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# In Search of Intuition

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**ABSTRACT**
What are intuitions? Stereotypical examples may suggest that they are the results of common intellectual reflexes. But some intuitions defy the stereotype: there are hard-won intuitions that take deliberate effort to have, improved intuitions that contravene how matters naively seem to us, and expertly guided intuitions in which an expert in some domain guides a novice toward having an intuition that, otherwise, he or she would not have had. I argue that reflection on these three phenomena motivates a conception of intuition that emphasizes its phenomenology over its aetiology, as well as its grounding in malleable problem-solving abilities.

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**KEYWORDS** intuition; expertise; dual process psychology; problem solving; mathematical intuition

## 1. Introduction

Intuition is commonly located by a structure of similarity and difference relations that it bears to sensory perception and explicit reasoning. First, intuitions and sensory perceptions are similar in some important psychological respects that distinguish them from explicit reasoning. Second, intuitions are different from sensory perceptions in that they have a wider subject matter than sensory perception, one similar in scope to explicit reasoning and other forms of cognition. This common ground leaves open much about the nature of intuition.

In this paper I highlight three phenomena associated with intuition that are relatively neglected in the contemporary literature, and that can be appealed to in motivating a conception of intuition that emphasizes its phenomenology and its grounding in problem-solving abilities. The three phenomena are as follows: hard-won intuitions that take deliberate effort to have, improved intuitions that contravene how matters naively seem to us, and expertly guided intuitions in which an expert in some domain guides a novice toward having an intuition that, otherwise, he or she would not have had. I describe these phenomena in section 2. In section 3, I contrast two approaches to explaining intuition's similarity and difference relations to sensory perception and explicit reasoning. One is associated with dual process theories of cognition. Another puts emphasis on phenomenology. I promote the phenomenological approach, giving weight to its ability to accommodate hard-won intuitions. Next, I turn to the contemporary literature on expert intuition, which also illuminates the nature of improvements in intuition that fall short of attaining expertise.

Expert intuition in a domain is a capacity for forming superior intuitions about that domain that is grounded in domain-related search strategies. This will take some spelling out and that is what I do in section 4. With this accomplished, it will be clear how to explain expertly guided intuitions. In section 5, I elaborate on my favoured understanding of an historically and philosophically significant example of improved intuition as a way of showing how to think about improvements in intuition more generally.

## 2. Three Phenomena

In this section I highlight three phenomena associated with intuition—hard-won intuitions, improved intuitions, and expertly guided intuitions. My discussion of them is provisional. I propose to take *prima facie* instances of them at face value, and to see what account of intuition emerges by trying to accommodate them.

The first two phenomena have been remarked on before, particularly in the literature on mathematical intuition. When the mathematician Felix Klein [1972: 904] wrote, 'But it should always be required that a mathematical subject not be exhausted until it has become intuitively evident', he was describing a goal for deliberate mathematical research. The intuitions that Klein implored mathematicians to pursue do not immediately pop into one's mind. Compare the proposition that *if* $a < b$ *and* $b < c$ *then* $a < c$ with the proposition that *if* $a < 1$ *then* $2–2a > 0$. The first is intuitively evident immediately. The second is intuitively evident after some conscious reflection. Klein had more sophisticated mathematical quarry in mind, but this example sufficiently illustrates how an intuition might be hard-won, requiring deliberate effort if it is to be had.

Consider the proposition that a curve can touch every point on a plane. When we first think about it, most of us have an intuition that this proposition is false. Curves are one-dimensional paths traced out by moving points. Planes are two-dimensional surfaces. If you try to imagine a curve touching every point on a plane, you are likely to imagine a point zigzagging about. But there are always gaps in its motion, however small. These gaps contain points on the plane that the curve fails to touch. So, the idea that a curve can touch every point on a plane is counterintuitive. It turns out, however, that a curve really can touch every point on a plane [Peano 1890]. On the basis of this and other examples of counterintuitive truths about continua, Hans Hahn concluded that 'intuition is a wholly unreliable guide' [1980: 100].

Benoit Mandelbrot took a different view [1983: 150]:

> This Essay demonstrates that Hahn is dead wrong. To tame his own examples, I find it necessary to train our present intuition to perform new tasks, but it does not suffer any discontinuous change of character. Hahn draws a mistaken diagnosis, and suggests a lethal treatment.

Mandelbrot is correct that appropriate training can make us comfortable with the facts being different from how they initially strike us. He is also correct that the real question is that of whether there is a 'discontinuous change of character'. Does our comfort with the initially counterintuitive facts come from improved intuitions, so that what was once counterintuitive is now intuitive? Hahn thinks that the answer is 'no', and draws a sceptical conclusion about intuition. Mandelbrot thinks that the answer is 'yes', and so he does not draw the sceptical conclusion but—like Kline—implores researchers to search for new improved intuitions.

The third phenomenon that I want to highlight has not previously been remarked on. Suppose that a novice and an expert at solving physics problems are presented with two inclined-plane problems. To the novice, they seem similar because they both involve inclined planes. To the expert, they seem different because one should be solved using Newton's laws and one should be solved using the conservation of energy (cf. Chi et al. [1981]). The novice has a novice intuition. The expert has an expert intuition. Now suppose that the expert points out the relevant features of the problems to the novice. Presumably after some explanation, the two problems will come to seem different to the novice. I would say that the novice now has an intuition with the same content as the expert's intuition—but that this intuition is the result of expert guidance, not expertise. One might think that the seeming that the novice has under expert guidance is not really an intuition, perhaps because it does not result from the right kind of psychological process. This depends on what counts as an intuition, an issue to which I will return below.

I think that perception differs from intuition on this point. Suppose that a novice and an expert birdwatcher spot the same bird. To the novice, it looks like a bird with an orange breast and a greyish back. To the birdwatcher it looks like an American Robin (cf. Tanaka and Taylor [1991]). The novice has a novice perceptual experience. The expert has an expert perceptual experience.[1] Now suppose that the expert points out the relevant features of the American Robin to the novice: in addition to the orange breast and greyish back, the bird's head is slightly darker, there are rings of white spots around its eyes, and there is a white patch on the lower belly. Presumably, the novice will see all of this when it is pointed out, and on the basis of seeing these features and background information about the markings of American Robins will agree that the bird is an American Robin. But the agreement is an agreement in *judgment*. While the novice's perceptual experience changes due to a change in the pattern of attention to the bird's features, this does not constitute a change in expertise-specific content. The novice's experience does not come to represent the bird as an American Robin. Over time with repeated experience it might, but that would be because the novice becomes an expert.

The three phenomena that I have highlighted are connected. Expertly guided intuitions will count as improved intuitions relative to the novice baseline, and improved intuitions will typically be hard-won intuitions, at least initially. I now consider what the attempt to accommodate these three phenomena might tell us about the nature of intuition.

## 3. Perception, Intuition, and Reasoning

In his Nobel Prize lecture, Daniel Kahneman [2003: 698] helpfully portrays his own ideas about intuition in Figure 1.

Kahneman's figure portrays both the familiar structure of similarity and difference relations that intuitions bear to sensory perceptions and explicit reasoning, as well as a distinctive strategy for explaining these relations.

Kahneman subscribes to a dual process theory of cognition, and identifies intuitions with impressions that result from System 1 as opposed to System 2 cognitive processes. An example of a System 1 cognitive process is heuristic attribute substitution. This occurs when you form an impression about one attribute—the target attribute—by

[1] I am assuming that perceptual expertise is a capacity that manifests itself in perceptual experiences with expertise-specific representational contents. This is not something that I can defend here.

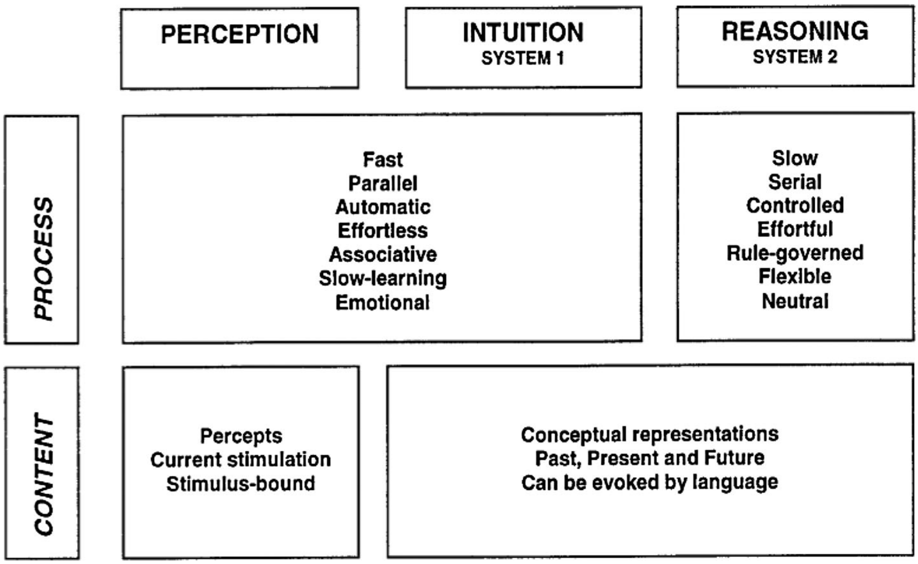



**Figure 1.** Kahneman's Dual Process View: Perception, Intuition, and Reasoning.

substituting the result of evaluating a more readily accessible attribute—the heuristic attribute. For example, you might form an impression about the probability of something by evaluating its representativeness: something evaluated as being highly representative will, via substitution, give the impression of being highly probable. Probability is the target attribute. Representativeness is the heuristic attribute. See Kahneman and Frederick [2002] and Kahneman [2011] for more details about the mechanisms of attribute substitution and more examples of target/heuristic attribute pairs.

The dual process conception of intuition explains, in the following way, the similarities and differences that intuitions bear to sensory perception and explicit reasoning. Sensory perception and intuition both result from reflex-like processes that tend to be fast, automatic, effortless, and opaque to introspection. Let us suppose, as is plausible, that sensory perception results from reflex-like processes in virtue of resulting from modular processing of sensory input.[2] Intuitions result from reflex-like processes in virtue of resulting from one or another System 1 cognitive processes, such as heuristic attribute substitution. Despite this difference, both sensory perceptions and intuitions contrast with explicit reasoning, which tends to be slow, controlled, effortful, and transparent to introspection. Processes that differ in their underlying mechanisms exhibit similar reflex-like characteristics at a higher level of abstraction. But intuition is different from sensory perception, in that its subject matter extends more widely

[2] My understanding of modularity derives from Fodor [1983]. The most fundamental characteristics of modular psychological processes, so understood, are two restrictions on the flow of information (cf. Scholl and Leslie [1999]). First, there is encapsulation: modular processes operate on proprietary input states and module-specific background information to the exclusion of other, even relevant, information. Second, there is opacity: only their output representations, not the representations formed and operated on during intermediate stages of processing, are accessible, or accessible without cost, to central cognitive operations such as memory storage, cognitively driven attention, and reasoning. Modular processing of sensory input is modular processing whose proprietary inputs are sensory registrations of stimuli, such as retinal registrations of incoming light. I refer to the reader to Deroy [2015] for a fuller up-to-date survey of the topic.

than what is currently given in stimulus-bound percepts. The subject matter of intuition is as wide as that of cognition, for, unlike sensory perception, intuition is the result of cognition, albeit System 1 cognition.

I do not accept the dual process conception of intuition. Figure 2 is structurally similar to Kahneman's figure, but different in the portrayed explanation of that structure.

On Kahneman's view, sensory perceptions and intuitions result from similar processes, but have different contents. My figure portrays sensory perceptions and intuitions as having similar phenomenology, but as resulting from different processes, and it doesn't portray anything about their contents. I will work backwards through these differences.

The first, and easiest to account for, is the failure to portray anything about the contents of sensory perceptions and intuitions. This is because I think that the facts about content that both Kahneman and I accept are explained by what the figure does portray. Sensory perceptions have the kind of content that results from modular processing of sensory input. This is the kind of content that elicits Kahneman's descriptors 'percept', 'current stimulation', and 'stimulus-bound'. Intuitions have the kind of content that results from the full range of central cognitive processes. This is the kind of content that elicits Kahneman's descriptors 'conceptual representations', 'past, present, future', and 'can be evoked by language'. It is, on its face, plausible that the kind of content that might result from central cognition is wider than the kind of content that might result from modular processing of sensory input.

The second difference is that, while Kahneman takes resulting-from-similar-processes to be the key similarity between sensory perception and intuition, I take resulting-from-different-processes to be the key difference between them. This is not just

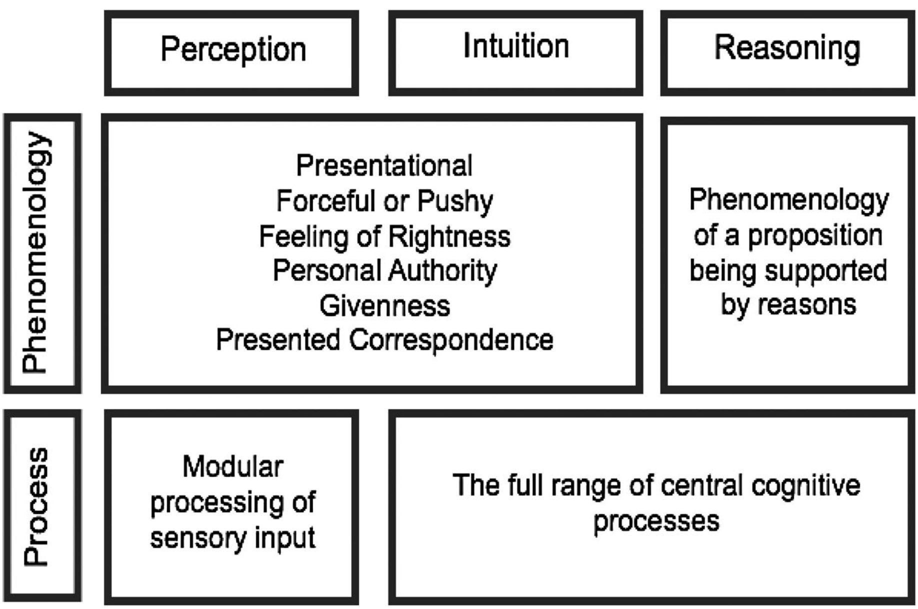



**Figure 2.** Phenomenological View of Perception, Intuition, and Reasoning.

because Kahneman is focusing on a level of analysis that abstracts from the differences between modular processing of sensory input and System 1 processing in favour of highlighting their common characteristic of being reflex-like. I reject the idea that intuitions must result from processes that are reflex-like (cf. Koksvik [2013]). Some intuitions result from processes that are fast, automatic, effortless, and opaque to introspection, but I do not think that all of them result from processes of this sort. As discussed in section 2, some intuitions are hard-won and take deliberate effort to have.

The third difference between Kahneman's figure and mine is that my figure portrays the key similarity between sensory perception and intuition as being phenomenological. I have included a number of descriptors inspired by the literature touching on this phenomenology: 'presentational' [Chudnoff 2013; Bengson 2015]; 'forceful or pushy' [Gödel 1953; Koksvik 2011]; 'feeling of rightness' [Thompson 2011]; 'personal authority' [Descartes 1984: 331; Mercier and Sperber 2017]; 'givenness' [Husserl 1982; Berghofer forthcoming]; 'presented correspondence' [Russell 1992; Bonjour 2005; Johnston 2006; Chudnoff 2013].[3]

To appreciate the justice of these descriptors, consider the following. Compare two ways of forming the impression that there is someone else with you on the beach. First, you draw this as a conclusion from your observation of fresh footprints in the sand. Second, you directly observe someone walking nearby. In the first case, the proposition is felt to be supported by reasons. This is reasoning, albeit reasoning from something that is perceived. In the second case, you do not need to attend to reasons supporting the proposition. It is a case of sensory perception. It is natural to describe your perceptual experience as presenting you with a state of affairs corresponding to the proposition that there is someone else on the beach with you. It thereby counts as presenting and not merely representing the proposition as true, forcing or pushing the proposition on you, endowing it with a feeling of rightness, giving you personal authority in asserting the proposition, or rendering the proposition given to you.

Similar contrast cases for intuition are bound to be more controversial, because the nature of intuition is controversial, and even among those who subscribe to a phenomenological conception of intuition there is disagreement about how to characterize its phenomenology. That there are such contrasts, however, is a commonplace in the mathematical literature (cf. the discussion of the principle of monotone sequences in Courant et al. [1996: 295–6]). For example, compare two ways of forming the impression that every non-empty set of natural numbers has a least element—for instance, the least element of {3, 19, 2, 6} is 2. First, you draw this as a conclusion from an inductive proof by strong mathematical induction.[4] Second, you think about how any such set is a selection from the number line; no matter how the selection is

[3] As the references to the literature show, the view that intuitions have some phenomenology is fairly widespread. But it is not universally accepted: Cappelen [2012], for example, expresses scepticism.

[4] The strong mathematical induction principle says: if whenever all the predecessors of a natural number n have a property P the number n also has the property P, then every natural number has the property P. We will let S be a subset of natural numbers without a least element and show that it must be empty. Let P be the property of not being a member of S. Now suppose all the predecessors of a natural number n have the property P. That is, all the predecessors of n are not members of S. Then n is also not a member of S because we have supposed that S has no least element and if n were in S then n would be its least element. So the property P meets the condition for applying strong induction. It follows that every natural number has the property P. That is, every natural number is not a member of S. But that is just to say S is empty. So, every non-empty subset of natural numbers must have a least element.

made, one of the elements selected has to be leftmost. In the first case, the proposition is felt to be supported by reasons. In the second case, the proposition is felt to be true on its own merits, not because of how it is supported by reasons. I do not think that the statements about selecting from the number line that I made in motivating the proposition just support it by giving independent reasons for believing it. They might do that, but they also do something else: namely, they enable an intuition of the proposition. For me and a significant number of others, such intuitions evoke some of the same phenomenological descriptors as sensory perception does.

As noted, there is a diversity of opinion among proponents of the phenomenological conception of intuition about how best to characterize the phenomenology that sensory perception and intuition are supposed to have in common. I see this diversity as analogous to the diversity of opinion that exists among proponents of the dual process conception of intuition about how best to characterize the processing features that sensory perception and intuition are supposed to have in common. In neither case do I count the mere diversity of opinion against the favoured conception of intuition. There are, however, two main disadvantages to the dual process conception of intuition.

As a preliminary point, note that the dual process conception of intuition does not conceive of intuition as a natural psychological kind. Resulting from a reflex-like process is not a natural psychological kind. Suppose that resulting from modular processing of sensory input is a natural psychological kind, and that resulting from System 1 processes is a natural psychological kind. Then resulting from a reflex-like process is a conjunction of two natural psychological kinds, not itself a natural psychological kind. Now consider the assumption that resulting from System 1 processes is a natural psychological kind. This is suspect. The kind is whatever is picked out by 'resulting from reflex-like processes but not resulting from modular processing of sensory input'. But there is little reason to think that this description will pick out a natural psychological kind. Rather, it probably picks out an assortment of psychological kinds, one example of which is resulting-from-heuristic-attribute-substitution, that have little but superficial properties in common. Glockner and Witteman [2010] and Thompson [2014] review a heterogenous assortment of psychological processes that fall under the description 'reflex-like processes but not modular processing of sensory input', and Mercier and Sperber [2017] press a related challenge to dual process conceptions of intuition. Evans and Stanovich [2013] offer a potential solution: the relevant natural psychological kind is that of resulting from autonomous processes. But this itself is a kind defined in negative terms.

The first main disadvantage of the dual process conception of intuition is that it fails to count both the reflex-like impressions on which Kahneman focuses and the hard-won impressions that Kline champions as intuitions. This would not count as a disadvantage if there were compelling reasons to think that one or both of these kinds of impression falls under a natural psychological kind that the other doesn't fall under. For example, if resulting from System 1 processes were a natural psychological kind, then perhaps it would make sense to isolate the reflex-like impressions on which Kahneman focuses, to call them intuitions, and to reserve another term for the hard-won impressions that Klein champions. But there are no compelling reasons to think that resulting from System 1 processes is a natural psychological kind. And there are reasons to group together, as intuitions, the reflex-like impressions on which Kahneman focuses and the hard-won impressions that Klein champions. Epistemic reflection suggests that the two impressions fall under a common epistemic kind. Each has the epistemic profile associated with intuition. This is the profile aptly characterized as seeing something for yourself. There is deferring to another's authoritative testimony. And there is following an argument where it leads. In neither case do you see something for yourself. Sensory perceptions and intuitions—including reflex-like and hard-won intuitions—make you see something for yourself. Mercier and Sperber [2017] call this personal authority, and it is precisely that feature of intuition that motivates Klein's injunction to make newly developed areas of mathematics intuitive. We have the proofs and can follow the arguments where they lead, but we want intuitions so that we can see for ourselves. That has a distinctive value, one that Mark Johnston [2006] calls 'better than mere knowledge'.

The second main disadvantage of the dual process conception of intuition is that it does not provide a setting in which to explain expertly guided intuitions. Recall the novice and the expert at solving physics problems. The expert has developed suitable intellectual reflexes that result in two presented inclined-plane problems seeming significantly different. The novice has not developed such intellectual reflexes. To the novice, the problems seem similar. Now suppose that the expert walks the novice through the relevant differences. It comes to seem to the novice, too, that the two inclined-plane problems are significantly different. But, on the dual process conception of intuition, this seeming would not count as an intuition because it is not the result of suitable intellectual reflexes. The novice would have to become an expert with such reflexes in order to have the intuition. This makes the case like the birdwatching case, which is unsurprising because perceptual experiences are results of reflex-like processing—namely, modular processing of sensory input. The phenomenological conception does not pose an analogous barrier to explaining expertly guided intuitions. It remains to show, however, just how they are possible. The phenomenological conception requires supplementation, to which I now turn.

## 4. Expert Intuition

Consider two people, one of whom knows how to get from A to B, and the other of whom does not. If the first leads the second, then the one who does not know manages to get from A to B even without knowing how to do it alone. There is nothing mysterious about that. Expertly guided intuition is analogous. Contemporary theories of expert intuition conceive of it as a capacity to get from A to B. And if someone with the capacity leads someone without the capacity, then the one without the capacity does manage to get from A to B even without the capacity to do it alone. I take up each point in turn.

When I say that contemporary theories of expert intuition conceive of it as a capacity to get from A to B, I have in mind two things. First, contemporary theories of expert intuition conceive of it as a capacity to solve problems [Kahneman and Klein 2009; Gobet and Chassy 2009]. Second, contemporary theories of problem-solving conceive of it as a process of representing and searching a problem space [Newell and Simon 1972; Davidson et al. 2003; Robertson 2016]. One and the same capacity—expert intuition—can be a capacity to form superior intuitions and a capacity to solve problems by representing and searching problem spaces. The capacity to solve physics problems by representing and searching problem spaces, for example, can also be a capacity to form impressions, and so can manifest itself in intuitions with physics specific content. It is important, however, that search strategies be broadly construed to include both constructing a representation of a problem space and searching it. Some expert intuitions are formed without searching, in the narrow sense. This happens when experts at solving physics problems form impressions of similarity and difference between problems without solving them [Chi et al. 1981]. I return below to the distinction between search strategies, construed broadly and construed narrowly.

The conception of problem-solving as a process of searching a problem space largely derives from Newell and Simon's [1972] pioneering work on the subject. It will be useful to sketch some key ideas. Consider an example problem—the three-disc Tower of Hanoi problem, pictured below. There are three pegs. On the first peg, there are three differently sized discs, arranged largest to smallest, bottom to top. The problem is to move this configuration onto the third peg while obeying three constraints: you can only move a single disc at a time; you can only move a disc if there is no disc above it; you cannot place a larger disc on top of a smaller disc. With reference to this problem, we can introduce four key concepts.

The *initial state* is the state in which you begin, in which the three discs are on the first peg. The *goal state* is the state at which it is your task to arrive, in which the three discs are on the third peg. To be confronted with a problem is simply to be in an initial state that is distinct from your goal state. There are *operators* that consist of actions along with conditions enabling them and constraints on their execution, such as the action of moving a disc, the condition that there is no disc above it, and the constraint that it not be placed on top of a smaller disc. The initial state and the operators generate a *problem space*. This is a network of states, including the initial state and the states that are accessible from it by applying the operators. The problem is soluble if the goal state is among these states. The three-disc Tower of Hanoi problem is soluble, and the problem space is not that large (see Figure 3).

Solving a problem is finding a path through the problem space from the initial state to the goal state. For most problems, the problem space is too large to do an exhaustive search for such a path. According to Newell and Simon, what human problem-solvers do in the face of such a problem space is to apply heuristics, and their program for a psychology of thinking and problem-solving is to study this process of heuristic search.

The foregoing begins to put some flesh on the idea that contemporary theories of expert intuition conceive of it as a capacity to get from A to B. Now let us consider the idea that, if someone with the capacity leads someone without the capacity, the one without the capacity does manage to get from A to B even without the capacity to do it alone. In a review of research on expertise, conducted over the several decades prior to its 2011 publication, Michelene Chi [2011] identifies three phases corresponding to three prevailing attitudes toward the relationship between expertise and problem-solving ability.

The first attitude takes expertise to improve search strategies, narrowly construed. By a search strategy, narrowly construed, I mean a strategy for searching a represented problem space. So, it does not include the construction of the representation. For example, there is a difference between forward search and backward search. In forward search, you apply operators to a given state to generate a new state. In backward search, you look for operators whose application could produce a given state. An early result on expertise was that experts solve problems by using forward search strategies, while novices solve the same problems by using backward search strategies [Larkin et al. 1980].

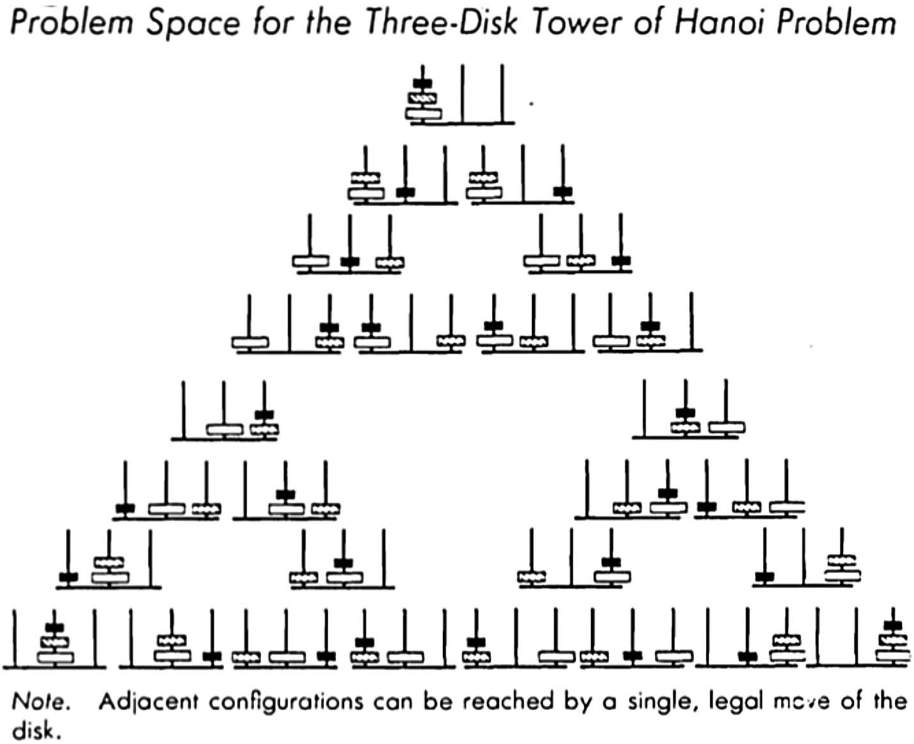



**Figure 3.** Reproduced from Anderson [1993: 35].

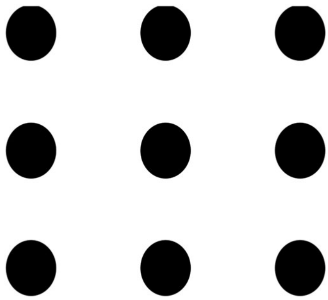
The second attitude takes expertise to improve problem-solving by providing a vast store of well-indexed knowledge about the domain of expertise. This attitude derives from de Groot's [1965] early work on chess expertise. De Groot found that experts do not differ significantly in their search strategies, narrowly construed. For example, in choosing the next move in a game, experts and novices search a similar number of options (width of search) and consider a similar number of consequences of each of those options (depth of search). What differentiates experts is that the options they consider are better. This suggests that expertise improves problem-solving ability largely through recognition of similarity to already known problem situations (cf. Klein [2017])

The third attitude takes expertise to improve the construction of problem spaces associated with given problems—that is, to improve the very representations of problem spaces. The classic result, already mentioned, comes from Chi's own research [Chi et al. 1981] showing that experts represent physics problems in terms of the underlying physical principles governing their solution, but that novices represent the same problems in terms of the problems' superficial characteristics. The importance of problem representation comes out very clearly when one considers the role of restructuring in insightful problem-solving.

Restructuring 'is a change in the problem-solver's mental representation of the problem' [Ohlsson 1984: 119]. As already discussed, when one confronts a problem one's representation of it includes information about an initial state, a goal state or some goal states if there is more than one possible solution, some operators, and some portion of the problem space. What changes in such a representation when restructuring occurs? The three-disc Tower of Hanoi problem is easily solved and requires no restructuring. Contrast the following so-called insight problems (the solutions of which appear at the end of the paper).

**Matchstick Problems**. 'I = II + II' and 'XI = III + III' are false arithmetical statements using Roman numerals. Change them into true arithmetical statements by moving only one 'matchstick' in each [Knoblich et al. 1999].

**Nine-Dot Problem**: Without lifting your pencil, connect the following nine dots by using four straight lines [Maier 1930].



These are more difficult than the three-disc Tower of Hanoi problem. The difficulty is not, as it is in chess, a matter of their problem spaces being too unwieldy. Rather, the difficulty is that the typical initial representation of the problems makes them unsolvable: the initial state and the operators that one typically takes to be available do not generate a problem space that includes the goal state. Hence, these problems require a change in how one represents them—that is, restructuring.

Typical problem-solvers approach matchstick problems by assuming that one can only move the matchsticks composing numerals and that numerals cannot be broken up. The constraints are imported from more familiar contexts in which one confronts arithmetical statements: there, the arithmetical operations are fixed, and writing down a true statement requires putting in the right numerals; and arithmetical reasoning works at a level where the numerals are units that cannot be further decomposed into independently manipulable parts. To solve the matchstick problems, one must have the insight that these constraints do not apply, and thus represent new operators corresponding to ways of moving the matchsticks that one did not initially consider (cf. Knoblich et al. [1999]). Typical problem-solvers approach the nine-dot problem by assuming that the lines must stay within the square determined by the outer perimeter of dots. To solve the problem, one must have the insight that this constraint does not apply, and thus represent new operators corresponding to ways of drawing lines that one did not initially consider (cf. Öllinger et al. [2014]). The foregoing answers our question about what changes in restructuring: the representational changes that constitute restructuring are changes in which operators that one represents are available.

Restructuring can be considered as the fast acquisition of expertise in a very small domain. It makes vivid how what initially seems impossible can come to seem very easy. And it illustrates just how significant a component of expert intuition might consist of improved problem representations, thereby supporting Chi's emphasis on the emergence and development of research on this aspect of expertise [Nokes and Chi 2010; Chi 2011].

The three attitudes toward the relationship between expertise and problem-solving ability are compatible. Expertise might improve problem-solving by improving search strategies, narrowly construed, and by providing a vast store of well-indexed knowledge, and by resulting in the representation of superior problem spaces. The point that I want to make here is that all three are accessible with guidance to novices. Novices will not represent superior problem spaces, recognize the kind of problem that they face as one with a proven strategy, or pursue superior strategies for finding new solutions on their own. But nothing bars them from doing all three when told what to do. None of these achievements requires representational or computational capacities that we cannot assume novices already have. A novice can think about the conservation of energy, can grasp the similarity between two chess board configurations, and can see the consequences of applying an operator to a state in a problem space when told to consider that particular operator, which they might not have thought to consider on their own. We cannot say something analogous when considering novice and expert perceivers. An expert birdwatcher has developed perceptual representational capacities over time that we cannot assume a novice birdwatcher has. Both can think about American Robins. But only the expert's perceptual system has developed what Pylyshyn [1999] calls a 'compiled transducer' that computes a perceptual representation of American Robbins on the basis of the appropriate stimulus array.

## 5. Improved Intuitions

In this section, I illustrate a general way of thinking about improved intuition, by reflecting on the dispute between Hahn and Mandelbrot that was mentioned in section 2. Hahn's view is that initially counterintuitive truths show intuition to be unreliable. Mandelbrot's view is that initially counterintuitive truths can become intuitive with the right training.

To appreciate the force of Mandelbrot's view, let us return to the possibility of a curve touching every point on the plane. Don't try to imagine visually tracking a moving point tracing out such a curve. Instead, think of matters in this way. Curves are continuous mappings of the unit interval. Such a mapping can be defined as the

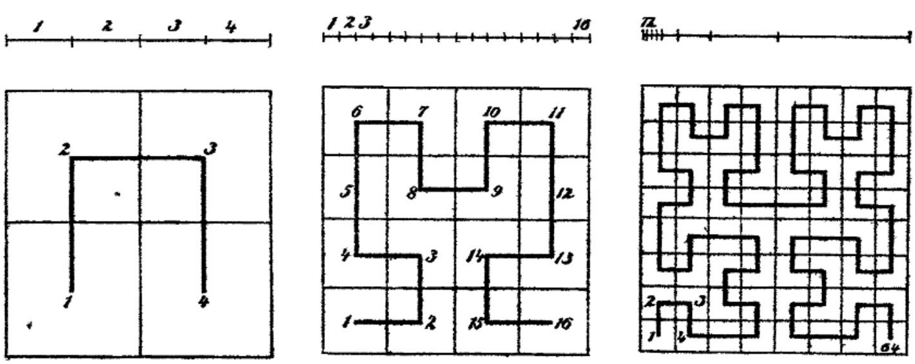


Figure 4. Hilbert Curve.

limit towards which an infinite sequence of mappings tend. Figure 4 shows the first few members of such a sequence, first presented by David Hilbert one year after Peano's discovery [Hilbert 1891].

You could visually track a point by moving along some of the paths in this sequence. But you couldn't visually track a point by moving along the limit path of this sequence. Still, it is a curve. And intuitively it does run through every point on the plane. Just think of the sequence of quadrants used to define the sequence of paths. At the limit, for each point on the plane there is a quadrant enclosing it. So, at the limit, for each point on the plane there is a point on the curve running through it. As Morris Kline writes, 'Since the subsquares and the parts of the unit segment both contract to a point as the subdivision continues, we can see intuitively that each point on the unit segment maps into one point on the square' [1972: 1019].

The irony is that Hahn himself provides an intuitive presentation of Hilbert's curve. His paper was written for a general audience, and so *all* of his examples are ones that he can get his reader to see with some elementary exposition. He does not just present the propositions and ask readers to take them on his word, or on the authority of the research publications in which they are presented. Nor does Hahn proceed to prove any of the counterintuitive claims that he discusses. In every case, he writes so as to get his reader to intuit for himself or herself the truth of the initially counterintuitive claim. What this suggests to me is that Hahn's view that the states of mind in which he puts his reader are not—as Mandelbrot and I would claim—intuitions derived from a theoretical view about intuition. And, indeed, Hahn does have a background theory of intuition, informing his view that mathematical results from the late-nineteenth century generated a crisis in intuition. The background theory is one that draws on Kant's theory of intuition.

Kant's theory of intuition is embedded in his overall transcendental philosophy [1998], which Hahn likely did not accept. The aspect of Kant's theory of intuition that Hahn did accept—as did many others writing about intuition at the time—is the idea that our capacity for intuitive representation is tied to our capacity for sensory representation. This notion of being 'tied to' is vague. When Kant spelled it out, he drew on elements of his overall transcendental philosophy. Hahn does not provide an alternative way to spell out what the tie between intuitive representation and sensory representation is. But it is clear that he thinks that there is some such tie which implies that, if we cannot visualize Hilbert's curve, then we cannot intuit its possibility. It is this conditional that does the work. So, we can leave open the details of the background theory. They must be such that there is some tie, between intuitive representation and sensory representation, that implies that if we cannot visualise Hilbert's curve then we cannot intuit its possibility. So, Hahn and Mandelbrot would agree that we cannot visualize Hilbert's curve. Hahn draws as a conclusion that we cannot intuit its possibility. Mandelbrot does not. Mandelbrot does not provide an alternative theory of intuition to justify his divergence from Hahn. But this hardly puts him in a worse position than Hahn.

What I want to suggest is that I have given a theory of intuition that supports Mandelbrot's view and undermines Hahn's view. The foregoing discussion of intuition does not support thinking that our capacity for intuitive representation is tied to our capacity for sensory representation. These are very different capacities. Our capacity for intuitive representation is cognitive and freely informed by central cognition. Our capacity for sensory representation is limited by modular processing of sensory input, and can only be influenced by central cognition in limited ways.

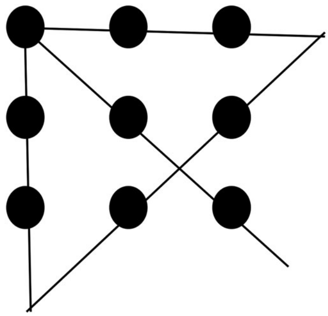
The discussion of expert intuition also provides a setting in which to think about the mechanisms that might enable an initially counterintuitive truth to become intuitive. First, it shows that the relevant intuition is not one that only a Hilbert or a Mandelbrot could have. Mathematical novices can have it, too. Second, it suggests ways in which a novice might come to have such an intuition. The intuition, recall, is that there is a possible curve that touches every point on a plane. One needs to search the space of possible curves to find one. Novice intuition fails because it considers only visualizable curves. This is like failing at a matchstick problem because you consider only certain manipulations of the matchsticks. The novice needs some insights to restructure his or her problem space. The operators that such a restructuring alters are not physical manipulations. They are mental manipulations of abstract representations. In particular, two insights are required. The constraint that the curve be visualizable must be dropped. And the option of defining the curve as the limit of an infinite sequence of other curves must be adopted. These are sophisticated moves. But they can be grasped with some exposition, such as Hahn's own. Once these mental manipulations required for bringing Hilbert's construction to mind at all become available, so too, it seems to me, does an intuition representing the possibility of a curve that touches every point on a plane.

There is an objection, likely to occur at this point, that is so natural that, although I believe that I have already dealt with it in substance, I want to mention it and reiterate the main point that I have in response to it. The objection is that, because grasping the possibility of a plane-filling curve takes sophisticated reasoning, really it should count as a kind of explicit reasoning, not as an intuition. The main point that I have in response to this objection is that there is no incompatibility between a mental state being both an intuition and the result of reasoning (cf. Koksvik [2013]). On the phenomenological conception of intuition that I have defended, a mental state counts as an intuition if it has the right phenomenology, and it doesn't matter whether it takes some kind of reasoning to get into the mental state. If a mental state doesn't have the phenomenology of presenting a proposition as true on its own merits, and instead is one that represents a proposition as supported by reasons, then it does not count as an intuition. This would be the situation of those who believe there are curves that touch every point on the plane because of a proof from a formula defining such a curve. But the nice thing about Hilbert's curve—as opposed to Peano's original curve—is that, in addition to such a proof, the possibility of the curve is evident on the basis of what is naturally described as an intuitive grasp of its construction (cf. Kline [1972: 1019], quoted above). I can add an analogy to this general point. Compare perception. Getting into a position to perceive that there is gold in the treasure chest might require a fair bit of reasoning: you have to navigate from your starting position by using the treasure map. Once you are in the right location, however, perception shows you that there is gold in the treasure chest. The connections between intuition, problem-solving, and search suggest that this can be read as a genuine structural analogy.[5]

[5] I presented earlier versions of this material at the Universities of Antioquia, Bergen, Connecticut, Fribourg, Graz, Hong Kong, and Liège, as well EAFIT University, Humboldt University, the Jean Nicod Institute, Regensburg University, and the Sorbonne University. I am grateful for the illuminating discussions I had at these different institutions. I would also like to thank Stephen Hetherington, Jennifer Nado, and two anonymous referees for this journal for very helpful written comments on earlier drafts.

## Solutions to Insight Problems

I = III - II
VI = III + III



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