2\sqrt{2}\,, \tag{132}$$

which are *exactly* the inequalities predicted by quantum mechanics, with the correct upper bounds at both ends. We have derived these inequalities entirely local-realistically however, by considering only the parallelization in the space of all possible measurement results, which in this case we took to be a unit 7-sphere. Moreover, we have derived the inequalities without necessitating any averaging procedure involving the results in the third direction $C_{\mathbf{N(z)}}(\lambda)$, and without needing to assume that the distribution of states $\rho(\lambda)$ remains uniform over Λ throughout the experiment.

So far we have only considered correlations among the equatorial points of the 7-sphere. These are sufficient for reproducing the three- and four-particle GHZ correlations, as we have shown in Ref. [4]. In general, however, we must also consider correlations among the non-equatorial points of the 7-sphere, by considering local functions of the form

$$S^7\ni A_{\mathbf{N(a)}}(\lambda)\,=\,\cos\alpha_{\mathbf{a}}\,+\,\{\boldsymbol{\mu}\cdot\mathbf{N(a)}\}\,\sin\alpha_{\mathbf{a}}\,=\,\pm\,1\;\text{ about }\;\widetilde{\mathbf{N}}(\mathbf{a})\in\mathrm{I\!R}^8 \tag{133}$$

$$\text{and}\quad S^7\ni B_{\mathbf{N(b)}}(\lambda)\,=\,\cos\beta_{\mathbf{b}}\,+\,\{\boldsymbol{\mu}\cdot\mathbf{N(b)}\}\,\sin\beta_{\mathbf{b}}\,=\,\pm\,1\;\text{ about }\;\widetilde{\mathbf{N}}(\mathbf{b})\in\mathrm{I\!R}^8, \tag{134}$$

which reduce to the equatorial points $\boldsymbol{\mu}\cdot\mathbf{N(a)}$ and $\boldsymbol{\mu}\cdot\mathbf{N(b)}$ for right angles (some intuition for the geometry and topology of the 7-sphere would be helpful here, as described, for example, in Ref. [21]). Note that $A_{\mathbf{N(a)}}(\lambda)B_{\mathbf{N(b)}}(\lambda)$ is again a non-equatorial point of the 7-sphere, exhibiting its closed-ness under multiplication. Conversely, any given point of the 7-sphere can always be factorized into any number of such non-equatorial points of the 7-sphere. Moreover, for the above non-equatorial points the expressions (67) and (68) for the parallelizing torsions yield

$$\mathcal{T}_{\mathbf{N(a)N(a')}} := \frac{1}{2}\left[\,A_{\mathbf{N(a)}}(\lambda),\, A_{\mathbf{N(a')}}(\lambda)\right] \,=\, -\,\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a'}}\,A_{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}}(\lambda) \tag{135}$$

$$\text{and}\quad \mathcal{T}_{\mathbf{N(b)N(b')}} := \frac{1}{2}\left[\,B_{\mathbf{N(b)}}(\lambda),\, B_{\mathbf{N(b')}}(\lambda)\right] \,=\, -\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b'}}\,B_{\mathbf{N(b)}\times_{\boldsymbol{\xi}_2}\mathbf{N(b')}}(\lambda)\,, \tag{136}$$

as can be readily checked. It is then straightforward to repeat the calculations from equation (122) onwards to arrive at the inequality

$$\begin{aligned} &|\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})| \\ &\qquad\leq\,2\sqrt{1-\left\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\right\}\cdot\left\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\right\}\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a'}}\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b'}}}\,. \end{aligned} \tag{137}$$

Then, by noticing that

$$-1\leq\,\left\{\mathbf{N(a)}\times_{\boldsymbol{\xi}_1}\mathbf{N(a')}\right\}\cdot\left\{\mathbf{N(b')}\times_{\boldsymbol{\xi}_2}\mathbf{N(b)}\right\}\sin\alpha_{\mathbf{a}}\,\sin\alpha_{\mathbf{a'}}\,\sin\beta_{\mathbf{b}}\,\sin\beta_{\mathbf{b'}}\,\leq +1\,, \tag{138}$$

we once again arrive at the inequalities

$$-\,2\sqrt{2}\;\leq\;\mathcal{E}(\mathbf{a},\,\mathbf{b})\,+\,\mathcal{E}(\mathbf{a},\,\mathbf{b'})\,+\,\mathcal{E}(\mathbf{a'},\,\mathbf{b})\,-\,\mathcal{E}(\mathbf{a'},\,\mathbf{b'})\;\leq\;+\,2\sqrt{2}\,. \tag{139}$$

Thus, it is not possible to exceed the upper bound on correlations set by quantum mechanics even when arbitrary, non-equatorial points (or a combination of equatorial and non-equatorial points) of the 7-sphere are considered.

V. CONCLUDING REMARKS — QUANTUM MUSIC OF THE CLASSICAL SPHERES

We have shown that the discipline of absolute parallelization in the manifold of all possible measurement results is what is responsible for the existence and strength of all quantum correlations. In particular, we have demonstrated that the discipline of absolute parallelization in a unit 3-sphere is responsible for the EPR-Bohm and Hardy type correlations, whereas the same in a unit 7-sphere is responsible for all GHZ type correlations. Moreover, we have proven that the upper bound of $2\sqrt{2}$ on the strength of all possible quantum correlations is derived from the maximum of parallelizing torsions within all possible norm-composing parallelizable manifolds. Consequently, no physically meaningful locally causal theory can predict correlations stronger than those predicted by quantum mechanics. Our results follow from the powerful mathematical theorems by Hurwitz, Cartan, Schouten, Wolf, Bott, Milnor, Adams, and others concerning the profound relationship between the absolute parallelizability of the only parallelizable spheres S^0, S^1, S^3, and S^7 and the existence of the real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. We have used the framework of Clifford or geometric algebra within which these division algebras are *real*, in every sense of the word. Moreover, we have proven our results purely local-realistically, without involving a single concept from quantum mechanics.

The logic of our argument runs as follows. Using the prototypical example of 3-sphere we first illustrated how the existence and strength of all super-linear correlations can be seen as stemming from the discipline of parallelization in the manifold of all possible measurement results. More precisely, we showed that the existence and strength of all quantum correlations can be understood as dictated by the discipline of parallelization in the codomain Σ of the corresponding Bell type functions $A_{\mathbf{n}}(\lambda) : \mathrm{I\!R}^3 \times \Lambda \rightarrow \Sigma$, regarded as the manifold of all possible measurement results for a given quantum system. A manifold is said to be parallelized if its Riemann curvature tensor vanishes identically. This is natural for the flat Euclidean spaces, but more general manifolds can also be parallelized by introducing a sufficiently non-vanishing torsion tensor [23]. This was recognized by Einstein, among others, in the context of his unified field theory, today known as *teleparallel gravity.* Once parallelized, there exists exquisite discipline among the points of the manifold, devoid of any singularities, discontinuities, or fixed points. Consequently, the parallelizing torsion responsible for this discipline provides a quantitative measure of the super-linear correlations among its points:

$$\text{Parallelizing Torsion }\;\mathcal{T}^{\;\gamma}_{\alpha\,\beta}\neq 0 \quad\Longleftrightarrow\quad \text{Quantum Correlations.}$$

That this is indeed the case can be checked by working out some examples. It can be checked, for example, that the parallelizing torsions in the 3- and 7-spheres do indeed reproduce exactly, not only the predictions of the rotationally invariant EPR-Bohm state, but also those of the rotationally non-invariant GHZ and Hardy states [4]. Moreover, parallelization in the codomain Σ of $A(\mathbf{n},\,\lambda)$ turns out to be equivalent to the completeness criterion of EPR. Thus parallelization in Σ is necessary not only for the existence and strength of the super-linear correlations, but also for the complete specification of all possible measurement results. The next step in our argument therefore is to show that the upper bound on the strength of all possible local-realistic correlations follows from the following two conditions:

$$\Sigma \text{ is a parallelized manifold; or, equivalently, } R^{\alpha}{}_{\beta\,\gamma\,\delta}=0; \text{ and} \tag{140}$$

$$\Sigma \text{ is norm-composing: } ||\,A_{\mathbf{a}}(\lambda)\,A_{\mathbf{a'}}(\lambda)\,||\;=\;||\,A_{\mathbf{a}}(\lambda)\,||\;||\,A_{\mathbf{a'}}(\lambda)\,||\,. \tag{141}$$

The physical significance of the second condition will become clear soon as a necessary and sufficient condition for maintaining local causality. Given these two conditions, the theorems mentioned above famously dictate that the only norm-composing parallelizable manifolds are the four spheres, S^0, S^1, S^3, and S^7. More precisely, the parallelizability of these four spheres is necessitated by the very existence of the four real (normed) division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$:

$$S^{k-1}\hookrightarrow \mathrm{I\!R}^k \text{ is parallelizable iff } \mathrm{I\!R}^k \text{ is a real division algebra.} \tag{142}$$

We are thus left with only the spheres S^0, S^1, S^3, and S^7 to analyze as possible codomains of the local functions $A_{\mathbf{n}}(\lambda)$. It is however easy to check that the parallelizing torsion $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ is identically zero for the sphere S^0 and S^1, and consequently the Bell-CHSH inequalities cannot be violated if these two spheres are taken as the codomains of $A_{\mathbf{n}}(\lambda)$. This is not surprising, because S^0 and S^1 are fictitious (or at least uninteresting) choices to begin with, with no real physical significance. The only physically meaningful choices for the codomain are thus the spheres S^3 and S^7, as we have demonstrated elsewhere with explicit examples [4][5]. As mentioned above, we have shown in Ref. [4] how parallelizations in the 3- and 7-spheres can exactly reproduce not only the predictions of the rotationally invariant EPR-Bohm state, but also those of the rotationally non-invariant GHZ and Hardy states. Detailed calculations then show that, with these two spheres as codomains, the upper bound of $2\sqrt{2}$ on the strength of all possible correlations cannot be exceeded, regardless of quantum mechanics. This is a consequence of the fact that the spheres S^3 and S^7 are the maximally disciplined of all nontrivially parallelizable manifolds. We are thus led to the following conclusion:

$$\text{Maximum of Torsion } \mathcal{T}^{\,\gamma}_{\alpha\,\beta}\neq 0 \quad\Longrightarrow\quad \text{The Upper Bound } 2\sqrt{2}.$$

In sum, the upper bound on the strength of all possible quantum correlations stems from the discipline of parallelization within the manifold Σ of all possible measurement results, irrespective of quantum mechanics. This discipline is characterized by the vanishing of the curvature tensor for Σ, $R^{\alpha}{}_{\beta\,\gamma\,\delta}=0$, with the maximum of parallelizing torsion $\mathcal{T}^{\,\gamma}_{\alpha\,\beta}$ entailing the upper bound on the strength of all possible bipartite correlations, for all possible manifolds Σ.

The above result depends, however, on *two* conditions: (140) and (141). It is therefore natural to ask whether the upper bound can be exceeded by relaxing either one of these conditions. It turns out that this may be logically possible, but not without compromising local causality and/or adapting some non-standard mathematics. To be sure, relaxing the parallelizability of Σ would not necessarily compromise local causality, but it would have the opposite effect—*i.e.*, instead of producing stronger-than-quantum correlations, an un-parallelized manifold (whether or not norm-composing) would produce weaker-than-quantum correlations; because—as we saw in Section II—it is the discipline of parallelization in Σ that makes the super-linear correlations possible. On the other hand, relaxing the composition law (141) would mean that the corresponding algebra would no longer be a division algebra (although Σ could still be parallelized), and that would certainly compromise local causality, because the factorizability condition (52) cannot be maintained within a non-division algebra [13]. Indeed, a loss of divisor would mean that Σ would not remain close under multiplication, and that would lead to violations of local causality. Relaxing the composition law is not really an option however, because it would require employing some non-standard mathematics. Indeed, given the towering significance of the theorems leading to (142) in mathematics, the upper bound of $2\sqrt{2}$ on the strength of possible correlations clearly cannot be exceeded without compromising some basic rules of mathematics [43].

We hope that—if not from our previous work [4][5][6][7][8]—from the results presented here it has become evident how hopelessly circular all Bell type arguments against local-realism are. We hope the fallacy in adapting the functions

$$A(\mathbf{n},\,\lambda):\mathrm{I\!R}^3\times\Lambda\longrightarrow\mathcal{I}\subseteq\mathrm{I\!R} \tag{143}$$

for the purposes of representing measurement results is now sufficiently transparent. Although employed by Bell himself [1], such functions conceal topologically unscrupulous treatment of the set of all possible measurement results, and hence commit to incompleteness in the accountings of such results from the start. One is thus beguiled by the siren of quantum non-locality from recognizing the incompleteness of quantum mechanics. The probabilistic counterparts of these functions—namely $P(A\,|\,\mathbf{n},\,\lambda)$—are especially deceptive in this regard, not the least because of their reliance on the topologically dubious vector-algebraic models of the physical space [5]. By contrast, our topologically sensitive analysis of the set of all possible measurement results allows us to complete the accountings by Bell, and leads us to conclude that there are no incompatibilities between local-realism and the predictions of quantum mechanics.

Finally, it has not escaped our notice that the tantalizing link uncovered here between quantum correlations and teleparallel gravity may provide a fresh new perspective in the quest for the future theory of quantum gravity.

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